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UNBOUNDED OPERATORS COMMUTING WITH RESTRICTED BACKWARD SHIFTS

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Abstract. The closed densely defined operators on a proper invariant subspace of the backward shift that commute with the restricted backward shift are shown to be coanalytic Toeplitz operators induced by functions in the Nevanlinna class. The result can be interpreted as a kind of commutant lifting theorem for unbounded operators. It extends, and in a certain sense completes, earlier work of Daniel Suárez.

1. Introduction

The venue for the prototypal version of the commutant lifting theorem is the classical Hardy space H^2 of the unit disk $\mathbb{D}$. In this setting the commutant lifting theorem states that a bounded operator on a coinvariant subspace of H^2 that commutes with the restricted backward shift is the restriction of a coanalytic Toeplitz operator. Is there a generalization to unbounded operators?

The question was prompted by the paper [7] of Daniel Suárez. Let u be an inner funciton, but not a finite Blaschke product. (This notation will remain fixed.) The subspace $K_u^2 = H^2 \ominus uH^2$ is the general proper infinite-dimensional subspace of H^2 that is invariant under S^*, the backward shift operator, the adjoint of the unilateral shift operator S. The compression of S to K_u^2 will be denoted by S_u; its adjoint S_u^* is the restriction of S^* to K_u^2.

In [7], Suárez characterized the closed densely defined operators on K_u^2 that commute with S_u^*, albeit rather indirectly. Among the operators in question are the restrictions to K_u^2 of the coanalytic Toeplitz operators with symbols in $\overline{H^2}$. Suárez showed that, at least for some u, the preceding operators do not exhaust the class.

Suárez in [7] does not contemplate Toeplitz operators whose symbols are not square integrable, although such operators are implicit in his work (and have arisen elsewhere). Specifically, each function in the Smirnov class induces an analytic and a coanalytic Toeplitz operator, and the restriction to K_u^2 of the coanalytic one is closed, densely defined, and commutes with S_u^*. But, as will be seen below, even these do not exhaust the class of closed densely defined operators that commute with S_u^*.

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Nevertheless, the question raised at the beginning of this introduction does have a positive answer, which is the main result in this paper. The answer involves an expansion of the class of coanalytic Toeplitz operators to include those induced by functions in the Nevanlinna class. These operators provide all of the closed densely defined operators that commute with S_u^*. Expressed in other terms, the main result says that a closed densely defined operator that commutes with S_u^* is a Nevanlinna function of S_u^*.

It is natural (perhaps obligatory) to wonder whether the commutant lifting theorem proven here for unbounded operators on the spaces K_u^2 is but a special case of a general theorem in the theory of operator dilations.

Sections 2 and 3 review, respectively, the needed facts about the spaces K_u^2 and about the Smirnov and Nevanlinna classes. Section 4 does the same for the analytic and coanalytic Toeplitz operators induced by functions in the Smirnov class. In Section 5 the coanalytic Toeplitz operators induced by functions in the Nevanlinna class are constructed and their basic properties derived.

Section 6 concerns the special case in which u is a Blaschke product with simple zeros, a case which can be dealt with independently of Suárez's analysis. The main result will be proved for this case, and the necessity of using Nevanlinna functions (not just Smirnov functions) as inducers of Toeplitz operators will be established. In fact, it will be shown that it does not even suffice to use holomorphic Nevanlinna functions as inducers.

In Section 7 Suárez's basic results will be rederived, in the form needed for the proof of the main result, which is accomplished in Section 8 through an extension of Suárez's analysis. In Section 9 the necessity of using Nevanlinna functions (not just holomorphic Nevanlinna functions) as inducers is proved for general u.

Notations

- The normalized Lebesgue measure of a measurable subset E of $\partial\mathbb{D}$ will be denoted by $|E|$.
- The Poisson integral of a finite Borel measure μ on $\partial\mathbb{D}$ will be denoted by $P\mu$.
- $K_u^\infty = K_u^2 \cap H^\infty$.
- The domain and graph of an operator T will be denoted by $\mathscr{D}(T)$ and $\mathscr{G}(T)$, respectively.
- $H^2 \oplus H^2$ will be denoted by H_2^2, and interpreted as the space of 2-by-1 column vectors with entries in H^2.

2. The space K_u^2

The spaces K_u^2 have a rich structure which has been under investigation for 40+ years; see for example the book [3] of N. K. Nikol'skiĭ. Only a few basic properties of K_u^2 are needed here; these are discussed in greater detail in [4].

The space K_u^2 is a reproducing kernel Hilbert space. The kernel function k_λ^u for the evaluation functional on K_u^2 at the point λ of $\mathbb{D}$ is given by $k_\lambda^u(z) = (1 -$

$\overline{u(\lambda)}u(z))/(1-\overline{\lambda}z)$.

The space K_u^2 carries a natural symmetry C, an antiunitary involution given by $(Cf)(z) = u(z)\overline{z}\overline{f(z)}$ $(f \in K_u^2,\ z \in \partial\mathbb{D})$. When convenient, Cf will be denoted alternatively by $\widetilde{f}$. In particular, $(\widetilde{k}_\lambda^u)(z) = \frac{u(z)-u(\lambda)}{z-\lambda}$.

For ψ in H^∞, the compression of the Toeplitz operator T_ψ to K_u^2 will be denoted by A_ψ. Its adjoint A_ψ^* is the restriction of $T_{\overline{\psi}}$ to K_u^2, and is denoted by $A_{\overline{\psi}}$. The operators A_ψ and $A_{\overline{\psi}}$ are not only adjoints of each other, they are C-transforms of each other: $A_{\overline{\psi}} = CA_\psi C$.

3. Nevanlinna and Smirnov Classes

The Nevanlinna class N (in $\mathbb{D}$) consists of all quotients ψ/χ where ψ and χ are in H^∞ and χ is not the zero function. The Smirnov class N^+ consists of such quotients in which χ is an outer function. The functions in N^+ are holomorphic in $\mathbb{D}$; those in N are meromorphic. The pole set of a function in N, if nonempty, constitutes a Blaschke sequence.

It is noted in [5] (Proposition 3.1) that each nonzero function φ in N^+ has what is called there a canonical representation, a unique expression of the form $\varphi = b/a$, where a and b are in H^∞, a is an outer function, $a(0) > 0$, and $|a|^2 + |b|^2 = 1$ almost everywhere on $\partial\mathbb{D}$. The proof of this is simple. Roughly, if $\varphi = \psi/\chi$ with ψ, χ as in the definition of N^+, one defines a to be the outer function that is positive at the origin and whose modulus on $\partial\mathbb{D}$ equals

$$\sqrt{\frac{|\chi|^2}{|\psi|^2 + |\chi|^2}},$$

after checking that the last function is log-integrable. The rest follows automatically.

It is a well-known consequence of Beurling's theorem that any family of inner functions has a greatest common divisor, an inner function that divides every function in the family and is divisible by every inner function with that property. For φ_1 and φ_2 in N^+, we let $\text{g.c.i.d.}(\varphi_1, \varphi_2)$ denote the normalized greatest common divisor of the inner factors of φ_1 and φ_2, the unique greatest common divisor whose first nonvanishing Taylor coefficient at the origin is positive.

With this said, the notion of canonical representation can be extended to the class N: each function φ in N can be written uniquely as $\varphi = b/va$, where b/a is a function in N^+, represented canonically, v is an inner function, and $\text{g.c.i.d.}(v, b) = 1$.

4. Toeplitz Operators with Symbols in N^+ and $\overline{N^+}$

The operators in question appear implicitly in the paper [7] of Suárez, mentioned earlier, and explicitly in the papers [1] of Henry Helson and [6] of Steven Seubert. Their basic properties were subsequently developed by the author in [5], and will be summarized in this section.

Throughout the section, let φ be a nonzero function in N^+, with canonical representation $\varphi = b/a$. The operator T_φ, the Toeplitz operator with symbol φ, is by definition the operator of multiplication by φ on the domain $\mathscr{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}$, which equals aH^2. The operator T_φ is closed and densely defined, and so has a closed and densely defined adjoint T_φ^*. The domain $\mathscr{D}(T_\varphi^*)$ is the de Branges–Rovnyak space $\mathscr{H}(b)$, and the graph $\mathscr{G}(T_\varphi^*)$ consists of all vectors $f \oplus g$ in H_2^2 satisfying $T_{\overline{b}}f = T_{\overline{a}}g$. The operator $T_{\overline{\varphi}}$, the Toeplitz operator with symbol $\overline{\varphi}$, is defined to be T_φ^*. (A rationale for the definition is presented in [5].)

The operator $T_{\overline{\varphi}}$ induces an operator on K_u^2, denoted by $A_{\overline{\varphi}}$, and defined by

$$A_{\overline{\varphi}} = T_{\overline{\varphi}} \mid \mathscr{D}(T_{\overline{\varphi}}) \cap K_u^2.$$

The operator $A_{\overline{\varphi}}$ is closed and densely defined, and its adjoint, denoted by A_φ, coincides with its transform under the conjugation C : $A_\varphi = CA_{\overline{\varphi}}C$, with domain $\mathscr{D}(A_\varphi) = C\mathscr{D}(A_{\overline{\varphi}})$. The definitions of $A_{\overline{\varphi}}$ and A_φ reduce to the usual ones in case φ is bounded: if φ is bounded then $A_{\overline{\varphi}}$ is the restriction $T_{\overline{\varphi}}$ to K_u^2, an A_φ is the compression of T_φ to K_u^2.

In the next section, the results summarized in the last paragraph will be extended to more general symbols.

5. Local Smirnov Classes

Suppose the function φ is in N but not in N^+; let $\varphi = b/va$ be its canonical representation. There is a natural way to define an associated analytic Toeplitz operator T_φ, namely, as the operator of multiplication by φ with domain $\mathscr{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}$. The operator is closed but not densely defined; in fact $\mathscr{D}(T_\varphi) = vaH^2$, which is dense in vH^2 but is not dense in H^2.

On the other hand, there seems to be no sensible way to define a corresponding coanalytic Toeplitz operator $T_{\overline{\varphi}}$. But if $\text{g.c.i.d.}(u,v) = 1$, such an operator can be defined on K_u^2, as will be shown in this section. These operators generalize the operators $A_{\overline{\varphi}}$ with φ in N^+, studied in [5], and the reasoning going into their construction is an adaptation of that used in [5].

The function φ in N will be said to belong to N_u^+ if it is the zero function, or if it has the form $\varphi = \psi/\chi$ where ψ and χ are in H^∞, $\chi \neq 0$, and $\text{g.c.i.d.}(u,\chi) = 1$. The space N_u^+ is closed under addition and multiplication. If $\varphi \neq 0$ and $\varphi = b/va$ in canonical form, then φ is in N_u^+ if and only if $\text{g.c.i.d.}(u,v) = 1$.

For the remainder of this section, we let $\varphi = b/va$ be a canonically represented nonzero function in N_u^+. The associated operator $A_{\overline{\varphi}}$ on K_u^2 alluded to in the preceding paragraph will be constructed in stages.

LEMMA 5.1. *$A_{\overline{v}\overline{a}}K_u^2$ is dense in K_u^2.*

Proof. The adjoint of $A_{\overline{v}\overline{a}}$ is A_{va}. Because $\text{g.c.i.d.}(u,va) = 1$, we have $uH^2 \cap vaH^2 = \{0\}$, so the operator A_{va} has a trivial kernel. Therefore its adjoint $A_{\overline{v}\overline{a}}$ has a dense range. □

We let $A_{\overline{\varphi}}^0$ be the operator with domain $A_{\overline{v}\overline{a}}K_u^2$ given by $A_{\overline{\varphi}}^0(A_{\overline{v}\overline{a}}h) = A_{\overline{b}}h$ $(h \in K_u^2)$. We let A_φ^0 be the C-transform of $A_{\overline{\varphi}}^0$: $A_\varphi^0 = CA_{\overline{\varphi}}^0C$ with domain $\mathscr{D}(A_\varphi^0) = C\mathscr{D}(A_{\overline{\varphi}}^0)$.

Define the operator $W : K_u^2 \oplus K_u^2 \to K_u^2 \oplus K_u^2$ by $W(f \oplus g) = g \oplus -f$. Thus, if A is a densely defined operator on K_u^2, then $\mathscr{G}(A^*)$ is the orthogonal complement of $W\mathscr{G}(A)$.

LEMMA 5.2. $A_\varphi^0 \subset (A_{\overline{\varphi}}^0)^*$ *and* $A_{\overline{\varphi}}^0 \subset (A_\varphi^0)^*$.

Proof. The two inclusions are C-transforms of each other, so it will suffice to prove the first one. This amounts to showing that $\mathscr{G}(A_\varphi^0)$ and $W\mathscr{G}(A_{\overline{\varphi}}^0)$ are orthogonal. For h_1 and h_2 in K_u^2, the vector $F_1 = A_{\overline{v}\,\overline{a}}h_1 \oplus A_{\overline{b}}h_1$ is a typical vector in $\mathscr{G}(A_{\overline{\varphi}}^0)$, and the vector $F_2 = CA_{\overline{v}\,\overline{a}}Ch_2 \oplus CA_{\overline{b}}Ch_2$ is a typical vector in $\mathscr{G}(A_\varphi^0)$. Since $CA_{\overline{v}\,\overline{a}}C = A_{va}$ and $CA_{\overline{b}}C = A_b$, we have

$$\begin{aligned} \langle F_1, WF_2 \rangle &= \langle A_{\overline{v}\,\overline{a}}h_1, A_b h_2 \rangle - \langle A_{\overline{b}}h_1, A_{va}h_2 \rangle \\ &= \langle A_{\overline{b}}A_{\overline{v}\,\overline{a}}h_1, h_2 \rangle - \langle A_{\overline{v}\,\overline{a}}A_{\overline{b}}h_1, h_2 \rangle \\ &= \langle A_{\overline{b}\,\overline{v}\,\overline{a}}h_1, h_2 \rangle - \langle A_{\overline{v}\,\overline{a}\,\overline{b}}h_1, h_2 \rangle = 0, \end{aligned}$$

the desired conclusion. □

LEMMA 5.3. (i) *The graph* $\mathscr{G}((A_\varphi^0)^*)$ *consists of all vectors* $f \oplus g$ *in* $K_u^2 \oplus K_u^2$ *such that* $A_{\overline{b}}f = A_{\overline{v}\,\overline{a}}g$.

(ii) *The graph* $\mathscr{G}((A_{\overline{\varphi}}^0)^*)$ *consists of all vectors* $g \oplus f$ *in* $K_u^2 \oplus K_u^2$ *such that* $A_b g = A_{va}f$.

Proof. The two statements are C-transforms of each other, so it will suffice to prove the first one. The typical vector in $\mathscr{G}(A_\varphi^0)$ equals $A_{va}h \oplus A_b h$ with h in K_u^2. Hence, the vector $f \oplus g$ in $K_u^2 \oplus K_u^2$ is in $\mathscr{G}((A_\varphi^0)^*)$ if and only if, for all h in K_u^2,

$$\begin{aligned} 0 &= \langle f \oplus g, W(A_{va}h \oplus A_b h) \rangle \\ &= \langle f, A_b h \rangle - \langle g, A_{va}h \rangle \\ &= \langle A_{\overline{b}}f - A_{\overline{v}\,\overline{a}}g, h \rangle, \end{aligned}$$

which happens if and only if $A_{\overline{b}}f = A_{\overline{v}\,\overline{a}}g$. □

We now make the definitions $A_{\overline{\varphi}} = (A_\varphi^0)^*$, $A_\varphi = (A_{\overline{\varphi}}^0)^*$. The operators $A_{\overline{\varphi}}$ and A_φ are C-transforms of each other.

LEMMA 5.4. *The operators* A_φ *and* $A_{\overline{\varphi}}$ *are adjoints of each other and are the respective closures of* A_φ^0 *and* $A_{\overline{\varphi}}^0$.

Proof. We have the orthogonal decompositions

$$\begin{aligned} K_u^2 \oplus K_u^2 &= \overline{\mathscr{G}(A_{\overline{\varphi}}^0)} \oplus W\mathscr{G}(A_\varphi) \\ &= \mathscr{G}(A_{\overline{\varphi}}) \oplus W\overline{\mathscr{G}(A_\varphi^0)}. \end{aligned} \tag{5.1}$$

By Lemma 5.2 we also have the inclusions $\mathscr{G}(A_{\overline{\varphi}}^0) \subset \mathscr{G}(A_{\overline{\varphi}})$, $\mathscr{G}(A_\varphi^0) \subset \mathscr{G}(A_\varphi)$. It is asserted that $\mathscr{G}(A_{\overline{\varphi}}^0) \oplus W\mathscr{G}(A_\varphi^0)$ is dense in $K_u^2 \oplus K_u^2$. Once this has been established, it will follow immediately from (5.1) and the preceding inclusions that $\mathscr{G}(A_{\overline{\varphi}}) = \overline{\mathscr{G}(A^0_{\overline{\varphi}})}$ and $\mathscr{G}(A_\varphi) = \overline{\mathscr{G}(A^0_\varphi)}$, and that $K^2_u \oplus K^2_u = \mathscr{G}(A_{\overline{\varphi}}) \oplus W\mathscr{G}(A_\varphi)$, in other words, that $A_\varphi = A^*_{\overline{\varphi}}$.

Let the vector $f \oplus g$ in $K^2_u \oplus K^2_u$ be orthogonal to $\mathscr{G}(A^0_{\overline{\varphi}}) \oplus W\mathscr{G}(A^0_\varphi)$. Then, by (5.1) and the preceding inclusions, $f \oplus g$ is in $W\mathscr{G}(A_\varphi) \cap \mathscr{G}(A_{\overline{\varphi}})$, implying by Lemma 5.3 that $A_{\overline{b}}f = A_{\overline{v}\,\overline{a}}g$ and $A_b g = -A_{va}f$. Applying the conjugation C to the first of these equalities, we get $A_b\widetilde{f} = A_{va}\widetilde{g}$. Hence the functions $vaf + bg$ and $va\widetilde{g} - b\widetilde{f}$ are in uH^2. The function

$$\widetilde{f}(vaf + bg) + g(va\widetilde{g} - b\widetilde{f}) = va(f\widetilde{f} + g\widetilde{g})$$

is therefore in uH^1. Since g.c.i.d.$(va, u) = 1$, the function $f\widetilde{f} + g\widetilde{g}$ is then also in uH^1. Almost everywhere on $\partial\mathbb{D}$ we have

$$\begin{aligned} f(z)\widetilde{f}(z) + g(z)\widetilde{g}(z) &= f(z)u(z)\overline{z}\overline{f(z)} + g(z)u(z)\overline{z}\,\overline{g(z)} \\ &= \overline{z}u(z)(|f(z)|^2 + |g(z)|^2). \end{aligned}$$

The nonnegative function $|f|^2 + |g|^2$ on $\partial\mathbb{D}$ is thus the boundary function of a function in H^1_0 and so is the zero function. Hence $f = g = 0$, which establishes the asserted density of $\mathscr{G}(A^0_{\overline{\varphi}}) \oplus W\mathscr{G}(A^0_\varphi)$ in $K^2_u \oplus K^2_u$. □

LEMMA 5.5. *Let ψ be in H^∞*

(i) $A_{\overline{\psi}}A_{\overline{\varphi}}f = A_{\overline{\varphi}}A_{\overline{\psi}}f$ *for f in $\mathscr{D}(A_{\overline{\varphi}})$.*

(ii) $A_{\overline{\psi}}A_{\overline{\varphi}}f = A_{\overline{\psi}\,\overline{\varphi}}f$ *for f in $\mathscr{D}(A_{\overline{\varphi}})$.*

Proof. (i) By Lemma 5.3, $\mathscr{G}(A_{\overline{\varphi}})$ consists of all vectors $f \oplus g$ in $K^2_u \oplus K^2_u$ such that $A_{\overline{b}}f = A_{\overline{v}\,\overline{a}}g$. Since $A_{\overline{\psi}}$ commutes with $A_{\overline{b}}$ and with $A_{\overline{v}\,\overline{a}}$, the desired conclusion is immediate.

(ii) We consider first the case where $\psi\varphi$ is in H^∞. Let f be in $\mathscr{D}(A_{\overline{\varphi}})$. By Lemma 5.4, $A_{\overline{\varphi}}$ is the closure of $A^0_{\overline{\varphi}}$. Take a sequence $(f_n)_1^\infty$ in $\mathscr{D}(A^0_{\overline{\varphi}})$ such that $f_n \to f$ and $A_{\overline{\varphi}}f_n \to A_{\overline{\varphi}}f$. Each f_n has the form $f_n = A_{\overline{v}\,\overline{a}}h_n$ with h_n in K^2_u, and $A_{\overline{\varphi}}f_n = A_{\overline{b}}h_n$. We have

$$\begin{aligned} A_{\overline{\psi}\,\overline{\varphi}}f &= \lim_{n\to\infty} A_{\overline{\psi}\,\overline{\varphi}}f_n = \lim_{n\to\infty} A_{\overline{\psi}\,\overline{\varphi}}A_{\overline{v}\,\overline{a}}h_n \\ &= \lim_{n\to\infty} A_{\overline{\psi}\,\overline{\varphi}\,\overline{v}\,\overline{a}}h_n = \lim_{n\to\infty} A_{\overline{\psi}}A_{\overline{b}}h_n \\ &= \lim_{n\to\infty} A_{\overline{\psi}}A_{\overline{\varphi}}f_n = A_{\overline{\psi}}A_{\overline{\varphi}}f. \end{aligned}$$

This establishes (ii) for the case where $\psi\varphi$ is in H^∞.

To establish (ii) in general we note that since g.c.i.d.$(va, u) = 1$, the operator A_{va} has a trivial kernel, and hence so does its C-transform $A_{\overline{v}\,\overline{a}}$. Therefore, it will suffice to show that $A_{\overline{v}\,\overline{a}}A_{\overline{\psi}\,\overline{\varphi}}f = A_{\overline{v}\,\overline{a}}A_{\overline{\psi}}A_{\overline{\varphi}}f$ for f in $\mathscr{D}(A_{\overline{\varphi}})$. But by the special case already established,

$$A_{\overline{v}\,\overline{a}}A_{\overline{\psi}\,\overline{\varphi}}f = A_{\overline{v}\,\overline{a}\,\overline{\psi}\,\overline{\varphi}}f,\; A_{\overline{v}\,\overline{a}}A_{\overline{\psi}}A_{\overline{\varphi}}f = A_{\overline{v}\,\overline{a}\,\overline{\psi}}A_{\overline{\varphi}}f = A_{\overline{v}\,\overline{a}\,\overline{\psi}\,\overline{\varphi}}f,$$

which completes the proof. □

COROLLARY. *$A_{\overline{\varphi}}f = A_{1/\overline{v}}A_{\overline{b}/\overline{a}}f$ for f in $\mathscr{D}(A_\varphi)$.*

Proof. The condition g.c.i.d.$(u,v)=1$ implies that the operator A_v is injective (as noted earlier). As $A_{\overline{v}}$ is both the C-transform and the adjoint of A_v, it is injective and has a dense range. Therefore, $A_{\overline{v}}^{-1}$ is a closed and densely defined operator. The function $1/v$ has the canonical representation $1/v = \frac{1/\sqrt{2}}{v/\sqrt{2}}$. Combining this with Lemma 5.3, one easily checks that $A_{1/\overline{v}} = A_{\overline{v}}^{-1}$.

By Lemma 5.5 we have, for f in $\mathscr{D}(A_{\overline{\varphi}})$, $A_{\overline{v}}A_{\overline{\varphi}}f = A_{\overline{b}/\overline{a}}f$. Letting $A_{1/\overline{v}} = A_{\overline{v}}^{-1}$ act on both sides of the preceding equality, we obtain the desired conclusion. □

LEMMA 5.6. *Suppose φ has the (perhaps noncanonical) representation $\varphi = \psi/\chi$, where ψ and χ are in H^∞ and* g.c.i.d.$(u,\chi)=1$.

(i) *$A_{\overline{\chi}}K_u^2 \subset \mathscr{D}(A_{\overline{\varphi}})$, and $A_{\overline{\varphi}}A_{\overline{\chi}}h = A_{\overline{\psi}}h$ for h in K_u^2.*

(ii) *$A_{\overline{\varphi}}$ is the closure of $A_{\overline{\varphi}} \mid A_{\overline{\chi}}K_u^2$.*

Proof. (i) Fix h in K_u^2. By Lemma 5.3, it will be enough to check that $A_{\overline{b}}A_{\overline{\chi}}h = A_{\overline{v}\,\overline{a}}A_{\overline{\psi}}h$. Since $b\chi = va\psi$, that equality is immediate.

(ii) Let $A_{\overline{\varphi}}^{00} = A_{\overline{\varphi}} \mid A_{\overline{\chi}}K_u^2$. By Lemma 5.4, the orthogonal complement of $\mathscr{G}(A_{\overline{\varphi}})$ is $W\mathscr{G}(A_\varphi)$. Hence it will suffice to show that any vector in $K_u^2 \oplus K_u^2$ that is orthogonal to $\mathscr{G}(A_{\overline{\varphi}}^{00})$ belongs to $W\mathscr{G}(A_\varphi)$. Let $f \oplus g$ be such a vector. Then for all h in K_u^2 we have

$$\begin{aligned} 0 &= \langle f \oplus g, A_{\overline{\chi}}h \oplus A_{\overline{\psi}}h\rangle = \langle f, A_{\overline{\chi}}h\rangle + \langle g, A_{\overline{\psi}}h\rangle \\ &= \langle A_\chi f + A_\psi g, h\rangle, \end{aligned}$$

implying that $A_\chi f + A_\psi g = 0$, which means $\chi f + \psi g$ is in uH^2. Because $\psi = b\chi/va$, it follows that $\chi\left(f + \frac{bg}{va}\right)$ is in uH^2, and hence that $\chi(bg+vaf)$ is in uH^2. Since g.c.i.d.$(u,\chi)=1$, this implies that $bg+vaf$ is in uH^2, so that $A_b g = -A_{va}f$. By Lemma 5.3, this means that $f \oplus g$ is in $W\mathscr{G}(A_\varphi)$, the desired conclusion. □

COROLLARY. *If χ is in H^∞ and* g.c.i.d.$(u,\chi)=1$, *then $A_{1/\overline{\chi}} = A_{\overline{\chi}}^{-1}$.*

Proof. By the reasoning in the proof of the corollary to Lemma 5.5, the operator $A_{\overline{\chi}}$ is injective and has a dense range, so $A_{\overline{\chi}}^{-1}$ is closed and densely defined. By Lemma 5.6, part (i) (the case $\psi = 1$), $A_{1/\overline{\chi}}A_{\overline{\chi}}$ is the identity operator. □

LEMMA 5.7. *Let φ_1 and φ_2 be two nonzero functions in N_u^+. Then $A_{\overline{\varphi}_1} = A_{\overline{\varphi}_2}$ if and only if u divides $\varphi_1 - \varphi_2$.*

Proof. Let φ_1 and φ_2 have the canonical representations $\varphi_1 = b_1/v_1a_1$ and $\varphi_2 = b_2/v_2a_2$. Let $A_{\overline{\varphi}_1}^{00} = A_{\overline{\varphi}_1} \mid A_{\overline{v}_1\overline{a}_1\overline{v}_2\overline{a}_2}K_u^2$ and $A_{\overline{\varphi}_2}^{00} = A_{\overline{\varphi}_2} \mid A_{\overline{v}_1\overline{a}_1\overline{v}_2\overline{a}_2}K_u^2$. By Lemma 5.6, $A_{\overline{\varphi}_1}$ is the closure of $A_{\overline{\varphi}_1}^{00}$ and $A_{\overline{\varphi}_2}$ is the closure of $A_{\overline{\varphi}_2}^{00}$. Hence it will suffice to show that $A_{\overline{\varphi}_1}^{00} = A_{\overline{\varphi}_2}^{00}$ if and only if u divides $\varphi_1 - \varphi_2$.

Let h be in K_u^2. By Lemma 5.6,

$$A_{\overline{\varphi}_1}A_{\overline{v}_1\overline{a}_1\overline{v}_2\overline{a}_2}h = A_{\overline{b}_1}A_{\overline{v}_2\overline{a}_2}h,\ A_{\overline{\varphi}_2}A_{\overline{v}_1\overline{a}_1\overline{v}_2\overline{a}_2}h = A_{\overline{b}_2}A_{\overline{v}_1\overline{a}_1}h.$$

The difference in the two images is

$$A_{\overline{b}_1\overline{v}_2\overline{a}_2-\overline{b}_2\overline{v}_1\overline{a}_1}h,$$

which is 0 for all h in K_u^2 if and only if $b_1v_2a_2 - b_2v_1a_1$ is in uH^∞. This is the desired conclusion, because

$$\varphi_1 - \varphi_2 = \frac{b_1v_2a_2 - b_2v_1a_1}{v_1a_1v_2a_2},$$

and g.c.i.d.$(u, v_1a_1v_2a_2) = 1$. □

Functional Calculus Interpretation of $A_{\overline{\varphi}}$

To close this section we note that $A_{\overline{\varphi}}$ can, in a natural sense, be interpreted as a function of S_u^*. The H^∞ functional calculus is applicable to S_u^*, as to any pure contraction. The operators $A_{\overline{\psi}}$ with ψ in H^∞ are just the H^∞ functions of S_u^*. Namely, for ψ in H^∞, $A_{\overline{\psi}} = \psi^*(S_u^*)$, where ψ^* is the H^∞ function whose coefficients at the origin are the complex conjugates of the coefficients of ψ (equivalently, $\psi^*(z) = \overline{\psi(\overline{z})}$). In particular, $A_{\overline{b}} = b^*(S_u^*)$ and $A_{\overline{v}\overline{a}} = (va)^*(S_u^*)$. By the corollary to Lemma 5.6, the operator $A_{\overline{v}\overline{a}}$ is injective, and $(A_{\overline{v}\overline{a}})^{-1} = A_{1/\overline{v}\overline{a}}$. And by Lemma 5.5, $A_{\overline{\varphi}}f = A_{1/\overline{a}\overline{v}}A_{\overline{b}}f$ for f in $\mathscr{D}(A_{\overline{\varphi}})$. Hence, for f in $\mathscr{D}(A_{\overline{\varphi}})$ we can write

$$A_{\overline{\varphi}}f = ((va)^*(S_u^*))^{-1}b^*(S_u^*)f.$$

Accordingly, it is natural to interpret $A_{\overline{\varphi}}$ as $\varphi^*(S_u^*)$, where $\varphi^* = b^*/(av)^*$ $(\varphi^*(z) = \overline{\varphi(\overline{z})})$.

The unbounded functional calculus for S_u^* obtained here is a special case of an unbounded functional calculus for fairly general contractions due to B. Sz.-Nagy and C. Foiaş; see Section 1 of Chapter IV in the book [8]. (I thank Hari Bercovici for alerting me to this.)

6. Blaschke Product with Simple Zeros

We consider in this section the special case in which the inner function u is a Blaschke product with simple zeros $z_1, z_2, \ldots$. This case can be handled without reliance on Suárez's results.

For each n, the kernel function $k_{z_n}^u$ for the evaluation functional at z_n on K_u^2 is just the kernel function k_{z_n} for the evaluation functional at z_n on H^2: $k_{z_n}(z) = 1/(1-\overline{z}_nz)$. The conjugate function $\widetilde{k}_{z_n}$ in K_u^2 is given by $\widetilde{k}_{z_n}(z) = u(z)/(z - z_n)$. The functions k_{z_n} span K_u^2, as do the functions $\widetilde{k}_{z_n}$. The sequences (k_{z_n}) and $(\widetilde{k}_{z_n})$ are biorthogonal:

$$\langle \widetilde{k}_{z_n}, k_{z_n}\rangle = \widetilde{k}_{z_m}(z_n) = \begin{cases} 0, & m \neq n \\ u'(z_n), & m = n. \end{cases}$$

We note the equality $\|\widetilde{k}_{z_n}\|_\infty = \|k_{z_n}\|_\infty = 1/(1-|z_n|)$.

The functions k_{z_n} are eigenvectors of S_u^* : $S_u^*(k_{z_n}) = \overline{z}_n k_{z_n}$. More generally, if ψ is in H^∞, then k_{z_n} is an eigenvector of $A_{\overline{\psi}}$ (the general bounded operator on K_u^2 commuting with S_u^*): $A_{\overline{\psi}} k_{z_n} = \overline{\psi(z_n)} k_{z_n}$. This conclusion extends to closed densely defined operators commuting with S_u^*.

LEMMA 6.1. *If A is a closed densely defined operator on K_u^2 that commutes with S_u^*, then each k_{z_n} is in $\mathscr{D}(A)$ and is an eigenvector of A.*

Proof. The graph $\mathscr{G}(A)$ is an invariant subspace of $S^* \oplus S^*$, hence is invariant under the weakly closed operator algebra generated by $S^* \oplus S^*$, which consists of the operators $T_{\overline{\psi}} \oplus T_{\overline{\psi}}$ with ψ in H^∞. Thus, if $f \oplus Af$ is in $\mathscr{G}(A)$ and ψ is in H^∞, then $A_{\overline{\psi}} f$ is in $\mathscr{D}(A)$ and $AA_{\overline{\psi}} f = A_{\overline{\psi}} Af$.

Fix n, and choose a function f in $\mathscr{D}(A)$ such that $\widetilde{f}(z_n) \neq 0$; such an f exists because $\mathscr{D}(A)$ is dense in K_u^2. Let $\psi = \widetilde{k}_{z_n}$, so that $\psi(z_m) = 0$ for $m \neq n$ and $\psi(z_n) = u'(z_n)$. We have

$$\begin{aligned} \langle \widetilde{k}_{z_m}, A_{\overline{\psi}} f \rangle &= \langle A_\psi \widetilde{f}, k_{z_m} \rangle = \langle \widetilde{f}, A_{\overline{\psi}} k_{z_m} \rangle \\ &= \psi(z_m) \widetilde{f}(z_m) = \begin{cases} 0, & m \neq n \\ u'(z_n) \widetilde{f}(z_n), & m = n. \end{cases} \end{aligned}$$

Thus $A_{\overline{\psi}} f$ is a nonzero vector orthogonal to $\widetilde{k}_{z_m}$ for every $m \neq n$, implying it is a nonzero scalar multiple of k_{z_n}. Hence k_{z_n} is in $\mathscr{D}(A)$.

Continuing to let $\psi = \widetilde{k}_{z_n}$, we note that if we replace the f chosen above by a general function in K_u^2, the reasoning above tells us that the range of $A_{\overline{\psi}}$ is the one-dimensional subspace spanned by k_{z_n}. In particular, since $AA_{\overline{\psi}} f = A_{\overline{\psi}} Af$, the function Ak_{z_n} is a scalar multiple of k_{z_n}. $\square$

COROLLARY. *If φ is in N_u^+ then, for each n, the kernel function k_{z_n} is an eigenvector of $A_{\overline{\varphi}}$, with eigenvalue $\overline{\varphi(z_n)}$.*

Proof. That k_{z_n} is in $\mathscr{D}(A_{\overline{\varphi}})$, and is an eigenvector of $A_{\overline{\varphi}}$, is given by Lemma 6.1. Let $\varphi = b/va$ be the canonical representation of φ. By Lemma 5.5,

$$\begin{aligned} \overline{v(z_n)}\,\overline{a(z_n)}\, A_{\overline{\varphi}} k_{z_n} &= A_{\overline{\varphi}} A_{\overline{v}\,\overline{a}} k_{z_n} = A_{\overline{\varphi}\,\overline{v}\,\overline{a}} k_{z_n} \\ &= A_{\overline{b}} k_{z_n} = \overline{b(z_n)} k_{z_n}, \end{aligned}$$

so that $A_{\overline{\varphi}} k_{z_n} = \overline{\varphi(z_n)} k_{z_n}$, as desired. $\square$

It will now be shown that a closed densely defined operator on K_u^2 that commutes with S_u^* is uniquely determined by its eigenvalues for the eigenvectors k_{z_n}, and that any sequence of eigenvalues is possible.

PROPOSITION 6.1. *Let (w_n) be a sequence of complex numbers. Then there is a unique closed densely defined operator A on K_u^2 that commutes with S_u^* and satisfies $Ak_{z_n} = \overline{w}_n k_{z_n}$ for all n.*

Proof. Existence. We let A_0 be the operator whose domain is the linear span of the kernel functions k_n, with $A_0 k_n = \overline{w}_n k_n$ for each n. It is clear that A_0 commutes with S_u^*, in other words, that $\mathscr{G}(A_0)$ is invariant under $S_u^* \oplus S_u^*$. We let $\widetilde{A_0}$ be the C-transform of A_0: the domain of $\widetilde{A_0}$ is the linear span of the functions $\widetilde{k}_{z_n}$, and $\widetilde{A_0}\widetilde{k}_{z_n} = w_n \widetilde{k}_{z_n}$ for each n.

We verify first that $A_0 \subset \widetilde{A}_0^*$. The vector $f \oplus g$ in $K_u^2 \oplus K_u^2$ lies in $\mathscr{G}(\widetilde{A}_0^*)$ if and only if $g \oplus -f$ is orthogonal to $\mathscr{G}(\widetilde{A_0})$, which happens if and only if $\langle g, \widetilde{k}_{z_m}\rangle - \langle f, \widetilde{A_0}\widetilde{k}_{z_m}\rangle = 0$ for all m. We have

$$\begin{aligned}\langle g, \widetilde{k}_{z_m}\rangle - \langle f, \widetilde{A_0}\widetilde{k}_{z_m}\rangle &= \langle k_{z_m}, \widetilde{g}\rangle - \langle A_0 k_{z_m}, \widetilde{f}\rangle \\ &= \overline{\widetilde{g}(z_m)} - \overline{w}_m \overline{\widetilde{f}(z_m)}.\end{aligned}$$

Hence, $f \oplus g$ is in $\mathscr{G}(\widetilde{A}_0^*)$ if and only if $\widetilde{g}(z_m) = w_m \widetilde{f}(z_m)$ for all m. For each n, the condition is satisfied by $k_{z_n} \oplus A_0 k_{z_n} = k_{z_n} \oplus \overline{w}_n k_{z_n}$: $w_m \widetilde{k}_{z_n}(z_m)$ and $(\overline{w}_n k_{z_n}))^{\sim}(z_m) = w_n \widetilde{k}_{z_n}(z_m)$ are both 0 for $m \neq n$ and obviously coincide for $m = n$. This gives the inclusion $A_0 \subset \widetilde{A}_0^*$.

Since A_0 has the closed extension $\widetilde{A}_0^*$, it is closable. We let A be the closure of A_0. The operator A has the required properties: it is closed, densely defined, commutes with S_u^* (being the closure of an operator that does), and satisfies $A k_{z_n} = \overline{w}_n k_{z_n}$ for all n. (That $A = \widetilde{A}_0^*$ follows from the uniqueness part of the lemma.)

Uniqueness. Let A be any operator with the required properties. It will be shown that A is the closure of A_0. We can assume with no loss of generality that the Blaschke product u is normalized, that is, that its first nonvanishing Taylor coefficient at the origin is positive. For each n let u_n be the (normalized) Blaschke product one obtains by deleting the first n Blaschke factors from the Blaschke product u; the zeros of u_n are $z_{n+1}, z_{n+2}, \ldots$. We have $u_n \to 1$ locally uniformly in $\mathbb{D}$ as $n \to \infty$.

Let $f \oplus g$ belong to $\mathscr{G}(A)$. Since $\mathscr{G}(A)$ is invariant under $S^* \oplus S^*$, it is invariant under $T_{\overline{u}_n} \oplus T_{\overline{u}_n}$ for each n. We have $T_{\overline{u}_n} k_{z_m} = \overline{u_n(z_m)} k_{z_m}$, which is 0 for $m > n$ and a nonzero scalar times k_{z_m} for $m \leqslant n$. For each n, then, the operator $T_{\overline{u}_n}$ maps K_u^2 onto the span of $k_{z_1}, \ldots, k_{z_n}$. The vectors $T_{\overline{u}_n} f \oplus T_{\overline{u}_n} g$ thus lie in $\mathscr{G}(A_0)$. From the local uniform convergence of the sequence (u_n) to 1 one easily checks that $u_n h \to h$ in norm for any h in H^2, implying that $T_{\overline{u}_n} h \to h$ weakly. Therefore $T_{\overline{u}_n} f \oplus T_{\overline{u}_n} g \to f \oplus g$ weakly, showing that $f \oplus g$ is in the closure of $\mathscr{G}(A_0)$. We can conclude that A is the closure of A_0. □

The special case of the main result under the present assumption on u is now easy to obtain.

PROPOSITION 6.2. *The closed densely defined operators on K_u^2 that commute with S_u^* are the operators $A_{\overline{\varphi}}$ with φ in N_u^+.*

The proposition follows immediately from Lemma 6.1 and its corolllary, Proposition 6.1, and the following lemma.

LEMMA 6.2. *Given a sequence (w_n) of complex numbers, there is a function φ in N_u^+ such that $\varphi(z_n) = w_n$ for each n.*

Proof. Take a sequence (ρ_n) of positive numbers such that $\Sigma_1^\infty \rho_n/(1-|z_n|) < \infty$ and $\Sigma_1^\infty \rho_n |w_n|/(1-|z_n|) < \infty$. Because $\|\widetilde{k}_{z_n}\|_\infty = 1/(1-|z_n|)$, the series $\Sigma_1^\infty \rho_n \widetilde{k}_{z_n}$ and $\Sigma_1^\infty \rho_n w_n \widetilde{k}_{z_n}$ converge uniformly in $\mathbb{D}$ to functions in K_u^∞, say to the functions χ and ψ, respectively. Because $\widetilde{k}_{z_n}(z_m)$ equals 0 for $m \neq n$ and equals $u'(z_n)$ for $m = n$, we have $\chi(z_n) = \rho_n u'(z_n)$, $\psi(z_n) = \rho_n w_n u'(z_n)$. In particular $\chi(z_n) \neq 0$ for all n, so $\text{g.c.i.d.}(u, \chi) = 1$. The function $\varphi = \psi/\chi$ is thus in N_u^+, and we have $\varphi(z_n) = w_n$ for all n, as desired. $\square$

If the sequence (w_n) in Lemma 6.2 grows too quickly, the interpolation in the lemma cannot be performed by a holomorphic function in N_u^+; in particular, it cannot be performed by a function in N^+. This is because holomorphic functions in N obey a simple size restriction, as given by the following lemma.

LEMMA 6.3. *If φ is a holomorphic function in N, then*

$$|\varphi(z)| = \exp\left(O\left(\frac{1}{1-|z|}\right)\right).$$

Proof. It will suffice to consider the case where φ is nowhere zero. In that case $\log|\varphi|$ is the Poisson integral of a finite real Borel measure on $\partial\mathbb{D}$, so is bounded from above by the Poisson integral of a finite positive Borel measure on $\partial\mathbb{D}$, say the measure μ. For z in $\mathbb{D}$,

$$(P\mu)(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|e^{i\theta}-z|^2} d\mu(e^{i\theta}).$$

The integrand in the integral above is bounded by $2/(1-|z|)$, so the integral is bounded by $2\|\mu\|/(1-|z|)$. $\square$

7. Suárez's Approach

In this section, Suárez's basic results from [7], in slightly modified form, will be rederived. Emphasis will be on those results from [7] needed to prove this paper's main result.

Throughout this section, A will denote a nonzero closed densely defined operator on K_u^2 that commutes with S_u^*. Suárez's starting point is to consider the orthogonal complement in H_2^2 of the graph $\mathscr{G}(A)$, which is an invariant subspace of $S \oplus S$ containing uH_2^2. By the vector-valued generalization of Beurling's theorem (see for example [2]), $H_2^2 \ominus \mathscr{G}(A) = MH_2^2$, where M is a two-by-two matrix inner function:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

where $m_{11}, m_{12}, m_{21}, m_{22}$ are in H^∞, and the boundary function of M is unitary at almost every point of $\partial\mathbb{D}$. The matrix M, in principle, contains complete information about A. For example, identifying M with its induced multiplication operator on H_2^2, we have $\mathscr{G}(A) = \ker M^*$, implying that a vector $f \oplus g$ in $K_u^2 \oplus K_u^2$ belongs to $\mathscr{G}(A)$ if and only if $T_{\overline{m}_{11}} f + T_{\overline{m}_{21}} g = 0$ and $T_{\overline{m}_{12}} f + T_{\overline{m}_{22}} g = 0$.

The matrix M will be called a Suárez matrix for A. One obtains the general Suárez matrix for A by multiplying any particular one from the right by a constant two-by-two unitary matrix.

Two properties of M are not specific to the corresponding operator A. First, because M is unitary at almost every point of $\partial\mathbb{D}$, we have $|m_{11}| = |m_{22}|$, $|m_{12}| = |m_{21}|$, and $|m_{11}|^2 + |m_{12}|^2 = 1$ almost everywhere on $\partial\mathbb{D}$. Second, because $\det M$ is a function in H^∞ with unimodular boundary values almost everywhere, it is an inner function.

Crucial for present purposes are various divisibility relations satisfied by the entries of M. These are worked out in the lemmas that follow.

LEMMA 7.1. g.c.i.d.(m_{11}, m_{12}) *divides* u.

Proof. As noted above, the subspace MH_2^2 contains uH_2^2. In particular, then,

$$uH^2 \subset \{m_{11}h_1 + m_{12}h_2 : h_1, h_2 \in H^2\}.$$

The closure of the vector subspace on the right side of the preceding inclusion is the S-invariant subspace generated m_{11} and m_{12}, whose corresponding inner function is g.c.i.d.(m_{11}, m_{12}). The divisibility of g.c.i.d.(m_{11}, m_{12}) into u follows. □

LEMMA 7.2. *The inner function* $\det M$ *is divisible by* u.

Proof. Let f be a function in $\mathscr{D}(A)$. Then $f \oplus Af$ is orthogonal to MH^2. Therefore, for all h_1 and h_2 in H^2 we have

$$\langle f, m_{11}h_1 + m_{12}h_2\rangle + \langle Af, m_{21}h_1 + m_{22}h_2\rangle = 0.$$

Let h be in H^2, and set $h_1 = m_{22}h$, $h_2 = -m_{21}h$ in the preceding equality to get

$$\langle f, (m_{11}m_{22} - m_{12}m_{21})h\rangle = 0.$$

As $\mathscr{D}(A)$ is dense in K_u^2, we can conclude that $(m_{11}m_{22} - m_{12}m_{21})H^2 \subset uH^2$, which implies that u divides $\det M$. □

LEMMA 7.3. g.c.i.d.$(m_{21}, m_{22}) = 1$.

Proof. Let $u_2 =$ g.c.i.d.(m_{21}, m_{22}). Since $\mathscr{G}(A)$ is a graph, it contains no vector of the form $0 \oplus g$ other than $0 \oplus 0$. For g in H^2, the condition that $0 \oplus g$ be in $\mathscr{G}(A)$, in other words, that $0 \oplus g$ be orthogonal to MH_2^2, is the condition that g be orthogonal to $m_{21}h_1 + m_{22}h_2$ for all h_1 and h_2 in H^2. The closure of $\{m_{21}h_1 + m_{22}h_2 : h_1, h_2$ in $H^2\}$ is u_2H^2, so we must have $u_2H^2 = H^2$, and $u_2 = 1$. □

LEMMA 7.4. $\ker A = K_{u_1}^2$, *where* $u_1 =$ g.c.i.d.(m_{11}, m_{12}).

Proof. The function f in K_u^2 belongs to $\ker A$ if and only if $f \oplus 0$ is orthogonal to MH_2^2, in other words, if and only if f is orthogonal to $m_{11}h_1 + m_{12}h_2$ for all h_1 and h_2 in H^2. The last condition just means that f is orthogonal to u_1H^2, in other words, that f is in $K_{u_1}^2$. □

LEMMA 7.5. g.c.i.d. $\big(\text{g.c.i.d.}(m_{11}, m_{21}), \text{g.c.i.d.}(m_{12}, m_{22})\big) = 1$.

Proof. We have

$$\begin{aligned}&\text{g.c.i.d.}\big(\text{g.c.i.d.}(m_{11}, m_{21}), \text{g.c.i.d.}(m_{12}, m_{22})\big)\\ &\quad= \text{g.c.i.d.}(m_{11}, m_{21}, m_{12}, m_{22})\\ &\quad= \text{g.c.i.d.}\big(\text{g.c.i.d.}(m_{11}, m_{12}), \text{g.c.i.d.}(m_{21}, m_{22})\big) = 1,\end{aligned}$$

because $\text{g.c.i.d.}(m_{21}, m_{22}) = 1$ by Lemma 7.2. □

LEMMA 7.6. *u and $\det M$ are codivisible.*

Proof. Let $u_0 = \det M$. On $\partial\mathbb{D}$, $1/u_0 = \overline{u}_0$ (a.e.), which together with the standard formula for the inverse of an invertible two-by-two matrix gives, on $\partial\mathbb{D}$,

$$M^{-1} = \overline{u}_0 \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}.$$

Since $MH_2^2 \supset uH_2^2$, we have $uM^{-1}H_2^2 \subset H_2^2$. So, for any functions h_1 and h_2 in H^2, the vector

$$\overline{u}_0 u\big((m_{22}h_1 - m_{12}h_2) \oplus (-m_{21}h_1 + m_{11}h_2)\big)$$

is in H_2^2. We thus have the inclusions

$$\overline{u}_0 u(m_{22}H^2 + m_{12}H^2) \subset H^2,\ \overline{u}_0 u(m_{21}H^2 + m_{11}H^2) \subset H^2.$$

By Lemma 7.3, $\text{g.c.i.d.}(m_{11}, m_{12}, m_{21}, m_{22}) = 1$, implying that $m_{22}H^2 + m_{12}H^2$ and $m_{21}H^2 + m_{11}H^2$ together span H^2. We can conclude that $\overline{u}_0 uH^2 \subset H^2$, i.e., $uH^2 \subset u_0H^2$. Hence u_0 divides u. By Lemma 7.2 u divides u_0. Thus u and u_0 are codivisible. □

LEMMA 7.7. *The inner functions* $\text{g.c.i.d.}(m_{11}, m_{21})$ *and* $\text{g.c.i.d.}(m_{12}, m_{22})$ *divide* u.

Proof. The inner functions in question obviously divide $\det M$, so they divide u by Lemma 7.6. □

LEMMA 7.8. *On* $\partial\mathbb{D}$, $m_{12} = -u\overline{m}_{21}$, $m_{22} = u\overline{m}_{11}$.

Proof. By Lemma 7.6, on $\partial\mathbb{D}$ we have the equality

$$M^{-1} = \overline{u} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}.$$

Because M is unitary valued on $\partial\mathbb{D}$, we also have there the equality

$$M^{-1} = \begin{pmatrix} \overline{m}_{11} & \overline{m}_{21} \\ \overline{m}_{12} & \overline{m}_{22} \end{pmatrix}.$$

Equating the two different expressions for M^{-1} gives one the desired equalities. □

According to Lemma 7.8, the functions m_{11} and m_{22} are conjugates of each other in the space K^2_{Su}, as are the functions m_{12} and $-m_{21}$. In particular, the entries of M belong to K^2_{Su}.

8. Main Result

THEOREM 8.1. *The closed densely defined operators on K^2_u that commute with S^*_u are the operators $A_{\overline{\varphi}}$ with φ in N^+_u.*

Some preliminaries will precede the proof. Throughout the section we let A denote a nonzero, closed, densely defined operator on K^2_u that commutes with S^*_u. To prove the theorem we need only show that A has the desired form (since we already know that operators of the desired form do commute with S^*_u).

Recall that if M and M' are two Suárez matrices for A, then M' is the product of M and a constant two-by-two unitary matrix, in that order. We shall say M and M' are disconnected if the unitary matrix in question has neither the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ nor the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. In that case, each column of M' is a linear combination of the columns of M, but not a scalar multiple of either of the columns of M.

Consider a Suárez matrix M for A:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

We know from Lemma 7.7 that the inner functions g.c.i.d.(m_{11}, m_{21}) and g.c.i.d.(m_{12}, m_{22}) divide u, and from Lemma 7.5 that they are relatively prime as inner functions. From Lemma 7.3 we know that g.c.i.d.$(m_{21}, m_{22}) = 1$. Let $w_1 =$ g.c.i.d.(m_{11}, m_{21}), $w_2 =$ g.c.i.d.(m_{12}, m_{22}). We can then write M as

$$M = \begin{pmatrix} w_1 p & w_2 q \\ w_1 r & w_2 s \end{pmatrix}, \tag{8.1}$$

where p, q, r, s are in H^∞, and the following properties hold:

(i) g.c.i.d.$(p, r) =$ g.c.i.d.$(q, s) =$ g.c.i.d.$(w_1 r, w_2 s) = 1$;

(ii) almost everywhere on $\partial\mathbb{D}$, $|p| = |s|$, $|q| = |r|$, and $|p|^2 + |r|^2 = |q|^2 + |s|^2 = 1$.

We call (8.1) the reduced form of M. The key step in the proof of the theorem will be to show that M can be so chosen that $w_1 = w_2 = 1$.

We shall say that two inner functions v_1 and v_2 are relatively prime modulo u if g.c.i.d.$(u, v_1, v_2) = 1$.

LEMMA 8.1. *Let M and $M^\#$ be two disconnected Suárez matrices for A, with reduced forms*

$$M = \begin{pmatrix} w_1 p & w_2 q \\ w_1 r & w_2 s \end{pmatrix}, \quad M^\# = \begin{pmatrix} w_1^\# p^\# & w_2^\# q^\# \\ w_1^\# r^\# & w_2^\# s^\# \end{pmatrix}.$$

Then the inner functions $w_1, w_2, w_1^\#, w_2^\#$ *are pairwise relatively prime modulo* u.

Proof. We already know that $\text{g.c.i.d.}(w_1, w_2) = \text{g.c.i.d.}(w_1^\#, w_2^\#) = 1$. Consider the pair w_1 and $w_1^\#$. Since M and $M^\#$ are disconnected, each column of $M^\#$ is a linear combination of the columns of M, with both coefficients in the linear combination being nonzero. In particular, $w_1^\# r^\#$ is such a linear combination of $w_1 r$ and $w_2 s$. So, if w_1 and $w_1^\#$ shared with u a nonconstant inner divisor, that divisor would also divide $w_2 s$, which is impossible because $\text{g.c.i.d.}(w_1 r, w_2 s) = 1$. The same reasoning handles the other cases. □

LEMMA 8.2. *A family of nonconstant, pairwise relatively prime, inner divisors of* u *is at most countable.*

Proof. Suppose u is a Blaschke product. A nonconstant inner divisor of u determines a nonempty subset of $u^{-1}(0)$, and two such divisors are relatively prime if and only if their corresponding subsets of $u^{-1}(0)$ are disjoint. Since a disjoint family of nonempty subsets of the countable set $u^{-1}(0)$ is at most countable, this settles the case where u is a Blaschke product.

Suppose u is a singular inner function, with corresponding singular measure σ. A nonconstant inner divisor of u then corresponds to a nonzero function in $L^\infty(\sigma)$. Two such inner divisors are relatively prime if and only if the product of their corresponding functions is the zero function in $L^\infty(\sigma)$, in which case the functions are orthogonal in $L^2(\sigma)$. Since $L^2(\sigma)$ is separable, any orthogonal family of nonzero functions in it is at most countable, which settles the case where u is a singular function.

The general case follows from the Blaschke and singular cases. □

Proof of Theorem 8.1. Let M^0 be a Suárez matrix for A, with reduced form

$$M^0 = \begin{pmatrix} w_1^0 p^0 & w_2^0 q^0 \\ w_1^0 r^0 & w_2^0 s^0 \end{pmatrix}.$$

For θ in $[-\pi, \pi]$, let

$$M^\theta = M^0 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

with reduced form

$$M^\theta = \begin{pmatrix} w_1^\theta p^\theta & w_2^\theta q^\theta \\ w_1^\theta r^\theta & w_2^\theta s^\theta \end{pmatrix}.$$

By Lemma 8.1, if θ' is not equal to θ or $-\theta$, the inner functions $w_1^\theta, w_2^\theta, w_1^{\theta'}, w_2^{\theta'}$ are pairwise relatively prime modulo u. By Lemma 8.2, there are at most countably many values of θ in the interval $[0, \pi)$ such that $\text{g.c.i.d.}(u, w_1^\theta) \neq 1$, and at most countably many values such that $\text{g.c.i.d.}(u, w_2^\theta) \neq 1$. Since all the inner functions w_1^θ and w_2^θ divide u, we have $w_1^\theta = w_2^\theta = 1$ except for at most countably many values of θ.

It has been shown, therefore, that A has a Suárez matrix M of the form

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

where g.c.i.d.(p,r) = g.c.i.d.(q,s) = g.c.i.d.$(r,s) = 1$, and, almost everywhere on $\partial\mathbb{D}$, $|p| = |s|$, $|q| = |r|$, and $|p|^2 + |r|^2 = |q|^2 + |s|^2 = 1$. It is asserted that also g.c.i.d.$(u,r) = 1$. In fact, since $u = \det M = ps - qr$, any nonconstant inner divisor shared by u and r would force either p and r or s and r to share a nonconstant inner divisor, contrary to the relations g.c.i.d.(p,r) = g.c.i.d.$(r,s) = 1$. By the same reasoning, g.c.i.d.$(u,s) = 1$.

The function $f \oplus g$ belongs to $\mathscr{G}(A)$ if and only if it lies in the kernel of M^*. Since

$$M^* = \begin{pmatrix} T_{\overline{p}} & T_{\overline{r}} \\ T_{\overline{q}} & T_{\overline{s}} \end{pmatrix},$$

the condition for $f \oplus g$ to be in $\mathscr{G}(A)$ is that $A_{\overline{p}}f + A_{\overline{r}}g = 0$ and $A_{\overline{q}}f + A_{\overline{s}}g = 0$. It is asserted that these two conditions imply each other. In fact, suppose the first equality holds. Because $ps - qr = u$, we have $A_{\overline{p}\,\overline{s}} = A_{\overline{q}\,\overline{r}}$. Therefore

$$A_{\overline{r}}A_{\overline{s}}g = -A_{\overline{s}\,\overline{p}}f = -A_{\overline{r}}A_{\overline{q}}f.$$

Because g.c.i.d.$(u,r) = 1$, the operator $A_{\overline{r}}$ has a trivial kernel, and we obtain $A_{\overline{q}}f + A_{\overline{s}}g = 0$, which is the second equality. The same reasoning gives the implication in the other direction.

The graph $\mathscr{G}(A)$ thus consists of all vectors $f \oplus g$ in $K_u^2 \oplus K_u^2$ such that $A_{\overline{p}}f = -A_{\overline{r}}g$. The function $\varphi = -p/r$ is in N_u^+, and because $|p|^2 + |r|^2 = 1$ almost everywhere on $\partial\mathbb{D}$, the numerator and denominator in the canonical representation of φ are $-p$ and r, respectively, to within a common unimodular scalar multiple. By Lemma 5.3, $A = A_{\overline{\varphi}}$. $\square$

The discussion at the end of Section 5 enables us to give a functional calculus restatement of Theorem 8.1: *The closed densely defined operators commuting with S_u^* are the operators $\varphi^*(S_u^*)$ with φ in N_u^+.*

9. Inadequacy of Coanalytic Symbols

THEOREM 9.1. *There is a closed densely defined operator on K_u^2 that commutes with S_u^* but is not of the form $A_{\overline{\varphi}}$ with φ a holomorphic function in N_u^+.*

The case where u is a Blaschke product is handled, essentially, in Section 6. Suppose u is a Blaschke product, and let u_0 be the Blaschke product with simple zeros such that $u_0^{-1}(0) = u^{-1}(0)$. Note that $N_u^+ = N_{u_0}^+$. Thus, each function φ in $N_{u_0}^+$ determines, as in Section 5, both an operator on $K_{u_0}^2$ and an operator on K_u^2, which we denote by $A_{\overline{\varphi},0}$ and $A_{\overline{\varphi}}$, respectively. The operator $A_{\overline{\varphi},0}$ is the restriction of $A_{\overline{\varphi}}$ to $K_{u_0}^2$. It was observed in Section 6 that φ can be so chosen that there is no holomorphic function φ_1 and $N_{u_0}^+$ such that $A_{\overline{\varphi},0} = A_{\overline{\varphi}_1,0}$. For such a φ, a fortiori, there is no holomorphic function φ_1 in N_u^+ such that $A_{\overline{\varphi}} = A_{\overline{\varphi}_1}$.

It thus only remains to prove Theorem 9.1 for the case where u has a nonconstant singular factor. Some preliminaries will precede the proof. The following known result will be used.

LEMMA 9.1. *Let ρ be a finite positive Borel measure on $\partial\mathbb{D}$ and τ a finite Borel measure absolutely continuous with respect to ρ. Let $P\rho$ and $P\tau$ be the Poisson integrals of ρ and τ. Then the ratio $P\tau/P\rho$ has nontangential limit $\frac{d\tau}{d\rho}$ almost everywhere with respect to ρ.*

One can derive the lemma starting from the Besicovich covering lemma. Here is a rough sketch. Given ρ and τ as above, one considers the relative maximal function $M_{\rho,\tau}$, whose value at a point $e^{i\theta}$ of $\partial\mathbb{D}$ is the supremum of $|\tau|(I)/\rho(I)$ taken over all subarcs I of $\partial\mathbb{D}$ centered at $e^{i\theta}$. Besicovich's lemma enables one to prove that $M_{\rho,\tau}$ is of weak-type $(1,1)$ relative to $L^1(\rho)$. Details can be found in the book of R. L. Wheeden and A. Zygmund [9]. Next one considers the corresponding relative radial maximal function $R_{\rho,\tau}$ and relative nontangential maximal functions $N_{\alpha,\rho,\tau}$ $\left(0<\alpha<\frac{\pi}{2}\right)$. The value of $R_{\rho,\tau}$ at a point $e^{i\theta}$ of $\partial\mathbb{D}$ is the supremum of $P|\tau|(re^{i\theta})/P\rho(re^{i\theta})$ for $0\leqslant r<1$. The value of $N_{\alpha,\rho,\tau}$ at $e^{i\theta}$ is the supremum of $P|\tau|(z)/P\rho(z)$ as z ranges over the nontangential approach region $\Gamma_\alpha(e^{i\theta})$, the interior of the convex hull of $e^{i\theta}$ and the disk $|z|\leqslant\sin\alpha$. The distribution function of the maximal function $R_{\rho,\tau}$, it turns out, is dominated by the distribution function of $M_{\rho,\tau}$, so it also is of weak-type $(1,1)$ relative to $L^1(\rho)$. A geometric argument shows that $N_{\alpha,\rho,\tau}$ is bounded by a constant times $R_{\rho,\tau}$, so $N_{\alpha,\rho,\tau}$ is also of weak-type $(1,1)$ relative to $L^1(\rho)$. Once that is established, a standard argument produces the result about nontangential limits.

Lemma 9.1 will be used in conjunction with the following result.

LEMMA 9.2. *Let E be a closed Lebesgue-null subset of $\partial\mathbb{D}$. Then there is a Blaschke sequence having E as its boundary cluster set, and that clusters nontangentially at each point of E.*

Proof. Take a decreasing sequence (ε_n) of numbers in $\left(0,\frac{1}{2}\right)$ such that $\Sigma_1^\infty\varepsilon_n<\infty$. For each n, find a relatively open subset G_n of $\partial\mathbb{D}$ such that $E\subset G_n$ and $|G_n|<\varepsilon_n$. The component arcs of G_n cover E, so E is covered by finitely many of those arcs, say by the arcs $I_{n,1},\dots,I_{n,m_n}$.

For each arc $I_{n,j}$, let $z_{n,j}$ be that point in $\mathbb{D}$ such that $z_{n,j}/|z_{n,j}|$ is the center of $I_{n,j}$ and $1-|z_{n,j}|=|I_{n,j}|$. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}(1-|z_{n,j}|)&=\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|I_{n,j}|\leqslant\sum_{n=1}^{\infty}|U_n|\\&<\sum_{n=1}^{\infty}\varepsilon_n<\infty.\end{aligned}$$

Hence $\beta=\{z_{n,j}:j=1,\dots,m_n,\ n=1,2,\dots\}$ is a Blaschke sequence.

Fix a point ζ in E, and fix n. Then ζ lies in one of the intervals $I_{n,1},\dots,I_{n,m_n}$, say in the interval $I_{n,j}$. We have

$$\begin{aligned}|\zeta-z_{n,j}|&\leqslant\left|\zeta-\frac{z_{n,j}}{|z_{n,j}|}\right|+1-|z_{n,j}|\\&<\pi|I_{n,j}|+1-|z_{n,j}|=(\pi+1)(1-|z_{n,j}|),\end{aligned}$$

giving $|\zeta - z_{n,j}|/(1-|z_{n,j}|) < \pi + 1$. It follows that $z_{n,j}$ lies in the Stoltz angle with vertex at ζ and opening $2\sec^{-1}(\pi+1)$. We can conclude that ζ is the nontangential limit of a subsequence of the Blaschke sequence β. Hence β clusters nontangentially at each point of E.

On the other hand, each arc $I_{n,j}$ contains a point of E, so each point $z_{n,j}$ is contained in such a Stoltz angle, implying that

$$\lim_{n\to\infty} \max\{\operatorname{dist}(z_{n,j}, E) : j = 1, \dots, m_n\} = 0.$$

Hence β clusters only at points of E. □

Proof of Theorem 9.1. As noted, we need only consider the case where u has a nonconstant singular factor. Let σ be the singular measure corresponding to that singular factor, and let E be a closed Lebesgue-null subset of $\partial\mathbb{D}$ such that $\sigma(E) > 0$. We use Lemma 9.2 to find a Blaschke sequence (z_n), without repetitions, whose boundary cluster set is E and that clusters nontangentially at each point of E. The conditions on the Blaschke sequence are preserved under suitably small perturbations, so we can assume no z_n is a zero of u. Let w be the Blaschke product with zero sequence (z_n). Then $1/w$ is in N_u^+. We shall assume that $A_{1/\overline{w}}$ can be written as $A_{\overline{\varphi}}$ with φ a holomorphic function in N_u^+, and obtain a contradiction.

We can write φ as $\varphi = \psi/\chi$, where ψ and χ are in H^∞, and χ has no zeros, and g.c.i.d.$(u,\chi) = 1$. We can assume without loss of generality that $\|\chi\|_\infty = 1$. By Lemma 5.7, u divides $\frac{1}{w} - \frac{\psi}{\chi}$ and hence divides $\chi - w\psi$, say $\chi - w\psi = u\omega$, where ω is in H^∞.

For each z_n we have $\chi(z_n) = u(z_n)\omega(z_n)$, so

$$\frac{|\chi(z_n)|}{|u(z_n)|} \leqslant \|\omega\|_\infty \ (n = 1, 2, \dots). \tag{9.1}$$

On the other hand, the absolute value of the reciprocal of the singular factor of u equals $\exp(P\sigma)$, so $\frac{1}{|u|} \geqslant \exp(P\sigma)$. And since g.c.i.d.$(u,\chi) = 1$, and χ has no zeros, and $\|\chi\|_\infty = 1$, we have $\frac{1}{|\chi|} = \exp(P\tau)$, where τ is a positive measure singular with respect to σ. Thus

$$\frac{|\chi|}{|u|} \geqslant \exp(P\sigma - P\tau).$$

By Lemma 9.1, with $\sigma + \tau$ playing the role of ρ, the ratio $P\tau/P\sigma$ has the nontangential limit 0 almost everywhere with respect to σ. It is well known (and also follows from Lemma 9.1, with $\rho = \sigma +$ Lebesgue measure) that $P\sigma$ has nontangential limit ∞ almost everywhere with respect to σ. Hence, there is a subsequence (z_{n_j}) of (z_n) such that $(P\sigma)(z_{n_j}) \to \infty$ and $(P\tau)(z_{n_j})/(P\sigma)(z_{n_j}) \to 0$. We have

$$\frac{|\chi(z_{n_j})|}{|u(z_{n_j})|} \geqslant \exp((P\sigma)(z_{n_j}) - (P\tau)(z_{n_j})),$$

and the right side tends to ∞ as $j \to \infty$, in contradiction to (9.1). □

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