ON A REDUCTION PROCEDURE FOR HORN
INEQUALITIES IN FINITE VON NEUMANN ALGEBRAS

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(communicated by C.-K. Li)

Abstract. We consider the analogues of the Horn inequalities in finite von Neumann algebras, which concern the possible spectral distributions of sums $a+b$ of self–adjoint elements a and b in a finite von Neumann algebra. It is an open question whether all of these Horn inequalities must hold in all finite von Neumann algebras, and this is related to Connes' embedding problem. For each choice of integers $1 \leqslant r \leqslant n$, there is a set T_r^n of Horn triples (I,J,K) of r–tuples of integers, and the Horn inequalities are in one–to–one correspondence with $\cup_{1 \leqslant r \leqslant n} T_r^n$.

We consider a property P_n, analogous to one introduced by Therianos and Thompson in the case of matrices, amounting to the existence of projections having certain properties relative to arbitrary flags, which guarantees that a given Horn inequality holds in all finite von Neumann algebras. It is an open question whether all Horn triples in T_r^n have property P_n. Certain triples in T_r^n can be reduced to triples in T_r^{n-1} by an operation we call *TT–reduction*. We show that property P_n holds for the original triple if property P_{n-1} holds for the reduced one.

A major part of this paper is devoted to showing that this operation of reduction preserves the value of the corresponding Littlewood–Richardson coefficients. We then characterize the TT–irreducible Horn triples in T_3^n, for arbitrary n, and for those LR–minimal ones (namely, those having Littlewood–Richardson coefficient equal to 1), we perform a construction of projections with respect to flags in arbitrary von Neumann algebras in order to prove property P_n for them. This shows that all LR–minimal triples in $\cup_{n \geqslant 3} T_3^n$ have property P_n, and so that the corresponding Horn inequalities hold in all finite von Neumann algebras.

Contents

1. Introduction and description of results

If A and B are Hermitian $n \times n$ matrices whose eigenvalues (repeated according to multiplicity) are $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ and $\beta_1 \geqslant \beta_2 \geqslant \cdots \beta_n$, respectively, what can the eigenvalues of $A + B$ be? In [7], A. Horn described sets T_r^n of triples (I, J, K) of subsets of $\{1, \ldots, n\}$, with $|I| = |J| = |K| = r$, and conjectured that a weakly decreasing real sequence $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$ can arise as the eigenvalues of $A + B$, for some A and B as above, if and only if

$$\sum_{i=1}^n \alpha_i + \sum_{j=1}^n \beta_j = \sum_{k=1}^n \gamma_k$$

and for each triple $(I, J, K) \in \bigcup_{r=1}^{n-1} T_r^n$, the so–called *Horn inequality*

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geqslant \sum_{k \in K} \gamma_k \tag{1}$$

holds. (We recall Horn's definition of the sets T_r^n in Section 3: see (26) and (27).) Horn's conjecture has been proved, due to work of Klyatchko, Tataro, Knutson and Tao. See the article [6] of Fulton.

The purpose of this paper is to prove that analogues of some of the Horn inequalities hold in all finite von Neumann algebras. This question was first considered by Bercovici and Li in [1] (see also [2]) and the following exposition is essentially from their papers. Let $\mathscr{M}$ be a von Neumann algebra with a fixed normal, faithful, tracial state τ. If $a = a^* \in \mathscr{M}$, the *eigenvalue function* of a is the non–increasing, right–continuous function $\lambda_a : [0, 1) \to \mathbf{R}$ given by

$$\lambda_a(t) = \sup\{x \in \mathbb{R} \mid \mu_a((x, \infty)) > t\},$$

where μ_a is the distribution of a, which is the Borel measure supported on the spectrum of a and satisfying

$$\tau(a^k) = \int_{\mathbb{R}} t^k \, d\mu_a(t) \qquad (k \geqslant 1).$$

For example, if

$$a = a^* \in M_n(\mathbf{C}) \hookrightarrow \mathscr{M} \tag{2}$$

has eigenvalues $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$, then

$$\lambda_a(t) = \alpha_j, \quad \frac{j-1}{n} \leqslant t < \frac{j}{n}, \quad j \in \{1, \ldots, n\}.$$

DEFINITION 1.1. Let $(I, J, K) \in T_r^n$ be a Horn triple. We say that *the Horn inequality corresponding to* (I, J, K) *holds* in $(\mathscr{M}, \tau)$ if

$$\int_{\omega_I} \lambda_a(t) \, dt + \int_{\omega_J} \lambda_b(t) \, dt \geqslant \int_{\omega_K} \lambda_{a+b}(t) \, dt \tag{3}$$

for all $a, b \in \mathscr{M}_{s.a.} := \{x \in \mathscr{M} \mid x = x^*\}$, where

$$\omega_I = \bigcup_{i \in I} \left[\frac{i-1}{n}, \frac{i}{n} \right)$$

and similarly for ω_J and ω_K.

Note that (3) becomes the usual Horn inequality (1) when a and b lie in the same copy of the $n \times n$ matrices, as in (2).

Bercovici and Li showed in [1] that the Horn inequalities corresponding to the Freede–Thompson inequalities (and certain generalizations of them) hold in all finite von Neumann algebras. In [2], they showed that if $(\mathcal{M}, \tau)$ satisfies Connes' embedding property, namely, if it embeds in the ultraproduct R^ω of the hyperfinite II_1–factor, or equivalently (assuming separable pre–dual), if all n–tuples of self–adjoints in $\mathcal{M}$ can be approximated in mixed moments by matrices, then all Horn inequalities hold in $(\mathcal{M}, \tau)$. Moreover, they showed that the set of possible triples $(\lambda_a, \lambda_b, \lambda_{a+b})$ for a and b self–adjoints in R^ω is characterized by the inequalities of the form (3). It is an important open question, known as *Connes' embedding problem,* whether all finite von Neumann algebras having separable pre–dual satisfy Connes' embedding property. In the converse direction, in [4] we showed that if certain versions of the Horn inequalities with matrix coefficients hold in all finite von Neumann algebras, then Connes' embedding problem has a positive answer. Seen in this light, it is quite interesting to learn about which Horn inequalities must hold in all finite von Neumann algebras. Some speculative observations about possible constructions of counter–examples to embeddability are found in Section 5.

One method of proving that the Horn inequality corresponding to a given Horn triple $(I, J, K) \in T_r^n$ holds in a finite von Neumann algebra $(\mathcal{M}, \tau)$ is to construct projections in $\mathcal{M}$ satisfying certain properties with respect to flags of projections in $(\mathcal{M}, \tau)$. We say (I, J, K) has *property P_n* if such projections can always be constructed. This is the analogue of the property of the same name found in [8]. We introduce a weaker, approximate version of this property, called property AP_n. (See the first part of Section 3 for details, but note that Definition 3.4 and Proposition 3.5 are for the symmetric reformulation of the Horn sets described there.) Bercovici and Li's proof [1] that certain Horn inequalities must hold in all finite von Neumann algebras was, to rephrase it, made by showing that they have property P_n. Following their proof, we show that if a triple (I, J, K) has property AP_n, then the corresponding Horn inequality holds in all finite von Neumann algebras.

In [8], Therianos and Thompson proved a reduction result, showing that the analogue of property P_n in $n \times n$ matrices for a given triple (I, J, K) can sometimes be deduced from the same analogue of property P_{n-1} for a related triple $(\widetilde{I}, \widetilde{J}, \widetilde{K})$. (See also [9].) They then used this reduction result and some explicit constructions of projections in matrices to show that Horn inequalities in $M_n(\mathbf{C})$ corresponding to triples in T_3^n hold for all n. We show (Lemma 3.6) that a similar reduction technique holds for properties P_n and AP_n in finite von Neumann algebras. Using this reduction result, though we were not able to prove that Horn inequalities in finite von Neumann algebras hold for all triples in $\bigcup_{n \geqslant 3} T_3^n$, we do show that they hold for all the LR–minimal triples in this set. The moniker LR–minimal refers to the Littlewood–Richardson coefficient of the triple (see Definition 3.8 and Lemma 3.9); it follows from Theorem 13 of [6] that the set of Horn inequalities coming from LR–minimal triples determines the remaining Horn inequalities, both in the case of matrices and of finite von Neumann algebras.

As a byproduct of our reduction technique, we also show that all the Horn inequalities corresponding to triples in $\bigcup_{r \in \{1,2\}, n \geqslant r} T_r^n$ hold in all finite von Neumann algebras, though this is can be more easily proved directly. As perhaps the most arduous part of our proof, we show (Proposition 3.10) that the reduction method referred to above preserves the Littlewood–Richardson coefficient.

Here is a brief description of the rest of this paper. In Section 2, we cover some preliminary and (mostly) well known facts about finite von Neumann algebras. In Section 3, we first describe minor reformulation of Horn's triples; the reformulated set is denoted $\widetilde{T}_r^n$, and is invariant under the obvious action of the group of permutations of three letters. Then we prove the analogue in finite von Neumann algebras of the reduction result from [8]. Triples that cannot be reduced are called irreducible, naturally enough. After introducing new notation $c^{(n)}(I,J,K)$ for Littlewood–Richardson coefficient of $(I,J,K) \in \widetilde{T}_r^n$ and observing the invariance of this quantity under permuting the arguments I, J and K, we prove that it is also invariant under the reduction method referred to above. We then characterize the irreducible triples in $\widetilde{T}_3^n$, compute their Littlewood–Richardson coefficients, and list the irreducible triples of minimal Littlewood–Richardson coefficient in $\widetilde{T}_4^n$, for $n \leqslant 9$. In Section 4, we exhibit a construction of projections in finite von Neumann algebras that suffices to prove that the Horn inequalities for all LR–minimal triples in $\bigcup_{n \geqslant 3} T_3^n$ hold in all finite von Neumann algebras. Merely because we like the argument involving almost invariant subspaces, we prove that property AP_6 holds for a certain element of $\widetilde{T}_3^6$ having Littlewood–Richardson coefficient equal to 2. Section 5, which is independent of the rest of the paper and can safely be skipped, contains some speculative remarks about how one might construct a non–embeddable finite von Neumann algebra.

2. Preliminaries concerning finite von Neumann algebras

In the following three subsections, we review some facts, introduce some notation and state some results that will be used later. While (most of) these are certainly not original, for convenience, we provide some proofs.

2.1. Two projections

Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a finite von Neumann algebra with a fixed normal, faithful, tracial state τ. Let $\mathrm{Proj}(\mathscr{M})$ denote the set of self–adjoint idempotents in $\mathscr{M}$, which are also called *projections* in $\mathscr{M}$. Many elementary but useful facts about projections in $\mathscr{M}$ follow from the standard description of the subalgebra generated by any two of them, which we now describe. Let $p, q \in \mathrm{Proj}(\mathscr{M})$. Recall that $p \wedge q$ denotes the projection onto the closed subspace $p\mathcal{H} \cap q\mathcal{H}$ and $p \vee q$ denotes the projection onto the closure of $p\mathcal{H} + q\mathcal{H}$. Let $\mathscr{A} = W^*(\{p,q,1\})$ be unital von Neumann algebra generated by p and q. Let $\mathfrak{A}$ denote the universal, unital C*–algebra generated by two projections P and Q. As is well–known,

$$\mathfrak{A} \cong \{f : [0,1] \to M_2(\mathbf{C}) \mid f \text{ continuous}, f(0), f(1) \text{ diagonal}\},$$

with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

We have a quotient map $\pi : \mathfrak{A} \to \mathscr{A}$ sending P to p and Q to q, and $\mathscr{A}$ is isomorphic to the weak closure of the image of the Gelfand–Naimark–Segal representation of $\mathfrak{A}$ arising from the trace $\tau \circ \pi$ on $\mathfrak{A}$. We thereby identify $\mathscr{A}$ with

$$\overset{p \wedge q}{\underset{\gamma_{11}}{\mathbf{C}}} \oplus \overset{p \wedge (1-q)}{\underset{\gamma_{10}}{\mathbf{C}}} \oplus L^\infty(\mu) \otimes M_2(\mathbf{C}) \oplus \overset{(1-p) \wedge q}{\underset{\gamma_{01}}{\mathbf{C}}} \oplus \overset{(1-p) \wedge (1-q)}{\underset{\gamma_{00}}{\mathbf{C}}}, \tag{4}$$

where $\gamma_{ij} \geqslant 0$, where μ is a measure concentrated on a subset of the open interval $(0,1)$, and where the notation in (4) means, for example, that $p \wedge q$ is the projection

$$p \wedge q = 1 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0$$

and $\tau(p \wedge q) = \gamma_{11}$. We have

$$p = 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0 \tag{5}$$

$$q = 1 \oplus 0 \oplus \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \oplus 1 \oplus 0 \tag{6}$$

and, if

$$a = \lambda_{11} \oplus \lambda_{10} \oplus \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \oplus \lambda_{01} \oplus \lambda_{00} \in \mathscr{A}$$

for $\lambda_{ij} \in \mathbf{C}$ and $f_{pq} \in L^\infty(\mu)$, then

$$\tau(a) = \lambda_{11}\gamma_{11} + \lambda_{10}\gamma_{10} + \frac{1}{2} \int (f_{11} + f_{22}) \, d\mu + \lambda_{01}\gamma_{01} + \lambda_{00}\gamma_{00}.$$

Thus, the total mass of μ is

$$|\mu| = 1 - \gamma_{11} - \gamma_{10} - \gamma_{01} - \gamma_{00}.$$

Of course, if in (4) some γ_{ij} or μ itself should be zero, then the corresponding summand should be understood to be absent. Inspecting this situation, we observe the following elementary result.

PROPOSITION 2.1.1. *We have*

$$p \vee q = 1 - (1-p) \wedge (1-q), \tag{7}$$

$$\tau(p \vee q) = \tau(p) + \tau(q) - \tau(p \wedge q) \tag{8}$$

$$\tau(p - (1-q) \wedge p) = \tau(q - (1-p) \wedge q). \tag{9}$$

And the following useful lemmas are also immediate.

LEMMA 2.1.2. *Then there is a projection $r \in \mathscr{A}$ such that $q \leqslant r + p$ and r is unitarily equivalent in $\mathscr{A}$ to $q - q \wedge p$. In particular, we have $\tau(r) = \tau(q) - \tau(q \wedge p)$.*

Proof. We let

$$r = 0 \oplus 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 1 \oplus 0.$$

LEMMA 2.1.3. *The projection onto the closure of $pq\mathcal{H}$ is equal to $p - p \wedge (1-q)$.*

Proof. Multiply the right–hand–sides of (5) and (6).

2.2. Affiliated operators

One of the virtues of a finite von Neumann algebra is that its set of affiliated operators forms an algebra. Here we briefly review this situation. Recall that a closed, densely defined, (possibly unbounded) operator X from $\mathcal{H}$ to itself is said to be *affiliated* with $\mathscr{M}$ if, letting $X = v|X|$ be the polar decomposition of X, we have $v \in \mathscr{M}$ and all spectral projections of the positive operator $|X|$ lie in $\mathscr{M}$. Thus, we have

$$|X| = \int_{[0,\infty)} t E_{|X|}(dt),$$

for a projection–valued measure E, taking Borel subsets of $[0,\infty)$ to elements of Proj($\mathscr{M}$). Since $\lim_{K \to +\infty} \tau(E_{|X|}([0,K]) = 1$, we easily see that, if $p \in \text{Proj}(\mathscr{M})$, then $p\mathcal{H} \cap \text{dom}(X)$ is dense in $p\mathcal{H}$, where $\text{dom}(X)$ denotes the domain of X. Thus, we see that if S and T are densely defined operators affiliated with $\mathscr{M}$, then $S + T$ and ST are densely defined and affiliated with $\mathscr{M}$.

We now define some terms and notation and make some observations that we will need later. Let X be a closed, densely defined operator from $\mathcal{H}$ to itself, having polar decomposition $X = v|X| = |X^*|v$ and where $E_{|X|}$ is the spectral measure of the positive operator $|X|$. The *kernel projection* kerproj(X) of X is the projection onto $\ker(X)$, and the *domain projection* of X is domproj(X) $= 1 - \text{kerproj}(X)$. Thus,

$$\text{kerproj}(X) = E_{|X|}(\{0\})$$
$$\text{domproj}(X) = E_{|X|}((0,+\infty)) = v^*v$$

and

$$X = X \cdot \text{domproj}(X).$$

The *range projection* of X is ranproj(X) $\in$ Proj($\mathscr{M}$) that is the projection onto the closure of the range of X. Thus,

$$\text{ranproj}(X) = E_{|X^*|}((0,+\infty)) = vv^*$$

and

$$X = \text{ranproj}(X) \cdot X.$$

Therefore, we have
$$\tau(\mathrm{domproj}(X)) = \tau(\mathrm{ranproj}(X)). \tag{10}$$

The *partial inverse* of X is the operator $Y = |Y^*|v^*$ where
$$|Y^*| = \int_{(0,\infty)} t^{-1} E_{|X|}(dt).$$

Thus,
$$XY = \mathrm{ranproj}(X) = \mathrm{domproj}(Y)$$
$$YX = \mathrm{domproj}(X) = \mathrm{ranproj}(Y).$$

Indeed, the restriction of X is an injective linear operator from $\mathrm{domproj}(X)\mathcal{H} \cap \mathrm{dom}(X)$ onto $\mathrm{ran}(X)$, and the restriction of Y to $\mathrm{ran}(X)$ is this operator's inverse.

Let
$$X^\sharp : \mathrm{Proj}(\mathcal{M}) \to \{q \in \mathrm{Proj}(\mathcal{M}) \mid q \leqslant \mathrm{ranproj}(X)\}$$

be the map defined by
$$X^\sharp(p) = \mathrm{ranproj}(Xp).$$

Clearly, $X^\sharp$ is order preserving and, moreover, if X and Z are operators affiliated with $\mathcal{M}$, then for any $p \in \mathrm{Proj}(\mathcal{M})$,
$$(XZ)^\sharp(p) = \mathrm{ranproj}(XZp) = \mathrm{ranproj}(X(Z^\sharp(p))) = X^\sharp Z^\sharp(p). \tag{11}$$

LEMMA 2.2.1. *Restricting $X^\sharp$ gives a bijection*
$$\{p \in \mathrm{Proj}(\mathcal{M}) \mid p \leqslant \mathrm{domproj}(X)\} \to \{q \in \mathrm{Proj}(\mathcal{M}) \mid q \leqslant \mathrm{ranproj}(X)\}. \tag{12}$$

Moreover, this bijection is trace preserving and a lattice isomorphism. Finally, for any $p \in \mathrm{Proj}(\mathcal{M})$, we have
$$X^\sharp(p) = X^\sharp\big(\mathrm{domproj}(X) - (1-p) \wedge \mathrm{domproj}(X)\big) \tag{13}$$
$$\tau(X^\sharp(p)) = \tau(p) - \tau(p \wedge \mathrm{kerproj}(X)). \tag{14}$$

Proof. Clearly, the restriction of $X^\sharp$ provides an order preserving map (12). Let Y be the partial inverse of X, described above. If $p \in \mathrm{Proj}(\mathcal{M})$ and $p \leqslant \mathrm{domproj}(X)$, then $YXp = p$ and consquently, using (11), we have $Y^\sharp X^\sharp(p) = p$. Similarly, if $q \in \mathrm{Proj}(\mathcal{M})$ and $q \leqslant \mathrm{ranproj}(X)$, then $XY = q$ and, consequently, $X^\sharp Y^\sharp(q) = q$. This shows that the restriction of $X^\sharp$ gives a bijection (12), whose inverse is the restriction of $Y^\sharp$ to $\{q \in \mathrm{Proj}(\mathcal{M}) \mid q \leqslant \mathrm{ranproj}(X)\}$.

To see that the bijection (12) is trace preserving, note that for a projection p with $p \leqslant \mathrm{domproj}(X)$, we have $\mathrm{domproj}(Xp) = p$, and by (10),
$$\tau(p) = \tau(\mathrm{ranproj}(Xp)) = \tau(X^\sharp(p)).$$

An order preserving bijection between lattices is necessarily a lattice isomorphism.

Now we will show (13). Using the form of the von Neumann algebra generated by two projections as described in §2.1, we find that for any $p, q \in \text{Proj}(\mathcal{M})$, we have

$$\text{ranproj}(qp) = q - (1 - p) \wedge q.$$

Therefore,

$$\begin{aligned}
X^\sharp(p) &= \text{ranproj}(Xp) = \text{ranproj}(X \, \text{domproj}(X)p) \\
&= \text{ranproj}(X(\text{domproj}(X) - (1 - p) \wedge \text{domproj}(X))) \\
&= X^\sharp(\text{domproj}(X) - (1 - p) \wedge \text{domproj}(X))
\end{aligned}$$

and this implies

$$\tau(X^\sharp(p)) = \tau(\text{domproj}(X) - (1 - p) \wedge \text{domproj}(X)). \tag{15}$$

Finally, (14) follows from (15) and (9).

The next result concerns what may be termed *almost invariant subspaces* of operators. We say $\mathcal{M}$ is *diffuse* if it has no minimal nonzero projections.

PROPOSITION 2.2.2. *Assume that $\mathcal{M}$ is diffuse. Let X be an operator affiliated with $\mathcal{M}$ and let $0 \leqslant t \leqslant \tau(\text{domproj}(X))$ and $\varepsilon > 0$. Then there are $p, q \in \text{Proj}(\mathcal{M})$ such that $p, q \leqslant \text{domproj}(X)$, $\tau(p) = t$, $\tau(q) \leqslant \varepsilon$ and*

$$X^\sharp(p) \leqslant p \vee q. \tag{16}$$

Proof. Let n be the least positive integer such that $t \leqslant n\varepsilon$. We will proceed by induction on n. If $n = 1$, then take any $p \in \text{Proj}(\mathcal{M})$ with $p \leqslant \text{domproj}(X)$ and $\tau(p) = t$ and let $q = X^\sharp(p)$. Then $\tau(q) = \tau(p) = t \leqslant \varepsilon$.

For the induction step, suppose $n \geqslant 2$ and $(n-1)\varepsilon < t \leqslant n\varepsilon$. By the induction hypothesis, there are $\tilde{p}, \tilde{q} \in \text{Proj}(\mathcal{M})$ with $\tilde{p} \leqslant \text{domproj}(X)$, $\tau(\tilde{p}) = t - \varepsilon$, $\tau(\tilde{q}) < \varepsilon$ and $X^\sharp(\tilde{p}) \leqslant \tilde{p} \vee \tilde{q}$. Replacing $\tilde{q}$ by $\tilde{q} - (\tilde{p} \wedge \tilde{q})$, if necessary, we may without loss of generality assume $\tilde{q} \wedge \tilde{p} = 0$. Adding something from $\text{domproj}(X) - (\tilde{p} \vee \tilde{q})$ to $\tilde{q}$, if necessary, we may also without loss of generality assume $\tau(\tilde{q}) = \varepsilon$. Now let $p = \tilde{p} \vee \tilde{q}$. Then $\tau(p) = t$ and

$$X^\sharp(p) = X^\sharp(\tilde{p}) \vee X^\sharp(\tilde{q}) \leqslant \tilde{p} \vee \tilde{q} \vee X^\sharp(\tilde{q}) = p \vee X^\sharp(\tilde{q}).$$

Let $q = X^\sharp(\tilde{q})$. Then $\tau(q) = \tau(\tilde{q}) = \varepsilon$ and (16) holds.

2.3. The complementary idempotents of two projections

In this subsection, we consider the idempotent affiliated operators associated to two projections e_1 and e_2 in a finite von Neumann algebra $\mathcal{M}$. We fix a normal faithful tracial state τ on $\mathcal{M}$ and, for convenience, we regard $\mathcal{M}$ as acting on $\mathcal{H} := L^2(\mathcal{M}, \tau)$ in the GNS–representation.

We define possibly unbounded operators $E(e_1,e_2)$ and $E(e_2,e_1)$, both with domain

$$(1-(e_1 \vee e_2))\mathcal{H} + e_1\mathcal{H} + e_2\mathcal{H}$$
$$= (1-(e_1 \vee e_2))\mathcal{H} + (e_1 - e_1 \wedge e_2)\mathcal{H} + (e_2 - e_1 \wedge e_2)\mathcal{H} + (e_1 \wedge e_2)\mathcal{H},$$

as follows. For ease of notation, we write E_1 for $E(e_1,e_2)$ and E_2 for $E(e_2,e_1)$. We set

$$E_i(\eta + \xi_1 + \xi_2 + \zeta) = \xi_i + \zeta$$

if $\eta \in (1-(e_1 \vee e_2))\mathcal{H}$, $\xi_j \in (e_j - e_1 \wedge e_2)\mathcal{H}$, $(j=1,2)$ and $\zeta \in (e_1 \wedge e_2)\mathcal{H}$. It is clear that E_i is well defined.

LEMMA 2.3.1.
(i) *Each operator E_i is closed, affiliated with the von Neumann algebra $W^*(\{1,e_1,e_2\})$ generated by e_1 and e_2, and idempotent.*

(ii) *We have*

$$\mathrm{ranproj}(E_i) = e_i, \tag{17}$$
$$\mathrm{kerproj}(E_i) = 1 - e_1 \vee e_2 + e_{i'} - e_1 \wedge e_2, \tag{18}$$
$$\mathrm{domproj}(E_i) = e_1 \vee e_2 - e_{i'} + e_1 \wedge e_2, \tag{19}$$

where $\{i,i'\} = \{1,2\}$, and

$$E_1 + E_2 = (e_1 \vee e_2) + (e_1 \wedge e_2). \tag{20}$$

(iii) *Let $f \in \mathcal{M}$ be a projection with $f \leqslant e_1 \vee e_2$. Then*

$$f \leqslant E_1^\sharp(f) \vee E_2^\sharp(f) \vee (e_1 \wedge e_2 - (1-f) \wedge e_1 \wedge e_2). \tag{21}$$

Proof. To show that E_i is closed, (taking $i=1$), if $h^{(n)} \in \mathrm{dom}(E_1)$ converges to $h \in \mathcal{H}$ and if $E_1(h^{(n)})$ converges to $y \in \mathcal{H}$, then we may write

$$h^{(n)} = \eta^{(n)} + \xi_1^{(n)} + \xi_2^{(n)} + \zeta^{(n)},$$

where $\eta^{(n)} = (1-(e_1 \vee e_2))h^{(n)}$, $\zeta^{(n)} = (e_1 \wedge e_2)h^{(n)}$ and where $\xi_j^{(n)} \in (e_j - e_1 \wedge e_2)\mathcal{H}$. We then have convergence:

$$\eta^{(n)} \to (1-(e_1 \vee e_2))h$$
$$\zeta^{(n)} \to (e_1 \wedge e_2)h$$
$$\xi_1^{(n)} = E_1(h^{(n)}) - \zeta^{(n)} \to y - (e_1 \wedge e_2)h \in (e_1 - e_1 \wedge e_2)\mathcal{H}.$$

Thus, we also get convergence

$$\xi_2^{(n)} \to z := (e_1 \vee e_2 - e_1 \wedge e_2)(h) - y \in (e_2 - e_1 \wedge e_2)\mathcal{H}.$$

Consequently, we have $h = (1 - (e_1 \vee e_2))h + y + z + (e_1 \wedge e_2)h$ and we conclude $E_1(h) = y$. So E_1 is closed.

By the analysis in section 2.1, we have

$$W^*(\{1, e_1, e_2\}) = \overset{e_1 \wedge e_2}{\underset{\gamma_{11}}{\mathbf{C}}} \oplus \overset{e_1 \wedge (1-e_2)}{\underset{\gamma_{10}}{\mathbf{C}}} \oplus L^\infty(\mu) \otimes M_2(\mathbf{C}) \oplus \overset{(1-e_1) \wedge e_2}{\underset{\gamma_{01}}{\mathbf{C}}} \oplus \overset{(1-e_1) \wedge (1-e_2)}{\underset{\gamma_{00}}{\mathbf{C}}},$$

for some measure μ on $(0,1)$, with

$$e_1 = 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0$$

$$e_2 = 1 \oplus 0 \oplus \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \oplus 1 \oplus 0.$$

Now compressing by the appropriate central projections, we easily see that E_1 and E_2 are limits in s.o.t. of elements of $W^*(\{1, e_1, e_2\})$, hence are affiliated with this von Neumann algebra and, in fact, can be written as

$$E_1 = 1 \oplus 1 \oplus \begin{pmatrix} 1 & -\sqrt{t/(1-t)} \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0 \tag{22}$$

$$E_2 = 1 \oplus 0 \oplus \begin{pmatrix} 0 & \sqrt{t/(1-t)} \\ 0 & 1 \end{pmatrix} \oplus 1 \oplus 0, \tag{23}$$

where this has the obvious meaning. It is clear from their definition that E_1 and E_2 are idempotent. This shows (i).

For (ii), we see from the definition that

$$\ker(E_i) = (1 - e_1 \vee e_2)\mathcal{H} + (e_{i'} - e_1 \wedge e_2)\mathcal{H},$$

so we get (18) and (19). Also, (17) is obvious, while (20) follows from (22) and (23).

For (iii), it is straightforward to see that $f\mathcal{H} \cap (e_1\mathcal{H} + e_2\mathcal{H})$ is dense in $f\mathcal{H}$, so letting r be the projection on the right–hand–side of (21), it will suffice to show $f\mathcal{H} \cap (e_1\mathcal{H} + e_2\mathcal{H}) \subseteq r\mathcal{H}$. Let $h \in f\mathcal{H} \cap (e_1\mathcal{H} + e_2\mathcal{H})$. Then $h = \xi_1 + \xi_2 + (e_1 \wedge e_2)h$, for $\xi_i \in (e_i - e_1 \wedge e_2)\mathcal{H}$. We have

$$\xi_i + (e_1 \wedge e_2)h = E_i(h) \in E_i^\sharp(f)\mathcal{H},$$

while using Lemma 2.1.3, we have

$$(e_1 \wedge e_2)h \in (e_1 \wedge e_2)f\mathcal{H} \subseteq (e_1 \wedge e_2 - (1-f) \wedge e_1 \wedge e_2)\mathcal{H}.$$

So $h \in r\mathcal{H}$.

3. Irreducible Horn triples

Horn's inequalities in the $n \times n$ matrices are of the form

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geqslant \sum_{k \in K} \gamma_k. \tag{24}$$

for certain triples (I,J,K) of subsets of $\{1,\ldots,n\}$. In [7], Horn defined sets T_r^n of triples (I,J,K) of subsets of $\{1,\ldots,n\}$ of the same cardinality r, by the following recursive procedure. By convention, a subset I of $\{1,\ldots,n\}$ is indexed in increasing order:

$$I = \{i_1,\ldots,i_r\}, \qquad i_1 < i_2 < \cdots < i_r. \tag{25}$$

Set

$$U_r^n = \left\{ (I,J,K) \,\middle|\, \sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \frac{r(r+1)}{2} \right\}. \tag{26}$$

When $r = 1$, set $T_1^n = U_1^n$. Otherwise, let

$$T_r^n = \left\{ (I,J,K) \in U_r^n \,\middle|\, \sum_{f\in F} i_f + \sum_{g\in G} j_g \leqslant \sum_{h\in H} k_h + \frac{p(p+1)}{2}, \right.$$
$$\left. \text{for all } p < r \text{ and } (F,G,H) \in T_p^r \right\}. \tag{27}$$

We will consider a reformulation of Horn's sets T_r^n, which was used also in [8]. Let σ_n be the permutation of $\{1,\ldots,n\}$ given by $\sigma_n(i) = n+1-i$. Thus, if I is indexed as in (25) and if we use the same convention for indexing $\sigma_n(I)$, namely

$$\sigma_n(I) = \{\tilde{i}_1,\ldots,\tilde{i}_r\}, \qquad \tilde{i}_1 < \tilde{i}_2 < \cdots < \tilde{i}_r,$$

then $i_j = n+1-\tilde{i}_{r+1-j}$. We let

$$\widetilde{T}_r^n = \{(\sigma_n(I),\sigma_n(J),K) \mid (I,J,K) \in T_r^n\}.$$

Reformulating Horn's definition, these sets are recursively defined as follows. Let $\widetilde{U}_r^n$ be the set consisting of triples (I,J,K) of subsets of $\{1,\ldots,n\}$ with $|I| = |J| = |K| = r$ by

$$\widetilde{U}_r^n = \left\{ (I,J,K) \,\middle|\, \sum_{i\in I} i + \sum_{j\in J} j + \sum_{k\in K} k = \frac{r(4n-r+3)}{2} \right\}. \tag{28}$$

If $r = 1$, then we have $\widetilde{T}_r^n = \widetilde{U}_r^n$, while for $r \in \{2,\ldots,n-1\}$, we have

$$\widetilde{T}_r^n = \left\{ (I,J,K) \in U_r^n \,\middle|\, \sum_{f\in F} i_f + \sum_{g\in G} j_g + \sum_{h\in H} k_h \geqslant \frac{p(4n-p+3)}{2}, \right.$$
$$\left. \text{for all } p < r \text{ and } (F,G,H) \in \widetilde{T}_p^r \right\}. \tag{29}$$

Now, for $(I,J,K) \in \widetilde{T}_r^n$, the corresponding Horn inequality is

$$\sum_{i\in I} \alpha_{n+1-i} + \sum_{j\in J} \beta_{n+1-j} \geqslant \sum_{k\in K} \gamma_k.$$

This reformulation of the Horn inequalities has certain advantages. As is apparent from the symmetry of (28) and (29), the set $\widetilde{T}_r^n$ is invariant under permuting the three

sets I, J and K. Moreover, Proposition 3.5 and the reduction procedure resulting from Lemma 3.6 are more natural in this alternative expression of the Horn inequalities.

In [8], S. Therianos and R.C. Thompson proved that many Horn inequalities in T_r^n can be reduced to inequalities in T_r^{n-1}. We will prove that similar results hold in finite von Neumann algebras.

For future use in this section, we record the following integration–by–parts formula for Riemann–Stieltjes integrals, which is well known and easily proved.

LEMMA 3.1. *Let $f : [0,1] \to \mathbf{R}$ be continuous and let $\lambda : [0,1] \to \mathbf{R}$ be monotone and assume λ is (one–sided) continuous at 0 and 1. Then the Riemann–Stieltjes integrals $\int_0^1 \lambda(t)df(t)$ and $\int_0^1 f(t)d\lambda(t)$ exist, and we have*

$$\int_0^1 \lambda(t)df(t) = \lambda(1)f(1) - \lambda(0)f(0) - \int_0^1 f(t)d\lambda(t).$$

DEFINITION 3.2. Let $\mathscr{M}$ be a diffuse, finite von Neumann algebra with a fixed faithful normal tracial state τ. A *flag* in $\mathscr{M}$ is a linearly ordered family $e = (e_t)_{0 \leqslant t \leqslant 1}$ of projections in $\mathscr{M}$ such that $\tau(e_t) = t$ for all t.

A *superflag* in $\mathscr{M}$ is a family $f = (f_t)_{0 \leqslant t \leqslant 1}$ of projections in $\mathscr{M}$ such that $f_s \leqslant f_t$ whenever $s \leqslant t$ and $\tau(f_t) \geqslant t$ for all $t \in [0,1]$.

PROPOSITION 3.3. *If $f = (f_t)_{0 \leqslant t \leqslant 1}$ is a superflag in $\mathscr{M}$, then there is a flag $e = (e_t)_{0 \leqslant t \leqslant 1}$ in $\mathscr{M}$ such that $e_t \leqslant f_t$ for all t.*

Proof. Let $\mathscr{S}$ be the set of sets of projections of $\mathscr{M}$ such that for any $S \in \mathscr{S}$ and any $t \in [0,1]$, $f_t \in S$, and for all $p, q \in S$, either $p \geqslant q$ or $q \geqslant p$.

The set $\mathscr{S}$ is a Zorn inductive set for the obvious order given by inclusion. Let $\widetilde{S}$ be a maximal element. The set of values $\tau(p), p \in \widetilde{S}$ is closed by maximality. Suppose, to obtain a contradiction, that this set is not all of $[0,1]$. Let $t \in [0,1]$ be a value that is not attained, and let

$$t_- = \sup\{\tau(p) \mid p \in \widetilde{S}, \tau(p) < t\}, \; t_+ = \inf\{\tau(p) \mid p \in \widetilde{S}, \tau(p) > t\},$$

so that we have $t_- < t < t_+$. Let $p_\pm$ in $\widetilde{S}$ such that $\tau(p_\pm) = t_\pm$. By elementary properties of finite diffuse von Neumann algebras, there is a projection $p \in \mathscr{M}$ between p_- and p_+ such that $\tau(p) = t$. This contradicts maximality of $\widetilde{S}$.

To construct the flag, for each t, let e_t be the unique $p \in \widetilde{S}$ such that $\tau(p) = t$.

Property P_n below is the von Neumann algebra analogue of Therianos and Thompson's property of the same name (which applied to matrices).

DEFINITION 3.4. Let r and n be positive integers with $r \leqslant n$. Consider a triple (I,J,K) of subsets of $\{1,\ldots,n\}$, each having cardinality r. Write

$$\begin{aligned} I &= \{i_1,\ldots,i_r\}, & i_1 < i_2 < \cdots < i_r \\ J &= \{j_1,\ldots,j_r\}, & j_1 < j_2 < \cdots < j_r \\ K &= \{k_1,\ldots,k_r\}, & k_1 < k_2 < \cdots < k_r. \end{aligned} \tag{30}$$

We say (I,J,K) has *property P_n* if whenever e, f and g are flags in any finite von Neumann algebra $(\mathscr{M},\tau)$, there exists a projection $p \in \mathscr{M}$ such that

$$\tau(p) = \frac{r}{n} \tag{31}$$

and for all $\ell \in \{1,2,\ldots,r\}$, we have

$$\tau(e_{\frac{i_\ell}{n}} \wedge p) \geqslant \frac{\ell}{n}, \qquad \tau(f_{\frac{j_\ell}{n}} \wedge p) \geqslant \frac{\ell}{n}, \qquad \tau(g_{\frac{k_\ell}{n}} \wedge p) \geqslant \frac{\ell}{n} \tag{32}$$

We say that (I,J,K) has *property AP_n* if whenever e, f and g are flags in any finite von Neumann algebra $(\mathscr{M},\tau)$, and whenever $\varepsilon > 0$, there is a projection $p \in \mathscr{M}$ such that

$$\tau(p) \leqslant \frac{r}{n} + \varepsilon$$

and for all $\ell \in \{1,2,\ldots,r\}$, the inequalities (32) hold.

The following result is analogous to well–known facts in $n \times n$ matrices. The proof in the case of property P_n was can easily be found in [1], and the approximate result follows straightforwardly. For convenience, we write a proof pointing to the appropriate parts of [1].

PROPOSITION 3.5. *If $(I,J,K) \in \widetilde{T}_r^n$ has property P_n or, more generally, property AP_n, then the Horn inequality corresponding to $(\sigma_n(I), \sigma_n(J), K)$ holds in every finite von Neumann algebra.*

Proof. First suppose that (I,J,K) has property P_n. Let $\widetilde{I} = \sigma_n(I)$, $\widetilde{J} = \sigma_n(J)$. We must show, given any finite von Neumann algebra $(\mathscr{M},\tau)$ and any $a,b \in \mathscr{M}_{s.a.}$, that we have

$$\int_{\omega_{\widetilde{I}}} \lambda_a(t)\,dt + \int_{\omega_{\widetilde{J}}} \lambda_b(t)\,dt \geqslant \int_{\omega_K} \lambda_{a+b}(t)\,dt. \tag{33}$$

But $\omega_{\widetilde{I}} = 1 - \omega_I := \{1 - t \mid t \in \omega_I\}$ and $\lambda_a(t) = -\lambda_{-a}(1-t)$, so letting $x = -a$, $y = -b$ and $z = a + b$, the inequality (33) becomes

$$\int_{\omega_I} \lambda_x(t)\,dt + \int_{\omega_J} \lambda_y(t)\,dt + \int_{\omega_K} \lambda_z(t)\,dt \leqslant 0,$$

which must be proved for all $x,y,z \in \mathscr{M}_{s.a}$ such that $x + y + z = 0$.

Let E_x, E_y and E_z be the spectral measures of x, y and z. As described on page 115 of [1], there are flags e, f and g in $\mathscr{M}$ such that

$$x = \int_0^1 \lambda_x(t)\,de_t, \qquad y = \int_0^1 \lambda_y(t)\,df_t, \qquad z = \int_0^1 \lambda_z(t)\,dg_t$$

where these integrals are the operator–valued analogues of Riemann–Stieltjes integrals, and, for all $t \in [0,1]$, we have

$$E_x((\lambda_x(t),\infty)) \leqslant e_t, \qquad E_y((\lambda_y(t),\infty)) \leqslant f_t, \qquad E_z((\lambda_z(t),\infty)) \leqslant g_t.$$

Consider the nondecreasing function W_I on $[0,1]$ which at t takes value equal to the Lebesgue measure of $\omega_I \cap [0,t]$. Then W_I is piecewise linear, has slope 1 on intervals $(\frac{i-1}{n}, \frac{i}{n})$ for $i \in I$ (thus, at points of ω_I) and has slope 0 elsewhere. Furthermore,

$$\int_{\omega_I} \lambda_x(t)\,dt = \int_0^1 \lambda_x(t)\,dW_I(t), \tag{34}$$

where the right–hand–side is the Riemann–Stieltjes integral.

Using that (I,J,K) has property P_n, let $p \in \mathscr{M}$ be a projection satisfying (31) and (32).Using (32), we get

$$\tau(p \wedge e_t) \geqslant W_I(t) \tag{35}$$

whenever $t = \frac{i}{n}$ with $i \in I$. Moreover, taking $0 \leqslant s \leqslant t \leqslant 1$ and using Proposition 2.1.1, we have

$$\begin{aligned}
\tau(p \wedge e_s) &= \tau((p \wedge e_t) \wedge e_s) \\
&= \tau(p \wedge e_t) + \tau(e_s) - \tau((p \wedge e_t) \vee e_s) \\
&\geqslant \tau(p \wedge e_t) + \tau(e_s) - \tau(e_t) \\
&= \tau(p \wedge e_t) - (t - s).
\end{aligned}$$

This implies both that $\tau(p \wedge e_t)$ is a continuous function of t and that (35) holds at all points $t \in \omega_I$ and, of course, at $t = 0$, where both sides are zero. However, since $W_I(t)$ is constant elsewhere and since $\tau(p \wedge e_t)$ is increasing, the inequality (35) holds for all $t \in [0,1]$. We define $\lambda_x(1)$ to make λ_x continuous from the right at 1. Using Lemma 3.1 and that we have

$$\begin{aligned}
W_I(0) &= 0 = \tau(p \wedge e_0) \\
W_I(1) &= \frac{r}{n} = \tau(p \wedge e_1)
\end{aligned}$$

we get

$$\begin{aligned}
\int_0^1 \lambda_x(t)\,dW_I(t) &= \lambda_x(1)\frac{r}{n} + \int_0^1 W_I(t)\,d(-\lambda_x)(t) \tag{36} \\
&\leqslant \lambda_x(1)\frac{r}{n} + \int_0^1 \tau(p \wedge e_t)\,d(-\lambda_x)(t) \\
&= \int_0^1 \lambda_x(t)\,d(\tau(p \wedge e_t)),
\end{aligned}$$

where the above inequality is because $-\lambda_x$ is nondecreasing and the inequality (35) holds. However, by Proposition 2.1 of [1], we have

$$\int_0^1 \lambda_x(t)\,d\tau(p \wedge e_t) \leqslant \tau(xp).$$

Putting this together with (34) and (36), we have

$$\int_{\omega_I} \lambda_x(t)\,dt \leqslant \tau(xp).$$

Arguing similarly for y and z, we get

$$\int_{\omega_I} \lambda_x(t)\,dt + \int_{\omega_J} \lambda_y(t)\,dt + \int_{\omega_K} \lambda_z(t)\,dt \leqslant \tau((x+y+z)p) = 0,$$

as required.

Now suppose (I,J,K) has property AP_n. Letting $\varepsilon > 0$, we may argue as above, except that instead of being able to choose p so that (31) and (32) are satisfied, in place of the equality (31) we may only assume

$$\tau(p) \leqslant \frac{r}{n} + \varepsilon.$$

Now instead of getting $\int_0^1 \lambda_x(t)\,dW_I(t) \leqslant \int_0^1 \lambda_x(t)\,d(\tau(p \wedge e_t))$ as we did in (36), we get

$$\int_0^1 \lambda_x(t)\,dW_I(t) \leqslant |\lambda_x(1)|\varepsilon + \int_0^1 \lambda_x(t)\,d(\tau(p \wedge e_t)).$$

Using $|\lambda_x(1)| \leqslant \|x\|$ and arguing as above, we get

$$\int_{\omega_I} \lambda_x(t)\,dt + \int_{\omega_J} \lambda_y(t)\,dt + \int_{\omega_K} \lambda_z(t)\,dt \leqslant \varepsilon(\|x\| + \|y\| + \|z\|).$$

Letting ε tend to zero yields the desired inequality.

The following lemma is an analogue for finite von Neumann algebras of Lemma 1 of [8]. We will use it to reduce the set of Horn inequalities that must be verified in finite von Neumann algebras.

Let

$$h_x(y) = \begin{cases} 0, & y \leqslant x \\ 1, & y > x. \end{cases}$$

LEMMA 3.6. *Let* $1 \leqslant r \leqslant n$ *be integers. Let* (I,J,K) *be a triple of subsets of* $\{1,\ldots,n\}$ *satisfying* (30)*and assume this triple has property* P_n, *respectively, property* AP_n. *Also set* $i_0 = j_0 = k_0 = 0$. *Suppose* $u,v,w \in \{0,1,\ldots,r\}$ *are such that*

$$i_u + j_v + k_w \leqslant n. \tag{37}$$

Set
$$i'_y = i_y + h_u(y)\,,\ j'_y = j_y + h_v(y)\,,\ k'_y = k_y + h_w(y) \qquad (y \in \{1,\ldots,r\}).$$
and let
$$I' = \{i'_1,\ldots,i'_r\}, \quad J' = \{j'_1,\ldots,j'_r\}, \quad K' = \{k'_1,\ldots,k'_r\}.$$
Then (I',J',K') *has property* P_{n+1}, *respectively, property* AP_{n+1}.

Proof. Let $(\mathscr{M},\tau)$ be a diffuse, finite von Neumann algebra and let e, f and g be any flags in $\mathscr{M}$. Suppose (I,J,K) has property P_n. From (37), we have

$$\tau(e_{\frac{i_u}{n+1}} \vee f_{\frac{j_v}{n+1}} \vee g_{\frac{k_w}{n+1}}) \leqslant \frac{n}{n+1}.$$

Let $q \in \mathcal{M}$ be a projection such that $\tau(q) = \frac{n}{n+1}$ and

$$e_{\frac{i_u}{n+1}} \vee f_{\frac{j_v}{n+1}} \vee g_{\frac{k_w}{n+1}} \leqslant q.$$

Then $q \wedge e_t = e_t$ if $t \leqslant \frac{i_u}{n+1}$ and, for all t, $\tau(q \wedge e_t) \geqslant t - \frac{1}{n+1}$. Similar results hold for f and g. Define

$$e'_t = \begin{cases} e_{\frac{nt}{n+1}}, & 0 \leqslant t \leqslant \frac{i_u}{n} \\ e_{\frac{nt+1}{n+1}} \wedge q, & \frac{i_u}{n} < t \leqslant 1 \end{cases}$$

$$f'_t = \begin{cases} f_{\frac{nt}{n+1}}, & 0 \leqslant t \leqslant \frac{j_v}{n} \\ f_{\frac{nt+1}{n+1}} \wedge q, & \frac{j_v}{n} < t \leqslant 1, \end{cases}$$

$$g'_t = \begin{cases} g_{\frac{nt}{n+1}}, & 0 \leqslant t \leqslant \frac{k_w}{n} \\ g_{\frac{nt+1}{n+1}} \wedge q, & \frac{k_w}{n} < t \leqslant 1. \end{cases}$$

Then in the cut–down von Neumann algebra $q\mathcal{M}q$, equipped with the rescaled trace $\frac{n+1}{n}\tau|_{q\mathcal{M}q}$, e', f' and g' are superflags. Invoking Proposition 3.3, let $\tilde{e}$, $\tilde{f}$ and $\tilde{g}$ be flags in $q\mathcal{M}q$ such that $\tilde{e}_t \leqslant e'_t$, $\tilde{f}_t \leqslant f'_t$ and $\tilde{g}_t \leqslant g'_t$ for all $t \in [0,1]$. Then we have

$$\tilde{e}_t = e'_t = e_{\frac{nt}{n+1}}, \qquad (0 \leqslant t \leqslant \frac{i_u}{n}) \tag{38}$$

$$\tilde{e}_t \leqslant e'_t = e_{\frac{nt+1}{n+1}} \wedge q, \qquad (\frac{i_u}{n} < t \leqslant 1). \tag{39}$$

By the assumption that (I,J,K) has property P_n, there is a projection $p \in q\mathcal{M}q$ such that

$$\frac{n+1}{n}\tau(p) \leqslant \frac{r}{n} \tag{40}$$

and, for all $y \in \{1,\ldots,r\}$, we have

$$\frac{n+1}{n}\tau(\tilde{e}_{\frac{i_y}{n}} \wedge p) \geqslant \frac{y}{n}; \; \frac{n+1}{n}\tau(\tilde{f}_{\frac{j_y}{n}} \wedge p) \geqslant \frac{y}{n}; \; \frac{n+1}{n}\tau(\tilde{g}_{\frac{k_y}{n}} \wedge p) \geqslant \frac{y}{n}. \tag{41}$$

We will show that p is the desired projection for (I',J',K') to have property P_{n+1}. We have $\tau(p) \leqslant \frac{r}{n+1}$. If $y \in \{1,\ldots,u\}$, then $i'_y = i_y$ and using (38) with $t = \frac{i_y}{n}$ and (41), we get

$$\tau(e_{\frac{i'_y}{n+1}} \wedge p) \geqslant \frac{y}{n+1}, \tag{42}$$

while if $y \in \{u+1,\ldots,r\}$, then $i'_y = i_y + 1$, so using (39) with $t = \frac{i_y}{n}$ and that $p \leqslant q$, we have

$$\tilde{e}_{\frac{i_y}{n}} \wedge p \leqslant e_{\frac{i'_y}{n+1}} \wedge p,$$

and from (41) we get (42) also in this case. In a similar manner, we get

$$\tau(f_{\frac{j'_y}{n+1}} \wedge p) \geqslant \frac{y}{n+1} \text{ and } \tau(g_{\frac{k'_y}{n+1}} \wedge p) \geqslant \frac{y}{n+1}$$

for all $y \in \{1, \ldots, r\}$. Thus, (I', J', K') has property P_{n+1}.

In the case that (I, J, K) has only property AP_n, the same argument applies, except that, given $\varepsilon > 0$, instead of (40) we get

$$\frac{n+1}{n} \tau(p) \leqslant \frac{r}{n} + \varepsilon$$

and this yields

$$\tau(p) \leqslant \frac{r}{n+1} + \frac{n}{n+1} \varepsilon.$$

REMARK 3.7. Lemma 3.6 provides a reduction procedure with respect to properties P_n and AP_n, in the following sense. Let $(I, J, K) \in \widetilde{T}_r^n$. Suppose there are $u, v, w \in \{0, \ldots, r\}$ such that all of the following four statements hold:

$$u = r \text{ or } i_{u+1} - i_u \geqslant 2 \tag{43}$$
$$v = r \text{ or } j_{v+1} - j_v \geqslant 2 \tag{44}$$
$$w = r \text{ or } k_{w+1} - k_w \geqslant 2 \tag{45}$$
$$i_u + j_v + k_w \leqslant n - 1, \tag{46}$$

where again we set $i_0 = j_0 = k_0 = 0$. Then Lemma 3.6 applies, and to verify that (I, J, K) has property P_n, respectively, AP_n, it will suffice to show that $(\widetilde{I}, \widetilde{J}, \widetilde{K})$ has property P_{n-1}, respectively, AP_{n-1}, where

$$\widetilde{I} = (\tilde{i}_1, \ldots, \tilde{i}_r), \qquad \widetilde{J} = (\tilde{j}_1, \ldots, \tilde{j}_r), \qquad \widetilde{K} = (\tilde{k}_1, \ldots, \tilde{k}_r) \tag{47}$$

are given by

$$\tilde{i}_p = \begin{cases} i_p, & 1 \leqslant p \leqslant u \\ i_p - 1, & u < p \leqslant r, \end{cases} \qquad \tilde{j}_p = \begin{cases} j_p, & 1 \leqslant p \leqslant v \\ j_p - 1, & v < p \leqslant r, \end{cases}$$

$$\tilde{k}_p = \begin{cases} k_p, & 1 \leqslant p \leqslant w \\ k_p - 1, & w < p \leqslant r. \end{cases} \tag{48}$$

In fact, we will only concern ourselves with this reduction procedure under the additional hypothesis

$$u + v + w = r, \tag{49}$$

which is quite natural because it insures that $(I, J, K) \in \widetilde{U}_r^n$ implies $(\widetilde{I}, \widetilde{J}, \widetilde{K}) \in \widetilde{U}_r^{n-1}$. In fact, we will soon show that $(I, J, K) \in \widetilde{T}_r^n$ implies $(\widetilde{I}, \widetilde{J}, \widetilde{K}) \in \widetilde{T}_r^{n-1}$ for this reduction procedure under the additional hypothesis (49), and, even more, that Littlewood–Richardson coefficients are preserved.

An important part of the solution of Horn's conjecture was to relate Horn's triples $(I, J, K) \in T_r^n$ to Littlewood–Richardson coefficients. If I is a set of r distinct positive integers, written as in (30), then we let

$$\rho_r(I) = (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1).$$

Note that $\rho_r(I) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ consists of integers satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0. \tag{50}$$

We let $\mathbf{N}_{0,\geqslant}^r$ denote the set of r–tuples $(\lambda_1, \ldots, \lambda_r)$ of integers satisfying (50), and note that ρ_r is a bijection from the set of subsets of $\mathbf{N}$ having cardinality r onto $\mathbf{N}_{0,\geqslant}^r$. For $n, r \in \mathbf{N}$, $n \geqslant r$, let

$$\Lambda_r^n = \{(\lambda, \mu, \nu) = (\rho_r(I), \rho_r(J), \rho_r(K)) \in (\mathbf{N}_{0,\geqslant}^r)^3 \mid (I, J, K) \in T_r^n\},$$

where T_r^n is the usual set of Horn triples. Using Thm. 12 of [6], we easily see

$$\Lambda_r^n = \left\{(\lambda, \mu, \nu) \in (\mathbf{N}_{0,\geqslant}^r)^3 \,\middle|\, \sum_{p=1}^r (\lambda_p + \mu_p) = \sum_{p=1}^r \nu_p, \quad \nu_1 \leqslant n - r, \quad c_{\lambda,\mu}^\nu \neq 0 \right\},$$

where $c_{\lambda,\mu}^\nu$ is the Littlewood–Richardson coefficient, which is a nonnegative integer. (See, for example, [5] for more about these.)

The map

$$\Phi_r^n : (I, J, K) \mapsto (\rho_r(\sigma_n(I)), \rho_r(\sigma_n(J)), \rho_r(K))$$

is an injective map from the set of triples of subsets of $\{1, 2, \ldots, n\}$, each with cardinality r, to $(\mathbf{N}_{0,\geqslant}^r)^3$ and restricts to a bijection from $\widetilde{T}_r^n$ onto Λ_r^n.

DEFINITION 3.8. Let (I, J, K) be a triple of subsets of $\{1, \ldots, n\}$, with $|I| = |J| = |K| = r$. The *Littlewood–Richardson coefficient* of (I, J, K), denoted $c^{(n)}(I, J, K)$, is equal to the Littlewood–Richardson coefficient $c_{\lambda,\mu}^\nu$, where $(\lambda, \mu, \nu) = \Phi_r^n((I, J, K))$.

As already remarked, if $(I, J, K) \in \widetilde{T}_r^n$ then all the triples

$$(I, K, J), \quad (J, I, K), \quad (J, K, I), \quad (K, I, J), \quad (K, J, I)$$

are also in $\widetilde{T}_r^n$. So at least the property $c^{(n)}(I, J, K) > 0$ is invariant under permuting the three sets I, J and K. We now show that the Littlewood–Richardson coefficient is itself invariant.

LEMMA 3.9. *The Littlewood–Richardson coefficient $c^{(n)}(I, J, K)$ is invariant under permutation of the three arguments.*

Proof. By definition, $c^{(n)}(I, J, K) = c_{\lambda,\mu}^\nu$ is the number of components of type V_ν that one finds in $V_\lambda \otimes V_\mu$, where V_λ, V_μ, V_ν are irreducible rational representations of $GL(r, \mathbb{C})$. In other words, it is

$$\dim \mathrm{Hom}_{GL(r,\mathbb{C})}(V_\nu, V_\lambda \otimes V_\mu).$$

The contragredient representation of V_ν is the representation of highest weights $(1 - k_1, \ldots, r - k_r)$. Following the representation theory conventions, we shall denote by $\bar{V}_\nu$ this representation. The fact that V_ν is irreducible implies by Schur's lemma that

$\bar{V}_\nu \otimes V_\nu$ contains one and only one copy of the trivial representation ε (of highest weight $(0,0,\ldots,0)$).

Observe also that the determinant representation is the representation of highest weight $(1,\ldots,1)$, and more generally, the power l of the determinant representation is the irreducible representation of highest weight $(l,\ldots,l)$. The fact that powers of the determinant representation are of dimension one implies that when tensored with any irreducible representation of highest weight $(x_1,\ldots,x_r)$, they yield an other irreducible representation of highest weight $(x_1+l,\ldots,x_r+l)$.

This implies that $\bar{V} \otimes \det^{n-r}$ has highest weight of type

$$(n+1-k_1-r,\ldots,n+1-k_r-r),$$

and that $\det^{n-r} \otimes \bar{V}_\nu \otimes V_\nu$ contains one and only one copy of the determinant representation $\det^{n-r}$.

We are interested in the dimension of the $GL(r,\mathbb{C})$ - Hom space

$$\mathrm{Hom}_{GL(r,\mathbb{C})}(V_\nu, V_\lambda \otimes V_{mu}):$$

from the above facts it turns out that this dimension is exactly the same as that of the dimension of

$$\mathrm{Hom}_{GL(r,\mathbf{C})}(\det^{n-r}, \det^{n-r} \otimes \bar{V}_\nu \otimes V_\lambda \otimes V_\mu).$$

The action by permutation of sets I,J,K in $\tilde{T}_r^n$ corresponds to the permutation of legs of the tensor $V_\lambda \otimes V_\mu \otimes (\det^{n-r} \otimes \bar{V}_\nu)$.

The fact that the fusion rules of tensor product of groups are abelian implies that the dimension of the Hom spaces are unchanged, so that $c^{(n)}(I,J,K)$ remains unchanged under permutation of indices.

We now show that the reduction procedure of Remark 3.7 preserves Littlewood–Richardson coefficients.

PROPOSITION 3.10. *Let $(I,J,K) \in \tilde{T}_r^n$ and suppose there are $u,v,w \in \{0,\ldots,r\}$ such that*

$$u+v+w = r \tag{51}$$
$$u = r \text{ or } i_{u+1} - i_u \geqslant 2$$
$$v = r \text{ or } j_{v+1} - j_v \geqslant 2$$
$$w = r \text{ or } k_{w+1} - k_w \geqslant 2$$
$$i_u + j_v + k_w \leqslant n-1, \tag{52}$$

where we set $i_0 = j_0 = k_0 = 0$. Let $\tilde{I}, \tilde{J}, \tilde{K}$ be as defined in (47) and (3.7).Then

$$c^{(n-1)}(\tilde{I},\tilde{J},\tilde{K}) = c^{(n)}(I,J,K).$$

Proof. Note that $\tilde{I}$, $\tilde{J}$ and $\tilde{K}$ are subsets of $\{1,\ldots,n-1\}$. Let

$$(\lambda,\mu,\nu) = \Phi_r^n(I,J,K) \text{ and } (\tilde{\lambda},\tilde{\mu},\tilde{\nu}) = \Phi_r^{n-1}(\tilde{I},\tilde{J},\tilde{K}). \tag{53}$$

Then for $p \in \{1, \ldots, r\}$ we have

$$\lambda_p = n - r - i_p + p \, , \ \mu_p = n - r - j_p + p \ \text{and} \ \nu_p = k_{r+1-p} - (r + 1 - p).$$

Let $a = u$, $b = v$ and $c = r - w$. Then (51) gives $c = a + b$. From (3.7) and (53), we get

$$\tilde{\lambda}_p = \begin{cases} \lambda_p - 1, & 1 \leqslant p \leqslant a, \\ \lambda_p, & a < p \leqslant r \end{cases}$$

$$\tilde{\mu}_p = \begin{cases} \mu_p - 1, & 1 \leqslant p \leqslant b, \\ \mu_p, & b < p \leqslant r \end{cases}$$

$$\tilde{\nu}_p = \begin{cases} \nu_p - 1, & 1 \leqslant p \leqslant c, \\ \nu_p, & c < p \leqslant r. \end{cases}$$

We must show

$$c_{\tilde{\lambda},\tilde{\mu}}^{\tilde{\nu}} = c_{\lambda,\mu}^{\nu}. \tag{54}$$

Since $c_{\lambda,\mu}^{\nu} = c_{\mu,\lambda}^{\nu}$ (see [6] or, indeed, Lemma 3.9), and since the statement of the lemma is invariant under interchanging the roles of λ and μ when we also interchange a and b, it follows that if we prove the lemma in some given case $a = a_0$ and $b = b_0$, then we may conclude that it also holds in the case $a = b_0$ and $b = a_0$.

The Littlewood–Richardson coefficient $c_{\lambda,\mu}^{\nu}$ is equal to the number of fillings of $\nu \backslash \lambda$ according to μ, as described on page 221 of Fulton's article [6]. Thus, if we let f_{ℓ}^k denote the number of times k appears in the ℓth row, then the fillings of $\nu \backslash \lambda$ according to μ are the choices of nonnegative integers $(f_{\ell}^k)_{1 \leqslant k \leqslant \ell}$ such that the following hold:

$$\lambda_{\ell} + \sum_{k=1}^{\ell} f_{\ell}^k = \nu_{\ell} \qquad (1 \leqslant \ell \leqslant r) \tag{55}$$

$$\sum_{\ell=k}^{r} f_{\ell}^k = \mu_k \qquad (1 \leqslant k \leqslant r) \tag{56}$$

$$\lambda_{\ell+1} + \sum_{k=1}^{p+1} f_{\ell+1}^k \leqslant \lambda_{\ell} + \sum_{k=1}^{p} f_{\ell}^k \qquad (0 \leqslant p < \ell < r) \tag{57}$$

$$\sum_{\ell=k+1}^{p+1} f_{\ell}^{k+1} \leqslant \sum_{\ell=k}^{p} f_{\ell}^k \qquad (1 \leqslant k \leqslant p < r). \tag{58}$$

Indeed, (56) is the condition Fulton lists as (iii), (57) is equivalent to Fulton's (ii), and (58) is equivalent to Fulton's (iv).

Suppose $(\tilde{f}_{\ell}^k)_{1 \leqslant k \leqslant \ell \leqslant r}$ is a filling of $\tilde{\nu} \backslash \tilde{\lambda}$ according to $\tilde{\mu}$ and let

$$f_{\ell}^k = \begin{cases} \tilde{f}_{\ell}^k + 1, & \text{if } 1 \leqslant k \leqslant b \text{ and } \ell = k + a \\ \tilde{f}_{\ell}^k, & \text{otherwise.} \end{cases} \tag{59}$$

We will show that the map

$$(\tilde{f}_\ell^k)_{1\leqslant k\leqslant\ell\leqslant r} \mapsto (f_\ell^k)_{1\leqslant k\leqslant\ell\leqslant r} \tag{60}$$

is a bijection from the set of fillings of $\tilde{v}\backslash\tilde{\lambda}$ according to $\tilde{\mu}$ onto the set of fillings of $v\backslash\lambda$ according to μ. It is straightforward to show that the "tilde" version of each of the equalities and inequalities (55)–(58) (i.e., where each λ, μ, v and f_ℓ^k is replaced by $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{v}$ and $\tilde{f}_\ell^k$, respectively) implies the "non–tilde" version of the same. Here we give further information about these implications:

$(55)_{11}$	$1\leqslant\ell\leqslant c$	$(55)_{00}$	$c<\ell\leqslant r$
$(56)_{11}$	$1\leqslant k\leqslant b$	$(56)_{00}$	$b<k\leqslant r$
$(57)_{11}$	$1\leqslant\ell\leqslant a$	$(57)_{11}$	$a<\ell<c, p\geqslant\ell-a$
$(57)_{00}$	$a<\ell\leqslant c, p<\ell-a$	$(57)_{01}$	$\ell=c<r, p\geqslant\ell-a=b$
$(57)_{00}$	$c<\ell<r$	$(58)_{11}$	$1\leqslant k<b, p\geqslant k+a$
$(58)_{00}$	$1\leqslant k\leqslant b, p<k+a$	$(58)_{01}$	$k=b, p\geqslant k+a=c$
$(58)_{00}$	$b<k<r.$		

The subscripts above indicate by how much the left– and right–hand–sides of the corresponding equations are incremented when changing from $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{v}$ and $\tilde{f}_\ell^k$ to λ, μ, v and f_ℓ^k, respectively. Thus, for example, the line containing $(58)_{01}$ indicates that when $k=b$ and $p\geqslant k+a$ and when we pass from

$$\sum_{\ell=k+1}^{p+1}\tilde{f}_\ell^{k+1} \leqslant \sum_{\ell=k}^{p}\tilde{f}_\ell^k$$

to the inequality (58) by substituting f_ℓ^k for $\tilde{f}_\ell^k$, the value of the right–hand–side increases by 1 while the value of the left–hand–side remains unchanged. The fact that the equalities and inequalities all remain valid when making these substitutions shows that the map (60) with f_ℓ^k defined by (59) is an injection from the set of fillings of $\tilde{v}\backslash\tilde{\lambda}$ according to $\tilde{\mu}$ into the set of fillings of $v\backslash\lambda$ according to μ.

To show that this map is onto is the same as showing that whenever $(f_\ell^k)_{1\leqslant k\leqslant\ell\leqslant r}$ is a filling of $v\backslash\lambda$ according to μ, then we have

$$f_{a+k}^k > 0 \qquad (k\in\{1,\ldots,b\}), \tag{61}$$

and if $c<r$, then

$$\lambda_{c+1}+\sum_{k=1}^{p+1}f_{c+1}^k < \lambda_c+\sum_{k=1}^{p}f_c^k \qquad (p\in\{b,b+1,\ldots,c-1\}) \tag{62}$$

and if $b>0$ and $c<r$, then

$$\sum_{\ell=b+1}^{p+1}f_\ell^{b+1} < \sum_{\ell=b}^{p}f_\ell^b \qquad (p\in\{c,c+1,\ldots,r-1\}), \tag{63}$$

where we see (61) from the definition (59) and we see (62) and (63) from the lines with $(57)_{01}$ and $(58)_{01}$, above. For enough values of a and b to prove the lemma, we will use (52) as well as (55)–(58) to show that the inequalities (61), (62) and (63) hold.

CASE 3.10.1. $a = b = 0$.

Then $(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) = (\lambda, \mu, \nu)$ and (54) holds trivially.

CASE 3.10.2. $b = 0$, $1 \leqslant a \leqslant r$.

If $a = r$, then there is nothing to check, so assume $a < r$. We have $a = c \in \{1, \ldots, r-1\}$ and (52) becomes

$$\nu_{a+1} < \lambda_a, \tag{64}$$

while $\tilde{f}_\ell^k = f_\ell^k$ for all k and ℓ. It will suffice to show that (64) implies

$$\lambda_{a+1} + \sum_{k=1}^{p+1} f_{a+1}^k < \lambda_a + \sum_{k=1}^{p} f_a^k \qquad (p \in \{0, 1, \ldots, a-1\}).$$

But we have

$$\lambda_{a+1} + \sum_{k=1}^{p+1} f_{a+1}^k \leqslant \lambda_{a+1} + \sum_{k=1}^{a+1} f_{a+1}^k = \nu_{a+1} < \lambda_a \leqslant \lambda_a + \sum_{k=1}^{p} f_a^k,$$

and Case 3.10.2 is proved.

CASE 3.10.3. $1 \leqslant b < r$ and $a = r - b$.

Then $c = r$. From (52) we get $n - r < \lambda_a + \mu_b$, so $\nu_1 < \lambda_a + \mu_b$ and therefore, using (55) and (56), we have

$$\lambda_1 + f_1^1 < \lambda_a + f_b^b + f_{b+1}^b + \cdots + f_r^b. \tag{65}$$

We must only verify that (61) holds. Suppose, for contradiction, that

$$f_{a+k'}^{k'} = 0 \tag{66}$$

for some $k' \in \{1, \ldots, b\}$. Then we get

$$f_b^b + f_{b+1}^b + \cdots + f_r^b \leqslant f_{b-1}^{b-1} + f_b^{b-1} + \cdots + f_{r-1}^{b-1} \tag{67}$$

$$\leqslant f_{b-2}^{b-2} + f_{b-1}^{b-2} + \cdots + f_{r-2}^{b-2} \tag{68}$$

$$\leqslant \cdots$$

$$\leqslant f_{k'}^{k'} + f_{k'+1}^{k'} + \cdots + f_{a+k'}^{k'} \tag{69}$$

$$= f_{k'}^{k'} + f_{k'+1}^{k'} + \cdots + f_{a+k'-1}^{k'} \tag{70}$$

$$\leqslant f_{k'-1}^{k'-1} + f_{k'}^{k'-1} + \cdots + f_{a+k'-2}^{k'-1} \tag{71}$$

$$\leqslant \cdots$$

$$\leqslant f_1^1 + f_2^1 + \cdots + f_a^1, \tag{72}$$

where in (67)–(69) we have used (58) with $k = b-1$ and $p = r-1$, then with $k = b-2$ and $p = r-2$, successively to $k = k'$ and $p = r-b+k' = a+k'$, where (70) results from (66) and where for (71)–(72) we used (58) with $k = k'-1$ and $p = a+k'-2$, then with $k = k'-2$ and $p = a+k'-3$, successively to $k = 1$ and $p = a$. But using (57) with $p = 1$ and, successively, $\ell = a-1, \ell = a-2, \ldots, \ell = 1$, we have

$$\lambda_a + f_a^1 + f_{a-1}^1 + \cdots + f_2^1 \leqslant \lambda_{a-1} + f_{a-1}^1 + f_{a-2}^1 + \cdots + f_2^1 \tag{73}$$

$$\leqslant \cdots$$

$$\leqslant \lambda_2 + f_2^1 \tag{74}$$

$$\leqslant \lambda_1, \tag{75}$$

which together with (67)–(72) gives

$$\lambda_a + f_b^b + f_{b+1}^b + \cdots + f_r^b \leqslant \lambda_1 + f_1^1.$$

Combining all of this with (65), we get

$$\lambda_1 + f_1^1 < \lambda_1 + f_1^1,$$

a contradiction. Thus, Case 3.10.3 is proved.

CASE 3.10.4. $1 \leqslant a \leqslant b$ and $a + b < r$.

Then (52) yields $\nu_1 + \nu_{a+b+1} < \lambda_a + \mu_b$, or

$$\lambda_1 + f_1^1 + \lambda_{a+b+1} + f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1} < \lambda_a + f_b^b + f_{b+1}^b + \cdots + f_r^b. \tag{76}$$

We now show that (61) must hold. Supposing for contradiction that we have $f_{k'+a}^{k'} = 0$ for some $k' \in \{1, \ldots, b\}$ and arguing as we did in (67)–(72), we get

$$f_b^b + f_{b+1}^b + \cdots + f_{b+a}^b \leqslant f_{b-1}^{b-1} + f_b^{b-1} + \cdots + f_{b+a-1}^{b-1}$$

$$\leqslant \cdots$$

$$\leqslant f_{k'}^{k'} + f_{k'+1}^{k'} + \cdots + f_{k'+a}^{k'}$$

$$= f_{k'}^{k'} + f_{k'+1}^{k'} + \cdots + f_{k'+a-1}^{k'}$$

$$\leqslant f_{k'-1}^{k'-1} + f_{k'}^{k'-1} + \cdots + f_{k'+a-2}^{k'-1}$$

$$\leqslant \cdots$$

$$\leqslant f_1^1 + f_2^1 + \cdots + f_a^1.$$

Using this in (76), we get

$$\lambda_1 + f_1^1 + \lambda_{a+b+1} + f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1}$$

$$< \lambda_a + f_a^1 + f_{a-1}^1 + \cdots + f_2^1 + f_1^1 \tag{77}$$

$$+ f_{a+b+1}^b + f_{a+b+2}^b + \cdots + f_r^b.$$

Using (73)–(75) in (77) yields

$$\lambda_{a+b+1} + f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1} < f^b_{a+b+1} + f^b_{a+b+2} + \cdots + f^b_r. \tag{78}$$

Adding $\lambda_r + f^1_r + f^2_r + \cdots + f^{b-1}_r$ to the right–hand–side of (78) and using (57) with $p = b-1$ and, successively, $\ell = r-1, \ell = r-2, \ldots, \ell = a+b+2$, we get

$$
\begin{aligned}
\lambda_{a+b+1} &+ f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1} \\
&< (\lambda_r + f^1_r + f^2_r + \cdots + f^{b-1}_r) + f^b_r + f^b_{r-1} + \cdots + f^b_{a+b+1} \\
&\leqslant (\lambda_{r-1} + f^1_{r-1} + f^2_{r-1} \cdots + f^{b-1}_{r-1}) + f^b_{r-1} + f^b_{r-2} + \cdots f^b_{a+b+1} \\
&\leqslant \cdots \\
&\leqslant \lambda_{a+b+1} + f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^b_{a+b+1}.
\end{aligned}
$$

From this, we get

$$f^{b+1}_{a+b+1} + f^{b+2}_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1} < 0,$$

which is a contradiction. Thus, (61) is proved.

We now show that (62) holds. If it fails for some $p = p' \in \{b, b+1, \ldots, b+a-1\}$, then we must have

$$\lambda_{a+b+1} + f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^{p'+1}_{a+b+1} = \lambda_{a+b} + f^1_{a+b} + f^2_{a+b} + \cdots + f^{p'}_{a+b}$$

and then from (76) we get

$$
\begin{aligned}
\lambda_1 + f^1_1 &+ (\lambda_{a+b} + f^1_{a+b} + f^2_{a+b} + \cdots + f^{p'}_{a+b}) \\
&+ (f^{p'+2}_{a+b+1} + f^{p'+3}_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1}) < \lambda_a + f^b_b + f^b_{b+1} + \cdots + f^b_r. \tag{79}
\end{aligned}
$$

Again using (58) in the familiar way, we obtain

$$
\begin{aligned}
f^b_b + f^b_{b+1} + \cdots f^b_{b+a-1} &\leqslant f^{b-1}_{b-1} + f^{b-1}_b + \cdots + f^{b-1}_{b+a-2} \\
&\leqslant \cdots \\
&\leqslant f^1_1 + f^1_2 + \cdots + f^1_a.
\end{aligned}
$$

With (79), this yields

$$
\begin{aligned}
\lambda_1 + f^1_1 + (\lambda_{a+b} + f^1_{a+b} + f^2_{a+b} + \cdots + f^{p'}_{a+b}) + (f^{p'+2}_{a+b+1} + f^{p'+3}_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1}) \\
< (\lambda_a + f^1_a + f^1_{a-1} + \cdots + f^1_2) + f^1_1 + (f^b_{a+b} + f^b_{a+b+1} + \cdots + f^b_r). \tag{80}
\end{aligned}
$$

Using (73)–(75) in (80), we get

$$
\begin{aligned}
(\lambda_{a+b} + f^1_{a+b} + f^2_{a+b} + \cdots + f^{p'}_{a+b}) + (f^{p'+2}_{a+b+1} + f^{p'+3}_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1}) \\
< f^b_{b+a} + f^b_{b+a+1} + \cdots + f^b_r. \tag{81}
\end{aligned}
$$

Adding $\lambda_r + f_r^1 + \cdots + f_r^{b-1}$ to the right–hand–side of (81) and using (57) with $p = b-1$ and, successively, $\ell = r-1, \ell = r-2, \ldots, \ell = a+b$, we get

$$(\lambda_{a+b} + f_{a+b}^1 + f_{a+b}^2 + \cdots + f_{a+b}^{p'}) + (f_{a+b+1}^{p'+2} + f_{a+b+1}^{p'+3} + \cdots + f_{a+b+1}^{a+b+1})$$
$$< \lambda_{a+b} + f_{a+b}^1 + f_{a+b}^2 + \cdots + f_{a+b}^b.$$

Thus, we get

$$(f_{a+b}^{b+1} + f_{a+b}^{b+2} + \cdots + f_{a+b}^{p'}) + (f_{a+b+1}^{p'+2} + f_{a+b+1}^{p'+3} + \cdots + f_{a+b+1}^{a+b+1}) < 0,$$

which is a contradiction, and (62) is proved.

Finally, we show that (63) holds. If it fails for some $p = p' \in \{a+b, a+b+1, \ldots, r-1\}$, then we have

$$f_{b+1}^{b+1} + f_{b+2}^{b+1} + \cdots + f_{p'+1}^{b+1} = f_b^b + f_{b+1}^b + \cdots + f_{p'}^b.$$

From this and (76), we have

$$\begin{aligned}
\lambda_1 + f_1^1 &+ \lambda_{a+b+1} + f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1} \\
&< \lambda_a + (f_{b+1}^{b+1} + f_{b+2}^{b+1} + \cdots + f_{p'+1}^{b+1}) + (f_{p'+1}^b + f_{p'+2}^b + \cdots + f_r^b).
\end{aligned} \tag{82}$$

Arguing as before, we have

$$\begin{aligned}
f_{b+1}^{b+1} + f_{b+2}^{b+1} + \cdots + f_{b+a}^{b+1} &\leqslant f_b^b + f_{b+1}^b + \cdots + f_{b+a-1}^b \\
&\leqslant \cdots \\
&\leqslant f_1^1 + f_2^1 + \cdots + f_a^1.
\end{aligned}$$

Using this in (82), we get

$$\begin{aligned}
\lambda_1 + f_1^1 &+ \lambda_{a+b+1} + f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1} \\
&< (\lambda_a + f_a^1 + f_{a-1}^1 + \cdots + f_2^1) + f_1^1 \\
&\quad + (f_{a+b+1}^{b+1} + f_{a+b+2}^{b+1} + \cdots + f_{p'+1}^{b+1}) + (f_{p'+1}^b + f_{p'+2}^b + \cdots + f_r^b).
\end{aligned}$$

Using this and (73)–(75), we get

$$\begin{aligned}
\lambda_{a+b+1} &+ f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1} \\
&< (f_{a+b+1}^{b+1} + f_{a+b+2}^{b+1} + \cdots + f_{p'+1}^{b+1}) + (f_{p'+1}^b + f_{p'+2}^b + \cdots + f_r^b). \tag{83}
\end{aligned}$$

adding $\lambda_r + f_r^1 + f^2 + r + \cdots + f_r^{b-1}$ to the right–hand–side of (83) and using (57) with $p = b-1$ and, successively, $\ell = r-1, \ell = r-2, \ldots, \ell = p'+1$, we get

$$\begin{aligned}
\lambda_{a+b+1} &+ f_{a+b+1}^1 + f_{a+b+1}^2 + \cdots + f_{a+b+1}^{a+b+1} \\
&< (f_{a+b+1}^{b+1} + f_{a+b+2}^{b+1} + \cdots + f_{p'+1}^{b+1}) + (\lambda_{p'+1} + f_{p'+1}^1 + f_{p'+1}^2 + \cdots + f_{p'+1}^b) \\
&= \lambda_{p'+1} + f_{p'+1}^1 + f_{p'+1}^2 + \cdots + f_{p'+1}^{b+1} + (f_{p'}^{b+1} + f_{p'-1}^{b+1} + \cdots + f_{a+b+1}^{b+1}).
\end{aligned}$$

Now using (57) with $p = b$ and, successively, $\ell = p'$, $\ell = p' - 1,\ldots,$ $\ell = a+b+1$, we get

$$\lambda_{a+b+1} + f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1} < \lambda_{a+b+1} + f^1_{a+b+1} + f^2_{a+b+1} + \cdots + f^{b+1}_{a+b+1}.$$

This implies

$$f^{b+2}_{a+b+1} + f^{b+3}_{a+b+1} + \cdots + f^{a+b+1}_{a+b+1} < 0,$$

which is a contradiction. Thus, Case 3.10.4 is proved.

We have now proved enough cases so that, if we also consider also the cases obtained from them by interchanging a and b, then the lemma is proved.

DEFINITION 3.11. Let $(I,J,K) \in \widetilde{T}^n_r$. We say (I,J,K) is *TT–reducible*, (or simply reducible) if the method of reduction described in Remark 3.7 can be performed, namely, if there are $u,v,w \in \{0,\ldots,r\}$ satisfying $u+v+w = r$ and such that (43)–(46) hold, (where we take $i_0 = j_0 = k_0 = 0$). Naturally enough, if (I,J,K) is not TT–reducible, then we may say it is *TT–irreducible* (or simply irreducible).

LEMMA 3.12. *Let $n \geqslant r \geqslant 2$ be integers. If $(I,J,K) \in \widetilde{T}^n_r$ is irreducible, then $i_r = j_r = k_r = n$.*

Proof. Suppose $(I,J,K) \in \widetilde{T}^n_r$ and $i_r < n$. We will show that (I,J,K) is reducible. In view of the symmetry of $\widetilde{T}^n_r$, this will suffice to prove the lemma.

Let $u = r$ and $v = w = 0$. Then (43) and (46) both hold. To show that (I,J,K) is reducible, it will suffice to show $j_1 \geqslant 2$, for then (44) will hold and by symmetry also (45) will hold. Inspecting (28), we must have $(r,1,r) \in \widetilde{U}^r_1 = \widetilde{T}^r_1$. Considering (29) and taking $p = 1$, we must have $i_r + j_1 + k_r \geqslant 2n+1$, so

$$j_1 \geqslant (2n+1) - i_r - k_r \geqslant (2n+1) - (n-1) - n = 2.$$

LEMMA 3.13. *Suppose $(I,J,K) \in \widetilde{T}^n_r$ satisfies $i_r = j_r = k_r = n$ and that there are $u,v,w \in \{0,1,\ldots,r\}$ such that $u+v+w = r$ and (46) holds, namely, $i_u + j_v + k_w \leqslant n-1$. Then (43)–(45) must hold.*

Proof. It will suffice to show that (43) holds. From (46), we have $u \leqslant r - 1$.

CASE 3.13.1. *$v \neq 0$ and $w = 0$.*

Then $(\{u+1\},\{v\},\{r\}) \in \widetilde{T}^r_1$, and from (29) we must have

$$i_{u+1} + j_v + k_r \geqslant 2n+1,$$

which yields

$$i_{u+1} - i_u = (i_{u+1} + j_v) - (i_u + j_v) \geqslant (n+1) - (n-1) = 2.$$

and (43) holds.

CASE 3.13.2. $v \neq 0$ *and* $w \neq 0$.

Then $(\{u+1,r\},\{v,r\},\{w,r\}) \in \widetilde{T}_2^r$, and from (29) we must have

$$i_{u+1} + i_r + j_v + j_r + k_w + k_r \geqslant 4n + 1,$$

which yields

$$i_{u+1} - i_u = (i_{u+1} + j_v + k_w) - (i_u + j_v + k_w) \geqslant (n+1) - (n-1) = 2.$$

and (43) holds.

The other case, $v = 0$ and $w \neq 0$, follows from symmetry considerations.

The above two lemmas imply the following.

PROPOSITION 3.14. *Let* $(I,J,K) \in \widetilde{T}_r^n$. *Then* (I,J,K) *is irreducible if and only if* $u,v,w \in \{0,1,\ldots,r\}$ *and* $u+v+w=r$ *implies*

$$i_u + j_v + k_w \geqslant n, \tag{84}$$

where we set $i_0 = j_0 = k_0 = 0$.

The next result describes the irreducible elements of $\widetilde{T}_3^n$ for arbitrary $n \geqslant 3$, which are particularly nice. Compare this to the first part of the proof of Theorem 1 of [8].

PROPOSITION 3.15. *Let* $1 \leqslant r \leqslant n$ *be integers and let* $(I,J,K) \in \widetilde{T}_r^n$.

 (i) If $r = 1$, *then* (I,J,K) *is irreducible if and only if* $n = 1$ *and*

$$(I,J,K) = (\{1\},\{1\},\{1\}).$$

 (ii) If $r = 2$, *then* (I,J,K) *is irreducible if and only if* $n = 2$ *and*

$$(I,J,K) = (\{1,2\},\{1,2\},\{1,2\}).$$

 (iii) If $r = 3$, *then* (I,J,K) *is irreducible if and only if*

$$(I,J,K) = (\{m,m+\ell,n\},\{m,m+\ell,n\},\{m,m+\ell,n\}) \tag{85}$$

 for some integers ℓ *and* m *satisfying* $1 \leqslant \ell \leqslant m$ *and* $2m+\ell = n$.

Proof. Part (i) follows immediately from Proposition 3.14.

Part (ii) follows easily from Proposition 3.14 because if $(I,J,K) \in \widetilde{T}_2^n$ irreducible, then

$$i_2 = j_2 = k_2 = n, \tag{86}$$

while we also have

$$i_1 + j_1 + k_1 = n + 1$$

from (28) and (86) and, again from Proposition 3.14 we get

$$i_1 + j_1 \geqslant n \,,\ i_1 + k_1 \geqslant n \text{ and } j_1 + k_1 \geqslant n. \tag{87}$$

Adding up (87), we get

$$2(n+1) = 2(i_1 + j_1 + k_1) \geqslant 3n,$$

so $n \leqslant 2$.

Now, for part (iii), suppose $r = 3$. $\widetilde{T}_1^3$ consists of the triples $(\{1\}, \{3\}, \{3\})$ and $(\{2\}, \{2\}, \{3\})$ and the four other triples obtained by permutations, while $\widetilde{T}_2^3$ consists of the triples $(\{1,2\}, \{2,3\}, \{2,3\})$ and $(\{1,3\}, \{1,3\}, \{2,3\})$ and their permutations. Thus, $\widetilde{T}_3^n$ is the set of triples (I, J, K) satisfying

$$i_1 + i_2 + i_3 + j_1 + j_2 + j_3 + k_1 + k_2 + k_3 = 6n \tag{88}$$

$$i_1 + i_2 + j_2 + j_3 + k_2 + k_3 \geqslant 4n+1, \qquad i_2 + i_3 + j_1 + j_2 + k_2 + k_3 \geqslant 4n+1,$$
$$i_2 + i_3 + j_2 + j_3 + k_1 + k_2 \geqslant 4n+1, \qquad i_1 + i_3 + j_1 + j_3 + k_2 + k_3 \geqslant 4n+1,$$
$$i_1 + i_3 + j_2 + j_3 + k_1 + k_3 \geqslant 4n+1, \qquad i_2 + i_3 + j_1 + j_3 + k_1 + k_3 \geqslant 4n+1,$$
$$i_1 + j_3 + k_3 \geqslant 2n+1, \qquad i_3 + j_1 + k_3 \geqslant 2n+1, \qquad i_3 + j_3 + k_1 \geqslant 2n+1,$$
$$i_2 + j_2 + k_3 \geqslant 2n+1, \qquad i_2 + j_3 + k_2 \geqslant 2n+1, \qquad i_3 + j_2 + k_2 \geqslant 2n+1.$$

One checks that all (I, J, K) of the form (85) belong to $\widetilde{T}_3^n$, because the above hold, and are irreducible, because if $u + v + w = 3$, then (84) holds.

Let $(I, J, K) \in \widetilde{T}_3^n$ be irreducible. Then, by Proposition 3.14,

$$i_3 = j_3 = k_3 = n \text{ and} \tag{89}$$

$$i_2 + j_1 \geqslant n \,, \qquad\qquad i_2 + k_1 \geqslant n \,, \qquad\qquad i_1 + j_2 \geqslant n \tag{90}$$
$$i_1 + k_2 \geqslant n \,, \qquad\qquad j_2 + k_1 \geqslant n \,, \qquad\qquad j_1 + k_2 \geqslant n.$$

But adding up (90), we get

$$2(i_1 + i_2 + j_1 + j_2 + k_1 + k_2) \geqslant 6n,$$

which, in light of (88) and (89) must be an equality. Thus, all of (90) must be equalities, and these imply $i_1 = j_1 = k_1 = m$ and $i_2 = j_2 = k_2 = m + \ell$ for some integers $m, \ell \geqslant 1$ satisfying $2m + \ell = n$. Using (84) with $u = v = w = 1$, we have $3m \geqslant n$, which implies $\ell \leqslant m$.

PROPOSITION 3.16. *Let $(I, J, K) \in \widetilde{T}_3^n$ be the irreducible Horn triple in* (85). *Then the Littlewood–Richardson coefficient $c^{(n)}(I, J, K)$ is equal to ℓ.*

Proof. Let $(\lambda, \mu, \nu) = \Phi_r^n(I, J, K)$. We have

$$\lambda = \mu = (m + \ell - 2, m - 1, 0), \qquad \nu = (2m + \ell - 3, m + \ell - 2, m - 1) \tag{91}$$

Figure 1: A typical filling of $\nu\backslash\lambda$ according to μ.

$\lambda_1 = m+\ell-2$			$f_1^1 = m-1$
$\lambda_2 = m-1$	f_2^1	f_2^2	
f_3^1	f_3^2		

and $c^{(n)}(I,J,K) = c^\nu_{\lambda,\mu}$ equals the number of fillings of $\nu\backslash\lambda$ according to μ, or, equivalently, the number of choices of nonnegative integers $f_1^1, f_2^1, f_2^2, f_3^1, f_3^2, f_3^3$ such that the following hold:

$$\lambda_1 + f_1^1 = \nu_1 \quad \lambda_2 + f_2^1 + f_2^2 = \nu_2 \qquad \lambda_3 + f_3^1 + f_3^2 + f_3^3 = \nu_3 \tag{92}$$

$$f_1^1 + f_2^1 + f_3^1 = \mu_1 \qquad f_2^2 + f_3^2 = \mu_2 \qquad\qquad\qquad f_3^3 = \mu_3 \tag{93}$$

$$\lambda_2 + f_2^1 \leqslant \lambda_1 \qquad \lambda_3 + f_3^1 \leqslant \lambda_2 \qquad \lambda_3 + f_3^1 + f_3^2 \leqslant \lambda_2 + f_2^1 \tag{94}$$

$$f_2^2 \leqslant f_1^1 \qquad f_2^2 + f_3^2 \leqslant f_1^1 + f_2^1 \qquad\qquad\quad f_3^3 \leqslant f_2^2. \tag{95}$$

Using also the values specified in (91), from $\lambda_1 + f_1^1 = \nu_1$ and, respectively, $f_3^3 = \mu_3$, we get

$$f_1^1 = m-1 \text{ and } f_3^3 = 0.$$

A typical filling is pictured in Figure 1. From (92) and (93), we get

$$f_2^1 + f_2^2 = \ell - 1 \qquad\qquad f_3^1 + f_3^2 = m - 1$$
$$f_2^1 + f_3^1 = \ell - 1 \qquad\qquad f_2^2 + f_3^2 = m - 1,$$

which yield $f_3^1 = f_2^2 = \ell - 1 - f_2^1$ and $f_3^2 = m - \ell + f_2^1$.

The filling is determined by the choice of $f_2^1 \in \{0,1,\ldots,\ell-1\}$ and each such choice leads to all the equalities and inequalities in (92)–(95) being satisfied. So we have $c^\nu_{\lambda,\mu} = \ell$.

Let us say that a Horn triple $(I,J,K) \in \widetilde{T}^n_r$ is *LR–minimal* (or simply minimal) if the Littlewood–Richardson coefficient $c^{(n)}(I,J,K)$ is equal to 1. It is known (see Theorem 13 of [6]) that the set of Horn inequalities corresponding to LR–minimal Horn triples determines all of the other Horn inequalities. For the purpose of showing that all Horn inequalities hold in all finite von Neumann algebras, it will suffice to verify that all the LR–minimal Horn inequalities hold in all finite von Neumann algebras.

COROLLARY 3.17. *Let $n \geqslant 3$ be an integer. Then $\widetilde{T}^n_3$ has an element that is an LR–minimal and irreducible Horn triple (I,J,K) if and only if $n = 2m+1$ is odd, and then the unique such triple is*

$$(I,J,K) = (\{m, m+1, n\}, \{m, m+1, n\}, \{m, m+1, n\}).$$

We were unable to find a nice characterization of the LR–minimal and irreducible Horn triples in $\widetilde{T}_4^n$. However, the complete list of such (up to permutation of I, J and K) for several values of n is given in Table 1. These were found using the Littlewood–Richardson Calculator package [3] of Anders Skovsted Buch and Maple.

Table 1: LR–minimal and irreducible triples in $\widetilde{T}_4^n$.

n	(I,J,K)
4	$(\{1,2,3,4\},\{1,2,3,4\},\{1,2,3,4\})$
5	$\emptyset$
6	$(\{1,3,4,6\},\{2,3,5,6\},\{2,3,5,6\}),\ (\{2,3,4,6\},\{2,3,4,6\},\{2,3,5,6\})$
7	$(\{2,3,5,7\},\{2,4,5,7\},\{3,4,5,7\}),\ (\{2,3,6,7\},\{2,4,5,7\},\{2,4,5,7\})$
8	$(\{1,4,5,8\},\{3,4,7,8\},\{3,4,7,8\}),\ (\{2,3,5,8\},\{3,5,6,8\},\{3,5,6,8\}),$
	$(\{2,3,6,8\},\{2,5,6,8\},\{3,5,6,8\}),\ (\{2,4,5,8\},\{3,4,6,8\},\{3,4,7,8\}),$
	$(\{2,4,5,8\},\{3,4,6,8\},\{3,5,6,8\}),\ (\{3,4,5,8\},\{3,4,5,8\},\{3,4,7,8\}),$
	$(\{3,4,5,8\},\{3,4,6,8\},\{3,4,6,8\})$
9	$(\{2,3,6,9\},\{3,6,7,9\},\{3,6,7,9\}),\ (\{2,5,6,9\},\{3,4,7,9\},\{3,6,7,9\}),$
	$(\{2,5,6,9\},\{3,4,7,9\},\{4,5,7,9\}),\ (\{2,5,6,9\},\{3,4,8,9\},\{3,5,7,9\}),$
	$(\{3,4,6,9\},\{3,5,6,9\},\{3,6,7,9\}),\ (\{3,4,6,9\},\{3,5,6,9\},\{4,5,7,9\}),$
	$(\{3,4,6,9\},\{3,5,7,9\},\{4,5,6,9\}),\ (\{3,4,7,9\},\{3,5,6,9\},\{4,5,6,9\}),$
	$(\{3,5,6,9\},\{3,5,6,9\},\{3,4,8,9\})$

4. Construction of projections

In this section, we exhibit a construct of projections which we use in combination with results of previous sections to prove that all of the LR–minimal Horn inequalities corresponding to triples in $\widetilde{T}_3^n$ for arbitrary n must hold in all finite von Neumann algebras.

LEMMA 4.1. *Let $\mathscr{M}$ be a finite von Neumann algebra with normal, faithful tracial state τ. Suppose $0 < \beta \leqslant \frac{2}{5}$ and $e_1, e_2, e_3 \in \mathscr{M}$ are projections with*

$$\tau(e_i) \geqslant \frac{1}{2} + \frac{\beta}{4}, \qquad (i \in \{1,2,3\}). \tag{96}$$

Then there is a projection $p \in \mathscr{M}$ satisfying $\tau(p) \leqslant \frac{3}{2}\beta$ and $\tau(p \wedge e_i) \geqslant \beta$ for all $i \in \{1,2,3\}$.

Proof. Let $q_0 = e_1 \wedge e_2 \wedge e_3$.

CASE 4.1.1. $\tau(q_0) \geqslant \beta$.

To prove the lemma in this case, we simply let $p \leqslant q_0$ be such that $\tau(p) = \beta$.

In the remaining cases, let

$$q_1 = (e_2 \wedge e_3) - q_0\,,\ q_2 = (e_1 \wedge e_3) - q_0 \text{ and } q_3 = (e_1 \wedge e_2) - q_0.$$

We clearly have

$$q_i \wedge q_j = (e_i - q_0) \wedge (e_j - q_0) = 0, \qquad (i \neq j)$$

and, using (8) and (96),

$$\tau(q_0) + \tau(q_i) = \tau(e_j \wedge e_k) \geqslant \frac{\beta}{2}, \tag{97}$$

where $\{i, j, k\} = \{1, 2, 3\}$. We assume, without loss of generality,

$$\tau(q_1) \geqslant \tau(q_2) \geqslant \tau(q_3). \tag{98}$$

CASE 4.1.2. $\tau(q_0) \leqslant \beta$ and $\tau(q_1) + \tau(q_2) \geqslant \beta - \tau(q_0)$

Take projections $q_2' \leqslant q_2$ and $q_3' \leqslant q_3$ such that

$$\tau(q_i') = \min(\tau(q_i), \frac{\beta}{2} - \frac{\tau(q_0)}{2}), \qquad (i \in \{2, 3\})$$

and let $q_1' \leqslant q_1$ be such that

$$\tau(q_1') + \tau(q_2') = \beta - \tau(q_0). \tag{99}$$

Then $\tau(q_3') \leqslant \tau(q_2')$, we have

$$\tau(q_0 + q_2' \vee q_3') \leqslant \beta$$
$$\tau(q_0 + q_1' \vee q_3') \leqslant \beta$$

and we may, therefore, choose projections

$$q_4 \leqslant e_1 - (q_0 + q_2' \vee q_3')$$
$$q_5 \leqslant e_2 - (q_0 + q_1' \vee q_3')$$

so that

$$\begin{aligned}\tau(q_4) &= \beta - \tau(q_0) - \tau(q_2') - \tau(q_3')\\ \tau(q_5) &= \beta - \tau(q_0) - \tau(q_1') - \tau(q_3').\end{aligned} \tag{100}$$

Let

$$p = q_0 \vee q_1' \vee q_2' \vee q_3' \vee q_4 \vee q_5.$$

Then
$$
\begin{aligned}
\tau(p) &\leqslant \tau(q_0) + \tau(q_1') + \tau(q_2') + \tau(q_3') + \tau(q_4) + \tau(q_5) \\
&= 2\beta - \tau(q_0) - \tau(q_3').
\end{aligned}
\tag{101}
$$

If $q_3' = q_3$, then using (97) we get $\tau(p) \leqslant \frac{3}{2}\beta$. On the other hand, if $q_3' \neq q_3$, then $\tau(q_3') = \frac{\beta}{2} - \frac{\tau(q_0)}{2}$ and from (101) we have $\tau(p) \leqslant \frac{3}{2}\beta - \frac{1}{2}\tau(q_0) \leqslant \frac{3}{2}\beta$.

But also
$$
\begin{aligned}
p \wedge e_1 &\geqslant q_0 + (q_2' \vee q_3') + q_4 \\
p \wedge e_2 &\geqslant q_0 + (q_1' \vee q_3') + q_5 \\
p \wedge e_3 &\geqslant q_0 + (q_1' \vee q_2'),
\end{aligned}
$$

which by (100) and (99) gives $\tau(p \wedge e_i) \geqslant \beta$ for all $i \in \{1,2,3\}$. This finishes the proof in Case 4.1.2.

CASE 4.1.3. $\tau(q_0) \leqslant \beta$ and $\tau(q_1) + \tau(q_2) \leqslant \beta - \tau(q_0)$.

Using (8), we have
$$
\begin{aligned}
\tau((e_1 - q_0) \vee (e_2 - q_0)) &= \tau(e_1) + \tau(e_2) - 2\tau(q_0) - \tau(q_3) \\
&\geqslant 1 + \frac{\beta}{2} - 2\tau(q_0) - \tau(q_3)
\end{aligned}
$$

and, using (8) again, we get
$$
\begin{aligned}
\tau((e_3 - (q_1 \vee q_2) - q_0) &\wedge ((e_1 - q_0) \vee (e_2 - q_0))) \\
&\geqslant (\tau(e_3) - \tau(q_1) - \tau(q_2) - \tau(q_0)) \\
&\quad + (1 + \frac{\beta}{2} - 2\tau(q_0) - \tau(q_3)) - (1 - \tau(q_0)) \\
&\geqslant \frac{1}{2} + \frac{3\beta}{4} - \sum_{i=1}^{3} \tau(q_i) - 2\tau(q_0) \\
&\geqslant \frac{1}{2} - \frac{\beta}{4} - \tau(q_1) - \tau(q_2) - \tau(q_0) \tag{102} \\
&\geqslant \beta - \tau(q_1) - \tau(q_2) - \tau(q_0), \tag{103}
\end{aligned}
$$

where (102) follows because the assumptions in this case and the ordering (98) imply $\tau(q_3) \leqslant \frac{\beta}{2} - \frac{\tau(q_0)}{2} \leqslant \beta - \tau(q_0)$ and (103) results from $\beta \leqslant \frac{2}{5}$.

Thus, we may take a projection
$$
f \leqslant (e_3 - (q_1 \vee q_2) - q_0) \wedge ((e_1 - q_0) \vee (e_2 - q_0))
$$

such that
$$
\tau(f) = \beta - \tau(q_1) - \tau(q_2) - \tau(q_0).
\tag{104}
$$

Let us write $E_1 = E(e_1 - q_0, e_2 - q_0)$ and $E_2 = E(e_2 - q_0, e_1 - q_0)$ for the idempotents defined in section 2.3. Let $r_1 = E_1^\sharp(f)$ and $r_2 = E_2^\sharp(f)$. By Lemma 2.3.1, we have $r_i \leqslant e_i - q_0$ and $\tau(r_i) \leqslant \tau(f)$ (for $i = 1, 2$) and

$$f \leqslant r_1 \vee r_2 \vee q_3. \tag{105}$$

Choose any projections

$$s_1 \leqslant e_1 - q_0 - (r_1 \vee q_2 \vee q_3)$$
$$s_2 \leqslant e_2 - q_0 - (r_2 \vee q_1 \vee q_3)$$

such that

$$\begin{aligned} \tau(s_1) &= \beta - \tau(q_0) - \tau(r_1 \vee q_2 \vee q_3) \\ \tau(s_2) &= \beta - \tau(q_0) - \tau(r_2 \vee q_1 \vee q_3). \end{aligned} \tag{106}$$

This is possible because

$$\tau(r_2 \vee q_1 \vee q_3) \leqslant \tau(f) + \tau(q_1) + \tau(q_3) = \beta - \tau(q_0) + \tau(q_3) - \tau(q_2) \leqslant \beta - \tau(q_0)$$

and, similarly, $\tau(r_1 \vee q_2 \vee q_3) \leqslant \beta - \tau(q_0)$. Let

$$p = q_0 \vee q_1 \vee q_2 \vee q_3 \vee r_1 \vee r_2 \vee s_1 \vee s_2.$$

We have

$$\tau(q_2 \vee q_3 \vee r_1 \vee s_1) = \tau(s_1) + \tau(r_1 \vee q_2 \vee q_3) = \beta - \tau(q_0),$$

so

$$\tau(q_0 \vee q_1 \vee q_2 \vee q_3 \vee r_1 \vee s_1) \leqslant \beta + \tau(q_1)$$

and

$$\begin{aligned} \tau(p) &\leqslant \tau(q_0 \vee q_1 \vee q_2 \vee q_3 \vee r_1 \vee s_1 \vee s_2) + \tau(r_2) - \tau(r_2 \wedge (q_1 \vee q_3)) \\ &\leqslant \tau(q_0 \vee q_1 \vee q_2 \vee q_3 \vee r_1 \vee s_1) + \tau(r_2) + \tau(s_2) - \tau(r_2 \wedge (q_1 \vee q_3)) \\ &\leqslant 2\beta - \tau(q_0) + \tau(q_1) + \tau(r_2) - \tau(r_2 \vee q_1 \vee q_3) - \tau(r_2 \wedge (q_1 \vee q_3)) \\ &= 2\beta - \tau(q_0) + \tau(q_1) - \tau(q_1 \vee q_3) \\ &= 2\beta - \tau(q_0) - \tau(q_3) \leqslant \frac{3}{2}\beta, \end{aligned}$$

where for the last inequality we used (97). On the other hand, we have

$$\begin{aligned} p \wedge e_1 &\geqslant q_0 \vee q_2 \vee q_3 \vee r_1 \vee s_1 = q_0 + (q_2 \vee q_3 \vee r_1) + s_1 \\ p \wedge e_2 &\geqslant q_0 \vee q_1 \vee q_3 \vee r_2 \vee s_2 = q_0 + (q_1 \vee q_3 \vee r_2) + s_2, \end{aligned}$$

so from (106) we get $\tau(p \wedge e_i) \geqslant \beta$ for $i = 1, 2$. Using (105), we have

$$p \wedge e_3 \geqslant q_0 \vee q_1 \vee q_2 \vee f = q_0 + (q_1 \vee q_2) + f,$$

so from (104) we have $\tau(p \wedge e_3) \geqslant \beta$. This finishes the proof in Case 4.1.3, and the lemma is proved.

The above lemma applies with $\beta = \frac{2}{n}$ to give the following.

EXAMPLE 4.2. Let $m \geqslant 2$ be an integer and let $n = 2m + 1$. Suppose e_1, e_2, e_3 are projections in a finite von Neumann algebra $\mathscr{M}$ with $\tau(e_i) \geqslant \frac{m+1}{n}$, ($i \in \{1,2,3\}$). Then there is a projection $p \in \mathscr{M}$ with $\tau(p) \leqslant \frac{3}{n}$ and with $\tau(p \wedge e_i) \geqslant \frac{2}{n}$ ($i \in \{1,2,3\}$).

THEOREM 4.3. *Let* $1 \leqslant r \leqslant n$ *be integers and let* $(I,J,K) \in \widetilde{T}_r^n$. *If either* $r \in \{1,2\}$ *or* $r = 3$ *and the triple* (I,J,K) *is LR–minimal, then the Horn inequality corresponding to* (I,J,K) *holds in all finite von Neumann algebras.*

Proof. By Proposition 3.5, it will suffice that each such (I,J,K) has property P_n. It follows from Lemma that every (I,J,K) can be reduced (as in Definition 3.11) to an irreducible triple, which will be LR–minimal if the original triple (I,J,K) is LR–minimal. By Lemma 3.6, it will, therefore, suffice to show that every irreducible triple $(I,J,K) \in \widetilde{T}_r^n$ with $r \in \{1,2\}$ has property P_n, and every irreducible and LR–minimal triple $(I,J,K) \in \widetilde{T}_3^n$ has property P_n.

By Proposition 3.15, for $r=1$ and $r=2$, we only need to verify that $(\{1\},\{1\},\{1\}) \in \widetilde{T}_1^1$ has property P_1 and $(\{1,2\},\{1,2\},\{1,2\}) \in \widetilde{T}_2^2$ has property P_2. But these facts are immediate. For $r = 3$, by Corollary 3.17, we need only see that the triple

$$(\{m,m+1,n\},\{m,m+1,n\},\{m,m+1,n\}) \tag{107}$$

has property P_n, where for integers $m \geqslant 1$ and $n = 2m + 1$. When $m = 1$, this is immediate from the definition. Take $m \geqslant 2$. Let e, f and g be flags in a finite von Neumann algebra $\mathscr{M}$, with specified trace τ. Then . It follows from Lemma 4.1 (see Example 4.2) that there is a projection p in $\mathscr{M}$ such that $\tau(p) = \frac{3}{n}$, and

$$\tau(e_{\frac{m+1}{n}} \wedge p) \geqslant \frac{2}{n}, \qquad \tau(f_{\frac{m+1}{n}} \wedge p) \geqslant \frac{2}{n}, \qquad \tau(g_{\frac{m+1}{n}} \wedge p) \geqslant \frac{2}{n}. \tag{108}$$

$$\tau(e_{\frac{n}{n}} \wedge p) = \tau(f_{\frac{n}{n}} \wedge p) = \tau(g_{\frac{n}{n}} \wedge p) = \tau(p) = \frac{3}{n}.$$

Since $\tau(e_{\frac{m}{n}}) = \tau(e_{\frac{m+1}{n}}) - \frac{1}{n}$ from (8) of Proposition 2.1.1 we get

$$\tau(e_{\frac{m}{n}} \wedge p) = \tau(e_{\frac{m}{n}} \wedge (e_{\frac{m+1}{n}} \wedge p)) = \tau(e_{\frac{m+1}{n}} \wedge p) + \tau(e_{\frac{m}{n}}) - \tau((e_{\frac{m+1}{n}} \wedge p) \vee e_{\frac{m}{n}}) \tag{109}$$

$$\geqslant \tau(e_{\frac{m+1}{n}} \wedge p) + \tau(e_{\frac{m}{n}}) - \tau(e_{\frac{m+1}{n}}) \tag{110}$$

$$= \tau(e_{\frac{m+1}{n}} \wedge p) - \frac{1}{n} \geqslant \frac{1}{n}, \tag{111}$$

and similarly

$$\tau(f_{\frac{m}{n}} \wedge p) \geqslant \frac{1}{n} \qquad \tau(g_{\frac{m}{n}} \wedge p) \geqslant \frac{1}{n}. \tag{112}$$

Now (108)–(112) taken together show that p satisfies the requirements of Definition 3.4 and the triple (107) has property P_n.

We would like to end this section with an argument that we discovered in an attempt to show that the triple

$$(\{2,4,6\},\{2,4,6\},\{2,4,6\}) \in \widetilde{T}_3^6 \tag{113}$$

has property AP_6. This triple is irreducible by Proposition 3.15 and by Proposition 3.16, the corresponding triple (λ, μ, ν) has Littlewood–Richardson coefficient equal to 2. The corresponding Horn inequality,

$$\alpha_1 + \alpha_3 + \alpha_5 + \beta_1 + \beta_3 + \beta_5 \geqslant \gamma_2 + \gamma_4 + \gamma_6,$$

is known to hold in all finite von Neumann algebras. Indeed, in the 6×6 matrices, by the ordering of eigenvalues, we have

$$\alpha_1 + \alpha_3 + \alpha_5 + \beta_1 + \beta_3 + \beta_5 \geqslant \frac{1}{2} \sum_{i=1}^{6} (\alpha_i + \beta_i) = \frac{1}{2} \sum_{i=1}^{6} \gamma_i \geqslant \gamma_2 + \gamma_4 + \gamma_6$$

and clearly a similar argument works in finite von Neumann algebras for integrals of eigenvalue functions. Nonetheless, it is an interesting question whether the triple (113) has property P_6, or at least AP_6. For the latter property, given arbitrary flags e, f and g in a finite von Neumann algebra and given $\varepsilon > 0$, we would need to find a projection p such that

$$\tau(p) \leqslant \frac{1}{2} + \varepsilon$$

$$\tau(e_{\frac{2}{6}} \wedge p) \geqslant \frac{1}{6}, \qquad \tau(f_{\frac{2}{6}} \wedge p) \geqslant \frac{1}{6}, \qquad \tau(g_{\frac{2}{6}} \wedge p) \geqslant \frac{1}{6} \qquad (114)$$

$$\tau(e_{\frac{4}{6}} \wedge p) \geqslant \frac{2}{6}, \qquad \tau(f_{\frac{4}{6}} \wedge p) \geqslant \frac{2}{6}, \qquad \tau(g_{\frac{4}{6}} \wedge p) \geqslant \frac{2}{6}.$$

The following lemma proves this, but under the added hypothesis that the projections from the flags appearing in (114) be in general position. Although we are not able to use this argument to prove that any further Horn inequalities hold in all finite von Neumann algebras (beyond those treated in Theorem 4.3), we hope that the construction of projections (and in particular, the use of "almost invariant subspaces" in the argument) may be of interest.

LEMMA 4.4. *Let* $e_1, e_2, e_3, f_1, f_2, f_3$ *be projections in a finite von Neumann algebra* $\mathscr{M}$ *with normal faithful tracial state* τ, *satisfying*

$$e_i \leqslant f_i, \qquad \tau(e_1) = \frac{1}{3}, \qquad \tau(f_i) = \frac{2}{3}, \qquad (1 \leqslant i \leqslant 3),$$

and let $\varepsilon > 0$. *Assume further that whenever* $\{i, j, k\} = \{1, 2, 3\}$, *we have*

$$e_i \wedge f_j = 0 \tag{115}$$
$$e_k \wedge (e_i \vee e_j) = 0. \tag{116}$$

Then there is a projection $p \in \mathscr{M}$ *such that*

$$\tau(p) \leqslant \frac{1}{2} + \varepsilon,$$

$$\tau(p \wedge e_i) \geqslant \frac{1}{6}, \qquad (1 \leqslant i \leqslant 3)$$

$$\tau(p \wedge f_i) \geqslant \frac{1}{3}, \qquad (1 \leqslant i \leqslant 3).$$

Proof. Throughout, we let $\{i, j, k\} = \{1, 2, 3\}$. Let us first show

$$\tau(f_i \wedge f_j) = \frac{1}{3}. \tag{117}$$

The inequality $\geqslant$ is clear from (8) in Proposition 2.1.1. On the other hand, $e_i \wedge f_j = e_i \wedge (f_i \wedge f_j)$, so again from (8), we get

$$\tau(e_i \wedge f_j) \geqslant \tau(e_i) + \tau(f_i \wedge f_j) - \tau(f_i) = \tau(f_i \wedge f_j) - \frac{1}{3},$$

so from (115) we get $\leqslant$ in (117). By (115), we also get

$$\tau(e_i \vee e_j) = \frac{2}{3}, \tag{118}$$

which in turn yields

$$\tau(f_k \wedge (e_i \vee e_j)) = \frac{1}{3}. \tag{119}$$

Indeed, $\geqslant$ is clear from (8) and (118), while from (116)

$$e_k \wedge (e_i \vee e_j) = e_k \wedge (f_k \wedge (e_i \vee e_j))$$

and (8) we get $\leqslant$ in (119). Let us write $E_i^j = E(e_i, e_j)$, etc., for the idempotents defined in Section 2.3. We have

$$\mathrm{domproj}(E_i^j) = e_i \vee e_j - e_j$$
$$\mathrm{kerproj}(E_i^j) = (1 - e_i \vee e_j) + e_j$$
$$\mathrm{ranproj}(E_i^j) = e_i.$$

Let $S_i^j = E_i^j \cdot (f_k \wedge (e_i \vee e_j))$ be the composition of operators. We have

$$\mathrm{kerproj}(E_i^j) \wedge (f_k \wedge (e_i \vee e_j)) = (\mathrm{kerproj}(E_i^j) \wedge (e_i \vee e_j)) \wedge f_k = e_j \wedge f_k = 0.$$

Thus, we have

$$\mathrm{domproj}(S_i^j) = f_k \wedge (e_i \vee e_j), \qquad \mathrm{ranproj}(S_i^j) = e_i,$$

and we have the picture in Figure 4, where the spokes represent projections of trace $\frac{1}{3}$.

Consider the operator

$$X = S_1^3 T_3^1 S_3^2 T_2^3 S_2^1 T_1^2.$$

Then X goes once around the wheel in Figure 4. Since we have

$$\mathrm{domproj}(T_1^2) = e_1$$
$$\mathrm{ranproj}(T_1^2) = f_3 \wedge (e_1 \vee e_2) = \mathrm{domproj}(S_2^1)$$
$$\mathrm{ranproj}(S_2^1) = e_2 = \mathrm{domproj}(T_2^3)$$
$$\mathrm{ranproj}(T_2^3) = f_1 \wedge (e_2 \vee e_3) = \mathrm{domproj}(S_3^2)$$
$$\mathrm{ranproj}(S_3^2) = e_3 = \mathrm{domproj}(T_3^1)$$
$$\mathrm{ranproj}(T_3^1) = f_2 \wedge (e_1 \vee e_3) = \mathrm{domproj}(S_1^3)$$
$$\mathrm{ranproj}(S_1^3) = e_1,$$

we see

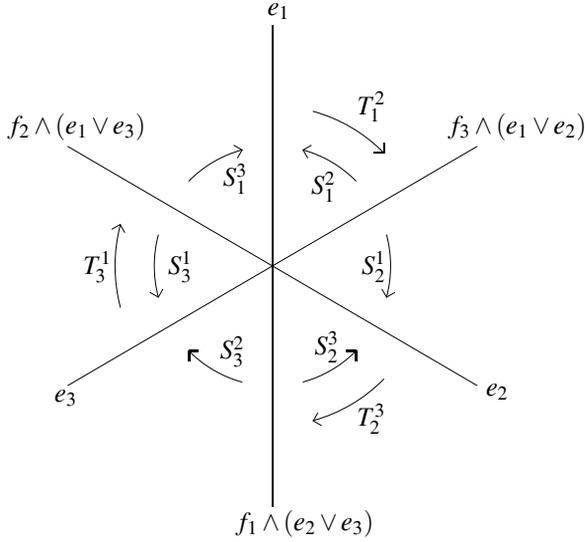$$\text{domproj}(X) = e_1 = \text{ranproj}(X).$$

Figure 2: *Some projections and operators.*

By Proposition 2.2.2, there is a projection $q_1 \leqslant e_1$ such that

$$\tau(q_1) = \frac{1}{6}, \qquad \tau(q_1 \vee X^{\sharp}(q_1)) \leqslant \frac{1}{6} + \varepsilon.$$

Let

$$r_3 = (T_1^2)^{\sharp}(q_1) \,, \; q_2 = (S_2^1)^{\sharp}(r_3) \,, \; r_1 = (T_2^3)^{\sharp}(q_2) \,, \; q_3 = (S_3^2)^{\sharp}(r_1) \,, \text{ and } r_2 = (T_3^1)^{\sharp}(q_3).$$

Thus $(S_1^3)^{\sharp}(r_2) = X^{\sharp}(q_1)$ and $\tau(q_i) = \tau(r_i) = \frac{1}{6}$, $(i = 1, 2, 3)$. Let

$$p = q_1 \vee q_2 \vee q_3 \vee X^{\sharp}(q_1).$$

Then

$$\tau(p) \leqslant \tau(q_1 \vee X^{\sharp}(q_1)) + \tau(q_2) + \tau(q_3) \leqslant \frac{1}{2} + \varepsilon$$

and $p \wedge e_i \geqslant q_i$, so

$$\tau(p \wedge e_i) \geqslant \frac{1}{6}, \quad (i = 1, 2, 3).$$

On the other hand, we have

$$(E_1^2)^{\sharp}(r_3) = (S_1^2)^{\sharp}(r_3) = q_1$$
$$(E_2^1)^{\sharp}(r_3) = (S_2^1)^{\sharp}(r_3) = q_2$$

and from Lemma 2.3.1, we get $r_3 \leqslant q_1 \vee q_2$. Similarly, we get $r_1 \leqslant q_2 \vee q_3$ and $r_2 \leqslant X^\sharp(q_1) \vee q_3$. Thus, for every $k \in \{1,2,3\}$, we have $r_k \leqslant p$ and $f_k \wedge p \geqslant q_k \vee r_k$. Since $q_k \wedge r_k \leqslant e_k \wedge (e_i \vee e_j) = 0$, where $\{i,j,k\} = \{1,2,3\}$, we have

$$\tau(f_k \wedge p) \geqslant \tau(q_k \vee r_k) = \tau(q_k) + \tau(r_k) = \frac{1}{3}$$

and the lemma is proved.

5. Possibilities for construction of a non–embeddable example

This section is speculative and can be skipped without compromising understanding of the rest of the paper.

Suppose you knew, (say, you met a time traveler from the future), that Connes' embedding problem has a negative answer and, even more, that the Horn inequality associated to a triple $(I,J,K) \in T_r^n$ fails to hold in some finite von Neumann algebra. How could you find and describe a finite von Neumann algebra where this Horn inequality fails? In this section, we describe an approach, though it is not one that would be guaranteed to work. We actually attempted to carry out this approach, without success, at the beginning of our work with Horn inequalities in finite von Neumann algebras. We did not benefit from an oracle of any sort, and we chose a Horn inequality (to try to violate in a finite von Neumann algebra) by simple guessing. (The particular one that we chose is, in fact, now known to hold in all finite von Neumann algebras, by results of this paper.)

We seek operators a and b whose distributions are, respectively,

$$\mu_a = \sum_{i=1}^{n} \delta_{\alpha_i} \, , \ \mu_b = \sum_{j=1}^{n} \delta_{\beta_j}$$

and we postulate that $a + b$ has distribution

$$\mu_{a+b} = \sum_{k=1}^{n} \delta_{\gamma_k} \, ,$$

for some real numbers

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n \, , \ \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n \ \text{and} \ \gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n, \qquad (120)$$

where the trace equality

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{n} \gamma_k \qquad (121)$$

holds and Horn's inequality (1) fails. After rescaling, we may suppose

$$1 + \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j = \sum_{k \in K} \gamma_k \qquad (122)$$

In fact, pick some specific values of $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$ such that (120), (121) and (122) all hold. Finding a finite von Neumann algebra in which such a and b can be found is equivalent to finding a positive trace τ on the algebra $\mathbf{C}\langle X, Y \rangle$ of polynomials in noncommuting variables X and Y such that $\tau(1) = 1$, and for all $k \in \mathbf{N}$, we have

$$\tau(X^k) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i^k \ , \ \tau(Y^k) = \frac{1}{n} \sum_{i=1}^{n} \beta_i^k \ , \text{ and} \tag{123}$$

$$\tau((X+Y)^k) = \frac{1}{n} \sum_{i=1}^{n} \gamma_i^k \ . \tag{124}$$

Indeed, such a trace will give rise, via the Gelfand–Naimark–Segal construction, to a Hilbert space and a representation of $\mathbf{C}\langle X, Y \rangle$ whose closure in the strong operator topology is the desired finite von Neumann algebra, with a and b being the images of X and Y under the representation.

For such a trace τ, the moments $\tau(X^k)$ and $\tau(Y^k)$ are, of course, specified by (123) above. It remains to choose values for mixed moments. In light of the trace property, this amounts to choosing values for all expressions of the form

$$\tau(X^{p_1} Y^{q_1} \cdots X^{p_\ell} Y^{q_\ell}) \tag{125}$$

for positive integers ℓ and $p_1, q_1, \ldots, p_\ell, q_\ell$. Of course, the trace condition implies that the value of (125) is unchanged by cyclically permuting the ℓ pairs $(p_1, q_1), \ldots, (p_\ell, q_\ell)$. For convenience, let us say that the expression (125) is in *canonical form* if $p_1 = \max_{1 \leqslant j \leqslant \ell} p_j$ and $q_1 = \max_{\{j | p_j = p_1\}} q_j$ and $p_2 = \max_{\{j | p_j = p_1, q_j = q_1\}} p_{j+1}$, *etc.*, and we choose values of the mixed moments (125) that are in canonical form.

Some linear relations between these moments are implied by the predetermined values found in (124). For example, taking $k = 2, 3, 4$, we get

$$2\tau(XY) = \tau((X+Y)^2) - \tau(X^2) - \tau(Y^2)$$
$$3(\tau(X^2 Y) + \tau(XY^2)) = \tau((X+Y)^3) - \tau(X^3) - \tau(Y^3)$$
$$4(\tau(X^3 Y) + \tau(XY^3)) + 2\tau(XYXY) = \tau((X+Y)^4) - \tau(X^4) - \tau(Y^4).$$

Finally, the positivity of τ is equivalent to the positive semidefinitenesss of every matrix of the form

$$\big(\tau(w_i^* w_j) \big)_{1 \leqslant i, j \leqslant n} \, ,$$

for every finite list $(w_1, \ldots, w_n)$ of distinct words in the free semigroup generated by X and Y, where w_i^* is the word w_i taken in reverse order.

Added in proofs. It has been more recently shown, in [10], that all Horn inequalities hold in all II_1–factors.

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