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REVISITING THE BOURGAIN–TZAFRIRI RESTRICTED INVERTIBILITY THEOREM

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Abstract. We will give some new techniques for working with problems surrounding the Bourgain-Tzafriri Restricted Invertibility Theorem. First we show that the parameters which work in the theorem for all $\|T\| \leqslant 2\sqrt{2}$ closely approximate the parameters which work for all operators. This yields a generalization of the theorem which simultaneously does restricted invertibility on a small partition of the vectors and yields a direct proof that the Bourgain-Tzafriri Conjecture is equivalent to the Feichtinger Conjecture. We also fill in two gaps in the theory involving the relationship between paving results for norm one operators with zero diagonal and restricted invertibility results.

1. Introduction

In 1987 Bourgain and Tzafriri [3] proved one of the most significant theorems in Banach space theory which is known as the *Restricted Invertibility Theorem*.

THEOREM 1.1. (Bourgain-Tzafriri Restricted Invertibility Theorem) *There are universal constants* $0 \leqslant c, A < 1$ *so that whenever* $T : \ell_2^n \to \ell_2^n$ *is a linear operator for which* $\|Te_i\| = 1$ *for* $1 \leqslant i \leqslant n$, *then there exists a subset* $\sigma \subset \{1, 2, \cdots, n\}$ *of cardinality* $|\sigma| \geqslant \frac{cn}{\|T\|^2}$ *so that*

$$\|\sum_{j\in\sigma} a_j Te_j\|^2 \geqslant A \sum_{j\in\sigma} |a_j|^2,$$

for all choices of scalars $\{a_j\}_{j\in\sigma}$.

In the original paper [3], using random methods, it is shown that $c = A = \frac{1}{10^{72}}$ works in the theorem. Joel Tropp [private communication] has informed us that modern methods applied to the original proof of Bourgain and Tzafriri can reduce these constants to $c \geqslant 1/128$ and $A \geqslant 1/(8\sqrt{2\pi})$.

In this paper, we will examine the relationship between the parameters c, A in the Restricted Invertibility Theorem and the norm of the operator $\|T\|$. In particular, we will show that whenever c, A hold in the theorem for all $\|T\| \leqslant 2\sqrt{2}$, then $c/8$, $A/2$

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work in the theorem for all T. In the process, we will see that the Restricted Invertibility Theorem can be extended to a *partition theorem*. That is, we will show that given any operator T of the type in the theorem, we can partition the basis into a fixed finite number of subsets $\{A_j\}_{j=1}^r$ so that $r = \frac{1}{\|T\|^2}$ and for each of these subsets we can choose a fixed percentage $0 < c$ of the vectors satisfying the required inequality. In particular, one of these sets has size $\frac{n}{\|T\|^2}$ and this one is the one given by the Restricted Invertibility Theorem. We will then see that this gives a direct proof that the Bourgain Tzafriri Conjecture and the Feichtinger Conjecture are equivalent (without having to go through the theory of C^*-algebras).

We will also address two gaps in the existing theory. First, we will show that restricted invertibility can be designed to yield $(1+\varepsilon)$-Riesz basic sequences. A special case of this was done earlier by Vershynin [12]. Next, we will give a direct method for passing paving results for norm one operators with zero diagonal to restricted invertibility results.

2. Preliminaries

In this section we give the notation to be used throughout the paper. A family of vectors $\{f_i\}_{i\in I}$ in a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is a *frame* if there are constants $0 < A \leqslant B < \infty$ (called the *lower, upper frame bounds* respectively) so that for every $f \in \mathbb{H}$

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2.$$

If we only assume the upper inequality, we call $\{f_i\}_{i\in I}$ a *B-Bessel sequence*. If $A = B$ this is an *A-tight frame* and if $A = B = 1$ it is a *Parseval frame*. If $\|f_i\| = 1$ for every $i \in I$ this is a *unit norm frame*. The *analysis operator* of the frame is $T(f) = \{\langle f, f_i\rangle\}_{i\in I}$. Its adjoint is the *synthesis operator* $T^* : \ell_2(I) \to \mathbb{H}$ and is given by $T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i$. Now $S = T^*T$ is the *frame operator* and is a positive, self-adjoint, and invertible operator on $\mathbb{H}$ given by:

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

It follows that for all $f \in \mathbb{H}$,

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2$$

If T is an invertible operator we call $\{f_i\}_{i\in I}$ a *Riesz basis*. In this case, for all $f = \{a_i\}_{i=1}^n$ we have

$$A\sum_{i\in I} |a_i|^2 \leqslant \|\sum_{i\in I} a_i Te_i\|^2 \leqslant B\sum_{i\in I} |a_i|^2.$$

The constants A, B are called the *Riesz basis bounds* of $\{f_i\}_{i\in I}$. We call this a $(1+\varepsilon)$-Riesz basic sequence if we have $A = 1-\varepsilon$, $B = 1+\varepsilon$. If $\{f_i\}_{i=1}^M$ is a frame for a finite dimensional space ℓ_2^n with frame operator S, let S have eigenvectors $\{g_i\}_{i=1}^n$

with respective eigenvalues $\|T\|^2 = \|S\| = B = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. It follows that for every $f = \sum_{i=1}^n a_i g_i$ we have

$$Sf = \sum_{i=1}^{n} \lambda_i a_i g_i \text{ and so } \langle Sf, f\rangle = \sum_{i=1}^{n} \lambda_i |a_i|^2.$$

A direct calculation yields:

$$\sum_{i=1}^{M} \|f_i\|^2 = \sum_{j=1}^{n} \lambda_j.$$

For a background on frame theory we refer the reader to [5, 10].

Now let $\{e_i\}_{i=1}^n$ be the unit vector basis of ℓ_2^n and let $T : \ell_2^n \to \ell_2^n$ be any operator on ℓ_2^n. Let $S : \ell_2^n \to \ell_2^n$ be the corresponding frame operator. That is,

$$Sf = \sum_{i=1}^{n} \langle f, Te_i\rangle Te_i.$$

For any $f = \sum_{i=1}^n a_i e_i$,

$$\begin{aligned} \langle Sf, f\rangle &= |\langle T^*Tf, f\rangle \\ &= |\langle Tf, Tf\rangle|^2 \\ &= \|Tf\|^2 \\ &= \|\sum_{i=1}^{n} a_i Te_i\|^2. \end{aligned}$$

If we let $P_{\ell,k}$ be the orthogonal projection of $\{g_j\}_{j=1}^n$ onto $\{g_j\}_{j=\ell}^k$ then $\{P_{\ell,k}Te_i\}_{i=1}^n$ is a frame for its span with frame bounds λ_k, λ_ℓ.

NOTATION 2.1. *If $T : \ell_2^n \to \ell_2^n$ is a linear operator and $\{e_i\}_{i=1}^n$ is the unit vector basis of ℓ_2^n, we denote by $D(T)$ the diagonal operator of the matrix of T with respect to the unit vectors. Also, we say that T has zero diagonal if it's matrix with respect to the unit vector basis has all zero's on the diagonal.*

3. An Extension of the restricted invertibility theorem

In this section we will generalize the restricted invertibility theorem from a *selection theorem* to a *partition theorem*. We will give some applications of this in later sections. For the proof we first recall a result of Berman, Halpern, Kaftal and Weiss [2].

PROPOSITION 3.1. *Let $(a_{ij})_{i,j=1}^n$ be a self-adjoint matrix with non-negative entries and zero diagonal. For every $r \in \mathbb{N}$ there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for every $j = 1, 2, \cdots, r$ and every $i \in A_j$*

$$\sum_{m \in A_j} a_{im} \leqslant \sum_{m \in A_\ell} a_{im}, \text{ for every } 1 \leqslant \ell \neq j \leqslant r. \tag{3.1}$$

We are now ready to generalize the Restricted Invertibility Theorem.

THEOREM 3.2. *Suppose c,A satisfy Theorem 1.1 for all $\|T\| \leqslant 2\sqrt{2}$. Then, for every $T:\ell_2^n \to \ell_2^n$ with $\|Te_i\|=1$ for all $i=1,2,\cdots,n$, there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ with $r=\|T\|^2$ and for every $j=1,2,\cdots,r$ there is a subset $B_j \subset A_j$ with $|B_j| \geqslant \frac{c}{8}|A_j|$ and $\{Te_i\}_{i\in B_j}$ is a Riesz basic sequence with lower Riesz basis bound $\frac{A}{2}$. In particular, for at least one $1 \leqslant j \leqslant r$ we have that $|A_j| \geqslant \frac{n}{\|T\|^2}$ and this set gives the conclusion of the restricted invertibility theorem.*

Proof. By Proposition 3.1, there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ with $r=\|T\|^2$ so that for every $j=1,2,\cdots,r$ and every $i\in A_j$ we have

$$\sum_{i\neq k\in A_j} |\langle Te_i, Te_k\rangle|^2 \leqslant \sum_{m\in A_\ell} |\langle Te_i, Te_m\rangle|^2, \text{ for all } \ell \neq j.$$

It follows that for all $i \in A_j$ we have

$$\begin{aligned}\sum_{i\neq k\in A_j} |\langle Te_i, Te_k\rangle|^2 &\leqslant \frac{1}{r}\sum_{\ell=1}^r \sum_{m\in A_\ell} |\langle Te_i, Te_m\rangle|^2 \\ &= \frac{1}{r}\sum_{m=1}^n |\langle Te_i, Te_m\rangle|^2 \\ &\leqslant \frac{1}{r}\|T\|^2\|Te_i\|^2 \\ &= \frac{\|T\|^2}{r} \leqslant 1.\end{aligned}$$

Now, for every $j=1,2,\cdots,r$ and for every $i\in A_j$ we have

$$\sum_{k\in A_j} |\langle Te_i, Te_k\rangle|^2 = |\langle Te_i, Te_i\rangle|^2 + \sum_{i\neq k\in A_j} |\langle Te_i, Te_k\rangle|^2 \leqslant 1+1=2.$$

For every $j=1,2,\cdots,r$ let $n_j=|A_j|$. Let S_j be the frame operator for $\{Te_i\}_{i\in A_j}$ with eigenvectors $\{g_m^j\}_{m=1}^{n_j}$ and respective eigenvalues $\lambda_1^j \geqslant \lambda_2^j \geqslant \cdots \geqslant \lambda_{n_j}^j$. Choose $1\leqslant \ell_j \leqslant n_j$ satisfying: $\lambda_{\ell_j}^j \geqslant 4 > \lambda_{\ell_j+1}^j$. Let P_j be the orthogonal projection of span $\{g_m^j\}_{m=1}^{n_j}$ onto span $\{g_m^j\}_{m=1}^{\ell_j}$. So, $\|(I-P_j)T\|^2 \leqslant \lambda_{\ell_j+1}^j < 4$.

Claim: $\|(I-P_j)Te_i\|^2 \geqslant \frac{1}{2}$ for all $i\in A_j$.

To prove the Claim, we check first for all $i \in A_j$,

$$\begin{aligned}\sum_{k\in A_j}|\langle P_jTe_i,P_jTe_k\rangle|^2 &= \sum_{m=1}^{\ell_j}\lambda_m^j|\langle Te_i,g_m^j\rangle|^2\\ &\leqslant \sum_{m=1}^{n_j}\lambda_m^j|\langle Te_i,g_m^j\rangle|^2\\ &= \sum_{k\in A_j}|\langle Te_i,Te_k\rangle|^2 \leqslant 2.\end{aligned}$$

Since $\lambda_{\ell_j}^j \geqslant 4$ we have

$$\begin{aligned}2 &\geqslant \sum_{k\in A_j}|\langle P_jTe_i,P_jTe_k\rangle|^2\\ &= \sum_{m=1}^{\ell_j}\lambda_m^j|\langle P_jTe_i,g_m^j\rangle|^2\\ &\geqslant 4\sum_{m=1}^{\ell_j}|\langle P_jTe_i,g_m^j\rangle|^2\\ &= 4\|P_jTe_i\|^2.\end{aligned}$$

Thus, for all $i\in A_j$ we have $\|P_jTe_i\|^2 \leqslant \frac{1}{2}$. This proves the claim since now $\|(I-P_j)Te_i\|^2 \geqslant \frac{1}{2}$.

For each $j=1,2,\cdots,r$ let

$$T_je_i = \frac{(I-P_j)Te_i}{\|(I-P_j)Te_i\|},\ \text{ for all } i\in A_j.$$

So $\|T_je_i\|=1$ for all $i\in A_j$. Now, for any scalars $\{a_i\}_{i\in A_j}$ we have

$$\begin{aligned}\|\sum_{i\in A_j}a_iT_je_i\| &= \|\sum_{i\in A_j}\frac{a_i}{\|(I-P_j)Te_i\|}(I-P_j)Te_i\|\\ &\leqslant \|(I-P_j)T\|\|\sum_{i\in A_j}\frac{a_i}{\|(I-P_j)Te_i\|}e_i\|\\ &\leqslant 2\left(\sum_{i\in A_j}\frac{|a_i|^2}{\|(I-P_j)Te_i\|^2}\right)^{1/2}\\ &\leqslant \frac{2}{\sqrt{1/2}}\left(\sum_{i\in A_j}|a_i|^2\right)^{1/2}\\ &= 2\sqrt{2}\left(\sum_{i\in A_j}|a_i|^2\right)^{1/2}.\end{aligned}$$

Hence, $\|T_j\| \leqslant 2\sqrt{2}$. By our assumption of the restricted invertibility theorem for $\|T\| \leqslant 2\sqrt{2}$, for every $j = 1,2,\cdots,r$ there is a subset $B_j \subset A_j$ with $|B_j| \geqslant \frac{c}{8}|A_j|$ giving for all scalars $\{a_i\}_{i\in B_j}$,

$$\|\sum_{i\in B_j} a_i T_j e_i\|^2 \geqslant A \sum_{i\in B_j} |a_i|^2.$$

Now, for any $\{a_i\}_{i\in B_j}$ we have

$$\begin{aligned}\|\sum_{i\in B_j} a_i T e_i\|^2 &= \|\sum_{i\in B_j} a_i \|(I-P_j)T_j e_i\| T_j e_i\|^2 \\ &\geqslant A\|\sum_{i\in B_j} |a_i|^2 \|(I-P_{\ell_j})T_j e_i\|^2 \\ &\geqslant \frac{A}{2}\sum_{i\in B_j} |a_i|^2.\end{aligned}$$

This completes the proof of the theorem.

Theorem 3.2 yields a very large total number of the vectors $\{Te_i\}_{i=1}^n$ which are divisible into sets of Riesz basic sequences.

COROLLARY 3.3. *Let c,A be as in Theorem 3.2. For every $n \in \mathbb{N}$ and every $T : \ell_2^n \to \ell_2^n$ with $\|Te_i\| = 1$ for all $i = 1,2,\cdots,n$, there exist disjoint subsets $\{B_j\}_{j=1}^r$ with $r = \|T\|^2$ satisfying:*

1. $|\cup_{j=1}^r B_j| \geqslant cn$.

2. For every $j = 1,2,\cdots,r$ and for every $\{a_i\}_{i\in B_j}$ we have

$$\|\sum_{i\in B_j} a_i T e_i\|^2 \geqslant \frac{A}{2}\sum_{i\in B_j} |a_i|^2.$$

By iterating the the Restricted Invertibility Theorem, one can obtain a number of disjoint sets giving the conclusion of Corollary 3.3. But we believe that the number of sets given in Corollary 3.3 is close to the minimal number.

We could significantly improve Theorem 3.2 and its applications by giving a positive solution to the following problem.

PROBLEM 3.4. *Given an operator $T : \ell_2^n \to \ell_2^n$ with $\|Te_i\| = 1$ for all $i = 1,2,\cdots,n$, does there exist a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ with $r = \|T\|^2$ so that letting $H_j =$ span $\{e_i\}_{i\in A_j}$ for all $j = 1,2,\cdots r$ yields*

$$\|T|_{H_j}\| \leqslant 2\sqrt{2}?$$

Unfortunately, there is little or no chance that Problem 3.4 has a positive solution because a positive solution to this problem would imply a positive solution to the famous, intractable 1959 Kadison-Singer Problem (See Section 5) - and most people today believe that the Kadison-Singer Problem has a negative answer.

4. Filling Two Gaps in the Literature

In this section we will fill in two gaps in the restricted invertibility theory. These involve the relationship between the parameters in the restricted invertibility theorem and the parameters in the following theorem of Bourgain-Tazfriri [3, 4]. For notation in the theorem, for $\sigma \subset \{1, 2, \cdots, n\}$ and $\{e_i\}_{i=1}^n$ the unit vector basis of ℓ_2^n, we denote by Q_σ the projection

$$Q_\sigma \left(\sum_{i=1}^n a_i e_i \right) = \sum_{i \in \sigma} a_i e_i.$$

THEOREM 4.1. *There is a $0 < c < 1$ and an $\varepsilon > 0$ so that for all natural numbers n, all norm one zero diagonal linear operators $T : \ell_2^n \to \ell_2^n$ there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ with $|\sigma| \geqslant cn$ and $\|Q_\sigma T Q_\sigma\| \leqslant 1 - \varepsilon$.*

The reason we want to have a direct connection between Theorems 1.1 and 4.1 is that we have no idea what the exact constants are in these two theorems at this time. It is possible that $c = 1 - \varepsilon = 1/2$ works in Theorem 4.1. It is easy to see that $c \leqslant 1/2$ by considering the matrix:

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

However, it is not known if this upper bound for c is necessary for Theorem 1.1. The point is, we would like to be able to derive the constants for one of these theorems and then use these bounds to compute the constants for the other theorem.

First, we need to recall another result of Bourgain and Tzafriri, Theorem 1.1 [3].

THEOREM 4.2. *There exists a constant $c > 0$ so that for all $0 < \delta < 1$, all $n > c^{-1}$ and all $T : \ell_2^n \to \ell_2^n$ with zero diagonal, and given any non-negative weights $\{\lambda_i\}_{i=1}^n$ with $\sum_{i=1}^n \lambda_i = 1$, there exists a $\sigma \subset \{1, 2, \cdots, n\}$ for which*

$$\sum_{i=1}^n \lambda_i \geqslant c\delta$$

and with $\|Q_\sigma T Q_\sigma\| \leqslant \sqrt{\delta} \|T\|$.

The following trivial corollary of Theorem 4.2 seems to have been overlooked in the literature. We believe, however, this result was certainly known to Bourgain and Tzafriri and it was just an oversight that they did not state it in their paper.

COROLLARY 4.3. *There exists a $0 < c < 1$ so that given $\varepsilon > 0$ and $T : \ell_2^n \to \ell_2^n$ is a zero diagonal linear operator, then there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ with $|\sigma| \geqslant c\varepsilon^2 n / \|T\|^2$ and $\|Q_\sigma T Q_\sigma\| \leqslant \varepsilon$.*

Proof. In Theorem 4.2, let $\lambda_i = 1/n$ for all $i = 1, 2, \cdots, n$ and let $\delta = \varepsilon^2 / \|T\|^2$. There there is a subset $\sigma \subset \{1, 2, \cdots, n\}$ with

$$\frac{|\sigma|}{n} = \sum_{i \in \sigma} \lambda_i \geqslant c\delta = \frac{c\varepsilon^2}{\|T\|^2},$$

and

$$\|Q_\sigma T Q_\sigma\| \leqslant \sqrt{\delta}\|T\| = \varepsilon.$$

Corollary 4.3 yields a standard method for moving paving results to restricted invertibility results. The following corollary was proved in weaker form by Vershynin [12] and later in stronger form by Vershynin [13]. For the sake of illustration, we give Vershynin's original proof.

PROPOSITION 4.4. *There exists a* $0 < c < 1$ *so that for every* $\varepsilon > 0$ *and every linear operator* $T : \ell_2^n \to \ell_2^n$ *with* $\|Te_i\| = 1$ *for all* $i = 1, 2, \cdots, n$*, there exists a subset* $\sigma \subset \{1, 2\cdots, n\}$ *satisfying*

$$|\sigma| \geqslant \frac{c\varepsilon^2 n}{\|T\|^2}$$

and $\{Te_i\}_{i\in\sigma}$ *is a* $(1+\varepsilon)$*-Riesz basic sequence.*

Proof. Given T as in the proposition, let $L = T^*T - D(T^*T)$. Then L has zero diagonal and by Corollary 4.3 there is a subset $\sigma \subset \{1, 2, \cdots, n\}$ so that $|\sigma| \geqslant c\varepsilon^2 n/\|T\|^2$ with $\|Q_\sigma T Q_\sigma\| < \varepsilon$. Now, for all families of scalars $\{a_i\}_{i\in\sigma}$ we have

$$\begin{aligned} \Big|\|\sum_{i\in\sigma} a_i T e_i\|^2 - \sum_{i\in\sigma}|a_i|^2\Big| &= \Big|\langle \sum_{i\in\sigma} a_i T e_i, \sum_{i\in\sigma} a_i T e_i\rangle - \sum_{i\in\sigma}|a_i|^2\Big| \\ &= \Big|\langle Q_\sigma(T^*T - I)Q_\sigma \sum_{i\in\sigma} a_i e_i, \sum_{i\in\sigma} a_i e_i\rangle\Big| \\ &\leqslant \|Q_\sigma L Q_\sigma\| \|\sum_{i\in\sigma} a_i e_i\|^2 \\ &\leqslant \varepsilon \|\sum_{i\in\sigma} a_i e_i\|^2. \end{aligned}$$

Now we will look at another gap in the literature of the restricted invertibility theory. As we have just seen, there is a connection between *paving* (See Section 5) and restricted invertibility. But at this point, we have no direct connection between the restricted invertibility constants and the constants for paving. The following result gives a fairly efficient method for moving the constants from one of these theorems to the other.

PROPOSITION 4.5. *Assume* $0 < c < 1$ *satisfies: For all zero diagonal linear operators* $T : \ell_2^n \to \ell_2^n$ *with* $\|T\| = 1$*, there is a subset* $\sigma \subset \{1, 2, \cdots, n\}$ *with* $|\sigma| \geqslant cn$ *and*

$$\|Q_\sigma T Q_\sigma\| \leqslant \frac{1-\varepsilon}{28}.$$

Then, for every $T : \ell_2^n \to \ell_2^n$ *with* $\|T\| \geqslant \sqrt{2}$ *and* $\|Te_i\| = 1$ *for all* $i = 1, 2, \cdots, n$*, there exists a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \cdots, n\}$ *with* $r = \|T\|^2$ *and for every* $j = 1, 2, \cdots, r$ *there is a subset* $B_j \subset A_j$ *with* $|B_j| \geqslant c|A_j|$ *and for all families of scalars* $\{a_i\}_{i\in B_j}$

$$\|\sum_{i\in A_j} a_i T e_i\|^2 \geqslant \frac{\varepsilon}{4}\sum_{i\in\sigma}|a_i|^2.$$

It follows that there is a $1 \leqslant j \leqslant r$ *with*

$$|B_j| \geqslant c|A_j| \geqslant \frac{cn}{\|T\|^2},$$

which is the set called for in the Restricted Invertibility Theory.

Proof. We follow the proof of Theorem 3.2 to find:

(1) A partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ so that if S_j is the frame operator for $\{Te_i\}_{i\in A_j}$, it has eigenvectors $\{g_m^j\}_{m=1}^{n_j}$ with $n_j = |A_j|$ and respective eigenvalues $\lambda_1^j \geqslant \lambda_2^j \geqslant \cdots \geqslant \lambda_{n_j}^j$.

(2) We choose $1 \leqslant \ell_j \leqslant n_j$ with

$$\lambda_{\ell_j}^j \geqslant 4 > \lambda_{\ell_j+1}^j$$

and we let P_j be the orthogonal projection of span $\{g_m^j\}_{m=1}^{n_j}$ onto span $\{g_m^j\}_{m=1}^{\ell_j}$.

(3) We have $\|(I-P_j)Te_i\|^2 \geqslant 1/2$ for all $i \in A_j$.

(4) If we let

$$T_je_i = \frac{(I-P_j)Te_i}{\|(I-P_j)Te_i\|}, \text{ for all } i \in A_j,$$

then $\|T_j\| \leqslant 2\sqrt{2}$ so $\|T_j^*T_j\| \leqslant 8$ and $D(T_j^*T_j) = I_{A_j}$. Now, a standard calculation yields

$$\|T_j^*T_j - D(T_j^*T_j)\| \leqslant 7.$$

Let

$$L_j = \frac{T_j^*T_j - D(T_j^*T_j)}{\|T_j^*T_j - D(T_j^*T_j)\|}.$$

By our assumption, there exist subsets $B_j \subset A_j$ with $|B_j| \geqslant c|A_j|$ and

$$\|Q_{B_j}L_jQ_{B_j}\| \leqslant \frac{1-\varepsilon}{28}.$$

That is,

$$\|Q_{B_j}(T_j^*T_j - D(T_j^*T_j)Q_{B_j}\| \leqslant \frac{1-\varepsilon}{28}\|T_j^*T_j - D(T_j^*T_j)\| \leqslant \frac{1-\varepsilon}{4}.$$

Now, for all families of scalars $\{a_i\}_{i\in B_j}$ we have

$$\begin{aligned}\|\sum_{i\in B_j} a_iTe_i\|^2 &\geqslant \|\sum_{i\in B_j} a_i(I-P_j)Te_i\|^2 \\ &= \langle \sum_{i\in B_j} a_i(I-P_j)Te_i, \sum_{i\in B_j} a_i(I-P_j)Te_i\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i \frac{(I-P_j)Te_i}{\|(I-P_j)Te_i\|}, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i \frac{(I-P_j)Te_i}{\|(I-P_j)Te_i\|} \rangle \\
&= \langle \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i T_j e_i, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i T_j e_i \rangle \\
&= T_j^* T_j \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i \rangle \\
&= \langle D(T_j^* T_j) \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i \rangle - \\
&\quad \langle (T_j^* T_j - D(T_j^* T_j)) \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i \rangle \\
&\geqslant \sum_{i\in B_j} \|(I-P_j)Te_i\|^2 |a_i|^2 - \\
&\quad \|Q_{B_j}(T_j^* T_j - D(T_j^* T_j)) Q_{B_j}\| \langle \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i, \sum_{i\in B_j} \|(I-P_j)Te_i\| a_i e_i \rangle \\
&\geqslant \frac{1}{4} \sum_{i\in B_j} |a_i|^2 - \frac{1-\varepsilon}{4} \sum_{i\in B_j} \|(I-P_j)Te_i\|^2 |a_i|^2 \\
&\geqslant \frac{1}{4} \sum_{i\in B_j} |a_i|^2 - \frac{1-\varepsilon}{4} \sum_{i\in B_j} |a_i|^2 \\
&= \frac{\varepsilon}{4} \sum_{i\in B_j} |a_i|^2.
\end{aligned}
$$

5. The Bourgain-Tzafriri Conjecture

The restricted invertibility theorem gave rise to what we call today the *Bourgain-Tzafriri Conjecture*.

THEOREM 5.1. *There is a universal constant $A > 0$ so that for every $B > 1$ there is a natural number $r = r(B)$ satisfying: For any natural number n, if $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|T\| \leqslant B$ and $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, then there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all choices of scalars $\{a_i\}_{i\in A_j}$ we have:*

$$\| \sum_{i\in A_j} a_i Te_i \|^2 \geqslant A \sum_{i\in A_j} |a_i|^2.$$

If we allow the constant A to depend upon the norm of the operator T above we call this the *weak Bourgain-Tzafriri Conjecture*. For years everyone believed that the Bourgain-Tzafriri Conjecture was equivalent to the famous 1959 Kadison-Singer Problem [11] but no one was quite able to give a formal proof of the equivalence. Now we have formal proofs that these two are in fact equivalent [8, 9].

PROBLEM 5.2. (Kadison-Singer Problem) *Does every pure state on the (abelian) von Neumann algebra $\mathscr{D}$ of bounded diagonal operators on ℓ_2 have a unique extension* *to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2?*

A *state* of a von Neumann algebra $\mathscr{R}$ is a linear functional f on $\mathscr{R}$ for which $f(I) = 1$ and $f(T) \geqslant 0$ whenever $T \geqslant 0$ (whenever T is a positive operator). The set of states of $\mathscr{R}$ is a convex subset of the dual space of $\mathscr{R}$ which is compact in the ω^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the *pure states* (of $\mathscr{R}$).

Most people today believe that the Kadison-Singer Problem has a negative answer. Indeed, even Kadison and Singer believd their problem had a negative answer when they posed it [11].

It was shown in 1979 by Anderson [1] that the Kadison-Singer Problem is equivalent to what is now known as Anderson's Paving Conjecture:

THEOREM 5.3. (Anderson's Paving Conjecture) *For $\varepsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of $\{1, \cdots, n\}$, such that*

$$\|Q_{A_j} T Q_{A_j}\| \leqslant \varepsilon \|T\| \quad \textit{for all } j = 1, 2, \cdots, r.$$

Another related conjecture is the Feichtinger Conjecture in Hilbert space frame theory (See [6, 7, 8]).

THEOREM 5.4. (The Feichtinger Conjecture) *Is every unit norm frame a finite union of Riesz basic sequences?*

It has been known for awhile [6] that the Feichtinger Conjecture is equivalent to the weak Bourgain-Tzafriri Conjecture. Recently [8] it was shown that weak BT is equivalent to the Kadison-Singer Problem. But this proof takes weak BT back into the theory of C^*-algebras and verifies KS directly. However, Theorem 3.2, shows that verifying the BT Conjecture for any operator is equivalent to verifying it for operators T with $\|T\| \leqslant 2\sqrt{2}$. i.e. This is a direct proof that the weak BT Conjecture, and hence the Feichtinger Conjecture, is equivalent to the Bourgain-Tzafriri Conjecture.

Next we will see that Problem 3.4 is stronger than the Kadison-Singer Problem. First, we state a weaker problem.

PROBLEM 5.5. *There exist a constant $4 < K$ so that for every $n \in \mathbb{N}$, every $T : \ell_2^n \to \ell_2^n$ with $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ with $r = r(\|T\|)$ so that if we let $H_j =$ span $\{Te_i\}_{i \in A_j}$ then for every $j = 1, 2, \cdots, r$ we have*

$$\|T|_{H_j}\| \leqslant K.$$

It is clear that a positive solution to Problem 3.4 implies a positive solution to Problem 5.5. Now we will show that Problem 5.5 is equivalent to the Kadison-Singer Problem. For this we need a result of Weaver [14] which gives another equivalent form of KS called KS_r'.

CONJECTURE 5.6. ($\mathrm{KS_r}'$) *There exists universal constants $B \geqslant 4$ and $\varepsilon > \sqrt{B}$ so that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| = 1$ for $i = 1, 2, \cdots, M$ and suppose for every $f \in \ell_2^n$,*

$$\sum_{i=1}^{M} |\langle f, f_i\rangle|^2 = B\|f\|^2. \tag{5.1}$$

Then, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $f \in \ell_2^n$ and all $j = 1, 2, \cdots, r$,

$$\sum_{i\in A_j} |\langle f, f_i\rangle|^2 \leqslant (B-\varepsilon)\|f\|^2.$$

THEOREM 5.7. *The following are equivalent:*

1. Problem 5.5.

3. The Kadison-Singer Problem.

Proof. $(1) \Rightarrow (2)$: We will check that Conjecture 5.6 holds. So let $B = K^2$, $r = B$ and for $n \in \mathbb{N}$ choose $\{f_i\}_{i=1}^M$ norm one vectors in ℓ_2^n satisfying

$$\sum_{i=1}^{M} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \text{ for all } f \in \ell_2^n.$$

Define $T : \ell_2^M \to \ell_2^n$ by $Te_i = f_i$ for all $i = 1, 2, \cdots, M$. Without loss of generality we may assume that the range of T is in ℓ_2^M (by applying a unitary operator if necessary). Now, for all $f \in \ell_2^M$,

$$\|T^* f\|^2 = \sum_{i=1}^{M} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2.$$

By our assumption in (1), there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that if $H_j =$ span $\{e_i\}_{i\in A_j}$ then for all $j = 1, 2, \cdots, r$

$$\|T|_{H_j}\| \leqslant K < K^2 - K = B - \sqrt{B}.$$

That is,

$$\|\left(T|_{H_j}\right)^* f\|^2 = \sum_{i\in A_j} |\langle f, f_i\rangle|^2 < (B-\sqrt{B})\|f\|^2.$$

Thus, Conjecture 5.6 holds and so KS holds.

$(2) \Rightarrow (1)$: Let $T : \ell_2^n \to \ell_2^n$ be an operator with $\|Te_i\| = 1$ for all $i = 1,2,\cdots n$. Let $L = T^*T - D(T^*T)$ where $D(T^*T)$ is the diagonal of the operator T^*T. Choose $0 < \varepsilon$ so that $1+\varepsilon\|L\| \leqslant 8$. By the Paving Conjecture, there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ with $r = f(\|L\|)$ so that for every $j = 1,2,\cdots,r$ we have

$$\|Q_{A_j} L Q_{A_j}\| \leqslant \varepsilon \|L\|.$$

Hence,

$$\|Q_{A_j} T^*T Q_{A_j}\| \leqslant 1+\varepsilon\|L\|.$$

Let $H_j =$ span $\{e_i\}_{i\in A_j}$ for all $j = 1,2,\cdots,r$ and let $T_j = T|_{H_j}$. For every $f \in \ell_2^n$ we have

$$\begin{aligned}
\|T_j f\|^2 &= \|TQ_{A_j} f\|^2 \\
&= \langle TQ_{A_j} f, TQ_{A_j} f\rangle \\
&= \langle Q_{A_j} T^*TQ_{A_j} f, f\rangle \\
&\leqslant \|Q_{A_j} T^*TQ_{A_j}\| \|f\|^2 \\
&\leqslant (1+\varepsilon\|L\|)\|f\|^2 \\
&\leqslant 8\|f\|^2.
\end{aligned}$$

It follows that $\|T_j\| \leqslant 2\sqrt{2}$ and so Problem 5.5 has a positive solution.

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