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A NOTE ON APPROXIMATE LIFTINGS

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Abstract. In this paper, we prove approximate lifting results in the C^*-algebra and von Neumann algebra settings. In the C^*-algebra setting, we show that two (weakly) semiprojective unital C*-algebras, each generated by n projections, can be glued together with partial isometries to define a larger (weakly) semiprojective algebra. In the von Neumann algebra setting, we prove lifting theorems for trace-preserving *-homomorphisms from abelian von Neumann algebras or hyperfinite von Neumann algebras into ultraproducts. We also extend a classical result of S. Sakai [16] by showing that a tracial ultraproduct of C*-algebras is a von Neumann algebra, which yields a generalization of Lin's theorem [12] on almost commuting selfadjoint operators with respect to $\|\cdot\|_p$ on any unital C*-algebra with trace.

1. Introduction

The key idea in this paper is the study of defining properties of (or tuples of) operators such that an operator that "almost" satisfies this property is "close" to an operator that actually does satisfy this property. In the setting of C*-algebras we would insist that the "closeness" be with respect to the norm, and in the finite von Neumann algebra sense "closeness" would be with respect to the 2-norm $\|\cdot\|_2$ defined in terms of the tracial state τ on the algebra, i.e., $\|x\|_2 = \tau(x^*x)^{1/2}$. A classic example of this phenomenon is the fact that if A is an operator such that $\|A - A^*\|$ and $\left\|A - A^2\right\|$ are small, then A is very close to a projection P. In fact, P can be chosen in the nonunital C*-algebra generated by A. It is also true that if $\mathscr{M}$ is a finite von Neumann algebra with a faithful normal trace τ and if $A \in \mathscr{M}$ such that $\|A - A^*\|_2$ and $\left\|A - A^2\right\|_2$ are small, then there is a projection $P \in W^*(A)$ (the von Neumann subalgebra generated by A in $\mathscr{M}$) such that $\|A - P\|_2$ is small.

In the C*-algebra setting these ideas are essentially the notion of weak semiprojectivity introduced by S. Eilers and T. Loring [5] in 1999, and semiprojectivity introduced by B. Blackadar [1] in 1985. These notions were studied by T. Loring [13] in terms of stable relations and D. Hadwin, L. Kaonga and B. Mathes [10] in terms of their noncommutative continuous functions.

The von Neumann algebra results appear in an ad hoc manner in various papers in the literature.

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Semiprojectivity and weak semiprojectivity also can be expressed in terms of liftings of representations into algebras of the form $\prod_1^\infty \mathscr{B}_n / \oplus_1^\infty \mathscr{B}_n$ or in terms of ultraproducts $\prod_1^\infty \mathscr{B}_n / \mathscr{J}$. It is in the theory of (tracial) ultraproducts of finite von Neumann algebras where many of these "approximate" results appear in the von Neumann algebra setting.

After the preliminary definitions and results (Section 2), we begin Section 3 with our results in the C*-algebra setting. Our main result is that two (weakly) semiprojective unital C*-algebras, each generated by n projections, can be glued together with partial isometries to define a larger (weakly) semiprojective algebra (Theorem 3.4).

In the von Neumann algebra setting (Section 4) we prove lifting theorems for trace-preserving *-homomorphisms from abelian von Neumann algebras (Corollary 4.5) or hyperfinite von Neumann algebras (Theorem 4.11) into ultraproducts. We also extend a classical result of S. Sakai [16] by showing (Theorem 4.1) that a tracial ultraproduct of C*-algebras is a von Neumann algebra. This result allows us to prove a hybrid result (Corollary 4.2), namely, an approximate result with respect to $\|\cdot\|_2$ on C*-algebras. For example, if $\varepsilon > 0$, then there is a $\delta > 0$ such that for any unital C*-algebra $\mathscr{A}$ with trace τ, we have that if u, v are unitaries in $\mathscr{A}$ with $\|uv - vu\|_2 < \delta$, then there are commuting unitaries u', v' in $C^*(u, v)$ (the unital C^*-subalgebra generated by u, v in $\mathscr{A}$) such that $\|u - u'\|_2 + \|v - v'\|_2 < \varepsilon$. With respect to the operator norm this fails even in the class of finite-dimensional C*-algebras [17].

2. Preliminaries

A C*-algebra $\mathscr{A}$ is *projective* if, for any *-homomorphism $\varphi : \mathscr{A} \to \mathscr{C}$, where $\mathscr{C}$ is a C^*-algebra, and every surjective *-homomorphism $\rho : \mathscr{B} \to \mathscr{C}$, there is a *-homomorphism $\overline{\varphi} : \mathscr{A} \to \mathscr{B}$ such that $\rho \circ \overline{\varphi} = \varphi$. A C^*-algebra $\mathscr{A}$ is *semiprojective* [1] if, for every *-homomorphism $\pi : \mathscr{A} \to \mathscr{B} / \overline{\cup_1^\infty \mathscr{J}_n}$, where $\{\mathscr{J}_n\}_{n=1}^\infty$ is an increasing family of ideals of a C^*-algebra $\mathscr{B}$, and with $\varphi_N : \mathscr{B} / \mathscr{J}_N \to \mathscr{B} / \overline{\cup_1^\infty \mathscr{J}_n}$ the natural quotient map, there exists a *-homomorphism $\pi_N : \mathscr{A} \to \mathscr{B} / \mathscr{J}_N$ such that $\pi = \varphi_N \circ \pi_N$. A C^*-algebra $\mathscr{A}$ is *weakly semiprojective* [13] if, for any given sequence $\{\mathscr{B}_n\}_{n=1}^\infty$ of C^*-algebras and a *-homomorphism $\pi : \mathscr{A} \to \prod_1^\infty \mathscr{B}_n / \oplus_1^\infty \mathscr{B}_n$, there exist functions $\pi_n : \mathscr{A} \to \mathscr{B}_n$ for all $n \geqslant 1$ and a positive integer N such that

(1) π_n is a *-homomorphism for all $n \geqslant N$, and

(2) $\pi(a) = [\{\pi_n(a)\}]$ for every $a \in \mathscr{A}$.

Equivalently, since $\prod_1^\infty \mathscr{B}_n / \oplus_1^\infty \mathscr{B}_n$ is isomorphic to $\prod_N^\infty \mathscr{B}_n / \oplus_N^\infty \mathscr{B}_n$, the conditions above say that there is a *-homomorphism $\rho : \mathscr{A} \to \prod_N^\infty \mathscr{B}_n$ such that $\pi(a) = \rho(a) + \oplus_N^\infty \mathscr{B}_n$ for every $a \in \mathscr{A}$.

These notions of projectivity make sense in two categories:

(1) the *nonunital category*, i.e., the category of C*-algebras with *-homomorphisms as morphisms, and

(2) the *unital category*, i.e., the category of unital C*-algebras with unital *-homomorphisms as morphisms.

These notions are drastically different in the different categories. For example, the 1-dimensional C*-algebra $\mathbb{C}$ is projective in the unital category, but not in the nonunital category, e.g., in the definition of projective C^*-algebra, let $\mathscr{B} = C_0((0,1])$, $\mathscr{C} = \mathbb{C}$ and $\rho(f) = f(1)$. However, if $\mathscr{A}$ is nonunital and projective (semiprojective, weakly semiprojective) in the nonunital category, and if $\mathscr{A}^+$ is the algebra obtained by adding a unit to $\mathscr{A}$, then $\mathscr{A}^+$ is projective (semiprojective, weakly semiprojective) in the unital category. In Loring's book [13] he only considers the nonunital category. In this paper we restrict ourselves to the unital category.

Suppose $\mathscr{S}$ is a subset of a unital C*-algebra $\mathscr{A}$. Let $C^*(\mathscr{S})$ denote the unital C*-subalgebra generated by $\mathscr{S}$ in $\mathscr{A}$.

In [10] the notions of semiprojectivity and weak semiprojectivity for finitely generated algebras were cast in terms of *noncommutative continuous functions*. The *-algebra of noncommutative continuous functions is basically the metric completion of the algebra of *-polynomials with respect to a family of seminorms. There is a functional calculus for these functions on any n-tuple of operators on any Hilbert space. Here is a list of a few of the basic properties of noncommutative continuous functions [10]:

(1) For each noncommutative continuous function φ there is a sequence $\{p_k\}$ of noncommutative *-polynomials such that for every tuple $(T_1,\ldots,T_n)$ we have

$$\|p_k(T_1,\ldots,T_n) - \varphi(T_1,\ldots,T_n)\| \to 0,$$

and the convergence is uniform on bounded n-tuples of operators.

(2) For any tuple $(T_1,\ldots,T_n)$, $C^*(T_1,\ldots,T_n)$ is the set of all $\varphi(T_1,\ldots,T_n)$ with φ a noncommutative continuous function.

(3) For any n-tuple $(A_1,\ldots,A_n)$ and any $S \in C^*(A_1,\ldots,A_n)$, there is a noncommutative continuous function φ such that $S = \varphi(A_1,\ldots,A_n)$ and $\|\varphi(T_1,\ldots,T_n)\| \leqslant \|S\|$ for all n-tuples $(T_1,\ldots,T_n)$.

(4) If $T_1,\ldots,T_n$ are elements of a unital C*-algebra $\mathscr{A}$ and $\pi : \mathscr{A} \to \mathscr{B}$ is a unital *-homomorphism, then

$$\pi(\varphi(T_1,\ldots,T_n)) = \varphi(\pi(T_1),\ldots,\pi(T_n))$$

for every noncommutative continuous function.

In [10] it was shown that the natural notion of relations used to define a C*-algebra generated by $a_1,\ldots,a_n$ are all of the form

$$\varphi(a_1,\ldots,a_n) = 0$$

for a noncommutative continuous function φ. In fact, it was also shown in [10] that given a unital C*-algebra $\mathscr{A}$ generated by $a_1,\ldots,a_n$, there is a single noncommutative continuous function φ such that $\mathscr{A}$ is isomorphic to the universal C*-algebra $C^*(x_1,\ldots,x_n|\varphi)$ with generators $x_1,\ldots,x_n$ and with the single relation $\varphi(x_1,\ldots,x_n) = 0$, where the map $x_j \mapsto a_j$ extends to a *-isomorphism. Such a noncommutative continuous function φ must be *null-bounded*, i.e., there is a number $r > 0$ such that $\|A_j\| \leqslant r$ for $1 \leqslant j \leqslant n$ whenever $\varphi(A_1,\ldots,A_n) = 0$. In this sense, every finitely generated C*-algebra is finitely presented. In particular, Theorem 14.1.4 in T. Loring's book [13] is true for all finitely generated C*-algebras.

For a finitely generated nonunital C*-algebra $\mathscr{A}$ there is a null-bounded noncommutative continuous function φ such that $\mathscr{A}$ is isomorphic to the universal nonunital C*-algebra $C_0^*(x_1,\ldots,x_n|\varphi)$ with generators $x_1,\ldots,x_n$ and with the single relation $\varphi(x_1,\ldots,x_n)=0$.

Here is a reformulation of the notions of semiprojectivity and weak semiprojectivity for finitely generated C*-algebras in terms of noncommutative continuous functions. We only state the result in the unital category.

PROPOSITION 2.1. [10] Suppose φ is a null-bounded noncommutative continuous function. Then

(1) $C^*(x_1,\ldots,x_n|\varphi)$ is weakly semiprojective if and only if there exist noncommutative continuous functions $\varphi_1,\ldots,\varphi_n$ such that for any $\varepsilon>0$, there exists $\delta>0$, such that for any operators $T_1,\cdots,T_n$ with $\|\varphi(T_1,\ldots,T_n)\|<\delta$, we have

(a) $\varphi(\varphi_1(T_1,\ldots,T_n),\ldots,\varphi_n(T_1,\ldots,T_n))=0$, and

(b) $\|T_j-\varphi_j(T_1,\ldots,T_n)\|<\varepsilon$.

(2) $C^*(x_1,\ldots,x_n|\varphi)$ is semiprojective if, in addition, we can choose $\varphi_1,\ldots,\varphi_n$ as in (1) so that $\varphi_j(A_1,\ldots,A_n)=A_j$ for $1\leqslant j\leqslant n$, whenever $\varphi(A_1,\ldots,A_n)=0$.

We call the functions $\varphi_1,\ldots,\varphi_n$ the *(weakly) semiprojective approximating functions* for φ.

For example, it is a classical result that has often been rediscovered that a selfadjoint operator A with $\|A-A^2\|$ sufficiently small is very close to a projection. More precisely, if

$$\|A-A^2\|<\varepsilon^2<1/9,$$

then

$$\sigma(A)\subset(-\varepsilon,\varepsilon)\cup(1-\varepsilon,1+\varepsilon).$$

So if $h:\mathbb{R}\to\mathbb{R}$ is defined by

$$h(t)=\begin{cases}0, & \text{if } t\leqslant\frac{1}{3}\\ 3t-1, & \text{if } \frac{1}{3}<t<\frac{2}{3}\\ 1, & \text{if } \frac{2}{3}\leqslant t\end{cases},$$

then $h(A)$ is a projection and $\|A-h(A)\|=\sup_{t\in\sigma(A)}|t-h(t)|<\varepsilon$. If A already is a projection, then $h(A)=A$. Thus the universal C*-algebra generated by a single projection is $C^*(x|\varphi)$ where $\varphi(x)=(x-x^*)^2+(x-x^2)^*(x-x^2)$, and defining $\varphi_1(x)=h\left(\frac{x+x^*}{2}\right)$ shows that $C^*(x|\varphi_1)$ is semiprojective.

Throughout this paper, all the C^*-algebras considered are unital and all *-homomorphism are unital.

3. C*-algebra Results

For simplicity we only consider finitely generated C^*-algebras throughout this section.

The main results in this section concern the (weak) semiprojectivity of C*-algebras defined in terms of partial isometries. We begin with some results that are elementary in the unital category.

LEMMA 3.1. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are separable unital C*-algebras. The following are true:*

(1) if $\mathscr{A}$ is (weakly) semiprojective, then $\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$ is (weakly) semiprojective;

(2) if $\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$ is weakly semiprojective, then $\mathscr{A}$ is weakly semiprojective;

(3) if $\mathscr{A}$ and $\mathscr{B}$ are projective (semiprojective, weakly semiprojective), then so is $\mathscr{A}*\mathscr{B}$;

(4) if $\mathscr{A}*\mathscr{B}$ is projective (weakly semiprojective, semiprojective), and there is a linear multiplicative functional α on $\mathscr{B}$, then $\mathscr{A}$ is projective (weakly semiprojective, semiprojective);

(5) $\mathscr{A}\oplus\mathscr{B}$ is (weakly) semiprojective if an only if both $\mathscr{A}$ and $\mathscr{B}$ are (weakly) semiprojective.

Proof. (1) Suppose $\mathscr{A}$ is weakly semiprojective, and $\mathscr{A}=C^*(x_1,\ldots,x_m|\varphi)$ with weakly semiprojective approximating functions $\varphi_1,\ldots,\varphi_m$. Since $\mathscr{M}_n(\mathbb{C})$ is semiprojective, we can assume that $\mathscr{M}_n(\mathbb{C})=C^*(y|\rho)$ with a semiprojective approximating function ρ_1. Hence

$$\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})=C^*(x_1,\ldots,x_m,y|\Phi)$$

where

$$\begin{aligned}\Phi(x_1,\ldots,x_m,y) &= \sum_{i=1}^{m}(x_iy-yx_i)^*(x_iy-yx_i)+\sum_{i=1}^{m}(x_iy^*-y^*x_i)^*(x_iy^*-y^*x_i)\\ &\quad+\varphi(x_1,\ldots,x_m)^*\varphi(x_1,\ldots,x_m)+\rho(y)^*\rho(y).\end{aligned}$$

Since matrix units $E_{i,j}(1\leqslant i,j\leqslant n)$ are in $\mathscr{M}_n(\mathbb{C})$, there exists a family of noncommutative continuous functions $\{\rho_{i,j}:1\leqslant i,j\leqslant n\}$ such that $E_{i,j}=\rho_{i,j}(y)$.

For any operators $T_1,\ldots,T_m,S$, and for $1\leqslant j\leqslant m$, let $\widehat{T_k}=\sum_{j=1}^{n}\rho_{j,1}(\rho_1(S))\cdot T_k\cdot\rho_{1,j}(\rho_1(S))$. Define functions $\{\Phi_k:1\leqslant k\leqslant m+1\}$ by

$$\Phi_k(T_1,\ldots,T_m,S)=\begin{cases}\varphi_k\left(\widehat{T_1},\ldots,\widehat{T_m}\right), & 1\leqslant k\leqslant m\\ \rho_1(S), & k=m+1\end{cases}$$

Given any $\varepsilon>0$, we will find some $\delta>0$ in the definition of weak semiprojectivity. Note that $\mathscr{M}_n(\mathbb{C})$ is semiprojective, there exists $\delta_1>0$, such that if $\|\rho(S)\|<\delta_1$, then

(1) $\rho(\Phi_{m+1}(T_1,\ldots,T_m,S))=\rho(\rho_1(S))=0$,

(2) $\|\rho_1(S)-S\|<\varepsilon$,

(3) $\rho_1(S)$ and $\rho_1(S)^*$ commute with all $\widehat{T_k}$.

Since $\mathscr{A}$ is weakly semiprojective, there exists $\delta_2>0$ such that, if $\|\varphi(\widehat{T_1},\cdots,\widehat{T_m})\|<\delta_2$, then

$$\varphi(\varphi_1(\widehat{T_1},\cdots,\widehat{T_m}),\cdots,\varphi_m(\widehat{T_1},\cdots,\widehat{T_m}))=0$$

and

$$\|\widehat{T_k} - \varphi_k(\widehat{T_1}, \cdots, \widehat{T_m})\| < \varepsilon.$$

Note that

$$\|\varphi(\widehat{T_1}, \cdots, \widehat{T_m})\| \leqslant \|\varphi(\widehat{T_1}, \cdots, \widehat{T_m}) - \varphi(T_1, \ldots, T_m)\| + \|\varphi(T_1, \ldots, T_m)\|,$$

there exists $\delta_3 > 0$ such that, if $\|T_k - \widehat{T_k}\| < \delta_3$, then

$$\|\varphi(\widehat{T_1}, \cdots, \widehat{T_m})\| \leqslant \|\varphi(\widehat{T_1}, \cdots, \widehat{T_m}) - \varphi(T_1, \ldots, T_m)\| < \frac{\delta_2}{2}.$$

In addition, if $\|\varphi(T_1, \ldots, T_m)\| < \frac{\delta_2}{2}$, then $\|\varphi(\widehat{T_1}, \cdots, \widehat{T_m})\| < \delta_2$. Furthermore,

$$\begin{aligned}
\|T_k - \widehat{T_k}\| &= \|T_k - \sum_{j=1}^{n} \rho_{j,1}(\rho_1(S)) \cdot T_k \cdot \rho_{1,j}(\rho_1(S))\| \\
&= \|\sum_{j=1}^{n} \rho_{j,1}(\rho_1(S)) \cdot \big(\rho_{1,j}(\rho_1(S)) \cdot T_k - T_k \cdot \rho_{1,j}(\rho_1(S))\big)\| \\
&\leqslant \sum_{j=1}^{n} \|\rho_{1,j}(\rho_1(S)) \cdot T_k - T_k \cdot \rho_{1,j}(\rho_1(S))\|.
\end{aligned}$$

Hence there exists $\delta_4 > 0$ such that, if $\|\rho_{1,j}(\rho_1(S)) \cdot T_k - T_k \cdot \rho_{1,j}(\rho_1(S))\| < \delta_4$, then $\|T_k - \widehat{T_k}\| < \delta_3$.

Note that $\rho_1(S) = \sum_{i,j=1}^{n} c_{ij} \cdot \rho_{ij}(\rho_1(S))$, where $c_{i,j}$'s are complex numbers. We have

$$\begin{aligned}
\|T_k\rho_1(S) - \rho_1(S)T_k\| &= \|T_k \sum_{i,j=1}^{n} c_{i,j} \cdot \rho_{ij}(\rho_1(S)) - \sum_{i,j=1}^{n} c_{i,j} \cdot \rho_{ij}(\rho_1(S))T_k\| \\
&\leqslant \sum_{i,j=1}^{n} |c_{ij}| \cdot \|T_k\rho_{ij}(\rho_1(S)) - \rho_{ij}(\rho_1(S))T_k\| \\
&= \sum_{i,j=1}^{n} |c_{ij}| \cdot \delta_4.
\end{aligned}$$

Let $\delta_5 = \sum_{i,j=1}^{n} |c_{ij}| \cdot \delta_4$. Since

$$\begin{aligned}
\|T_k\rho_1(S) - \rho_1(S)T_k\| &= \|T_k(\rho_1(S) - S + S) - (\rho_1(S) - S + S)T_k\| \\
&\leqslant 2\|T_k\| \cdot \|\rho_1(S) - S\| + \|T_kS - ST_k\|,
\end{aligned}$$

there exists $\delta_6 > 0$ such that, if $\|\rho_1(S) - S\| < \delta_6$ and $\|T_kS - ST_k\| < \delta_6$, then $\|T_k\rho_1(S) - \rho_1(S)T_k\| < \delta_5$.

By the fact that $\mathscr{M}_n(\mathbb{C}) = C^*(y|\rho)$ is semiprojective, there exists $\delta_7 > 0$ such that if $\|\varphi(S)\| < \delta_7$, then $\|\rho_1(S) - S\| < \delta_6$.

Note that $\|\varphi(T_1, \ldots, T_m)\|, \|\rho(S)\|, \|T_iS - ST_i\|, \|T_iS^* - S^*T_i\|$ are all less than or equal to $\sqrt{\|\Phi(T_1, \ldots, T_m, S)\|}$. Put $\delta = \min\{\delta_1^2, (\delta_2/2)^2, \delta_6^2, \delta_7^2\}$, then $\Phi_1, \ldots, \Phi_{m+1}$ are weakly semiprojective approximating functions for Φ.

If $\mathscr{A}$ is semiporjective and $\varphi_1,\ldots,\varphi_m$ are semiprojective approximating functions for φ, it is clear that $\Phi_1,\ldots,\Phi_{m+1}$ are semiprojective approximating functions for Φ.

(2) Suppose $\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$ is weakly semiprojective. Let $\pi:\mathscr{A}\to\prod_1^\infty\mathscr{B}_k/\oplus_1^\infty\mathscr{B}_k$ be a unital *-homomorphism. Then $\rho=\pi\otimes id$ is a unital *-homomorphism from $\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$ to $(\prod_1^\infty\mathscr{B}_k/\oplus_1^\infty\mathscr{B}_k)\otimes\mathscr{M}_n(\mathbb{C})=\prod_1^\infty(\mathscr{B}_k\otimes\mathscr{M}_n(\mathbb{C}))/\oplus_1^\infty(\mathscr{B}_k\otimes\mathscr{M}_n(\mathbb{C}))$. Since $\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$ is weakly semiprojective, there is a positive integer N and maps $\rho_k:\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})\to\mathscr{B}_k$ such that, for $k\geqslant N$, ρ_k is a unital *-homomorphism and, for every $x\in\mathscr{A}\otimes\mathscr{M}_n(\mathbb{C})$,

$$\rho(x)=[\{\rho_k(x)\}].$$

It follows that there is a sequence $\{U_k\}$ of unitary elements ($U_k\in\mathscr{B}_k$) such that $\|U_k-1\|\to 0$ and $\left\|1\otimes T-U_k^*\rho_k(1\otimes T)U_k\right\|\to 0$ for every $T\in\mathscr{M}_n(\mathbb{C})$. Therefore, for $k\geqslant N$ and $A\in\mathscr{A}$, $U_k^*\rho_k(A\otimes 1)U_k$ is in the commutant of $1\otimes\mathscr{M}_n(\mathbb{C})$, which is $\mathscr{B}_k\otimes 1$. Hence there are representations $\pi_k:\mathscr{A}\to\mathscr{B}_k$ such that $\pi_k(A)\otimes 1=U_k^*\rho_k(A\otimes 1)U_k$ for every $A\in\mathscr{A}$. Clearly, $\pi(A)=[\{\pi_k(A)\}]$ for every $A\in\mathscr{A}$.

(3) This is obvious from the defining properties of the free product in the unital category.

(4) We give a proof for the projective case; the other cases are handled similarly. Suppose $\mathscr{C}$ is a unital C*-algebra with an ideal $\mathscr{J}$ and $\pi:\mathscr{A}\to\mathscr{C}/\mathscr{J}$ is a *-homomorphism. Define a unital *-homomorphism $\sigma:\mathscr{B}\to\mathscr{C}/\mathscr{J}$ by $\sigma(x)=\alpha(x)\cdot 1$. Thus there is a unital *-homomorphism $\rho:\mathscr{A}*\mathscr{B}\to\mathscr{C}/\mathscr{J}$ such that $\rho|\mathscr{A}=\pi$ and $\rho|\mathscr{B}=\sigma$. Since $\mathscr{A}*\mathscr{B}$ is projective, ρ lifts to a *-homomorphism $\tau:\mathscr{A}*\mathscr{B}\to\mathscr{C}$. Thus $\tau|\mathscr{A}$ is the required lifting of π.

(5) Suppose $\mathscr{A}=C^*(x_1,\ldots,x_m|\varphi)$ is weakly semiprojective with weakly semiprojective approximating functions $\varphi_1,\ldots,\varphi_m$, and $\mathscr{B}=C^*(y_1,\ldots,y_n|\psi)$ is weakly semiprojective with weakly semiprojective approximating functions $\psi_1,\ldots,\psi_n$. Then

$$\mathscr{A}\oplus\mathscr{B}=C^*(x_1,\ldots,x_m,y_1,\ldots,y_n,p|\Phi),$$

where

$$\begin{aligned}&\Phi(x_1,\ldots,x_m,y_1,\ldots,y_n,p)\\ =\;&\varphi(x_1,\ldots,x_m)^*\varphi(x_1,\ldots,x_m)+\psi(y_1,\ldots,y_n)^*\psi(y_1,\ldots,y_n)\\ &+(p-p^*)^*(p-p^*)+(p-p^2)^*(p-p^2)+\sum_{j=1}^m(px_j-x_jp)^*(px_j-x_jp)\\ &+\sum_{j=1}^m(px_j-x_j)^*(px_j-x_j)+\sum_{j=1}^n(py_j-y_jp)^*(py_j-y_jp).\end{aligned}$$

Let f be a continuous function on $\mathbb{R}$ such that $f(t)=0$ when $t\leqslant\frac{1}{4}$ and $f(t)=1$ when $t\geqslant\frac{3}{4}$. Define $\Phi_1,\ldots,\Phi_{m+n+1}$ on $\mathscr{A}\oplus\mathscr{B}$ by

$$\Phi_i(x_1,\ldots,x_m,y_1,\ldots,y_n,p)=\begin{cases}f(p)\varphi_i(x_1,\ldots,x_m)f(p), & 1\leqslant i\leqslant m\\ (1-f(p))\psi_{i-m}(y_1,\ldots,y_n)(1-f(p)), & m+1\leqslant i\leqslant m+n\\ f(p), & i=m+n+1\end{cases}$$

For any $\varepsilon > 0$, and any operators $S_1,\dots,S_m,T_1,\dots,T_n,Q$, there exists $\delta_1 > 0$ such that, if $\|P-P^*\| < \delta_1$ and $\|Q-Q^2\| < \delta_1$, then $f(Q)$ is a projection and $\|Q-f(Q)\| < \varepsilon$. In addition, there exists $\delta_2 > 0$ such that, if $\|\varphi(S_1,\dots,S_m)\| < \delta_2$, then

$$\|S_j - \varphi_j(S_1,\dots,S_m)\| < \varepsilon \quad \text{for all} \quad 1 \leqslant j \leqslant m$$

and

$$\varphi\left(\varphi_1(S_1,\dots,S_m),\dots,\varphi_m(S_1,\dots,S_m)\right) = 0;$$

also there exists $\delta_3 > 0$ such that, if $\|\psi(T_1,\dots,T_n)\| < \delta_3$, then

$$\|T_k - \psi_k(T_1,\dots,T_n)\| < \varepsilon \quad \text{for all} \quad 1 \leqslant k \leqslant n$$

and

$$\psi\left(\psi_1(T_1,\dots,T_n),\dots,\psi_n(T_1,\dots,T_n)\right) = 0.$$

Let

$$\widehat{S_j} = f(Q)\varphi_j(S_1,\dots,S_m)f(Q) \quad \text{for all} \;\; 1 \leqslant j \leqslant m$$

and

$$\widehat{T_k} = (1-f(Q))\psi_k(T_1,\dots,T_n)(1-f(Q)) \quad \text{for all} \;\; 1 \leqslant k \leqslant n.$$

Then $\widehat{S_j}f(Q) = f(Q)\widehat{S_j} = \widehat{S_j}$ and $\widehat{T_k}f(Q) = f(Q)\widehat{T_k} = 0$.

Choose $\delta = \min\{\delta_1^2,\delta_2^2,\delta_3^2\}$, then $\Phi_1,\dots,\Phi_{m+n+1}$ are weakly semiprojective approximating functions for Φ.

Conversely, suppose $\mathscr{A}\oplus\mathscr{B} = C^*(x_1\oplus y_1,\dots,x_n\oplus y_n|\varphi)$ is weakly semiprojective with weakly semiprojective approximating functions $\varphi_1,\dots,\varphi_n$. Since $\varphi(A\oplus B) = \varphi(A)\oplus\varphi(B)$, it is clear that $\mathscr{A} = C^*(x_1,\dots,x_n|\varphi)$ and $\mathscr{B} = C^*(y_1,\dots,y_n|\varphi)$, and $\mathscr{A}$ and $\mathscr{B}$ are all weakly semiprojective with weakly semiprojective approximating functions $\varphi_1,\dots,\varphi_n$.

It is not hard to prove the semiprojective case using the similar idea. $\square$

REMARK 3.2. (1) Statement (5) of Lemma 3.1 is not true in the nonunital case, and statement (1) is not true for projectivity in the unital case, e.g., if S is the unilateral shift operator, then the C*-algebra generated by the image of $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ in the Calkin algebra is isomorphic to $\mathscr{M}_2(\mathbb{C})$, but it cannot be lifted to a representation of $\mathscr{M}_2(\mathbb{C})$ in $\mathscr{B}(\mathscr{H})$.

(2) In general the different types of projectivity are not preserved under tensor products even when the algebras are very nice. For example, if X is the unit circle, then $C(X)$, which is isomorphic to the universal C*-algebra generated by a unitary operator, is projective, but $C(X)\otimes C(X) = C^*(x,y|x,y \text{ unitary}, xy-yx=0)$ is not weakly semiprojective [17].

(3) In the nonunital category, statement (4) of Lemma 3.1 always holds, because the 0 functional is allowed.

The following lemma is a key ingredient to our main results in this section.

LEMMA 3.3. *There exists a noncommutative continuous function $\phi(x,y,z)$ such that, for any C*-algebra $\mathscr{A}$ and any $\varepsilon > 0$, there exists $\delta > 0$, such that, whenever $P,Q,A \in \mathscr{A}$ with P and Q projections and $\|A^*A - P\| < \delta$, $\|AA^* - Q\| < \delta$, we have*

(1) $\|\phi(P,Q,A) - A\| < \varepsilon$,

(2) $\phi(P,Q,A)^*\phi(P,Q,A) = P$ and $\phi(P,Q,A)\phi(P,Q,A)^* = Q$,

(3) $\phi(P,Q,A) = A$ whenever $A^*A = P$ and $AA^* = Q$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous defined by $f(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \frac{1}{4} \\ \frac{1}{\sqrt{t}}, & \frac{3}{4} \leqslant t \leqslant \frac{5}{4} \end{cases}$, and define $\phi(P,Q,A) = f(QAPA^*Q)QAP$. $\square$

The following is our main theorem in this section. Suppose $\mathscr{A}$ is a unital C*-algebra generated by partial isometries $V_1,\dots,V_n$ and C* $(V_1^*V_1,\dots,V_n^*V_n)$ and C* $(V_1V_1^*,\dots,V_nV_n^*)$ are both (weakly) semiprojective or C* $(V_1^*V_1,\dots,V_n^*V_n,V_1V_1^*,\dots,V_nV_n^*)$ is (weakly) semiprojective. Does it follow that $\mathscr{A}$ is weakly semiprojective? We prove this is true when the only relations on $V_1,\dots,V_n$ are those on $V_1^*V_1,\dots,V_n^*V_n,V_1V_1^*,\dots,V_nV_n^*$.

THEOREM 3.4. *Suppose φ and ψ are null-bounded noncommutative continuous functions. The following are true:*

(1) Suppose $C^*(P_1,\dots,P_n|\ \varphi)$ and $C^*(Q_1,\dots,Q_n|\psi)$ are (weakly) semiprojective and $P_1,\dots,P_n$, $Q_1,\dots,Q_n$ are projections. Then the universal C-algebra $\mathscr{A} = C^*(X_1,\dots,X_n|\Phi)$ with the relation Φ defined by*

$$\begin{aligned}\Phi(X_1,\dots,X_n) =& \varphi(X_1^*X_1,\dots,X_n^*X_n)^*\varphi(X_1^*X_1,\dots,X_n^*X_n)\\ &+\psi(X_1X_1^*,\dots,X_nX_n^*)^*\psi(X_1X_1^*,\dots,X_nX_n^*)\end{aligned}$$

is (weakly) semiprojective.

(2) If $C^*(P_1,\dots,P_n,Q_1,\dots,Q_n|\psi)$ is (weakly) semiprojective and $P_1,\dots,P_n,Q_1,\dots,Q_n$ are projections, then $C^*(X_1,\dots,X_n|\Psi)$ is (weakly) semiprojective, where the relation Ψ is defined by

$$\begin{aligned}&\Psi(X_1,\dots,X_n)\\ =&\psi(X_1^*X_1,\dots,X_n^*X_n,X_1X_1^*,\dots,X_nX_n^*)^*\psi(X_1^*X_1,\dots,X_n^*X_n,X_1X_1^*,\dots,X_nX_n^*)\end{aligned}$$

Proof. (1) Let $\varphi_1,\dots,\varphi_n$ be weakly semiprojective approximating functions for φ, and $\psi_1,\dots,\psi_n$ be weakly semiprojective approximating functions for ψ.

Define the functions $\Phi_1,\dots,\Phi_n$ by

$$\Phi_i(X_1,\dots,X_n) = \phi(\varphi_i(P_1,\dots,P_n),\psi_i(Q_1,\dots,Q_n),V_i),$$

where ϕ is the noncommutative continuous function in Lemma 3.3.

Given any $\varepsilon > 0$. Let δ_0 be ε defined in Lemma 3.3.

Since $C^*(P_1,\dots,P_n|\ \varphi)$ is weakly semiprojective, there exists $\delta_1 > 0$ such that, for $1 \leqslant i \leqslant n$ and any operators $A_1,\dots,A_n$ with $\|\varphi(A_1^*A_1,\dots,A_n^*A_n)\| < \delta_1$, $\varphi_i(A_1^*A_1,\dots,A_n^*A_n)$ is projection and $\|A_i^*A_i - \varphi_i(A_1^*A_1,\dots,A_n^*A_n)\| < \delta_0$.

Since $C^*(Q_1,\dots,Q_n|\psi)$ is weakly semiprojective, there exists $\delta_2 > 0$ such that, if

$\|\varphi(A_1A_1^*,\ldots,A_nA_n^*)\| < \delta_2$, then, for $1 \leqslant i \leqslant n$, $\psi_i(A_1A_1^*,\ldots,A_nA_n^*)$ is projection and $\|A_iA_i^* - \psi_i(A_1A_1^*,\ldots,A_nA_n^*)\| < \delta_0$.

Put $\varphi_i(A_1^*A_1,\ldots,A_n^*A_n), \psi_i(A_1A_1^*,\ldots,A_nA_n^*), A_i$ to P,Q,A in Lemma 3.3, we have that

$$\phi(\varphi_i(A_1^*A_1,\ldots,A_n^*A_n), \psi_i(A_1A_1^*,\ldots,A_nA_n^*), A_i) \ (= \Phi_i(A_1,\ldots,A_n))$$

is a partial isometry from $\varphi_i(A_1^*A_1,\ldots,A_n^*A_n)$ to $\psi_i(A_1A_1^*,\ldots,A_nA_n^*)$.

Let $\delta = \min\{\delta_1^2,\delta_2^2\}$. We prove that $\Phi_1,\ldots,\Phi_n$ are weakly semiprojective approximating functions for Φ.

Use the similar idea and Lemma 3.3, we can prove the weakly semiprojective case.

(2) Similar to the proof of (1). □

EXAMPLE 3.5. Suppose

$$\begin{aligned}\mathscr{M}_2(C) &= C^*\left(P_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix}, P_2 = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}, P_3 = \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\\ &= C^*\left(P_1,P_2,P_3|\varphi\right)\end{aligned}$$

and

$$\begin{aligned}\mathscr{M}_3(C) &= C^*\left(Q_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, Q_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, Q_3 = \begin{pmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}\right)\\ &= C^*\left(Q_1,Q_2,Q_3|\psi\right).\end{aligned}$$

Then the universal C*-algebra generated by partial isometries V_1,V_2,V_3 such that

$$\varphi(V_1^*V_1, V_2^*V_2, V_3^*V_3) = 0$$

and

$$\psi\left(V_1V_1^*, V_2V_2^*, V_3V_3^*\right) = 0$$

is semiprojective.

We also can apply our results to the generalized version of the noncommutative unitary construction of K. McClanahan [15].

PROPOSITION 3.6. If $\mathscr{A} = C^*\left(x_1,\ldots x_n|\varphi\right)$ is (weakly) semiprojective, then the universal C*-algebra $\mathscr{E}$ generated by $\left\{a_{ijk} : 1 \leqslant i,j \leqslant m, 1 \leqslant k \leqslant n\right\}$, subject to $\varphi\left(\left(a_{ij1}\right),\ldots,\left(a_{ijn}\right)\right) = 0$ is (weakly) semiprojective.

Proof. Suppose $\varphi_1,\ldots,\varphi_n$ are (weakly) semiprojective approximating functions for φ. Define functions $\{\Phi_{i,j,k} : 1 \leqslant i,j \leqslant m, 1 \leqslant k \leqslant n\}$ by

$$\Phi_{i,j,k}\left(\{a_{s,t,l}\}_{s,t,l}\right) = f_{i,j}\left(\varphi_k\left(\left(a_{s,t,1}\right),\ldots,\left(a_{s,t,n}\right)\right)\right),$$

where $f_{i,j} : \mathscr{M}_m(\mathbb{C}) \mapsto \mathbb{C}$ such that for any $m \times m$ matrix A, $A = (f_{i,j}(A))$. It is clear that $\{\Phi_{i,j,k} : 1 \leqslant i,j \leqslant m, 1 \leqslant k \leqslant n\}$ are (weakly) semiprojective approximating functions for Φ. □

COROLLARY 3.7. *Suppose $C^*(P_1,\ldots,P_n|\varphi)$ and $C^*(Q_1,\ldots,Q_n|\psi)$ are (weakly) semiprojective, where $P_1,Q_1,\ldots,P_n,Q_n$ are projections. Suppose m is a positive integer and $\mathscr{A}$ is the universal C*-algebra generated by $\{a_{ijk}: 1 \leqslant i,j \leqslant m, 1 \leqslant k \leqslant n\}$ subject to*

$$\varphi\left(\left(a_{ij1}\right)^*\left(a_{ij1}\right),\ldots,\left(a_{ijn}\right)^*\left(a_{ijn}\right)\right)=0,$$

$$\psi((a_{ij1})(a_{ij1})^*,\ldots,(a_{ijn})(a_{ijn})^*)=0.$$

Then $\mathscr{A}$ is (weakly) semiprojective.

We also can define projectivity in terms of noncommutative continuous functions. A unital C*-algebra $C^*(b_1,\ldots,b_n|\varphi)$ is *projective* in the unital category if, for any unital C*-algebra $\mathscr{A}$ and any ideal $\mathscr{J}$ in $\mathscr{A}$, and any $x_1,\ldots,x_n \in \mathscr{A}/\mathscr{J}$ with $\varphi(x_1,\ldots,x_n)=0$, there exist elements $a_1,\ldots,a_n$ in $\mathscr{A}$ such that $x_i = a_i + \mathscr{J}$ and $\varphi(a_1,\ldots,a_n)=0$.

REMARK 3.8. (1) The universal C*-algebra generated by A such that $\|A\| \leqslant r$ (r is a positive number) is projective. The universal C*-algebra generated by $\{A_n\}_{n=1}^{\infty}$ such that $\|A_n\| \leqslant r_n$ for some numbers r_n is projective. Thus every separable unital C*-algebra is isomorphic to $\mathscr{A}/\mathscr{J}$, where $\mathscr{A}$ is a projective C*-algebra and $\mathscr{J}$ is an ideal of $\mathscr{A}$.

(2) If $\{\mathscr{A}_n\}_{n=1}^{\infty}$ is a sequence of projective C*-algebras, then the free product $*_n\mathscr{A}_n$ is projective.

(3) If $\{\mathscr{A}_n\}_{n=1}^{\infty}$ is a sequence of separable unital semiprojective algebras, then $*_n\mathscr{A}_n$ may not be weakly semiprojective. For example, $\mathscr{M}_2(\mathbb{C}) * \mathscr{M}_3(\mathbb{C}) * \mathscr{M}_4(\mathbb{C}) * \cdots$ is not weakly semiprojective, but each $\mathscr{M}_n(\mathbb{C})$ is semiprojective.

Although weakly semiprojective C*-algebras need not be finitely generated, identity representation on such algebras must be a pointwise limit of representations into finitely generated subalgebras.

PROPOSITION 3.9. Suppose $\mathscr{A}$ is separable and weakly semiprojective and $\{\mathscr{A}_n\}_{n=1}^{\infty}$ is an increasing sequence of finitely generated C*-subalgebras whose union is dense in $\mathscr{A}$. Then there is a positive integer N and unital *-homomorphisms $\pi_n : \mathscr{A} \to \mathscr{A}_n$ for all $n \geqslant N$ such that

$$\|x-\pi_n(x)\| \to 0$$

for every $x \in \mathscr{A}$.

Proof. It follows from the hypothesis that $\operatorname{dist}(x,\mathscr{A}_n) \to 0$ for every $x \in \mathscr{A}$. Thus, for each $x \in \mathscr{A}$, there is a $x_n \in \mathscr{A}_n$ such that $\|x-x_n\| \leqslant \operatorname{dist}(x,\mathscr{A}_n)+\frac{1}{n}$. Define a unital *-homomorphism $\pi : \mathscr{A} \to \prod_1^{\infty}\mathscr{A}_n / \oplus_1^{\infty}\mathscr{A}_n$ by

$$\pi(x) = [\{x_n\}].$$

The desired result follows easily from the weak semiprojectivity of $\mathscr{A}$. $\square$

DEFINITION 3.10. A unital C^*-algebra $\mathscr{A}$ is called GCR if for any irreducible representation π from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$, $\mathscr{K}(\mathscr{H}) \subseteq \pi(\mathscr{A})$.

LEMMA 3.11. *If $\mathscr{A}$ is a unital GCR C^*-algebra, then there exists a positive integer n and a representation $\pi : \mathscr{A} \to \mathscr{M}_n(\mathbb{C})$ that is onto.*

Proof. Suppose $\mathscr{J}$ is a maximal ideal in $\mathscr{A}$. Then $\mathscr{A}/\mathscr{J}$ is a simple C*-algebra. Let $\pi : \mathscr{A}/\mathscr{J} \to \mathscr{B}(\mathscr{H})$ be an irreducible representation. Then $\pi(\mathscr{A}/\mathscr{J})' = \mathbb{C}1$. It follows that $\mathscr{K}(\mathscr{H}) \subseteq \pi(\mathscr{A}/\mathscr{J})$ is a closed ideal, therefore $\mathscr{H}$ is finite-dimensional. □

From the above lemma, it is not hard to see that if $\mathscr{A}$ is a simple infinite-dimensional C*-algebra, then $\mathscr{A}$ cannot be a subalgebra of a GCR algebra.

COROLLARY 3.12. *If $\mathscr{A}$ is a unital simple infinite-dimensional C*-algebra that is a subalgebra of a direct limit of subalgebras of GCR C*-algebras, then $\mathscr{A}$ is not weakly semiprojective.*

Proof. It follows from the proof of Proposition 3.9 that there is a *-homomorphism $\pi : \mathscr{A} \to \prod_1^\infty \mathscr{A}_n / \oplus_1^\infty \mathscr{A}_n$, where $\{\mathscr{A}_n\}_{n=1}^\infty$ as defined in Proposition 3.9. Assume via contradiction that $\mathscr{A}$ is weakly semiprojective. Then there is a representation $\pi_n : \mathscr{A} \to \mathscr{A}_n$ for some positive integer n. Since $\mathscr{A}_n$ is a subalgebra of a GCR algebra, it follows that $\mathscr{A}_n$, and hence $\mathscr{A}$, has a finite-dimensional representation. Since $\mathscr{A}$ is simple, every representation of $\mathscr{A}$ is one-to-one, which implies that $\mathscr{A}$ is finite-dimensional, a contradiction. □

REMARK 3.13. The Cuntz algebra is weakly semiprojective, hence, the Cuntz algebra cannot be embedded into a direct limit of subalgebras of GCR C*-algebras. The irrational rotation algebra $\mathscr{A}_\theta$ is not weakly semiprojective, since it can be embedded into the direct limit of subalgebras of GCR C*-algebras.

We conclude this section with an observation concerning the reduced free group C*-algebra, $C_r^*(\mathbb{F}_n)$.

PROPOSITION 3.14. $C_r^*(\mathbb{F}_n)$ is not weakly semiprojective.

Proof. U. Haagerup and S. Thorbjørnsen [8] proved that there was a map

$$\pi : C_r^*(\mathbb{F}_n) \to \prod_{n\geqslant 1} \mathscr{M}_n(\mathbb{C}) / \oplus_{n\geqslant 1} \mathscr{M}_n(\mathbb{C}).$$

Since $C_r^*(\mathbb{F}_n)$ is infinite-dimensional and simple, $C_r^*(\mathbb{F}_n)$ has no finite-dimensional representation. Hence $C_r^*(\mathbb{F}_n)$ cannot be weakly semiprojective. □

4. Finite von Neumann Algebras and Trace Norms

When we talked about weak semiprojectivity in C*-algebras we described it in terms of mappings into algebras $\prod_1^\infty \mathscr{B}_n / \oplus_1^\infty \mathscr{B}_n$ being "liftable". There is another way to describe this by replacing the $\prod_1^\infty \mathscr{B}_n / \oplus_1^\infty \mathscr{B}_n$ construction with ultraproducts.

Suppose $\mathbb{I}$ is an infinite set and ω is an ultrafilter on $\mathbb{I}$, i.e., ω is a family of subset of $\mathbb{I}$ such that
(1) $\emptyset \notin \omega$,
(2) If $A, B \in \omega$, then $A \cap B \in \omega$,
(3) For every subset A in $\mathbb{I}$, either $A \in \omega$ or $\mathbb{I} \setminus A \in \omega$.

One example of an ultrafilter is obtained by choosing an element ι in $\mathbb{I}$ and letting ω be the collection of all subsets of $\mathbb{I}$ that contain ι. Such an ultrafilter is called *principle* ultrafilter and ultrafilter not of this form are called *free*. We call an ultrafilter ω *nontrivial* if it is free and there is a sequence $\{E_n\}_{n=1}^\infty$ in ω whose intersection is empty. We can always choose $E_1 = \mathbb{I}$ and, by replacing E_n with $\cap_{k=1}^n E_k$, we can assume that $\{E_n\}_{n=1}^\infty$ is decreasing. Throughout this paper we will only use nontrivial ultrafilters.

Suppose $\{\mathscr{A}_i : i \in \mathbb{I}\}$ is a family of C*-algebras and ω is a nontrivial ultrafilter on $\mathbb{I}$. Then

$$\mathscr{J} = \left\{ \{A_i\} \in \prod_{i \in \mathbb{I}} \mathscr{A}_i : \lim_{i \to \omega} \|A_i\| = 0 \right\}$$

is a norm-closed two-sided ideal in $\prod_{i \in \mathbb{I}} \mathscr{A}_i$, and we call the quotient $\left(\prod_{i \in \mathbb{I}} \mathscr{A}_i\right)/\mathscr{J}$ the *C*-algebraic ultraproduct* of the $\mathscr{A}_i$'s and denote it by $\prod^\omega \mathscr{A}_i$. For an introduction to ultraproducts see [7]. It is easily verified that a C*-algebra $\mathscr{A}$ is weakly semiprojective if and only if, given a unital *-homomorphism $\pi : \mathscr{A} \to \prod^\omega \mathscr{A}_i$ there are functions $\pi_i : \mathscr{A} \to \mathscr{A}_i$ for each $i \in \mathbb{I}$ such that, eventually along ω, π_i is a unital *-homomorphism and such that, for ever $a \in \mathscr{A}$,

$$\pi(a) = [\{\pi_i(a)\}]_\omega.$$

We now want to look at analogue of weak semiprojectivity for finite von Neumann algebras with faithful tracial states. Suppose $\mathscr{A}$ is a C*-algebra with a tracial state τ. As in the GNS construction there is a seminorm $\|\cdot\|_{2,\tau}$ on $\mathscr{A}$ defined by $\|a\|_{2,\tau} = \tau(a^*a)^{1/2}$. More generally, if $1 \leqslant p < \infty$, we define $\|a\|_{p,\tau} = (\tau((a^*a)^{p/2}))^{1/p}$. Since $C^*(a^*a)$ is isomorphic to $C(X)$, where X is the spectrum of a^*a, there is a probability measure μ such that $\tau(f(a^*a)) = \int_X f d\mu$ for every $f \in C(X)$. Thus, for $1 \leqslant p < \infty$,

$$\|a\|_{p,\tau} = 0 \text{ if and only if } \|a\|_{2,\tau} = 0. \tag{1}$$

If there is no confusion, we can simply use $\|\cdot\|_2$ and $\|\cdot\|_p$ to denote $\|\cdot\|_{2,\tau}$ and $\|\cdot\|_{p,\tau}$ respectively.

Suppose $\{(\mathscr{A}_i, \tau_i) : i \in \mathbb{I}\}$ is a family of C*-algebras $\mathscr{A}_i$ with tracial states τ_i. We can define a trace ρ on $\prod_{i \in \mathbb{I}} \mathscr{A}_i$ by

$$\rho(\{a_i\}) = \lim_{i \to \omega} \tau_i(a_i).$$

The set $\mathscr{J}_2 = \{\{A_i\} \in \prod_{i\in\mathbb{I}} \mathscr{A}_i : \lim_{i\to\omega} \|A_i\|_2 = 0\}$ is a closed two-sided ideal in $\prod_{i\in\mathbb{I}} \mathscr{A}_i$, and we call the quotient $\left(\prod_{i\in\mathbb{I}} \mathscr{A}_i\right)/\mathscr{J}_2$ the *tracial ultraproduct* of the $\mathscr{A}_i$'s, and we denote it by $\prod^\omega (\mathscr{A}_i, \tau_i)$. There is a natural faithful trace τ on $\prod^\omega (\mathscr{A}_i, \tau_i)$ defined by

$$\tau([\{a_i\}]_\omega) = \lim_{i\to\omega} \tau_i(a_i).$$

$\prod^\omega (\mathscr{A}_i, \tau_i)$ is the representation of $\prod_{i\in\mathbb{I}} \mathscr{A}_i$ using the GNS construction with ρ. By Equation (1) we see that $\prod^\omega (\mathscr{A}_i, \tau_i) = \left(\prod_{i\in\mathbb{I}} \mathscr{A}_i\right)/\mathscr{J}_p$, where

$$\mathscr{J}_p = \left\{ \{A_i\} \in \prod_{i\in\mathbb{I}} \mathscr{A}_i : \lim_{i\to\omega} \|A_i\|_p = 0 \right\}.$$

One immediate consequence of $\mathscr{J}_p = \mathscr{J}_2$ is the fact that, on a bounded subset of any $(\mathscr{A}, \tau)$ the norms $\|\cdot\|_2$ and $\|\cdot\|_p$ generate the same topology.

It was shown by S. Sakai [16] that a tracial ultraproduct of finite factors is a von Neumann algebra and is, in fact, a factor. However, it is true that any tracial ultraproduct of von Neumann algebras is a von Neumann algebra. Here we prove that any tracial ultraproduct of C*-algebras is a von Neumann algebra.

THEOREM 4.1. *Suppose $\{\mathscr{A}_i\}_{i\in\mathbb{I}}$ is a family of C^*-algebras with a tracial state τ_i on each $\mathscr{A}_i$ and ω is a nontrivial ultrafilter on $\mathbb{I}$. Then the tracial ultraproduct $\prod^\omega (\mathscr{A}_i, \tau_i)$ of $\{\mathscr{A}_i\}_{i\in\mathbb{I}}$ is a von Neumann algebra.*

Proof. Let $\mathscr{A} = \prod^\omega (\mathscr{A}_i, \tau_i)$. Note that $\mathrm{Ball}(\overline{\mathscr{A}}^{*-SOT}) = \overline{\mathrm{Ball}(\mathscr{A})}^{\|\cdot\|_2}$, i.e., the closure of the unit ball of $\mathscr{A}$ in strong operator topology is equal to the $\|\cdot\|_2$-closure of the unite ball of $\mathscr{A}$.

Suppose $T \in \overline{\mathrm{Ball}(\mathscr{A})}^{\|\cdot\|_2}$. Then for any positive integer n, there exists $A_n \in \mathrm{Ball}(\mathscr{A})$ such that $\|T - A_n\|_2 \leqslant \frac{1}{4^n}$. Write $A_n = [\{A_{ni}\}]_\omega$ with each $A_{ni} \in \mathrm{Ball}(\mathscr{A}_i)$.

Since ω is nontrivial, there is a family $\{E_n\}$ of elements of ω such that

$$I = E_1 \supseteq E_2 \supseteq \cdots \quad \text{and} \quad \cap_n E_n = \emptyset.$$

Let

$$F_n = \{i \in E_n : \forall 1 \leqslant k \leqslant n, \|A_{ki} - A_{ni}\|_2 < \frac{1}{4^n} + \frac{1}{4^k}\}.$$

Let $X_i = A_{ki}$ for $i \in F_k \setminus F_{k+1}$. For any $i \in F_n$, there exists some $k \geqslant n$ such that $i \in F_k \setminus F_{k+1}$ and

$$\|A_{ni} - X_i\|_2 = \|A_{ni} - A_{ki}\|_2 \leqslant \frac{1}{4^n} + \frac{1}{4^k} \leqslant \frac{2}{4^n} \leqslant \frac{1}{2^n}.$$

Let $X = [\{X_i\}]_\omega$. Then $X \in \mathrm{Ball}(\mathscr{A})$ and

$$\|A_n - X\|_2 \leqslant \frac{1}{2^n}.$$

Hence $T = X \in \mathrm{Ball}(\mathscr{A})$. This implies that $\mathscr{A} = \prod^\omega (\mathscr{A}_i, \tau_i)$ is a von Neumann algebra. $\square$

The next Theorem gives a generalization of Lin's theorem for $\|\cdot\|_p$ on C*-algebras with trace. When $p=2$, it was proved for finite factors in [9].

THEOREM 4.2. *For every $\varepsilon>0$ and every $1\leqslant p<\infty$, there exists $\delta>0$ such that, for any C^*-algebra $\mathscr{A}$ with trace τ, and $A_1,\ldots,A_n\in \mathrm{Ball}(\mathscr{A})$ with $\|A_jA_j^*-A_j^*A_j\|_p<\delta$ and $\|A_jA_k-A_kA_j\|_p<\delta$, there exists $B_1,\ldots,B_n\in \mathrm{Ball}(\mathscr{A})$ so that $B_jB_j^*=B_j^*B_j$, $B_jB_k=B_kB_j$ and $\sum_{j=1}^n\|A_j-B_j\|_p<\varepsilon$.*

Proof. Assume the statement is false. Then there is an $\varepsilon>0$ such that, for every positive integer k, there is a unital C*-algebra $\mathscr{A}_k$ with trace τ_k and elements $A_{k,1},\ldots,A_{k,n}$ with $\|A_{k,j}A_{k,j}^*-A_{k,j}^*A_{k,j}\|_p<\frac{1}{k}$ and $\|A_{k,j}A_{k,i}-A_{k,i}A_{k,j}\|_p<\frac{1}{k}$, so that for all $B_1,\ldots,B_n\in\mathrm{Ball}(\mathscr{A})$ with $B_jB_j^*=B_j^*B_j$ and $B_jB_k=B_kB_j$ we have $\sum_{j=1}^n\|A_j-B_j\|_p^p\geqslant\varepsilon$. The tracial ultraproduct $\mathscr{A}=\prod^\omega\mathscr{A}_i=(\prod_{i\in\mathbb{I}}\mathscr{A}_i)/\mathscr{J}_p$ is a von Neumann algebra and $\{A_j=[\{A_{k,j}\}]_\omega:1\leqslant j\leqslant n\}$ is a family of commuting normal operators. Hence, by the proof of Theorem 5.5 in [2], there is a selfadjoint operator $C\in\mathscr{A}$ and bounded continuous functions $f_1,\ldots,f_n:\mathbb{R}\to\mathbb{C}$ such that $A_j=f_j(C)$ for $1\leqslant j\leqslant n$. Write $C=[\{C_k\}]_\omega$ with each $C_k=C_k^*$. Define $B_{k,j}=f_j(C_k)$ for $1\leqslant j\leqslant n$ and $k\in\mathbb{N}$. Then $A_j=[\{B_{k,j}\}]_\omega$ for $1\leqslant j\leqslant n$ and $\{B_{k,j}:1\leqslant j\leqslant n\}$ is a family of commuting normal operators. So

$$\varepsilon\leqslant\lim_{k\to\omega}\sum_{j=1}^n\|A_{k,j}-B_{k,j}\|_p=\sum_{j=1}^n\|A_j-f_j(C)\|_p=0,$$

which is a contradiction. □

REMARK 4.3. Suppose K is a compact nonempty subset of $\mathbb{C}$ that is a continuous image of $[0,1]$. It follows from Proposition 39 in [10] that there is a noncommutative continuous function α such that, for every operator T with $\|T\|\leqslant1$ we have $\alpha(T)$=0 if and only if T is normal and the spectrum of T is contained in K. If, in Corollary 4.2, we add the condition that $\|\alpha(A_1)\|_2<\delta$, then we can choose B_1 so that its spectrum is contained in K. In particular, if we add $\|1-A_1^*A_1\|_2<\delta$, we can choose B_1 to be unitary.

The next theorem shows (see Corollary 4.5) that, unlike in the C*-algebra case, commutative C*-algebras are "weakly semiprojective" in the "diffuse von Neumann algebra" sense.

THEOREM 4.4. *Suppose, for $i\in\mathbb{I}$, $\mathscr{M}_i$ is a diffuse von Neumann algebra with faithful trace τ_i and $\mathscr{A}$ is a commutative countably generated von Neumann subalgebra of the ultraproduct $\prod^\omega(\mathscr{M}_i,\tau_i)$. Then, for every i, there is a trace-preserving *-homomorphism $\pi_i:\mathscr{A}\to\mathscr{M}_i$ such that, for every $a\in\mathscr{A}$,*

$$a=[\{\pi_i(a)\}]_\omega.$$

Proof. Suppose P is a projection in $\prod^\omega(\mathscr{M}_i,\tau_i)$. It is well-known that P can be written as $P=[\{A_i\}]_\omega$ with each A_i a projection. Since $\tau(A_i)\to\tau(P)$ as $i\to\omega$ and since each $\mathscr{M}_i$ is diffuse, we can, for each i, find a projection $P_i \in \mathscr{M}_i$ so that $\tau_i(P_i) = \tau(P)$ and either $P_i \leqslant A_i$ or $A_i \leqslant P_i$. Since $\|A_i - P_i\|_2 = \sqrt{|\tau_i(P_i) - \tau_i(A_i)|} \to 0$, we have $P = [\{P_i\}]_\omega$. Hence, every projection in $\prod^\omega(\mathscr{M}_i, \tau_i)$ can be lifted to projections with the same trace.

Next suppose $P = [\{P_i\}]_\omega, Q = [\{Q_i\}]_\omega$ are projections in $\prod^\omega(\mathscr{M}_i, \tau_i)$ such that $P \leqslant Q$, and, for every i, $P_i \leqslant Q_i$ and $\tau_i(P_i) = \tau(P)$ and $\tau_i(Q_i) = \tau(Q)$. Suppose E is a projection in $\prod^\omega(\mathscr{M}_i, \tau_i)$ and $P < E < Q$. Applying what we just proved to the projection $E - P$ in the ultraproduct

$$(Q-P)\left(\prod^\omega(\mathscr{M}_i, \tau_i)\right)(Q-P) = \prod^\omega(Q_i - P_i)\mathscr{M}_i(Q_i - P_i),$$

We can find projections $E_i \in \mathscr{M}_i$ so that $P_i \leqslant E_i \leqslant Q_i$, $\tau_i(E_i) = \tau(E)$ and $E = [\{E_i\}]_\omega$. Since $\mathscr{A}$ is countably generated and commutative, we know from von Neumann's Theorem that $\mathscr{A}$ is generated by a single selfadjoint T with $0 \leqslant T \leqslant 1$. Since $\prod^\omega(\mathscr{M}_i, \tau_i)$ is diffuse, the chain $\left\{\chi_{[0,s)}(T) : 0 \leqslant s \leqslant 1\right\}$ can be extended to a chain $\{P(t) : t \in [0,1]\}$ such that $\tau(P(t)) = t$ for $0 \leqslant t \leqslant 1$. Repeatedly using the result above we can find projections $P_i(t)$ for each i and each rational $t \in [0,1]$ such that $\tau_i(P_i(t)) = t$ and $P(t) = [\{P_i(t)\}]_\omega$, and such that $P_i(s) \leqslant P_i(t)$ for all i and $s \leqslant t$. Hence, for each $t \in [0,1]$ and each $i \in I$, we can define

$$P_i(t) = \sup\{P_i(s) : s \leqslant t, s \in \mathbb{Q}\} = \inf\{P_i(s) : s \geqslant t, s \in \mathbb{Q}\}.$$

Then we must have $P(t) = [\{P_i(t)\}]_\omega$ for every $t \in [0,1]$. For each i, the map $P(t) \mapsto P_i(t)$ extends to a trace-preserving *-homomorphism $\rho_i : \{P(t) : t \in [0,1]\}'' \to \mathscr{M}_i$, and we can let $\pi_i = \rho_i|\mathscr{A}$. $\square$

COROLLARY 4.5. *Suppose $\mathscr{M}_i$ is a diffuse von Neumann algebra for every $i \in \mathbb{I}$ and $\mathscr{A}$ commutative countably generated unital C*-algebra and $\pi : \mathscr{A} \to \prod^\omega(\mathscr{M}_i, \tau_i)$ is a unital *-homomorphism. Then, for every i, there is a *-homomorphism $\pi_i : \mathscr{A} \to \mathscr{M}_i$ such that*

(1) $\pi(a) = [\{\pi_i(a)\}]_\omega$ for every $a \in \mathscr{A}$, and

(2) $\tau_i \circ \pi_i = \tau \circ \pi$ for every $i \in \mathbb{I}$.

COROLLARY 4.6. *Suppose $\mathscr{A}$ is a commutative von Neumann algebra with a faithful normal trace τ_0 and $\mathscr{A}$ is generated by countably many elements $A_1, A_2, \ldots$. For every $\varepsilon > 0$ and every positive integer n, there exist a positive integer N and $\delta > 0$ such that, if $\mathscr{M}$ is a diffuse von Neumann algebra with a faithful trace τ and $T_1, \ldots, T_n \in \mathscr{M}$ satisfying*

$$|\tau(m(T_1, \ldots, T_n)) - \tau_0(m(A_1, \ldots, A_n))| < \delta$$

*for any *-monomial m with $\deg m \leqslant N$, then there exists a *-homomorphism $\pi : \mathscr{A} \to \mathscr{M}$ such that*

(1) $\tau \circ \pi = \tau_0$,

(2) $\sum_{i=1}^n \|T_i - \pi(A_i)\| < \varepsilon$.

If in the preceding corollary we let $\mathscr{A}$ be a von Neumann algebra generated by a single Haar unitary, then we obtain the following result.

COROLLARY 4.7. *For every* $\varepsilon > 0$ *there is a positive integer* N *and* $\delta > 0$ *such that, for every diffuse finite von Neumann algebra with trace* τ *and every* $U \in \mathrm{Ball}(\mathscr{M})$, *if* $\left|\tau\left(U^k\right)\right| < \delta$ *for* $1 \leqslant k \leqslant N$ *with* $\|1 - U^*U\|_2 < \delta$, *then there is a Haar unitary* $V \in \mathscr{M}$ *such that* $\|U - V\|_2 < \varepsilon$.

REMARK 4.8. It follows from Theorem 4.4 that the hypothesis in Theorem 4 in [11] that $w_1, w_2, \ldots$ are Haar unitaries can be replaced by the assumption that they are unitaries. In particular, if $\mathscr{M}$ is a von Neumann algebra with a faithful trace τ, and $\mathscr{N}$ is a diffuse subalgebra of $\mathscr{M}$, and $\{v_n\}$ is a sequence of Haar unitaries in $\mathscr{M}$ and $\{w_n\}$ is a sequence of unitaries in $\mathscr{N}$, and $\|w_n - v_n\|_2 \to 0$, then there exists a sequence $\{u_n\}$ of Haar unitaries in $\mathscr{N}$ such that $\|u_n - v_n\|_2 \to 0$.

We next give an analogue of Theorem 4.4 with $\mathscr{A}$ hyperfinite instead of commutative, but with each of the $\mathscr{M}_i$'s a II_1 factor.

LEMMA 4.9. [3] *Let* $\mathscr{M}$ *be a separable factor and* ω *a nontrivial ultrafilter. Let* $E = [\{E_i\}]_\omega$ *and* $F = [\{F_i\}]_\omega$ *be equivalent projections in* $\prod^\omega \mathscr{M}$ *with* E_i*'s and* F_i*'s projections. Suppose* V *is a partial isometry from* E *to* F. *Then* $V = [\{V_i\}]_\omega$, *where* V_i *is a partial isometry from* E_i *to* F_i.

LEMMA 4.10. *Suppose each* $\mathscr{M}_i$ *is a* II_1 *factor with the trace* τ_i *and* $\mathscr{A} \subseteq \mathscr{B}$ *are finite-dimensional C*-subalgebras of the ultraproduct* $\prod^\omega (\mathscr{M}_i, \tau_i)$, *and suppose for every* i, *there is a trace-preserving homomorphism* $\pi_i : \mathscr{A} \to \mathscr{M}_i$ *such that, for every* $a \in \mathscr{A}$, $a = [\{\pi_i(a)\}]_\omega$. *Then, for every* i, *there is a trace-preserving homomorphism* $\rho_i : \mathscr{B} \to \mathscr{M}_i$ *such that,*

(1) for every $b \in \mathscr{B}$,

$$b = [\{\rho_i(b)\}]_\omega,$$

and

(2) for every i, $\rho_i|_{\mathscr{A}} = \pi_i$.

Proof. To avoid a notational nightmare, we will describe the proof for a specific example. It will be easy to see how this technique applies universally. Suppose $\mathscr{A}$ is isomorphic to $\mathscr{M}_2 \oplus \mathscr{M}_3$ and $\mathscr{B}$ is isomorphic to $\mathscr{M}_4 \oplus \mathscr{M}_5$ where the inclusion $\mathscr{A} \subset \mathscr{B}$ identifies $A \oplus B$ with $(A \oplus A) \oplus (A \oplus B)$. Let $\{e_{st} : 1 \leqslant s,t \leqslant 4\}$ denote matrix units for $\mathscr{M}_4 \oplus 0$ and $\{f_{st} : 1 \leqslant s,t \leqslant 5\}$ denote matrix units for $0 \oplus \mathscr{M}_5$. Then

$$\mathscr{S}_1 = \{e_{11} + e_{33} + f_{11}, e_{12} + e_{34} + f_{12}, e_{21} + e_{43} + f_{21}, e_{22} + e_{44} + f_{22}\}$$

is a set of matrix units for $\mathscr{M}_2 \oplus 0$ and

$$\mathscr{S}_2 = \{f_{33}, f_{34}, f_{35}, f_{43}, f_{44}, f_{45}, f_{53}, f_{54}, f_{55}\}$$

is a set of matrix units for $0 \oplus \mathscr{M}_3$. We have π_i is defined on $\mathscr{S}_1 \cup \mathscr{S}_2$. We want to extend π_i to all of the matrix units for $\mathscr{B}$. However, $\{e_{11} + e_{33} + f_{11}, e_{22} + e_{44} + f_{22}, f_{33}, f_{55}\}$

is a commuting family that is contained in $span(\{e_{ss} : 1 \leqslant s \leqslant 4\} \cup \{f_{ss} : 1 \leqslant s \leqslant 5\})$, and, using the techniques in the proof of Theorem 4.4, we can extend π_i to a trace-preserving *-homomorphism ρ_i on $span(\{e_{ss} : 1 \leqslant s \leqslant 4\} \cup \{f_{ss} : 1 \leqslant s \leqslant 5\})$. Using the fact that

$$e_{11}(e_{12} + e_{34} + f_{12}) = e_{12},$$

we naturally can define

$$\rho_i(e_{12}) = \rho_i(e_{11})\pi_i(e_{12} + e_{34} + f_{12}).$$

The definition of ρ_i for the remaining matrix units in $\mathscr{B}$ is immediately obtained using Lemma 4.9. □

THEOREM 4.11. *If each $\mathscr{M}_i$ is a II_1 factor with the trace τ_i and $\mathscr{A}$ is a countably generated hyperfinite von Neumann subalgebra of the ultraproduct $\prod^\omega(\mathscr{M}_i, \tau_i)$, then, for every i, there is a trace-preserving homomorphism $\pi_i : \mathscr{A} \to \mathscr{M}_i$ such that, for every $a \in \mathscr{A}$,*

$$a = [\{\pi_i(a)\}]_\omega.$$

Proof. There is an increasing sequence $\{\mathscr{A}_n\}$ of finite-dimensional C*-subalgebras of $\mathscr{A}$ whose union $\mathscr{D}$ is $\|\cdot\|_2$-dense in $\mathscr{A}$ such that $\mathscr{A}_1 = \mathbb{C} \cdot 1$. Using Lemma 4.10, for every i, there is a trace-preserving homomorphism $\pi_i : \mathscr{D} \to \mathscr{M}_i$ such that, for every $a \in \mathscr{A}$,

$$a = [\{\pi_i(a)\}]_\omega.$$

However, since each π_i is an isometry in $\|\cdot\|_2$, we can extend π_i uniquely to an isometry (i.e., trace-preserving) linear map (still called π_i) from $\mathscr{A}$ to $\mathscr{M}_i$. Since multiplication and the map $x \to x^*$ are $\|\cdot\|_2$-continuous on bounded sets, it follows that $\pi_i : \mathscr{A} \to \mathscr{M}_i$ is a *-homomorphism, and that

$$a = [\{\pi_i(a)\}]_\omega$$

holds for every $a \in \mathscr{A}$. □

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