Operators
and
Matrices

Volume 3, Number 1 (2009), 145–153

NORMALIZED NUMERICAL RANGES OF SOME OPERATORS

L. Z. GEVORGYAN

(*communicated by C.-K. Li*)

Abstract. We describe the normalized numerical ranges of certain operators. First, the case of a normal operator, acting in a two dimensional space is considered in detail, leading to the general Kantorovich inequality. Then we pass to the finite dimensional case and settle the problem when the Gustafson-Seddighin two component property, failing in general, takes place. In the next part finite and infinite dimensional Jordan cells are investigated.We conclude with the description of the normalized numerical range of a two-dimensional Toeplitz matrix.

0. Let $(H, \langle \bullet, \bullet \rangle)$ be a Hilbert space and A be a bounded linear operator, acting in H. One of the most familiar ways of solving the equation

$$Ax = b$$

is the (Richardson's) iterative method

$$x_{n+1} = x_n - \alpha_n (Ax_n - b), \quad n \in \mathbb{Z}^+,$$

where x_0 is an initial guess and α_n is a numerical parameter. The sequence $\{x_n\}$ converges to the solution if $\|\alpha_n A - I\| < 1$.

Denote $m(A) = \inf\limits_{t \in \mathbb{C}} \|tA - I\|$ and $p(A) = \inf\limits_{Ax \neq \theta} \frac{|\langle Ax, x\rangle|}{\|Ax\| \cdot \|x\|}$.

Recall that the set

$$W_n(A) = \left\{ \frac{\langle Ax, x\rangle}{\|Ax\| \cdot \|x\|} : Ax \neq \theta \right\}$$

is said [1] to be the normalized numerical range of the operator A. From this definition easily is deduced the equality $W_n(cA) = e^{i \arg c} W_n(A)$ for any nonzero c.

In [2] it has been shown that $m^2(A) + p^2(A) = 1$, so the condition $p(A) > 0$ is very important for the convergence of different iterative methods of solution of Hilbert space operator equations.

Let $L \subset H$ be a subspace, invariant with respect to A and denote by A_L the restriction of A on L. Evidently $W_n(A_L) \subset W_n(A)$, so description of normalized numerical ranges of different finite dimensional operators may be useful. For a normal operator in a two-dimensional Hilbert space such description is given in [2]. In the

Mathematics subject classification (2000): 47A12, 47B15, 47N40.

Keywords and phrases: normalized numerical range, convergence rate, Kantorovich inequality.

Paper OaM-03-07

present note we supply more details and show how $p(A)$ may be found for normal (and not only normal) operators and consider finite and infinite dimensional Jordan cells.

1. Let A be a normal operator in two dimensional space, defined by the matrix

$$A = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \quad M \neq 0, N \neq 0 \tag{1}$$

and $u = \{\alpha, \beta\} \in \mathbb{C}^2$. Then

$$\frac{\langle Au, u\rangle}{\|Au\| \cdot \|u\|} = \frac{M|\alpha|^2 + N|\beta|^2}{\sqrt{|M|^2|\alpha|^2 + |N|^2|\beta|^2} \cdot \sqrt{|\alpha|^2 + |\beta|^2}}.$$

Denoting the quotient $|\alpha|^2 / |\beta|^2$ by t, we get

$$\frac{\langle Au, u\rangle}{\|Au\| \cdot \|u\|} = \frac{Mt + N}{\sqrt{|M|^2 t + |N|^2} \cdot \sqrt{t+1}}.$$

The plot of this function is the curve $r = r(t), t \in [0; +\infty]$ (the value at the point $t = +\infty$ is assumed as the limit of this expression), joining the points $\operatorname{sgn} N = e^{i \arg N} = e^{i\psi}$ and $\operatorname{sgn} M = e^{i \arg M} = e^{i\phi}$. Its curvature depends, particularly, on the quotient $|M| / |N|$ (note that at $|M| = |N|$ one gets the segment, joining mentioned above points and at $\arg M = \arg N$ - the segment, joining the points $\operatorname{sgn} M$ and $2\frac{\sqrt{|M|\cdot|N|}}{|M|+|N|} \cdot \operatorname{sgn} M$ is run twice in opposite directions).

PROPOSITION 1. *For operator A from (1) one has*

$$p = \frac{\sqrt{|M| \cdot |N|}}{|M| + |N|} \cdot |\operatorname{sgn} M + \operatorname{sgn} N| = 2\frac{\sqrt{|M| \cdot |N|}}{|M| + |N|} \cdot \left|\cos\frac{\varphi - \psi}{2}\right|. \tag{2}$$

Proof. Straightforward calculations show that this curve lies at the minimal distance p from the coordinate system origin (the summit of the curve, corresponding to $t = |N| / |M|$), defined by (2). □

This formula may be called Kantorovich general inequality, as for the first time it has been proved [5] for positive operators $(M > 0, N > 0)$. Note that at $\varphi = \psi + \pi$ the curve is reduced to the diameter of the unit circle. The modulus sign may omitted, if the angles φ and ψ are chosen continuously, such that $|\varphi - \psi| \leqslant \pi$, the last condition is supposed to be satisfied in what follows.

Formula (2) in an inconvenient form is cited in [4] (formula (3.6-6)). To be fair, we mention that this shortcoming is corrected in the next paper [6]. It should be noted that assertions in both papers, concerning this problem are either incomplete or erroneous. First, the equality

$$\inf_{Ax \neq \theta} \frac{|\langle Ax, x\rangle|}{\|Ax\| \cdot \|x\|} = 1,$$

according to the Schwartz inequality, means that for any x, $Ax \neq \theta$

$$\inf_{Ax\neq\theta} \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|} = \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|} = \sup_{Ax\neq\theta} \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|}.$$

The second part concerns normal operators, acting in a finite dimensional space. To formulate corresponding assertion, introduce some notations. Denote by $\mathscr{L}$ the set of all two-dimensional subspaces, generated by the eigenelements of A. In the cited above paper is stated that

$$\inf_{Ax\neq\theta} \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|} = \inf_{L\in\mathscr{L}} \inf_{x\in L} \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|}.$$

The next example shows that this is not always the case.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{1+i\sqrt{3}}{2} \end{pmatrix}$$

and $x = \{1\,;\,1\,;\,1\}$. Then $Ax \neq 0$ and $\langle Ax, x\rangle = 0$. It is easy to see that

$$\inf_{x\in L_k} \frac{|\langle Ax, x\rangle|}{\|Ax\|\cdot\|x\|} = 0.5, \quad k = 1,2,3.$$

The minima are attained on the elements $\{1\,;\,1\,;\,0\}$, $\{1\,;\,0\,;\,1\}$ and $\{0\,;1\,;\,1\}$.

This example also shows that Theorem 3.11 from [6] saying that for a unitary operator U with countable spectrum lying on the (mimimal) arc with endpoints $\exp(i\beta_1)$ and $\exp(i\beta_2)$ the equality

$$\inf_{Ux\neq\theta} \frac{|\langle Ux, x\rangle|}{\|Ux\|\cdot\|x\|} = \left|\cos\frac{\beta_1-\beta_2}{2}\right|$$

holds, in general, is not true.

Note that this theorem is valid without restriction on operator's spectrum, but only when $|\beta_1-\beta_2| \leqslant \pi$. Evidently, in this case the modulus sign may be omitted. If $|\beta_1-\beta_2| > \pi$, then the infimum is equal to zero.

PROPOSITION 2. *For operator A from (1) the normalized numerical range is symmetric with respect to the midpoint perpendicular of the segment* $[\operatorname{sgn} M\,;\,\operatorname{sgn} N]$.

Proof. Turning the whole picture by the angle $\frac{\varphi-\psi}{2}$, one may assume that the points M and N are $ae^{i\alpha}$ and $be^{-i\alpha}$ respectively. Then

$$\begin{cases} x(t) = \dfrac{at+b}{\sqrt{a^2t+b^2}\cdot\sqrt{1+t}}\cos\alpha \\[2ex] y(t) = \dfrac{at-b}{\sqrt{a^2t+b^2}\cdot\sqrt{1+t}}\sin\alpha \end{cases}$$

Change the variable $s=\frac{\sqrt{a}\sqrt{t}-\sqrt{b}}{\sqrt{a}\sqrt{t}+\sqrt{b}}$, mapping the segment $[0;+\infty]$ onto $[-1;1]$. Then $t=\frac{b}{a}\left(\frac{1+s}{1-s}\right)^2$ and $1+t=\frac{a(1-s)^2+b(1+s)^2}{a(1-s)^2}$, implying

$$\begin{cases} x=\dfrac{2\sqrt{ab}\left(1+s^2\right)}{\sqrt{\left(a\left(1+s\right)^2+b\left(1-s\right)^2\right)\left(a\left(1-s\right)^2+b\left(1+s\right)^2\right)}}\cos\alpha \\ y=\dfrac{4\sqrt{ab}s}{\sqrt{\left(a\left(1+s\right)^2+b\left(1-s\right)^2\right)\left(a\left(1-s\right)^2+b\left(1+s\right)^2\right)}}\sin\alpha \end{cases}$$

The first of these functions is even and the second is odd, which proves our assertion. □

Using this fact, we can establish an interesting property of the normalized numerical range of a normal operator, acting in a two-dimensional unitary space. As it is well known, the symmetry with respect to the axis of a unit vector v is defined by the operator, generated by the correspondence $r\mapsto 2\langle r,v\rangle\, v-r$. Recall also that the inner product (at the canonical identification of $\mathbb{R}^2$ and $\mathbb{C}$) is defined by the formula $\langle r,v\rangle=\operatorname{Re}(r\overline{v})$. We have $v=e^{i\frac{\varphi+\psi}{2}}$, so $2\langle r,v\rangle\, v-r=2\operatorname{Re}\left(re^{-i\frac{\varphi+\psi}{2}}\right)e^{i\frac{\varphi+\psi}{2}}-r.$ Carrying out calculations and using the identity $2e^{i\frac{\varphi+\psi}{2}}\cos\frac{\varphi-\psi}{2}=e^{i\varphi}+e^{i\psi}$, we get

$$\frac{|M|\,te^{i\psi}+|N|\,e^{i\varphi}}{\sqrt{|M|^2\,t+|N|^2}\cdot\sqrt{1+t}}.$$

So the normalized numerical ranges of operators $A=\begin{pmatrix} M & 0\\ 0 & N\end{pmatrix}$ and $B=\begin{pmatrix} |M|\,e^{i\arg N} & 0\\ 0 & |N|\,e^{i\arg M}\end{pmatrix}$ coincide.

If one of the eigenvalues, e.g. N is equal to zero, then

$$\frac{\langle Au,u\rangle}{\|Au\|\cdot\|u\|}=\frac{M\,|\alpha|}{|M|\cdot\sqrt{|\alpha|^2+|\beta|^2}}=\frac{|\alpha|}{\sqrt{|\alpha|^2+|\beta|^2}}\operatorname{sgn}M,\quad(\alpha\neq 0),$$

meaning that $W_n(A)$ is the half-segment $(0;\operatorname{sgn}M]$.

Denote by SpA the spectrum of A. Let for the operator A the following equality be satisfied

$$\|\alpha A-I\|=\sup_{z\in SpA}|\alpha z-1|\quad\text{for all}\quad\alpha \tag{3}$$

(this condition, particularly, is satisfied for normal operators).

EXAMPLE 2. Let S be the operator of the simple unilateral shift. For any $\alpha,\beta\in\mathbb{C}$ one has

$$\|\alpha S+\beta I\|=\sup_{|z|\leqslant 1}|\alpha z+\beta|=|\alpha|+|\beta|.$$

In [3] it has been shown (particularly, for normal operators) that $m(A) < 1$ if and only if the coordinate system's origin O does not belong to the convex hull of SpA. If this condition is satisfied, then there exists a unique circle $D(C,R)$ with the center at C and of radius R, containing SpA, leaving outside O and having the least ratio R/OC among all circles, satisfying the above conditions. This circle will be referred as optimal. It has been proved that the optimal for any convex compact subset F circumference contains at least two points of F. One has the equality

$$m(A) = R/OC. \tag{4}$$

This formula implies the general form of the Kantorovich inequality

$$\inf_{Ax\neq\theta} \frac{|\langle Ax, x\rangle|}{\|Ax\| \cdot \|x\|} = \sqrt{1 - \frac{R^2}{OC^2}}. \tag{5}$$

The optimal for the segment $[\lambda_1;\, \lambda_2]$ circle has the center at

$$C = \frac{|\lambda_1| + |\lambda_2|}{\operatorname{sgn}\overline{\lambda}_1 + \operatorname{sgn}\overline{\lambda}_2}$$

and the radius is

$$R = \left| \frac{\lambda_1 - \lambda_2}{\operatorname{sgn}\lambda_1 + \operatorname{sgn}\lambda_2} \right|,$$

so the ratio

$$\frac{R}{OC} = \frac{|\lambda_1 - \lambda_2|}{|\lambda_1| + |\lambda_2|}. \tag{6}$$

The last equality and formula (5) lead to formula (2).

It is easy to see that the Gustafson-Seddighin two component property will take place if and only if the optimal for SpA circle is optimal for at least a segment, joining two eigenvalues of A. This, in his turn is equivalent to the condition that the optimal for some two eigenvalues circle contains SpA. On the other hand, if the optimal for SpA circle D is not optimal for a segment $[\lambda_1;\, \lambda_2] \subset SpA$, then the boundary of D contains at least a third point from SpA. So, to find the optimal for SpA circle the following algorithm may be proposed. First, using formula (6) the quotient should be calculated for different pairs of eigenvalues and their maximum should be found. If the corresponding circle does not contain SpA, then all circles, passing through three vertices of the convex hull of SpA should be traced. One of them will be the optimal one.

The general Kantorovich inequality holds not only for normal operators with the spectrum $SpA = [\lambda_1;\, \lambda_2]$, but for any operator, satisfying condition (3) and having the spectrum, containing the points λ_1 and λ_2 and contained in the optimal for the segment $[\lambda_1;\, \lambda_2]$ circle.

For the operator, considered in the Example 2 we have

$$\|t(S + \alpha I) - I\| = |t| + |t\alpha - 1|.$$

For $|\alpha| > 1$ one gets

$$m(S+\alpha I) = \inf_{t\in\mathbb{C}} \{|t| + |\alpha t - 1|\} = \frac{1}{|\alpha|}.$$

According to the Kantorovich general inequality

$$m(S+\alpha I) = \frac{|\alpha| + 1 - (|\alpha| - 1)}{|\alpha| + 1 + |\alpha| - 1} = \frac{1}{|\alpha|}.$$

2. Let the operator A be nilpotent with the index 2, i. e. $A^2 = 0$. As it is well known, any Hilbert space H may be represented as the orthogonal sum $H = N(A^*) \oplus \overline{R}(A)$, where $N(A)$ is the null-space, and $R(A)$ is the range of the operator A. Let $h = f + g$, $f \in N(A^*)$, $g \in \overline{R}(A)$. Evidently $Ah = Af$, $\langle Ah, h\rangle = \langle Af, f+g\rangle = \langle f, A^*(f+g)\rangle = \langle f, A^*g\rangle = \langle Af, g\rangle$, implying (at $f \notin N(A)$)

$$\frac{\langle Ah, h\rangle}{\|Ah\| \cdot \|h\|} = \frac{\langle Af, g\rangle}{\|Af\| \cdot \sqrt{\|f\|^2 + \|g\|^2}}.$$

Changing f by λf, $\lambda \in \mathbb{C}$, we get

$$\frac{\lambda \langle Af, g\rangle}{|\lambda| \cdot \|Af\| \cdot \sqrt{|\lambda|^2 \cdot \|f\|^2 + \|g\|^2}}.$$

Considering values $|\lambda| = 1$, we see that $W_n(A)$ has a circular symmetry. Letting $|\lambda| \to \infty$ we conclude that it contains some circle. Choosing g parallel to Af, we get

$$\frac{\|g\|}{\sqrt{\|f\|^2 + \|g\|^2}},$$

showing that the circle has radius equal to 1. From here we may deduce the following

PROPOSITION 3. *The normalized numerical range of the operator*

$$J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the open unit circle $\{z : |z| < 1\}$.

Evidently the normalized numerical range of $n-$ dimensional Jordan cell J_n will be the same.

Note [2] that the normalized numerical range of the unilateral shift S is the open unit circle and for S^* - the closed unit circle.

Let now

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $u=\{\alpha,\beta\}$, where $\alpha,\beta\in\mathbb{C}$. Then $Au=\{\alpha+\beta,\beta\}$ and

$$\frac{\langle Au,u\rangle}{\|Au\|\cdot\|u\|}=\frac{|\alpha|^2+\beta\overline{\alpha}+|\beta|^2}{\sqrt{|\alpha+\beta|^2+|\beta|^2}\cdot\sqrt{|\alpha|^2+|\beta|^2}}.$$

Denoting $\alpha=\overline{z}\beta$, we get

$$\frac{\langle Au,u\rangle}{\|Au\|\cdot\|u\|}=\frac{1+z+|z|^2}{\sqrt{|1+z|^2+1}\cdot\sqrt{|z|^2+1}}$$

or

$$\frac{\langle Au,u\rangle}{\|Au\|\cdot\|u\|}=\frac{1+x+x^2+y^2+iy}{\sqrt{(x+1)^2+y^2+1}\cdot\sqrt{x^2+y^2+1}}\overset{def}{=}w(x,y).$$

Fixing x or y, we get coordinate curves $w=f(y)$ or $w=g(x)$. The boundary of this region is the caustic of the set of coordinate curves. Using the derivatives, it is easy to check that the point, where the minimum of $g(x)$ is attained, does not depend on y and takes place at $x=-\frac{1}{2}$. Another proof of this fact may derived from the formula

$$w(x,y)=\frac{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}+y^2+iy}{\sqrt{\left(\frac{1}{2}+x+\frac{1}{2}\right)^2+y^2+1}\cdot\sqrt{\left(\frac{1}{2}-\left(x+\frac{1}{2}\right)\right)^2+y^2+1}}.$$

Therefore, the boundary of this domain is the curve, which is obtained from the general formula for $w=f(y)$ putting the extremal value $x=-\frac{1}{2}$. Find now the image of the vertical straight line $z=-0.5+iy,\ y\in\mathbb{R}$

$$f(y)=\frac{0.75+y^2+iy}{1.25+y^2}.$$

Direct calculations show that

$$\frac{0.75+y^2+iy}{1.25+y^2}=0.8+0.2\cos t+\frac{i}{\sqrt{5}}\sin t,\ \ t\in(0;\ 2\pi).$$

It is easy to see that $\inf\limits_{Au\neq\theta}\frac{|\langle Au,u\rangle|}{\|Au\|\cdot\|u\|}$ is attained on the element $u=\{1;-2\}$ and is equal to 0.6, and $\sup\limits_{Au\neq\theta}\frac{|\langle Au,u\rangle|}{\|Au\|\cdot\|u\|}$ is attained on the element $u=\{1;0\}$ and is equal to 1.

More tedious calculations prove the following

PROPOSITION 4. *The normalized numerical range of the operator*

$$A=\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$$

is the domain, bounded by the ellipse

$$z=\frac{4}{4+|k|^2}+\frac{1}{4+|k|^2}\cos t+i\frac{1}{\sqrt{4+|k|^2}}\sin t,\ \ t\in[0;\ 2\pi).$$

Hence the normalized numerical range of the operator

$$J_2 + \lambda I_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

will be the domain, bounded by the ellipse

$$z = \left(\frac{4|\lambda|^2}{4|\lambda|^2+1} + \frac{1}{4|\lambda|^2+1}\cos t + i\frac{1}{\sqrt{4|\lambda|^2+1}}\sin t \right) \exp(i\arg\lambda), \quad t \in [0;\ 2\pi).$$

Consider now the operator $I + S^*$, where S is the operator of the simple unilateral shift. The subspace, generated by the first m elements $\{e_k\}_1^m$ of the basis, shifted by S is invariant with respect to $I + S^*$ and its restriction on that subspace has the matrix

$$I_m + J_m = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Evidently $W_n(I_2 + J_2) \subset W_n(I_3 + J_3) \subset \cdots \subset W_n(I + S^*)$.

PROPOSITION 5. *For infinite dimensional Jordan cell* $I + S^*$

$$W_n(I + S^*) = \{z : |z| \leqslant 1, \operatorname{Re} z > 0\}.$$

Proof. Easily may be checked that any point from the circle $\{\lambda : |\lambda - 1| < 1\} = \left\{\lambda : |\lambda|^2 < 2\operatorname{Re}\lambda\right\}$ is an eigenvalue of the operator $I + S^*$ and the corresponding eigenelement has the form $h_\lambda = \sum\limits_{n=1}^{\infty} (\lambda - 1)^{n-1} e_n$. Consider the element $u = h_\lambda - h_1$.

We have

$$\|u\|^2 = \langle h_\lambda - h_1, h_\lambda - h_1 \rangle = \|h_\lambda\|^2 - 1,$$
$$(I + S^*)u = \lambda h_\lambda - h_1,$$
$$\|(I + S^*)u\|^2 = |\lambda|^2 \cdot \|h_\lambda\|^2 - 2\operatorname{Re}\lambda + 1,$$
$$\langle (I + S^*)u, u \rangle = \lambda \cdot \|h_\lambda\|^2 - \lambda.$$

Therefore

$$M(u) = \frac{\langle (I + S^*)u, u \rangle}{\|(I + S^*)u\| \cdot \|u\|} = \frac{\lambda\left(\|h_\lambda\|^2 - 1\right)}{\sqrt{|\lambda|^2 \cdot \|h_\lambda\|^2 - 2\operatorname{Re}\lambda + 1} \cdot \sqrt{\|h_\lambda\|^2 - 1}}.$$

We also have $\|h_\lambda\|^2 = \frac{1}{1-|\lambda-1|^2} = \frac{1}{2\operatorname{Re}\lambda - |\lambda|^2}$. Fixing $\arg\lambda = \alpha$, $\left(|\alpha| < \frac{\pi}{2}\right)$, consider the segment of the straight line, lying in the above mentioned circle. Letting λ approach to the boundary of the circle, we get $M(u) \to e^{i\alpha}$ and $M(u) \to 0$. As

$$\frac{\langle (I + S^*)h_\lambda, h_\lambda \rangle}{\|(I + S^*)h_\lambda\| \cdot \|h_\lambda\|} = \frac{\lambda\,\|h_\lambda\|^2}{|\lambda| \cdot \|h_\lambda\|^2} = \operatorname{sgn}\lambda,$$

hence

$$W_n(I + S^*) = \{z : |z| \leqslant 1, \operatorname{Re} z > 0\}.$$

□

PROPOSITION 6. *The normalized numerical range of the operator*

$$B = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \quad M \neq 0,\, N \neq 0$$

is an ellipse.

Proof. Let $u = \{\alpha, \beta\}$. Then

$$\frac{\langle Bu, u\rangle}{\|Bu\| \cdot \|u\|} = \frac{M\beta\overline{\alpha} + N\alpha\overline{\beta}}{\sqrt{|M\beta|^2 + |N\alpha|^2} \cdot \sqrt{|\alpha|^2 + |\beta|^2}}.$$

After transformations, described before Proposition 4 this quotient may be reduced to the form

$$\frac{mz + \overline{z}}{\sqrt{m^2 + |z|^2} \cdot \sqrt{1 + |z|^2}} \cdot \exp\left(i\frac{\arg M + \arg N}{2}\right), \; m = \frac{|M|}{|N|}, \; z \in \mathbb{C}. \qquad (7)$$

The first factor is equal to

$$\frac{mre^{it} + re^{-it}}{\sqrt{m^2 + r^2} \cdot \sqrt{1 + r^2}} = \frac{r\,(m+1)\cos t + ir\,(m-1)\sin t}{\sqrt{m^2 + r^2} \cdot \sqrt{1 + r^2}}.$$

This curve (at fixed r) is an ellipse with semi-axes $\frac{r(m+1)}{\sqrt{m^2+r^2}\cdot\sqrt{1+r^2}}$ and $\frac{r|m-1|}{\sqrt{m^2+r^2}\cdot\sqrt{1+r^2}}$, each of which attains its maximum value at $r = \sqrt{m}$. Therefore the boundary of this domain is the curve, which results if the substitution $z = \sqrt{m}\exp(it)$ is carried out in (7). Hence, the domain, bounded by the ellipse

$$w(t) = \left(\cos t + i\frac{|M| - |N|}{|M| + |N|}\sin t\right) \cdot \exp\left(i\frac{\arg M + \arg N}{2}\right), \; t \in [0\,;\, 2\pi)$$

is the normalized numerical range of the operator B. □

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(Received June 5, 2008)

L. Z. Gevorgyan
State Engineering University of Armenia
Department of Mathematics
e-mail: levgev@hotmail.com