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AN ELEMENTARY PROOF OF VOICULESCU'S ASYMPTOTIC FREENESS FOR RANDOM UNITARY MATRICES

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Abstract. D. Voiculescu [2] proved that a standard family of independent random unitary $k\times k$ matrices and a constant $k\times k$ unitary matrix is asymptotically free as $k\to\infty$. This result was a key ingredient in Voiculescu's proof [3] that his free entropy is additive when the variables are free. In this paper, we give a very elementary proof of a more detailed version of this result [2]. We have not yet recaptured Voiculescu's strengthened version [4].

1. Preliminaries

The theory of free probability and free entropy was introduced by D. Voiculescu in the 1980's, and has become one of the most powerful and exciting new tools in the theory of von Neumann algebras. D. Voiculescu [2] proved that a standard family of independent random unitary $k\times k$ matrices and a constant $k\times k$ unitary matrix is asymptotically free as $k\to\infty$. To prove this result, Voiculescu used his noncommutative central limit theorem and the fact that the unitaries in the polar decomposition of a family of standard Gaussian random matrices form a standard family of independent unitary $k\times k$ random matrices. Voiculescu used this result and a Lipschitz property and facts about Levy families to prove that the Haar measure of certain sets of tuples of $k\times k$ matrices converges to 1 as $k\to\infty$ (see the remarks after Theorem 3.9 in [2]). Later, D. Voiculescu [4], using similar techniques, strengthened his asymptotic result by removing restrictions on the type of constant matrices.

In this paper, we give a very elementary proof of Voiculescu's asymptotic result in [2] that uses only the basic properties of Haar measure and the definition of unitary matrix. A simple application of Chebychev's inequality yields the result about the measures of sets converging to 1 (see Corollary 7).

Let $\mathscr{M}_k(\mathbb{C})$ be the $k\times k$ full matrix algebra with entries in $\mathbb{C}$. For $1\leqslant i,j\leqslant k$, define $f_{ij}:\mathscr{M}_k(\mathbb{C})\to\mathbb{C}$ so that any element a in $\mathscr{M}_k(\mathbb{C})$ is the matrix $(f_{ij}(a))$, i.e., $f_{ij}(a)$ is the (i,j)-entry of a. Define the normalized trace τ_k on $\mathscr{M}_k(\mathbb{C})$ by

$$\tau_k(a)=\frac{1}{k}\sum_{i=1}^{k}f_{ii}(a),\qquad \text{for any } a\in\mathscr{M}_k(\mathbb{C}).$$

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A $k\times k$ matrix u is a unitary matrix if and only if

$$\sum_{i=1}^{k}|f_{ij_1}(u)|^2=\sum_{j=1}^{k}|f_{i_1j}(u)|^2=1,\quad \text{for } 1\leqslant i_1,j_1\leqslant k,\quad \text{and}$$

$$\sum_{i=1}^{k}f_{ij_1}(u)\overline{f_{ij_2}(u)}=\sum_{j=1}^{k}f_{i_1j}(u)\overline{f_{i_2j}(u)}=0,\quad \text{whenever } i_1\neq i_2 \text{ and } j_1\neq j_2.$$

Let $\mathscr{U}_k$ be the group of all unitary matrices in $\mathscr{M}_k(\mathbb{C})$. Since $\mathscr{U}_k$ is a compact group, there exists a unique normalized Haar measure μ_k on $\mathscr{U}_k$. In addition,

$$\int_{\mathscr{U}_k}f(u)d\mu_k(u)=\int_{\mathscr{U}_k}f(vu)d\mu_k(u)=\int_{\mathscr{U}_k}f(uv)d\mu_k(u),$$

for every continuous function $f:\mathscr{U}_k\to\mathbb{C}$ and $v\in\mathscr{U}_k$.

By the translation-invariance of μ_k, we have the following lemmas (also see Lemma 12, Lemma 13 and Lemma 14 in [1]).

LEMMA 1. *If $g:\mathbb{C}^n\to\mathbb{C}^n$ is a continuous function, σ and ρ are permutations of $\{1,2,\ldots,k\}$, then*

$$\int_{\mathscr{U}_k}g\left(f_{i_1j_1}(u),f_{i_2j_2}(u),\ldots,f_{i_nj_n}(u)\right)d\mu_k(u)$$
$$=\int_{\mathscr{U}_k}g(f_{\sigma(i_1),\rho(j_1)}(u),f_{\sigma(i_2),\rho(j_2)}(u),\ldots,f_{\sigma(i_n),\rho(j_n)}(u))d\mu_k(u).$$

LEMMA 2. *If $\int_{\mathscr{U}_k}f_{i_1j_1}(u)\cdots f_{i_mj_m}(u)\overline{f_{s_1t_1}(u)}\cdots\overline{f_{s_rt_r}(u)}d\mu_k(u)\neq 0$, then*
1. $m=r$,
2. $(i_1,i_2,\ldots,i_m)$ is a permutation of $(s_1,s_2,\ldots,s_r)$,
3. $(j_1,j_2,\ldots,j_m)$ is a permutation of $(t_1,t_2,\ldots,t_r)$.

LEMMA 3. *If d is the maximum cardinality of the sets $\{i_1,\ldots,i_n\}$, $\{j_1,\ldots,j_n\}$, $\{s_1,\ldots,s_r\}$ and $\{t_1,\ldots,t_r\}$, then, for every positive integer $k\geqslant d$,*

$$\left|\int_{\mathscr{U}_k}f_{i_1j_1}(u)\cdots f_{i_nj_n}(u)\overline{f_{s_1t_1}(u)}\cdots\overline{f_{s_rt_r}(u)}d\mu_k(u)\right|\leqslant\frac{1}{P(k,d)},$$

where $P(k,d)=k(k-1)\cdots(k-d+1)$.

2. Main result

If $f:\mathscr{F}\to\mathbb{C}$, let $\|f\|_\infty=\sup\{|f(x)|:x\in\mathscr{F}\}$.

LEMMA 4. *Let n,m,k be positive integers. Let F,G be finite subsets of $\mathbb{N}$ with $n=Card(F)$ and $m=Card(G)$. Suppose $\{f_i,g_j : i\in F, j\in G\}$ is a family of mappings from $\{1,\ldots,k\}=H$ to $\mathbb{C}$ such that $\sum_{a=1}^{k} f_i(a)=0$ for $i\in F$. Then*

$$\left|\sum_{\sigma:F\cup G\overset{1-1}{\to}H}\prod_{i\in F} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right| \leqslant k^{m+\frac{n}{2}}(n+m)^n\prod_{i\in F}\|f_i\|_\infty\prod_{j\in G}\|g_j\|_\infty .$$

Proof. The proof is by induction on n. When $n=0$, the obvious interpretation of the inequality is

$$\left|\sum_{\sigma}\prod_{j\in G} g_j(\sigma(j))\right| \leqslant k^m\prod_{j\in G}\|g_j\|_\infty ,$$

and it holds since the number of functions $\sigma : G\overset{1-1}{\to}H$ is no more than k^m.

Suppose the lemma holds for n. For $n+1$, let $E=F\setminus\{b\}$ be a subset of F, where $b\in F$. Then the cardinality of E is n. We can define a one-to-one mapping $\sigma : F\cup G\to H$ by defining the one-to-one mapping $\sigma : E\cup G\to H$ and choosing $s\notin\sigma(E\cup G)$ to be $\sigma(b)$. Then

$$\begin{aligned}
&\left|\sum_{\sigma:F\cup G\overset{1-1}{\to}H}\prod_{i\in F} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right| \\
&=\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{s\notin\sigma(E\cup G)} f_b(s)\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right| \\
&=\Bigg|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{s=1}^{k} f_b(s)\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j)) \\
&\quad -\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{s\in\sigma(E\cup G)} f_b(s)\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\Bigg| \\
&\quad \left(\text{using } \sum_{s=1}^{k} f_b(s)=0\right) \\
&=\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{s\in\sigma(E\cup G)} f_b(s)\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right| \\
&=\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{t\in E\cup G} f_b(\sigma(t))\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right| \\
&\leqslant\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{t\in E} f_b(\sigma(t))\right)\prod_{i\in E} f_i(\sigma(i))\prod_{j\in G} g_j(\sigma(j))\right|
\end{aligned}$$

$$+\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\sum_{t\in G}f_b(\sigma(t))\right)\prod_{i\in E}f_i(\sigma(i))\prod_{j\in G}g_j(\sigma(j))\right|$$

$$\leqslant\sum_{t\in E}\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\prod_{i\in E,i\neq t}f_i(\sigma(i))\right)(f_bf_t)(\sigma(t))\prod_{j\in G}g_j(\sigma(j))\right|$$

$$+\sum_{t\in G}\left|\sum_{\sigma:E\cup G\overset{1-1}{\to}H}\left(\prod_{i\in E,i\neq t}f_i(\sigma(i))\right)(f_bf_t)(\sigma(t))\prod_{j\in G}g_j(\sigma(j))\right|$$

(using induction on the quantities inside the absolute value signs and viewing f_bf_t as a single function)

$$\leqslant n((m+1)+(n-1))^{n-1}k^{\frac{n-1}{2}+m+1}\prod_{i\in F}\|f_i\|_\infty\prod_{j\in G}\|g_j\|_\infty$$

$$+m(m+n)^nk^{\frac{n}{2}+m}\prod_{i\in F}\|f_i\|_\infty\prod_{j\in G}\|g_j\|_\infty$$

$$\leqslant(m+n+1)^{n+1}k^{\frac{n+1}{2}+m}\prod_{i\in F}\|f_i\|_\infty\prod_{j\in G}\|g_j\|_\infty.$$

□

Let $\mathscr{U}_k^n$ denote the direct product of n copies of $\mathscr{U}_k$, and μ_k^n denote the corresponding product measure. We will use $\overrightarrow{u}$ to denote a tuple $(u_1,\ldots,u_n)$ in $\mathscr{U}_k^n$.

The following lemma is a vastly improved estimate over Lemma 14 in [1] since it is independent of the maximum cardinality of the indices in the integral. We require the elementary inequalities $m^m\leqslant 2^{m^2}$ and $\frac{1}{P(k,m)}\leqslant\frac{m^m}{k^m}$ for positive integers $m\leqslant k$.

LEMMA 5. *Suppose m is a positive integer. For every positive integers k,n with $k\geqslant m$, and for all tuples $(i_1,\ldots,i_m,j_1,\ldots,j_m)$ with each element taking from $\{1,\ldots,k\}$, and $(\iota_1,\ldots,\iota_m,\eta_1,\ldots,\eta_m)$ with each element taking from $\{1,\ldots,n\}$,*

$$\left|\int_{\mathscr{U}_k^n}f_{i_1j_1}(u_{\iota_1})\cdots f_{i_mj_m}(u_{\iota_m})\overline{f_{s_1t_1}(u_{\eta_1})}\cdots\overline{f_{s_mt_m}(u_{\eta_m})}d\mu_k^n(\vec{u})\right|\leqslant\frac{4^{m^2}}{k^m}.$$

Proof. For $1\leqslant j\leqslant n$, let $T_j=\{1\leqslant\lambda\leqslant m:\iota_\lambda=j\}$ and $T_j^*=\{1\leqslant\lambda\leqslant m:\eta_\lambda=j\}$. Then

$$\int_{\mathscr{U}_k^n}f_{i_1j_1}(u_{\iota_1})\cdots f_{i_mj_m}(u_{\iota_m})\overline{f_{s_1t_1}(u_{\eta_1})}\cdots\overline{f_{s_mt_m}(u_{\eta_m})}d\mu_k^n(\vec{u})$$
$$=\prod_{j=1}^n\int_{\mathscr{U}_k}(\prod_{\lambda\in T_j}f_{i_\lambda j_\lambda}(u_j)\prod_{\lambda\in T_j^*}\overline{f_{s_\lambda t_\lambda}(u_j)})d\mu_k(u_j).$$

Hence, we can assume that $n=1$. Moreover, in view of the Cauchy-Schwarz inequality, it is sufficient to prove that

$$I=\int_{\mathscr{U}_k}\left|f_{i_1j_1}(u)\right|^2\left|f_{i_2j_2}(u)\right|^2\cdots\left|f_{i_mj_m}(u)\right|^2d\mu_k(u)\leqslant\frac{4^{m^2}}{k^m}. \quad (1)$$

Let d be the maximum cardinality of the sets $\{i_1,\ldots,i_m\}$ and $\{j_1,\ldots,j_m\}$. By replacing u with u^*, which does not alter the integral but interchanges i's with j's, we can assume that d is the cardinality of $\{i_1,\ldots,i_m\}$. Then $1 \leqslant d \leqslant m$. Let $B_{d,k}$ be the largest integral of the type in (1) with $d = Card(\{i_1,\ldots,i_m\})$.

If $d = m$, then, by Lemma 3, the integral in (1) is at most $\frac{1}{P(k,m)}$, and $\frac{1}{P(k,m)} \leqslant \frac{m^m}{k^m} \leqslant \frac{4^{m^2}}{k^m}$.

Now we will prove that $B_{d,k} \leqslant 2^m B_{d+1,k}$ whenever $1 \leqslant d < m$. For $1 \leqslant d < m$, assume that the integral I in (1) equals $B_{d,k}$. Since $d < m$, at least two of $i_1,\ldots,i_m$ must be the same. From Lemma 1, we can assume that $1 \leqslant i_1,\ldots,i_m \leqslant d$ and $1 = i_1 = i_2 = \cdots = i_s$ and $1 \notin \{i_{s+1},\ldots,i_m\}$. Since $k \geqslant m > d$, we can define a unitary matrix v with 1 on the diagonal except in the $(1,1)$ and (k,k) positions, with $\frac{1}{\sqrt{2}}$ in the $(1,1),(k,1),(k,k)$ positions and $-\frac{1}{\sqrt{2}}$ in the $(1,k)$ position. Since the integral remains unchanged when we replace the variable u with vu, we obtain

$$\begin{aligned}
B_{d,k} &= \frac{1}{2^s}\int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left|f_{1j_\beta}(u)+f_{kj_\beta}(u)\right|^2\prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u)\\
&= \frac{1}{2^s}\int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left(\left|f_{1j_\beta}(u)\right|^2+\overline{f_{1j_\beta}(u)}f_{kj_\beta}(u)+f_{1j_\beta}(u)\overline{f_{kj_\beta}(u)}+\left|f_{kj_\beta}(u)\right|^2\right)\cdot\\
&\quad \prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u)\\
&= \frac{1}{2^s}\int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left|f_{1j_\beta}(u)\right|^2\prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u)\\
&\quad +\frac{1}{2^s}\int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left|f_{kj_\beta}(u)\right|^2\prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u)+\frac{1}{2^s}\int_{\mathcal{U}_k}\Delta d\mu_k(u),
\end{aligned}$$

where Δ is a summation of 4^s-2 terms with each of them having both an $f_{1*}(u)$ and an $f_{k*}(u)$ factor (with or without conjugation signs) and the maximum cardinality of the indices in each term is $d+1$, which implies $\left|\int_{\mathcal{U}_k}\Delta d\mu_k(u)\right| \leqslant (4^s-2)B_{d+1,k}$.

Since

$$\begin{aligned}
B_{d,k} &= \int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left|f_{1j_\beta}(u)\right|^2\prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u)\\
&= \int_{\mathcal{U}_k}\prod_{\beta=1}^{s}\left|f_{kj_\beta}(u)\right|^2\prod_{\alpha=s+1}^{m}\left|f_{i_\alpha j_\alpha}(u)\right|^2 d\mu_k(u),
\end{aligned}$$

we have

$$B_{d,k} \leqslant \frac{1}{2^s}\left(B_{d,k}+B_{d,k}\right)+\frac{1}{2^s}(4^s-2)B_{d+1,k}.$$

Therefore

$$B_{d,k} \leqslant 2^m B_{d+1,k}.$$

It follows that $B_{d,k} \leqslant 2^{m(m-d)} B_{m,k} \leqslant \frac{2^{m^2}}{P(k,m)} \leqslant \frac{2^{m^2} m^m}{k^m} \leqslant \frac{4^{m^2}}{k^m}$ when $k \geqslant m$ and $1 \leqslant d \leqslant m$. □

For any positive integer m, let $B(m)$ be the *Bell number* of m, i.e., the number of equivalence relations on a set with cardinality m. Suppose $\mathscr{M}$ is a von Neumann algebra with a faithful tracial state τ and $\mathscr{U}(M)$ is the set of all unitary elements in $\mathscr{M}$ and $\overrightarrow{u} = (u_1, \ldots, u_n) \in \mathscr{U}(\mathscr{M})^n$. Let $\mathbb{F}_n$ be a free group with standard generators $h_1, \ldots, h_n$. Then there is a homomorphism $\rho : \mathbb{F}_n \to \mathscr{U}(\mathscr{M})$ such that $\rho(h_j) = u_j$. We use the notation $\rho(g) = g(\overrightarrow{u}) = g(u_1, \ldots, u_n)$.

D. Voiculescu [2] proved that a standard family of independent random unitary $k \times k$ matrices and a constant $k \times k$ unitary matrix are asymptotically free as $k \to \infty$. The following theorem gives a very elementary proof of a more detailed version of D. Voiculescu's result. The constants in the following theorem are far from best possible, but they are, at least, explicit.

THEOREM 6. *Suppose $M > 0$ and m, k are positive integers with $k \geqslant m$. For every reduced words $g_1, \ldots, g_w \in \mathbb{F}_n \backslash \{e\}$ with $\sum_{i=1}^{w} \text{length}(g_i) = m$, and commuting normal $k \times k$ matrices $x_1, \ldots, x_w$ with trace 0 and $\|x_i\| \leqslant M$ for all $1 \leqslant i \leqslant w$, we have*

1.
$$\left| \int_{\mathscr{U}_k^n} \tau_k \left(g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w \right) d\mu_k^n(\vec{u}) \right| \leqslant \frac{B(m) \cdot 2^{m^2} \cdot (Mw)^w}{k},$$

2.
$$\int_{\mathscr{U}_k^n} \left| \tau_k \left(g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w \right) \right|^2 d\mu_k^n(\vec{u}) \leqslant \frac{B(2m) \cdot 4^{m^2} \cdot (2Mw)^{2w}}{k^2},$$

3. *if* $\varepsilon > 0$ *and* $k > \frac{2 \cdot B(m) \cdot 2^{m^2} \cdot (Mw)^w}{\varepsilon}$, *then*

$$\mu_k^n \left(\{ \overrightarrow{v} \in \mathscr{U}_k^n : \left| \tau_k \left(g_1(\vec{v}) x_1 g_2(\vec{v}) x_2 \cdots g_w(\vec{v}) x_w \right) \right| \geqslant \varepsilon \} \right) \leqslant \frac{4 \cdot B(2m) \cdot 4^{m^2} \cdot (2M\omega)^{2\omega}}{k^2 \varepsilon^2}.$$

Proof. Since $x_1, \ldots, x_w$ are commuting normal matrices, there is a unitary matrix v such that, for $1 \leqslant j \leqslant w$, $v x_j v^* = a_j$ is diagonal. Since τ_k is tracial and

$$g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w = v^* \left(g_1(v\vec{u}v^*) a_1 g_2(v\vec{u}v^*) a_2 \cdots g_w(v\vec{u}v^*) a_w \right) v,$$

we have

$$\tau_k \left(g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w \right) = \tau_k \left(g_1(v\vec{u}v^*) a_1 g_2(v\vec{u}v^*) a_2 \cdots g_w(v\vec{u}v^*) a_w \right).$$

Thus, by the translation-invariance of μ_k^n, we can assume that $x_1, \ldots, x_w$ are all diagonal matrices.

Proof of the first statement. Write $g_1(\vec{u}) = u_{s_1}^{\varepsilon_1} \cdots u_{s_{m_1}}^{\varepsilon_{m_1}},\ g_2(\vec{u}) = u_{s_{m_1+1}}^{\varepsilon_{m_1+1}} \cdots u_{s_{m_2}}^{\varepsilon_{m_2}}, \ldots,$ $g_w(\vec{u}) = u_{s_{m_{w-1}+1}}^{\varepsilon_{m_{w-1}+1}} \cdots u_{s_{m_w}}^{\varepsilon_{m_w}}$ with each $\varepsilon_j \in \{-1,1\}$ and $s_j \in \{1,\ldots,n\}$ and with the property that $s_j = s_{j+1}$ implies $\varepsilon_j = \varepsilon_{j+1}$ unless $j \in \{m_1,\ldots,m_w\}$. Note that $m_w = m$ since $\sum \mathrm{length}(g_i) = m$. Also write $x_j = \mathrm{diag}(\gamma_j(1),\ldots,\gamma_j(k))$ for $1 \leqslant j \leqslant w$.

Define $\dot{+}$ on $\{1,\ldots,m_w = m\}$ by $s\dot{+}1 = \begin{cases} 1, & s = m_w \\ s+1, & 1 \leqslant s \leqslant m_w - 1 \end{cases}$. Then we have

$$\int_{\mathscr{U}_k^n} \tau_k\left(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w\right) d\mu_k^n(\vec{u})$$
$$= \frac{1}{k}\sum_{1\leqslant i_1,\ldots,i_{m_w+1}=i_1\leqslant k}\left(\prod_{\nu=1}^{w}\gamma_\nu\left(i_{m_\nu\dot{+}1}\right)\right)\int_{\mathscr{U}_k^n}\prod_{j=1}^{m_w} f_{i_ji_{j+1}}\left(u_{s_j}^{\varepsilon_j}\right) d\mu_k^n(\vec{u}).$$

Let $E = \{1,2,\ldots,m_w\}$. We can represent a choice of $1 \leqslant i_1,\ldots,i_{m_w} \leqslant k$ by a function $\alpha : E \to H = \{1,\ldots,k\}$. Thus we can replace the sum $\sum\limits_{1\leqslant i_1,\ldots,i_{m_w+1}=i_1\leqslant k}$ with $\sum\limits_{\alpha:E\to H}$ in the above equation. That is

$$(I =)\,\frac{1}{k}\sum_{\alpha:E\to H}\left(\prod_{\nu=1}^{w}\gamma_\nu\left(\alpha(m_\nu\dot{+}1)\right)\right)\int_{\mathscr{U}_k^n}\prod_{j=1}^{m_w} f_{\alpha(j),\alpha(j\dot{+}1)}\left(u_{s_j}^{\varepsilon_j}\right) d\mu_k^n(\vec{u}).$$

It is enough to restrict sums to the functions α such that the integral

$$I(\alpha) = \int_{\mathscr{U}_k^n}\prod_{j=1}^{m_w} f_{\alpha(j),\alpha(j\dot{+}1)}\left(u_{s_j}^{\varepsilon_j}\right) d\mu_k^n(\vec{u}) \neq 0.$$

We call such function α *good*, thus

$$I = \frac{1}{k}\sum_{\substack{\alpha:E\to H\\ \alpha \text{ is good}}}\left(\prod_{\nu=1}^{w}\gamma_\nu\left(\alpha(m_\nu\dot{+}1)\right)\right) I(\alpha).$$

Since α is good, Lemma 2 tells us that m_w must be even and exactly half of the ε_j's are 1 and the other half are -1. Combining Lemma 5 and the fact $m_w = m$, we know that

$$|I(\alpha)| \leqslant \frac{4^{(m_w/2)^2}}{k^{m_w/2}} = \frac{4^{(m/2)^2}}{k^{m/2}} \leqslant \frac{2^{m^2}}{k^{m/2}}. \tag{2}$$

Moreover, since α is good, Lemma 2 says that if $j \in E$ but $j \notin \{1\dot{+}m_1,\ldots,1\dot{+}m_w\}$, then $\alpha(j) = \alpha(j')$ for some $j' \neq j$.

Next we define an equivalence relation $\sim_\alpha$ on E by saying $i \sim_\alpha j$ if and only if $\alpha(i) = \alpha(j)$. Note that if $\beta : E \to H$, then the relations $\sim_\alpha$ and $\sim_\beta$ are equal if and only if there is a permutation $\sigma : H \to H$ such that $\beta = \sigma \circ \alpha$. We define an equivalence relation $\approx$ on the set of all good functions by

$$\alpha \approx \beta \text{ if and only if } \sim_\alpha = \sim_\beta .$$

It is clear that

$$\alpha \approx \beta \Longrightarrow I(\alpha) = I(\beta).$$

If $j \in E$, let $[j]_\alpha$ denote the $\sim_\alpha$-equivalence class of j, and let E_α denote the set of all such equivalence classes. We can construct all of the functions β equivalent to α in terms of injective functions

$$\sigma : E_\alpha \stackrel{1\text{-}1}{\to} H$$

by defining

$$\beta(j) = \sigma([j]_\alpha).$$

Let A be a set that contains exactly one function α from each $\approx$-equivalence class of good functions. Then we can write

$$\begin{aligned} |I| &= \left| \frac{1}{k} \sum_{\substack{\alpha : E \to H \\ \alpha \text{ is good}}} \left(\prod_{\nu=1}^{w} \gamma_\nu (\alpha(m_\nu \dot{+} 1)) \right) I(\alpha) \right| \\ &= \left| \frac{1}{k} \sum_{\alpha \in A} I(\alpha) \sum_{\beta \approx \alpha} \prod_{\nu=1}^{w} \gamma_\nu (\beta(m_\nu \dot{+} 1)) \right| \\ &= \frac{1}{k} \left| \sum_{\alpha \in A} |I(\alpha) \sum_{\sigma : E_\alpha \stackrel{1\text{-}1}{\to} H} \prod_{\nu=1}^{w} \gamma_\nu (\sigma([m_\nu \dot{+} 1]_\alpha)) \right| \\ &\leqslant \frac{1}{k} \sum_{\alpha \in A} |I(\alpha)| \left| \sum_{\sigma : E_\alpha \stackrel{1\text{-}1}{\to} H} \prod_{\nu=1}^{w} \gamma_\nu (\sigma([m_\nu \dot{+} 1]_\alpha)) \right|. \end{aligned} \tag{3}$$

Also we know that

$$Card(A) \leqslant B(m). \tag{4}$$

We only need to focus on $\left| \sum_{\sigma : E_\alpha \stackrel{1\text{-}1}{\to} H} \prod_{\nu=1}^{w} \gamma_\nu (\sigma([m_\nu \dot{+} 1]_\alpha)) \right|$. Let

$$F_\alpha = \{[m_\nu \dot{+} 1]_\alpha : 1 \leqslant \nu \leqslant w, Card([m_\nu \dot{+} 1]_\alpha) = 1\},$$

$$G_\alpha = \{[m_\nu \dot{+} 1]_\alpha : 1 \leqslant \nu \leqslant w, Card([m_\nu \dot{+} 1]_\alpha) > 1\},$$

$$K_\alpha = E_\alpha \setminus (F_\alpha \cup G_\alpha).$$

Since the product $\prod_{\nu=1}^{w} \gamma_\nu (\sigma([m_\nu \dot{+} 1]_\alpha))$ is determined once σ is defined on $F_\alpha \cup G_\alpha$, it follows that this product is repeated at most $P(k, \mathrm{card}(K_\alpha))$ times. Hence we have

$$\left| \sum_{\sigma : E_\alpha \stackrel{1\text{-}1}{\to} H} \prod_{\nu=1}^{w} \gamma_\nu (\sigma([m_\nu \dot{+} 1]_\alpha)) \right|$$

$$\leqslant P(k,\operatorname{card}(K_\alpha))\left|\sum_{\sigma:F_\alpha\cup G_\alpha\overset{1-1}{\to}H}\prod_{\nu=1}^{w}\gamma_\nu\left(\sigma\left([m_\nu\dot{+}1]_\alpha\right)\right)\right|$$

$$\leqslant k^{\operatorname{card}(K_\alpha)}\left|\sum_{\sigma:F_\alpha\cup G_\alpha\overset{1-1}{\to}H}\prod_{\nu=1}^{w}\gamma_\nu\left(\sigma\left([m_\nu\dot{+}1]_\alpha\right)\right)\right|. \tag{5}$$

If $a=[m_\nu\dot{+}1]_\alpha\in F_\alpha$, from the definition of F_α, it is clear that ν is unique. Then define $f_a(\sigma(a))=\gamma_\nu(\sigma(a))$. By $\tau_k(x_i)=0$ for all $1\leqslant i\leqslant w$, it follows that $\sum_{s=1}^{k} f_a(s)=0$. If $b=[m_\nu\dot{+}1]_\alpha\in G_\alpha$, from the definition of G_α, the cardinality r of b is greater than 1. Then define $g_b(\sigma(b))=(\gamma_\nu(\sigma(b)))^r$. Therefore

$$\begin{aligned}
&\left|\sum_{\sigma:F_\alpha\cup G_\alpha\overset{1-1}{\to}H}\prod_{\nu=1}^{w}\gamma_\nu\left(\sigma\left([m_\nu\dot{+}1]_\alpha\right)\right)\right|\\
&=\left|\sum_{\sigma:F_\alpha\cup G_\alpha\overset{1-1}{\to}H}\prod_{a\in F_\alpha} f_a(\sigma(a))\prod_{b\in G_\alpha} g_b(\sigma(b))\right|\\
&\quad(\text{letting } F=F_\alpha, G=G_\alpha \text{ and using Lemma 4 })\\
&\leqslant k^{[\operatorname{card}(F_\alpha)/2]+\operatorname{card}(G_\alpha)}w^wM^w.
\end{aligned}\tag{6}$$

As we mentioned before that $\operatorname{card}([j]_\alpha)=1$ implies $[j]_\alpha\in F_\alpha$, we see that

$$[\operatorname{card}(F_\alpha)/2]+\operatorname{card}(G_\alpha)+\operatorname{card}(K_\alpha)\leqslant\operatorname{card}(E)/2=m_w/2. \tag{7}$$

Combining inequalities (2), (3), (4), (5), (6) and (7) together, we have

$$|I|\leqslant\frac{1}{k}B(m)\cdot 2^{m^2}\cdot(Mw)^w.$$

Proof of the second statement. Notice that

$$\begin{aligned}
&\left|\tau_k\left(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w\right)\right|^2\\
&=\frac{1}{k^2}\sum_{1\leqslant i_1,\ldots,i_{m_w+1}=i_1\leqslant k}\left(\prod_{\nu=1}^{w}\gamma_\nu\left(i_{m_\nu+1}\right)\right)\prod_{j=1}^{m_w} f_{i_ji_{j+1}}\left(u_{s_j}^{\varepsilon_j}\right)\cdot\\
&\qquad\sum_{1\leqslant l_1,\ldots,l_{m_w+1}=l_1\leqslant k}\left(\prod_{\lambda=1}^{w}\overline{\gamma_\lambda\left(l_{m_\lambda+1}\right)}\right)\prod_{t=1}^{m_w}\overline{f_{l_tl_{t+1}}\left(u_{s_t}^{\varepsilon_t}\right)}.
\end{aligned}$$

Define $\dot{+}$ on the set $\{1,2,\ldots,2m_w\}$ by

$$x\dot{+}1=\begin{cases}1, & x=m_w\\ m_w+1, & x=2m_w\\ x+1, & 1\leqslant x\leqslant m_w-1 \text{ or } m_w+1\leqslant x\leqslant 2m_w-1\end{cases}.$$

Let $E=\{1,2,\ldots,2m_w\}$ and $H=\{1,\ldots,k\}$. Then we have

$$\int_{\mathscr{U}_k^n}\left|\tau_k\left(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w\right)\right|^2 d\mu_k^n(\vec{u})$$
$$=\frac{1}{k^2}\sum_{\alpha:E\to H}\left(\prod_{\nu=1}^{w}\gamma_\nu\left(\alpha(m_\nu\dot{+}1)\right)\right)\left(\prod_{\lambda=1}^{w}\overline{\gamma_\lambda\left(\alpha((m_\lambda\dot{+}1)+m_w)\right)}\right)\cdot$$
$$\int_{\mathscr{U}_k^n}\prod_{j=1}^{m_w}f_{\alpha(j)\alpha(j\dot{+}1)}\left(u_{s_j}^{\varepsilon_j}\right)\prod_{t=1}^{m_w}\overline{f_{\alpha(t+m_w)\alpha((t\dot{+}1)+m_w)}\left(u_{s_t}^{\varepsilon_t}\right)}.$$

The rest of the proof is similar to the proof of the first statement.

Proof of the third statement. The third statement follows from statement 1 and statement 2 and Chebychev's inequality. The proof is similar to the proof of Theorem 2 in [1]. □

The following corollary is a direct consequence of the third statement of Theorem 6.

COROLLARY 7. *Suppose M,m,k are positive integers. Let $\mathscr{D}$ be a finite set of commuting normal matrices with trace 0 in $\mathscr{M}_k(\mathbb{C})$ and $\|x\|\leqslant M$ for all $x\in\mathscr{D}$. Let*

$$\mathscr{E}=\{(g_1,\ldots,g_r,x_1,\ldots,x_r): r\in\mathbb{N}, g_1,\ldots,g_r \text{ are reduced words in } \mathbb{F}_n\setminus\{e\}$$
$$\text{such that } \sum_{i=1}^{r}\mathrm{length}(g_i)\leqslant m, \text{ and } x_1,\ldots,x_r\in\mathscr{D}\}.$$

If $\mathfrak{e}=(g_1,\ldots,g_r,x_1,\ldots,x_r)\in\mathscr{E}$ and $\overrightarrow{v}\in\mathscr{U}_k^n$, define $\mathfrak{e}(\overrightarrow{v})=g_1(\overrightarrow{v})x_1\cdots g_r(\overrightarrow{v})x_r$. Then

$$\mu_k^n\left(\bigcap_{\mathfrak{e}\in\mathscr{E}}\{\overrightarrow{v}:|\tau_k(\mathfrak{e}(\overrightarrow{v}))|<\varepsilon\}\right)\geqslant 1-\frac{4\cdot\mathrm{card}(\mathscr{E})\cdot B(2m)\cdot 4^{m^2}\cdot(2Mr)^{2r}}{k^2\varepsilon^2}.$$

Lemma 5.1 [3] follows directly from the corollary above.

Let $\mathscr{M}$ be a von Neumann algebra with a tracial state τ and $X_1,X_2,\ldots,X_n$ be elements in $\mathscr{M}$. For any $R,\varepsilon>0$, and positive integers m and k, define $\Gamma_R(X_1,\ldots,X_n;m,k,\varepsilon)$ to be the subset of $\mathscr{M}_k(\mathbb{C})^n$ consisting of all $(x_1,\ldots,x_n)$ in $\mathscr{M}_k(\mathbb{C})^n$ such that $\|x_j\|\leqslant R$ for $1\leqslant j\leqslant n$, and

$$\left|\tau_k(x_{i_1}^{\eta_1}\cdots x_{i_q}^{\eta_q})-\tau(X_{i_1}^{\eta_1}\cdots X_{i_q}^{\eta_q})\right|<\varepsilon,$$

for all $1\leqslant i_1,\ldots,i_q\leqslant n$, all $\eta_1,\ldots,\eta_q\in\{1,*\}$ and all q with $1\leqslant q\leqslant m$.

Suppose $\overrightarrow{U}$ is a n-tuple in $\mathscr{M}$ and, for each positive integer k, $\overrightarrow{u_k}$ is a n-tuple in $\mathscr{M}_k(\mathbb{C})$, then we say $\overrightarrow{u_k}$ *converges to* $\overrightarrow{U}$ *in distribution* if $p(\overrightarrow{u_k})\to p(\overrightarrow{U})$ for all $*$-monomials p.

COROLLARY 8. *Let M,m be positive integers and $\varepsilon > 0$. Suppose $\mathscr{M}$ is a von Neumann algebra with a faithful trace τ. Suppose $X_1,\ldots,X_s$ are commuting normal operators in $\mathscr{M}$, $U_1,\ldots,U_n$ are free Haar unitary elements in $\mathscr{M}$ and $\{X_1,\ldots,X_s\}$, $\{U_1,\ldots,U_n\}$ are free. For any positive integer k, let $\{x(k,1),\ldots,x(k,s)\}$ be a set of commuting normal $k\times k$ matrices such that $\sup_{k,j}\|x(k,j)\| \leqslant M$ and*

$$(x(k,1),\ldots,x(k,s)) \to (X_1,\ldots,X_s)$$

in distribution as $k\to\infty$.

If

$$\Omega_k = \{(v_1,\ldots,v_n) \in \mathscr{U}_k^n : (x(k,1),\ldots,x(k,s),v_1,\ldots,v_n) \in \Gamma_M(X_1,\ldots,X_s,U_1,\ldots,U_n;m,k,\varepsilon)\},$$

then

$$\lim_{k\to\infty} \mu_k^n(\Omega_k) = 1.$$

Lemma 5.2 [3] follows directly from the corollary above.

We end this paper with one last corollary.

COROLLARY 9. *Let M,m be positive integers and $\varepsilon > 0$. Suppose $\mathscr{M}$ is a von Neumann algebra with a faithful trace τ. Suppose $X_1,\ldots,X_s$ are free normal operators in $\mathscr{M}$. Suppose $\{x(k,1),\ldots,x(k,s)\}$ is a set of normal $k\times k$ matrices such that $\sup_{k,j}\|x(k,j)\| \leqslant M$ and, for $1\leqslant j\leqslant s$, $x(k,j)\to X_j$ in distribution as $k\to\infty$.*

If

$$\Theta_k = \{(v_1,\ldots,v_s) \in \mathscr{U}_k^s : (v_1^* x(k,1)v_1,\ldots,v_s^* x(k,s)v_s) \in \Gamma_M(X_1,\ldots,X_s;m,k,\varepsilon)\},$$

then

$$\lim_{k\to\infty} \mu_k^n(\Theta_k) = 1.$$

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