Operators and Matrices
Volume 3, Number 2 (2009), 215–225

ON THE POLAR DECOMPOSITION OF THE DUGGAL TRANSFORMATION AND RELATED RESULTS

SAJI MATHEW AND M. S. BALASUBRAMANI

(*Communicated by T. Ando*)

Abstract. Let $T = U|T|$ be the polar decomposition of a bounded operator T on a Hilbert space. The transformation $\Delta(T) = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation, and $\Gamma(T) = |T|U$ is called the Duggal transformation of T. We discuss Aluthge transformation and Duggal transformation of binormal operators and centered operators. We obtain results about the polar decomposition of Duggal transformation. We give necessary and sufficient conditions for $\Gamma(T)$ to have the polar decomposition $\Gamma(T) = \Gamma(U)|\Gamma(T)|$. As a consequence we get $\Gamma(T) = \Gamma(U)|\Gamma(T)|$ to be the polar decomposition of $\Gamma(T)$ if T is binormal.

1. Introduction

In what follows, an operator means a bounded operator on a complex Hilbert space $\mathscr{H}$. An operator T is said to be positive (denoted $T \geqslant 0$) if $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$.

It is well known that every operator T can be decomposed into $T = U|T|$ with a partial isometry U, and $|T| = (T^*T)^{1/2}$. U is determined uniquely by the kernel condition $\ker(U) = \ker(T)$, then this decomposition is called the polar decomposition. In [1], Aluthge defined a transformation $\Delta(T) = |T|^{1/2}U|T|^{1/2}$ which was later called the *Aluthge transformation.* Aluthge transformation is very useful in the study of non-normal operators. Moreover, for each non-negative integer n, Jung, Ko, Pearcy defined the n^{th} Aluthge transform $\Delta^n(T)$ in [8] as $\Delta^n(T) = \Delta(\Delta^{n-1}(T))$ and $\Delta^0(T) = T$. In [6], Ito, Yamazaki, Yanagida obtained several results about the polar decomposition of Aluthge transformation.

In [4], Foias, Jung, Ko, Pearcy defined a transformation $\Gamma(T) = |T|U$ called the *Duggal transformation.* For each non-negative integer n, the n^{th} Duggal transformation $\Gamma^n(T)$ can be defined as $\Gamma^n(T) = \Gamma(\Gamma^{n-1}(T))$ and $\Gamma^0(T) = T$.

An operator T is said to be *binormal* if $[\,|T|, |T^*|\,] = 0$, where $[A,B] = AB - BA$. The operator T is said to be *centered* if the following sequence

$$\ldots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \ldots$$

is commutative.

Mathematics subject classification (2000): 47A05, 47B20.
Keywords and phrases: Duggal transformation, Aluthge transformation, polar decomposition, binormal operators, centered operators.

Paper OaM-03-12

Binormal and centered operators were defined by Campbell [2] and Morrel- Muhly [9], respectively. Relations among these classes and that of quasinormal operators are easily obtained as follows.

$$\text{quasinormal} \subset \text{centered} \subset \text{binormal}.$$

In [7], Ito, Yamazaki, Yanagida obtained results on the polar decomposition of the product of two operators and of Aluthge transformation. They also studied properties and characterizations of binormal and centered operators from the view point of polar decomposition and Aluthge transformation.

In [6], Ito, Yamazaki, Yanagida gave an example of a binormal, invertible operator T such that the Aluthge transformation $\Delta(T)$ is not binormal. In this paper, first we show that if T is a binormal, invertible operator, then the Duggal transformation $\Gamma(T)$ is binormal. We discuss some consequences of applying Aluthge transformation and Duggal transformation successively on an invertible operator T.

We show that if T is an invertible operator with polar decomposition $T = U|T|$, then the polar decomposition of $\Gamma(T)$ is $\Gamma(T) = U|\Gamma(T)|$. We give necessary and sufficient conditions for $\Gamma(T)$ to have the polar decomposition $\Gamma(T) = \Gamma(U)|\Gamma(T)|$.

Let $T = U|T|$ be the polar decomposition of T. A theorem in [7] says that T is binormal if and only if $\Delta(T) = \Delta(U)|\Delta(T)|$ is the polar decomposition of the Aluthge transformation $\Delta(T)$. We discuss a similar situation of Duggal transformations. We prove that if T is binormal, then $\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$. We also give necessary and sufficient condition for $\Gamma(T)$ to have the polar decomposition $\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$. In section 4, we give an alternate proof of a theorem in [7] which deals with Aluthge transformations.

2. Aluthge and Duggal transformations of binormal operators

LEMMA 2.1. *Let $T = U|T|$ be the polar decomposition of T. The following hold.*

i. $\Gamma(T) = U^*TU$.

ii. If T is invertible, then $\Delta(T)$ and $\Gamma(T)$ are invertible, and in that case, $\Delta(T) = |T|^{1/2}T|T|^{-1/2}$ and $\Gamma(T) = |T|\,T\,|T|^{-1}$.

THEOREM 2.2. [6] *Let $T = U|T|$ be the polar decomposition of T. Then $\Delta(T) = U|\Delta(T)|$ if and only if T is binormal. (This assertion does not mean that when T is binormal, $\Delta(T) = U|\Delta(T)|$ is the polar decomposition of $\Delta(T)$).*

THEOREM 2.3. [6] *Let $T = U|T|$ be the polar decomposition of T. If T is binormal, then $\Delta(T) = U^*UU|\Delta(T)|$ is the polar decomposition of $\Delta(T)$.*

THEOREM 2.4. *Let T be invertible and suppose that $T = U|T|$ is the polar decomposition of T. If T is binormal, then $\Delta(T) = U|\Delta(T)|$ is the polar decomposition of $\Delta(T)$.*

Proof. Since T is invertible, U is unitary. By Theorem 2.3, the proof follows. $\square$

If S,T and V are operators with $S = V^*TV$, and V is unitary, then we have $|S| = V^*|T|V$, $|S|^{1/2} = V^*|T|^{1/2}V$. Further, if $T = U|T|$ is the polar decomposition of T, then the polar decomposition of S is $S = (V^*UV)|S|$ and hence $\Delta(S) = V^*\Delta(T)V$.

REMARK 2.5. The binormality of T does not imply the binormality of the Aluthge transformation $\Delta(T)$. As shown in [6], if

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

then A is a binormal operator, and the Aluthge transformation $\Delta(A)$ is not binormal. Note that A is invertible. However, the equivalent case of Duggal transformation is different.

THEOREM 2.6. *Let $T = U|T|$ is the polar decomposition of the operator T, and U a coisometry. Then T is binormal if and only if $\Gamma(T)$ is binormal.*

Proof. We have $\Gamma(T) = U^*TU$. Therefore, $(\Gamma(T))^*\Gamma(T) = U^*|T|^2U \geqslant 0$, and $|\Gamma(T)| = U^*|T|U$. On the other hand, $\Gamma(T)(\Gamma(T))^* = U^*|T^*|^2U \geqslant 0$, and $|(\Gamma(T))^*| = U^*|T^*|U$. Hence, if T is binormal, then $\Gamma(T)$ is also binormal. Conversely, if $\Gamma(T)$ is binormal, then we have $U^*|T^*||T|U = U^*|T||T^*|U$. Multiplying U and U^* with both sides, we obtain $|T^*||T| = |T||T^*|$, that is, T is binormal. $\square$

THEOREM 2.7. *Let T be invertible. If T is binormal, then $\Gamma(\Delta(T)) = \Delta(\Gamma(T))$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since T is invertible, U is unitary. By Lemma 2.1, $\Gamma(T) = U^*TU$. Therefore, $\Delta(\Gamma(T)) = U^*\Delta(T)U$.

By Theorem 2.4, $\Delta(T) = U|\Delta(T)|$ is the polar decomposition of $\Delta(T)$, and therefore, again by Lemma 2.1, $\Gamma(\Delta(T)) = U^*\Delta(T)U$. $\square$

The following characterization of centered operators from the viewpoint of the polar decomposition and the Aluthge transformation can be seen in [6].

THEOREM 2.8. [6] *Let T be an operator. Then $\Delta^n(T)$ is binormal for all $n \geqslant 0$ if and only if T is a centered operator.*

THEOREM 2.9. *Let T be invertible and centered. Then $\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T))$ for all $n \geqslant 0$.*

Proof. Since T is centered, T is binormal. Therefore, the result is true for $n = 1$, by Theorem 2.7. Suppose that the result is true for $n = m-1$. Then $\Gamma(\Delta^{m-1}(T)) = \Delta^{m-1}(\Gamma(T))$.

Now, $\Delta^{m-1}(T)$ is invertible since T is invertible. By Theorem 2.8, $\Delta^{m-1}(T)$ is binormal. Therefore, by Theorem 2.7, $\Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))]$. Hence,

$$\begin{aligned}\Gamma(\Delta^m(T)) &= \Gamma[\Delta(\Delta^{m-1}(T))]\\ &= \Delta[\Gamma(\Delta^{m-1}(T))]\\ &= \Delta[\Delta^{m-1}(\Gamma(T))]\\ &= \Delta^m(\Gamma(T)).\end{aligned}$$

The theorem follows by induction. □

THEOREM 2.10. *Let T be invertible and binormal. Then $\Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))$ for all $n \geqslant 0$.*

Proof. The result is true for $n = 1$, by Theorem 2.7.

Suppose that the result is true for $n = m-1$. Then $\Delta(\Gamma^{m-1}(T)) = \Gamma^{m-1}(\Delta(T))$.

Since T is invertible and binormal, by Lemma 2.1 and Theorem 2.6, $\Gamma^n(T)$ is invertible and binormal for all $n \geqslant 0$. Since $\Gamma^{m-1}(T)$ is binormal and invertible, by Theorem 2.7, $\Delta[\Gamma(\Gamma^{m-1}(T))] = \Gamma[\Delta(\Gamma^{m-1}(T))]$. Hence,

$$\begin{aligned}\Delta(\Gamma^m(T)) &= \Delta[\Gamma(\Gamma^{m-1}(T))]\\ &= \Gamma[\Delta(\Gamma^{m-1}(T))]\\ &= \Gamma[\Gamma^{m-1}(\Delta(T))]\\ &= \Gamma^m(\Delta(T)).\end{aligned}$$

The theorem follows by induction. □

THEOREM 2.11. *Let T be invertible and centered. Then*

$$\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))$$

for all $m,n \geqslant 0$.

Proof. Every centered operator is binormal. Therefore the proof follows by Theorem 2.8 and Theorem 2.10. □

3. Polar decomposition of Aluthge and Duggal transformations

The following result shows the polar decomposition of the product of two operators.

THEOREM 3.1. [5] *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. If T and S are doubly commutative (i.e., $[T\ ,\ S] = [T\ ,\ S^*] = 0$), then*

$$TS = UV|TS|$$

is the polar decomposition of TS.

The following is a generalization of this result.

THEOREM 3.2. [7] *Let* $T = U|T|$, $S = V|S|$ *and* $|T|\,|S^*| = W|\,|T|\,|S^*|\,|$ *be the polar decompositions. Then*

$$TS = UWV|TS|$$

is also the polar decomposition.

This theorem can be used to obtain the polar decomposition of the Duggal transformation of an invertible operator as follows.

THEOREM 3.3. *Let* T *be invertible. If* $T = U|T|$ *is the polar decomposition of* T, *then*

$$\Gamma(T) = U|\Gamma(T)|$$

is the polar decomposition of $\Gamma(T)$.

Proof. Since T is invertible, U is unitary and $|T|$ is invertible. We have, $|U| = |U^*| = I$, the identity operator, because $UU^* = U^*U = I = I^2$ and $I \geqslant 0$. Since $\ker I = \ker U = \ker U^* = \{0\}$, we see that $U = UI$ is the polar decomposition of U. We have $|T|^*\,|T| = |T|^2$ and $|T| \geqslant 0$. Therefore, $|\,|T|\,| = |T|$. Since $|T|$ is invertible, $\ker|T| = \ker I = \{0\}$, and hence $|T| = I|\,|T|\,|$ is the polar decomposition of $|T|$. Also $|T|\,|U^*| = |T|$. Replacing T and S by $|T|$ and U respectively in Theorem 3.2, we obtain $|T|U = IIU|\,|T|U\,|$ is the polar decomposition of $|T|U$, or in other words, $\Gamma(T) = U|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$. $\square$

LEMMA 3.4. *If* U *is a partial isometry, then* $|U| = U^*U$ *and* $U = U\,|U|$ *is the polar decomposition of* U. *Also,* $\Gamma(U) = \Delta(U) = U^*UU$.

Proof. Since U^*U is a projection, $(U^*U)^2 = U^*U$ and $U^*U \geqslant 0$. Therefore, $|U| = (U^*U)^{1/2} = U^*U$. Since U^*U is the support of U, we have $UU^*U = U$, i.e., $U\,|U| = U$. The kernel condition for the polar decomposition is satisfied automatically. Hence $U = U\,|U|$ is the polar decomposition of U.

Since $|U| = U^*U = (U^*U)^2$ and $U^*U \geqslant 0$, we have $|U|^{1/2} = U^*U$. Therefore, $\Gamma(U) = |U|\,U = U^*UU$ and $\Delta(U) = |U|^{1/2}\,U\,|U|^{1/2} = (U^*U)U(U^*U) = U^*UU$. $\square$

Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. The following theorem gives an equivalent condition so that $TS = UV\,|TS|$ becomes the polar decomposition.

THEOREM 3.5. [7] *Let* $T = U|T|$ *and* $S = V|S|$ *be the polar decompositions. Then* $|T|\,|S^*| = |S^*|\,|T|$ *if and only if*

$$TS = UV\,|TS|$$

is the polar decomposition.

We use this theorem to prove the following result on the polar decomposition of the Duggal transformation of an operator.

THEOREM 3.6. *Let $T = U|T|$ be the polar decomposition of T. Then $\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$ if and only if $|T|\ |U^*| = |U^*|\ |T|$.*

Proof. Since U is a partial isometry, $(\ker U)^{\perp}$ is the initial space of U. Since U^*U is the support of both T and U, we see that $\operatorname{ran}(U^*U) = (\ker U)^{\perp} = (\ker T)^{\perp}$.

Also, U^*U is self-adjoint. Therefore, $\ker|T| = \ker T = [\operatorname{ran}(U^*U)]^{\perp} = \ker(U^*U)$. Every projection is a partial isometry, in particular, U^*U is a partial isometry. Further, $(U^*U)\ |T| = |T|\ (U^*U) = |T|$. Hence $|T| = (U^*U)\ |T|$ is the polar decomposition of $|T|$, recalling $|\ |T|\ | = |T|$.

By Lemma 3.4, $U = U\ |U|$ is the polar decomposition of U and $\Gamma(U) = U^*UU$.

Replacing T and S in Theorem 3.5 by $|T|$ and U respectively, we see that

$$|\ |T|\ | \cdot |U^*| = |U^*| \cdot |\ |T|\ |$$

if and only if

$$|T|U = U^*UU\ |\ |T|U\ |$$

is the polar decomposition of $|T|U$.
Hence

$$|T| \cdot |U^*| = |U^*| \cdot |T|$$

if and only if

$$\Gamma(T) = U^*UU\ |\Gamma(T)|$$

is the polar decomposition of $\Gamma(T)$.
Therefore

$$|T| \cdot |U^*| = |U^*| \cdot |T|$$

if and only if

$$\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$$

is the polar decomposition of $\Gamma(T)$. □

Let $T = U|T|$ be the polar decomposition of T. A theorem in [7] says that T is binormal if and only if $\Delta(T) = \Delta(U)\,|\Delta(T)|$ is the polar decomposition of the Aluthge transformation $\Delta(T)$. In the following two theorems we discuss the similar situation of Duggal transformations.

THEOREM 3.7. *Let $T = U|T|$ be the polar decomposition of T. If T is binormal, then $\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$.*

Proof. Let $F = UU^*$. Then F is the support of T^*. If T is binormal, then $\ker|T^*|$ is invariant under $|T|$. Therefore, $(\ker|T^*|)^{\perp}$ is invariant under $|T|$. But $(\ker|T^*|)^{\perp} = (\ker T^*)^{\perp} = \operatorname{ran} F$. Therefore, $F\,|T|\,F = |T|\,F$, and hence $|T|\,F = F\,|T|$. It follows that $|T|\,|U^*| = |U^*|\,|T|$. By Theorem 3.6, $\Gamma(T) = \Gamma(U)\ |\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$. □

THEOREM 3.8. *Let $T = U|T|$ be the polar decomposition of T, and E,F the initial and final projections, respectively, of the partial isometry U. If $\Gamma(T) = \Gamma(U)\,|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$, then $EF = FE$, or equivalently, U is binormal.*

Proof. We have, $E = U^*U$ and $F = UU^*$. If $\Gamma(T) = \Gamma(U)\,|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$, then by Theorem 3.6, $|T|\,|U^*| = |U^*|\,|T|$. Thus, $|T|\,F = F\,|T|$, and therefore, $\operatorname{ran}|T|$ is invariant under F. Hence $\overline{\operatorname{ran}|T|}$ is invariant under F. But $\overline{\operatorname{ran}|T|} = (\ker T)^{\perp} = \operatorname{ran} E$. Hence $EF = FE$.

Next, U is binormal, if and only if, $|U|\,|U^*| = |U^*|\,|U|$, if and only if, $EF = FE$. □

REMARK 3.9. Let T be an operator, and $T = U|T|$ the polar decomposition of T. Let E,F be the initial and final projections, respectively, of the partial isometry U. By Lemma 3.4, $\Gamma(U) = \Delta(U) = U^*UU$. Theorem 3.1 in [7] says that if U is a partial isometry, then the following assertions are mutually equivalent: (i) U is binormal, (ii) $\Delta(U)$ is a partial isometry, (iii) U^2 is a partial isometry. Thus if $\Gamma(T) = \Gamma(U)\,|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$, then the following hold.

i. $|T|\,|U^*| = |U^*|\,|T|$.

ii. $EF = FE$.

iii. U is binormal.

iv. $\Gamma(U)$ is a partial isometry.

v. $\Delta(U)$ is a partial isometry.

vi. U^2 is a partial isometry.

On the other hand, if T is binormal, then $\Gamma(T) = \Gamma(U)\,|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$, and this in turn implies each of the above statements.

THEOREM 3.10. *Let $T = U|T|$ be the polar decomposition of T. If $\Gamma^n(T) = \Gamma^n(U)\,|\Gamma^n(T)|$ is the polar decomposition of $\Gamma^n(T)$ for all $n = 0,1,2,\ldots$, then $|U^*|$ commutes with every $|\Gamma^n(T)|$, or in other words,* $\operatorname{ran} U$ *is invariant under every* $|\Gamma^n(T)|$.

Proof. Let $E = U^*U$ and $F = UU^*$. We note that $\operatorname{ran} U = \operatorname{ran} F$.

First we show that for all $n = 1,2,\ldots$, $|\Gamma^n(T)| = (\Gamma^{n-1}(U))^*\,|\Gamma^{n-1}(T)|\,(\Gamma^{n-1}(U))$. Since $\Gamma^{n-1}(T) = \Gamma^{n-1}(U)\,|\Gamma^{n-1}(T)|$ and $\Gamma^n(T) = \Gamma^n(U)\,|\Gamma^n(T)|$ are polar decompositions, by Theorem 3.6,

$$|(\Gamma^{n-1}(U))^*| \cdot |\Gamma^{n-1}(T)| = |\Gamma^{n-1}(T)| \cdot |(\Gamma^{n-1}(U))^*|.$$

Therefore, $\Gamma^{n-1}(U)\,(\Gamma^{n-1}(U))^*$ commutes with $|\Gamma^{n-1}(T)|$. Since $\Gamma^{n-1}(U)$ is a partial isometry, $\Gamma^{n-1}(U)\,(\Gamma^{n-1}(U))^*\,\Gamma^{n-1}(U) = \Gamma^{n-1}(U)$. Therefore,

$$\begin{aligned}(\Gamma^n(T))^*\,\Gamma^n(T) &= (|\Gamma^{n-1}(T)|\,\Gamma^{n-1}(U))^*\,|\Gamma^{n-1}(T)|\,\Gamma^{n-1}(U)\\ &= (\Gamma^{n-1}(U))^*\,|\Gamma^{n-1}(T)|^2\,\Gamma^{n-1}(U)\end{aligned}$$

$$
\begin{aligned}
&= (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)|^2 \Gamma^{n-1}(U) (\Gamma^{n-1}(U))^* \Gamma^{n-1}(U) \\
&= (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) \\
&= ((\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U))^2.
\end{aligned}
$$

Since $(\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) \geqslant 0$, we see that

$$|\Gamma^n(T)| = (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U).$$

Next we show that if n is a positive integer, then $|\Gamma^n(T)|$ commutes with $|(\Gamma(U))^*|$ if and only if $|\Gamma^n(T)|$ commutes with $|U^*|$. Since $\Gamma(T) = \Gamma(U)|\Gamma(T)|$ is the polar decomposition of $\Gamma(T)$, by Theorem 3.8, we have $EF = FE$. Since $(\Gamma(U))^*$ is a partial isometry, we have $(\Gamma(U)(\Gamma(U))^*)^2 = \Gamma(U)(\Gamma(U))^*$. Also $\Gamma(U)(\Gamma(U))^* \geqslant 0$. Therefore,

$$|(\Gamma(U))^*| = (\Gamma(U)(\Gamma(U))^*)^{1/2} = \Gamma(U)(\Gamma(U))^* = U^*UUU^*U^*U = EFE = EF.$$

Since $U, \Gamma(U), \Gamma^2(U), \ldots$ are partial isometries, we have

$$\Gamma(U) = U^*UU, \Gamma^2(U) = (\Gamma(U))^* \Gamma(U)\Gamma(U), \ldots.$$

It follows that

$$\Gamma(U)E = \Gamma(U), \Gamma^2(U)E = \Gamma^2(U), \ldots.$$

In particular, $\Gamma^{n-1}(U)E = \Gamma^{n-1}(U)$ and therefore $E(\Gamma^{n-1}(U))^* = (\Gamma^{n-1}(U))^*$. Hence

$$
\begin{aligned}
|\Gamma^n(T)||(\Gamma(U))^*| &= (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U)EF \\
&= (\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U)F \\
&= |\Gamma^n(T)|F
\end{aligned}
$$

and

$$
\begin{aligned}
|(\Gamma(U))^*||\Gamma^n(T)| &= EF(\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) \\
&= FE(\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) \\
&= F(\Gamma^{n-1}(U))^* |\Gamma^{n-1}(T)| \Gamma^{n-1}(U) \\
&= F|\Gamma^n(T)|.
\end{aligned}
$$

Thus $|\Gamma^n(T)|$ commutes with $|(\Gamma(U))^*|$ if and only if $|\Gamma^n(T)|$ commutes with F. Hence $|\Gamma^n(T)|$ commutes with $|(\Gamma(U))^*|$ if and only if $|\Gamma^n(T)|$ commutes with $|U^*|$.

Finally we show that $|\Gamma^n(T)||U^*| = |U^*||\Gamma^n(T)|$ for all $n = 0, 1, 2, \ldots$. Let n be any nonnegative integer. Since $\Gamma^{n+1}(T) = \Gamma^{n+1}(U)|\Gamma^{n+1}(T)|$ is the polar decomposition of $\Gamma^{n+1}(T)$, by Theorem 3.6,

$$|\Gamma^n(T)||(\Gamma^n(U))^*| = |(\Gamma^n(U))^*||\Gamma^n(T)|,$$

i.e., $|\Gamma(\Gamma^{n-1}(T))|$ commutes with $|(\Gamma(\Gamma^{n-1}(U)))^*|$. Since $\Gamma^{n-1}(T) = \Gamma^{n-1}(U)|\Gamma^{n-1}(T)|$ is the polar decomposition of $\Gamma^{n-1}(T)$, by applying to $\Gamma^{n-1}(T)$ what we proved in the previous paragraph we see that $|\Gamma^n(T)|$ commutes with $|(\Gamma^{n-1}(U))^*|$. Since $\Gamma^n(T) = \Gamma^2(\Gamma^{n-2}(T))$ and $\Gamma^{n-2}(T) = \Gamma^{n-2}(U)|\Gamma^{n-2}(T)|$ is the polar decomposition of $\Gamma^{n-2}(T)$, by applying similar argument to $\Gamma^{n-2}(T)$, we see that $|\Gamma^n(T)|$ commutes with $|(\Gamma^{n-2}(U))^*|$. Proceeding like this we obtain $|\Gamma^n(T)|$ commutes with $|U^*|$. □

4. Semigroup properties of factors in the polar decomposition and some applications

Ito, Yamazaki, and Yanagida in [7] proves Theorem 4.1 stated below using properties of the polar decomposition of the product of two operators. In this section we give an alternate proof of this theorem using semigroup properties of factors in the polar decomposition proved by Catepillan and Szymanski in [3].

THEOREM 4.1. [7] *Let $T = U|T|$ be the polar decomposition of T. Then the following are equivalent.*

i. *T is centered.*

ii. *$\Delta^n(T) = \Delta^n(U)\,|\Delta^n(T)|$ is the polar decomposition for all nonnegative integer n.*

iii. *$T^n = U^n\,|T^n|$ is the polar decomposition for all natural number n.*

THEOREM 4.2. [7] *Let $T = U|T|$ be the polar decomposition of T. Then T is binormal if and only if $\Delta(T) = \Delta(U)\ |\Delta(T)|$ is the polar decomposition of $\Delta(T)$.*

DEFINITION 4.3. Let S be a commutative semigroup with unit, $\mathscr{H}$ a Hilbert space, $\mathscr{L}(\mathscr{H})$ the space of all bounded operators on $\mathscr{H}$, and $\pi : S \to \mathscr{L}(\mathscr{H})$ be a mapping. The mapping π is called a *semigroup homomorphism* if $\pi(s+t) = \pi(s)\pi(t)$, $s,t \in S$ and $\pi(0) = I$. The mapping π is called a *centered homomorphism* if it is a semigroup homomorphism, and the set $\{\pi(s)^*\pi(s), \pi(t)\pi(t)^*,\ s,t \in S\}$ is commutative.

Note that here $\mathscr{L}(\mathscr{H})$ is considered as a semigroup under the operation of multiplication of operators.

THEOREM 4.4. [3] *Let $\pi : S \to \mathscr{L}(\mathscr{H})$ be a semigroup homomorphism. Let*

$$\pi(s) = \theta(s)\mu(s)$$

be the polar decomposition of the operator $\pi(s)$, $s \in S$.

i. *If π is centered, then θ is a semigroup homomorphism.*

ii. *Assume additionally that for each $s,t \in S$ there exists $r \in S$ such that $s = t + r$ or $t = s + r$. If θ is a semigroup homomorphism, then π is centered.*

The semigroup $\mathbb{N}$ satisfies the additional condition in (ii) of Theorem 4.4, but the semigroup $\mathbb{N}\times\mathbb{N}$ does not.

Alternate proof of Theorem 4.1. Let $\pi : \mathbb{N} \to \mathscr{L}(\mathscr{H})$ be defined by

$$\pi(n) = T^n, n \in \mathbb{N}.$$

Then π is a semigroup homomorphism. Let

$$\pi(n) = \theta(n)\mu(n)$$

be the polar decomposition of the operator $\pi(n), n \in \mathbb{N}$. Now, $\theta(1) = U$ and $\mu(1) = |T|$.

Assume that T is centered. Then the set

$$\{\pi(n)^*\pi(n), \pi(m)\pi(m)^*, n,m \in \mathbb{N}\} = \{(T^n)^*T^n, T^m(T^m)^*, n,m \in \mathbb{N}\}$$

is commutative, and therefore, π is a centered homomorphism. By Theorem 4.4, $\theta : \mathbb{N} \to \mathscr{L}(\mathscr{H})$ is a semigroup homomorphism. Therefore, for every $n \in \mathbb{N}$, $U^n = (\theta(1))^n = \theta(n)$. Hence the polar decomposition of $T^n = \pi(n)$ is $T^n = \theta(n)\mu(n) = \theta(n)|\pi(n)| = U^n|T^n|$ for every $n \in \mathbb{N}$. This proves (i) $\Rightarrow$ (iii).

Assume that $T^n = U^n|T^n|$ is the polar decomposition for all natural number n. Then $\theta(n) = U^n$ for all $n \in \mathbb{N}$. Therefore, for $n,m \in \mathbb{N}$, $\theta(n+m) = U^{n+m} = U^nU^m = \theta(n)\theta(m)$, and hence θ is a semigroup homomorphism. By Theorem 4.4, π is a centered homomorphism. It follows that T is centered. This proves (iii) $\Rightarrow$ (i).

The rest of the proof is essentially the same as that in [7]. We give it here for completeness.

To prove (i) $\Rightarrow$ (ii). Assume that T is centered. By Theorem 2.8, $\Delta^n(T)$ is binormal for all $n \in \mathbb{N}$. By Theorem 4.2,

$$\Delta(T) = \Delta(U)\,|\Delta(T)| \text{ is the polar decomposition of } \Delta(T). \tag{1}$$

Next since $\Delta(T)$ is binormal and since (1) holds, we have by Theorem 4.2,

$$\Delta^2(T) = \Delta^2(U)\,|\Delta^2(T)| \text{ is the polar decomposition of } \Delta^2(T).$$

Repeating this method, we have $\Delta^n(T) = \Delta^n(U)\,|\Delta^n(T)|$ is the polar decomposition for all nonnegative integer n.

To prove (ii) $\Rightarrow$ (i). Assume that $\Delta^n(T) = \Delta^n(U)\,|\Delta^n(T)|$ is the polar decomposition for all nonnegative integer n. By Theorem 4.2, $\Delta^n(T)$ is binormal for all $n \in \mathbb{N}$. Hence T is centered by Theorem 2.8. □

REMARK 4.5. Let $T = U|T|$ be the polar decomposition of T. If T is centered, then for all $n = 0,1,2,\ldots$, $\Gamma(U^n)$ is binormal and $\Gamma(T^n) = \Gamma(U^n)\,|\Gamma(T^n)|$ is the polar decomposition of $\Gamma(T^n)$.

Acknowledgements. The authors are grateful to the referees for their helpful suggestions and comments that improved this paper a lot. The first author acknowledges the financial support of University Grants Commission, India.

REFERENCES

[1] A. ALUTHGE, *On p-hyponormal operators for $0 < p < 1$*, Integral Equations Operator Theory, 13:307–315, 1990.

[2] S. L. CAMPBELL, *Linear operators for which T^*T and $TT*$ commute*, Proc. Amer. Math. Soc., 34:177–180, 1972.

[3] X. CATEPILLAN AND W. SZYMANSKI, *Semigroup properties of factors in the polar decomposition or De Moivre's formula*, Proc. Amer. Math. Soc., 126(12):3521–3526, 1998.

[4] C. FOIAS, I. B. JUNG, E. KO, AND C. PEARCY, *Complete contractivity of maps associated with the Aluthge and Duggal transforms.*, Pacific J. Math, 209(2):249–259, 2003.

[5] T. FURUTA, *On the polar decomposition of an operator*, Acta Sci. Math. (Szeged), 46:261–268, 1983.

[6] M. ITO, T. YAMAZAKI AND M. YANAGIDA, *On the polar decomposition of the Aluthge transformation and related results*, J. Operator Theory, 51:303–319, 2004.

[7] M. ITO, T. YAMAZAKI AND M. YANAGIDA, *On the polar decomposition of the product of two operators and its applications*, Integral Equations Operator Theory, 49(4):461–472, 2004.

[8] I. B. JUNG, E. KO AND C. PEARCY, *Aluthge transforms of operators*, Integral Equations Operator Theory, 37:437–448, 2000.

[9] B. B. MORREL AND P. S. MUHLY, *Centered operators*, Studia Math., 51:251–263, 1974.

(Received January 5, 2008)

Saji Mathew
Dept. of Mathematics
St. Joseph's College
Devagiri, Calicut
Kerala, India
e-mail: sajimathew@fastmail.fm
sajimathe@gmail.com

M. S. Balasubramani
Dept. of Mathematics
University of Calicut
Kerala, India 673 635
e-mail: msbaala@gmail.com