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A CHARACTERIZATION OF HILBERT C^*–MODULES OVER FINITE DIMENSIONAL C^*–ALGEBRAS

LJILJANA ARAMBAŠIĆ, DAMIR BAKIĆ AND MOHAMMAD SAL MOSLEHIAN

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Abstract. We show that the unit ball of a full Hilbert C^*-module is sequentially compact in a certain weak topology if and only if the underlying C^*-algebra is finite dimensional. This provides an answer to the question posed in J. Chmieliński et al [Perturbation of the Wigner equation in inner product C^*-modules, J. Math. Phys. 49 (2008), no. 3, 033519].

1. Introduction and preliminaries

Let $\mathscr{A}$ be a C^*-algebra. A linear space $\mathscr{M}$ that is an algebraic left $\mathscr{A}$-module with $\lambda(ax) = a(\lambda x) = (\lambda a)x$ for $x \in \mathscr{M}$, $a \in \mathscr{A}$, $\lambda \in \mathbb{C}$, is called a *pre-Hilbert $\mathscr{A}$-module* (or an *inner product $\mathscr{A}$-module*) if there exists an $\mathscr{A}$-valued inner product on $\mathscr{M}$, i.e., a mapping $\langle \cdot,\cdot \rangle : \mathscr{M} \times \mathscr{M} \to \mathscr{A}$ satisfying

(i) $\langle \lambda x + y, z\rangle = \lambda \langle x,z\rangle + \langle y,z\rangle$;

(ii) $\langle ax, y\rangle = a\langle x,y\rangle$;

(iii) $\langle x,y\rangle^* = \langle y,x\rangle$;

(iv) $\langle x,x\rangle \geqslant 0$;

(v) $\langle x,x\rangle = 0 \Leftrightarrow x = 0$,

for all $x,y,z \in \mathscr{M}$, $a \in \mathscr{A}$, $\lambda \in \mathbb{C}$. Conditions (i) and (iii) yield the fact that the inner product is conjugate-linear with respect to the second variable. It follows from the definition that $\|x\|_{\mathscr{M}} := \sqrt{\|\langle x,x\rangle\|_{\mathscr{A}}}$ is a norm on $\mathscr{M}$, whence $\mathscr{M}$ becomes a normed left $\mathscr{A}$-module. A pre-Hilbert $\mathscr{A}$-module $\mathscr{M}$ is called a *Hilbert C^*-module* if it is complete with respect to this norm. We say that a Hilbert $\mathscr{A}$-module $\mathscr{M}$ is *full* if the linear subspace $\langle \mathscr{M},\mathscr{M}\rangle$ of $\mathscr{A}$ generated by $\{\langle x,y\rangle : x,y \in \mathscr{M}\}$ is dense in $\mathscr{A}$. The simplest examples are usual Hilbert spaces as Hilbert $\mathbb{C}$-modules, and C^*-algebras as Hilbert C^*-modules over themselves via $\langle a,b\rangle = ab^*$.

The concept of a Hilbert C^*-module has been introduced by Kaplansky [6] and Paschke [11]. For more information we refer the reader e.g. to monographs [7, 9].

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Despite a formal similarity of definitions, it is well known that Hilbert C^*-modules may lack many properties familiar from Hilbert space theory. In fact, it turns out that properties of a C^*-module reflect (or originate from) the properties of the underlying C^*-algebra.

A particularly well behaved class is the class of Hilbert C^*-modules over C^*-algebras of compact operators. There are several nice characterizations of such modules (see e.g. [1, 4, 5, 8, 13]). In our proofs we make use of orthonormal bases which exist only in Hilbert C^*-modules over C^*-algebras of compact operators (see [1, 2]). Recall that a system of vectors $\{\varepsilon_i : i \in I\}$ in a Hilbert $\mathscr{A}$-module $\mathscr{M}$ is said to be an *orthonormal basis for* $\mathscr{M}$ if it satisfies the following conditions:

1. $p_i := \langle \varepsilon_i, \varepsilon_i \rangle \in \mathscr{A}$ is a projection such that $p_i \mathscr{A} p_i = \mathbb{C} p_i$ for every $i \in I$;
2. $\langle \varepsilon_i, \varepsilon_j \rangle = 0$ for every $i, j \in I, i \neq j$;
3. $\{\varepsilon_i : i \in I\}$ generates a norm-dense submodule of $\mathscr{M}$.

If $\{\varepsilon_i : i \in I\}$ is an orthonormal basis for $\mathscr{M}$ then the reconstruction formula $x = \sum_{i \in I} \langle x, \varepsilon_i \rangle \varepsilon_i$ holds for every $x \in \mathscr{M}$, with the norm convergence. Since all orthonormal bases for a Hilbert $\mathscr{A}$-module $\mathscr{M}$ have the same cardinality (see [2]), it makes sense to define the *orthogonal dimension* of $\mathscr{M}$, denoted by $\dim_{\mathscr{A}} \mathscr{M}$, as the cardinal number of any of its orthonormal bases.

Various specific properties of Hilbert C^*-modules turn out to be particularly useful in applications. An interesting example of investigations of this type is a recent study of the stability of the Wigner equation (see [3] and the references therein). In particular, the main result in [3] is obtained for Hilbert C^*-modules satisfying the following condition:

[H] For each norm-bounded sequence (x_n) in $\mathscr{M}$, there exist a subsequence (x_{n_k}) of (x_n) and an element $x_0 \in \mathscr{M}$ such that the sequence $(\langle x_{n_k}, y \rangle)$ converges to $\langle x_0, y \rangle$ in norm for any $y \in \mathscr{M}$.

Notice that in case of a Hilbert space, condition [H] is clearly satisfied: this is simply the fact that the unit (and hence each) ball in a Hilbert space is weakly sequentially compact.

It is proved in [3, Proposition 2.1] that a Hilbert $\mathscr{A}$-module $\mathscr{M}$ satisfies condition [H] whenever the underlying C^*-algebra is finite dimensional. In this note we prove the converse, i.e., we show that condition [H] is an exclusive property of the class of Hilbert C^*-modules over finite dimensional C^*-algebras. In this way we obtain a new characterization of such modules and answer the question posed in [3] concerning condition [H].

2. The result

For a Hilbert space H we denote by $\mathbb{B}(H)$ and $\mathbb{K}(H)$ the C^*-algebras of all bounded, respectively compact operators acting on H. We begin with a proposition that reduces the discussion to the class of C^*-algebras of compact operators.

PROPOSITION 2.1. *Suppose that* $\mathscr{M}$ *is a full Hilbert* C^**-module over a* C^**-algebra* $\mathscr{A}$*, which satisfies condition* [H]. *Then* $\mathscr{A}$ *is isomorphic to a* C^**-algebra of (not necessarily all) compact operators acting on some Hilbert space.*

Proof. Let us fix $y \in \mathscr{M}$. Consider the map $T_y : \mathscr{M} \to \mathscr{M}$ given by $T_y(x) = \langle y,x\rangle y$. Obviously, T_y is a bounded anti-linear operator.

Let (x_n) be a norm-bounded sequence in $\mathscr{M}$ and let (x_{n_k}) be a subsequence of (x_n) such that, for some $x_0 \in \mathscr{M}$, $\lim_{k\to\infty} \langle x_{n_k}, y\rangle = \langle x_0, y\rangle$ for all $y \in \mathscr{M}$. Then $\lim_{k\to\infty}\langle y, x_{n_k}\rangle y = \langle y, x_0\rangle y$ for all $y \in \mathscr{M}$. This can be restated in the following way: for each norm-bounded sequence (x_n) in $\mathscr{M}$, the sequence $(T_y(x_n))$ has a convergent subsequence. Hence, T_y is a compact operator. Moreover, by the hypothesis, this is true for each $y \in \mathscr{M}$.

By [1, Proposition 1], $(4) \Rightarrow (1)$, there is a faithful representation $\pi : \mathscr{A} \to \mathbb{B}(H)$ of $\mathscr{A}$ on some Hilbert space H such that $\pi(\langle y,y\rangle) \in \mathbb{K}(H)$. This holds for every $y \in \mathscr{M}$, so, by polarization, $\pi(\langle x,y\rangle) \in \mathbb{K}(H)$ for all $x,y \in \mathscr{M}$, and therefore $\pi(\mathscr{A}) \subseteq \mathbb{K}(H)$.

By the preceding proposition, condition [H] can only be satisfied in Hilbert C^*-modules over C^*-algebras of compact operators. (Here, and in the sequel, we identify $\mathscr{A}$ with $\pi(\mathscr{A})$, where π is the representation from the preceding proof.) However, even if the underlying algebra is a C^*-algebra of compact operators, one still cannot conclude that condition [H] is satisfied.

We demonstrate this fact in the following two examples.

EXAMPLE 2.2. Consider a separable infinite dimensional Hilbert space H with an orthonormal basis (ε_n). We shall regard $\mathbb{K}(H)$ as a Hilbert C^*-module over itself via the inner product $\langle a,b\rangle = ab^*$. Let us show that $\mathbb{K}(H)$ does not satisfy [H].

For $n \in \mathbb{N}$, denote by p_n the orthogonal projection onto $\mathrm{span}\{\varepsilon_1,\dots,\varepsilon_n\}$. Obviously, the sequence (p_n) is norm-bounded.

Suppose that there exist a subsequence (p_{n_k}) and a compact operator $a \in \mathbb{K}(H)$ such that $\lim_{k\to\infty}\langle p_{n_k}, y\rangle = \langle a,y\rangle$ for all $y \in \mathbb{K}(H)$. This means $p_{n_k}y^* \to ay^*$ for all $y \in \mathbb{K}(H)$, which in turn gives us $p_{n_k}y\xi \to ay\xi$ for all $y \in \mathbb{K}(H)$ and for all $\xi \in H$. In particular, for every $n \in \mathbb{N}$, we can take $y = p_n - p_{n-1}$ (that is the orthogonal projection to the one-dimensional subspace spanned by ε_n) and $\xi = \varepsilon_n$. Then the preceding relation yields $a\varepsilon_n = \varepsilon_n$ for all $n \in \mathbb{N}$; i.e., a is the identity operator. Since $\dim H = \infty$, this is not a compact operator. Thus, the assumed property [H] leads to a contradiction.

Recall that, by [2, Example 2], $\dim_{\mathbb{K}(H)} \mathbb{K}(H) = \dim H$.

Our following example shows that even a Hilbert $\mathbb{K}(H)$-module $\mathscr{M}$ such that $\dim_{\mathbb{K}(H)} \mathscr{M} < \infty$ need not have property [H].

EXAMPLE 2.3. (cf. [2, Example 1]) Let H be a Hilbert space. For $\xi,\eta \in H$ define $\langle \xi,\eta\rangle = e_{\xi,\eta} \in \mathbb{K}(H)$, where $e_{\xi,\eta}(\nu) = (\nu|\eta)\xi$. Also, for $a \in \mathbb{K}(H)$, define a left action on $\xi \in H$ in a natural way as the action of the operator a on the vector ξ.

In this way H becomes a left Hilbert $\mathbb{K}(H)$-module. Notice that the resulting norm coincides with the original norm on H.

We also know that $\dim_{\mathbb{K}(H)} H = 1$. Indeed, if ε is an arbitrary unit vector then each $\xi \in H$ admits a representation of the form $\xi = \langle \xi, \varepsilon \rangle \varepsilon$ (because $\langle \xi, \varepsilon \rangle \varepsilon = e_{\xi,\varepsilon}(\varepsilon) = (\varepsilon|\varepsilon)\xi = \xi$). This means that $\{\varepsilon\}$ is an orthonormal basis for H, regarded as a $\mathbb{K}(H)$-module.

Notice that the entire preceding discussion was independent on the (usual) dimension of the underlying space H. Suppose now that H is a separable infinite dimensional Hilbert space. We claim that then H, as a Hilbert $\mathbb{K}(H)$-module, does not satisfy [H].

To see this, let us fix an orthonormal basis (ε_n) for H. The sequence (ε_n) is obviously norm-bounded. Suppose that there exist a subsequence (ε_{n_k}) of (ε_n) and $\varepsilon_0 \in H$ such that $\lim_{k\to\infty}\langle \varepsilon_{n_k}, \xi \rangle = \langle \varepsilon_0, \xi \rangle$ for all $\xi \in H$. In particular, this would imply $\lim_{k\to\infty}\langle \varepsilon_{n_k}, \varepsilon_1 \rangle = \langle \varepsilon_0, \varepsilon_1 \rangle$, i.e., $\|e_{\varepsilon_{n_k},\varepsilon_1} - e_{\varepsilon_0,\varepsilon_1}\| \to 0$. But, $\|e_{\varepsilon_{n_k},\varepsilon_1} - e_{\varepsilon_0,\varepsilon_1}\| = \sup_{\|\eta\|=1}\|e_{\varepsilon_{n_k},\varepsilon_1}(\eta) - e_{\varepsilon_0,\varepsilon_1}(\eta)\| = \sup_{\|\eta\|=1}\|(\eta|\varepsilon_1)(\varepsilon_{n_k} - \varepsilon_0)\| = \|\varepsilon_{n_k} - \varepsilon_0\|$ and the last expression obviously does not converge to 0 as $k \to \infty$.

REMARK 2.4. Suppose that $\mathscr{M}$ is an arbitrary Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ of compact operators. It is well known that there is a family $(H_j), j \in J$, of Hilbert spaces such that $\mathscr{A} = \bigoplus_{j\in J}\mathbb{K}(H_j)$. Furthermore, it then follows that $\mathscr{M} = \bigoplus_{j\in J}\mathscr{M}_j$, where $\mathscr{M}_j = \overline{\mathbb{K}(H_j)\mathscr{M}}$ (i.e., $\mathscr{M}$ is an outer direct sum of $\mathscr{M}_j$'s, where each $\mathscr{M}_j$ is a full Hilbert $\mathbb{K}(H_j)$-module).

Now, by [2, Theorem 3] and the preceding example, we conclude that if there exists $j_0 \in J$ such that $\dim H_{j_0} = \infty$ then $\mathscr{M}_{j_0}$ cannot satisfy [H]. Consequently, $\mathscr{M}$ does not satisfy [H]. Namely, if $\dim \mathscr{M}_{j_0} = d$ (here d can be an arbitrary cardinal number), then, by Theorem 3 from [2], $\mathscr{M}_{j_0}$ is an orthogonal sum of d copies of ${}_{\mathbb{K}(H_{j_0})}H_{j_0}$, and, by Example 2.3, just one copy of ${}_{\mathbb{K}(H_{j_0})}H_{j_0}$ is enough to ruin property [H].

From the preceding discussion we conclude that if $\mathscr{M}$ is a full Hilbert C^*-module satisfying [H], then $\mathscr{M}$ is necessarily a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ of compact operators. Moreover, $\mathscr{A}$ has to be of the form $\mathscr{A} = \bigoplus_{j\in J}\mathbb{K}(H_j)$ and each H_j must be finite dimensional. If, moreover, J is of finite cardinality, then $\mathscr{A}$ is finite dimensional. Next we show that if $\mathrm{card}J = \infty$ with $\dim H_j < \infty$ for all $j \in J$, then again $\mathscr{M}$ cannot satisfy [H].

First, in this situation, since J as a set of infinite cardinality contains a countable subset J', $\mathscr{M} = \bigoplus_{j\in J}\mathscr{M}_j$ can be written as the orthogonal sum of the form $\mathscr{M} = \left(\bigoplus_{j\in J'}\mathscr{M}_j\right) \oplus \left(\bigoplus_{j\in J\setminus J'}\mathscr{M}_j\right)$. Thus, $\mathscr{M}$ contains, as an orthogonal summand, a submodule of the form $\mathscr{M}' = \bigoplus_{n\in\mathbb{N}}\mathscr{M}_n$, where each $\mathscr{M}_n$ is a module over $\mathbb{K}(H_n)$ and $\dim H_n < \infty$. Moreover, each $\mathscr{M}_n$ is, by [2, Theorem 3], unitarily equivalent to the orthogonal sum of $d_n = \dim \mathscr{M}_n$ copies of ${}_{\mathbb{K}(H_n)}H_n$, i.e., $\mathscr{M}_n \simeq \bigoplus_1^{d_n}{}_{\mathbb{K}(H_n)}H_n$.

If we take just one copy of each ${}_{\mathbb{K}(H_n)}H_n$, we conclude that $\mathscr{M}'$ (and hence $\mathscr{M}$) contains, as an orthogonal summand, a submodule of the form $\mathscr{M}'' \simeq \bigoplus_{n=1}^{\infty}{}_{\mathbb{K}(H_n)}H_n$. It is now enough to prove that $\mathscr{M}''$ does not satisfy [H] and this can be argued essentially in the same way as in Example 2.3.

Observe that $\mathscr{M}''$ is also a Hilbert C^*-module over a direct sum $\bigoplus_{n=1}^{\infty}\mathbb{K}(H_n) \subset \mathbb{K}(H)$, where $H = \bigoplus_{n=1}^{\infty}H_n$ is an infinite dimensional Hilbert space. For each $n \in \mathbb{N}$

take a unit vector $\varepsilon_n \in H_n \subset H$. Let $x_n = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, 0, \ldots)$, $n \in \mathbb{N}$. Notice that $\langle x_n, x_n \rangle = \sum_{i=1}^{n} e_{\varepsilon_i, \varepsilon_i}$. Since this is an orthogonal projection onto an n-dimensional subspace of H, we have $\|x_n\| = 1$; thus, (x_n) is a norm-bounded sequence in $\mathscr{M}''$. Suppose now that there exists a subsequence (x_{n_k}) and $x_0 = (\xi_1, \xi_2, \ldots) \in \mathscr{M}''$ such that $\lim_{k\to\infty} \langle x_{n_k}, y \rangle = \langle x_0, y \rangle$ for all $y \in \mathscr{M}''$. Inserting $y = (\varepsilon_1 - \xi_1, 0, 0, \ldots)$ we obtain $\|\langle x_{n_k}, y \rangle - \langle x_0, y \rangle\| = \|\langle \varepsilon_1 - \xi_1, \varepsilon_1 - \xi_1 \rangle\| \to 0$, which implies $\xi_1 = \varepsilon_1$. Similarly, for $y = (0, \varepsilon_2 - \xi_2, 0, \ldots)$ we obtain $\xi_2 = \varepsilon_2$ and, proceeding in the same way, $\xi_n = \varepsilon_n$ for all $n \in \mathbb{N}$. This gives us $x_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$, which is impossible since this sequence does not belong to $\mathscr{M}''$.

After all, combining the preceding discussion with Proposition 2.1 from [3], we get our main result.

THEOREM 2.5. *A full Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ satisfies condition* [H] *if and only if $\mathscr{A}$ is a finite dimensional C^*-algebra.*

REMARK 2.6. We may ask ourselves if one could replace condition [H] with a weaker one:

[H'] For each norm-bounded sequence (x_n) in $\mathscr{M}$ and for every $y \in \mathscr{M}$ there exists a subsequence (x_{n_k}) of (x_n) such that the sequence $(\langle x_{n_k}, y \rangle)$ converges in norm.

Observe that [H'] is sufficient to prove Proposition 2.1, so full Hilbert C^*-modules with property [H'] have to be over C^*-algebras of compact operators. Also, it is obvious that [H'] is fulfilled in every Hilbert C^*-module over a finite dimensional C^*-algebra. However, our next example shows that [H'] does not characterize these Hilbert modules; in other words, [H'] is not sufficient for [H].

Consider a separable infinite dimensional Hilbert space H and the C^*-algebra $\mathscr{A} \subset \mathbb{K}(H)$ of all diagonal (with respect to a fixed orthonormal basis) operators with diagonal entries converging to 0. Let $\mathscr{M} = \mathscr{A}$. Then $\mathscr{A}$ is a Hilbert C^*-module whose underlying C^*-algebra $\mathscr{A}$ is infinite dimensional. By the preceding theorem, the Hilbert C^*-module $\mathscr{A}$ cannot satisfy [H].

On the other hand, since $\mathscr{A}$ is a Hilbert C^*-module over the (commutative) C^*-algebra $\mathscr{A}$ of compact operators, by [1, Theorem 4] (see also its proof), all mappings $T_y : \mathscr{A} \to \mathscr{A}$ given by $T_y(x) = \langle y, x \rangle y$ are compact. But here we have $T_y(x) = yx^*y = x^*y^2$ for all $y \in \mathscr{A}$. In particular, taking self-adjoint y we get that $x \mapsto x^*y$ is compact for every positive $y \in \mathscr{A}$, and since positive elements of a C^*-algebra span the whole C^*-algebra, we get that the operator $x \mapsto x^*y = \langle y, x \rangle$ is compact for every $y \in \mathscr{A}$. This shows that our Hilbert C^*-module $\mathscr{A}$ satisfies [H'].

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Ljiljana Arambašić
Department of Mathematics
University of Zagreb
Bijenička cesta 30
10000 Zagreb
Croatia
e-mail: arambas@math.hr

Damir Bakić
Department of Mathematics
University of Zagreb
Bijenička cesta 30
10000 Zagreb
Croatia
e-mail: bakic@math.hr

Mohammad Sal Moslehian
Department of Pure Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159
Mashhad 91775
Iran
Center of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad Iran
e-mail: moslehian@ferdowsi.um.ac.ir
moslehian@ams.org