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HYPERINVARIANT, CHARACTERISTIC AND MARKED SUBSPACES

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Abstract. Let V be a finite dimensional vector space over a field K and f a K-endomorphism of V. In this paper we study three types of f-invariant subspaces, namely hyperinvariant subspaces, which are invariant under all endomorphisms of V that commute with f, characteristic subspaces, which remain fixed under all automorphisms of V that commute with f, and marked subspaces, which have a Jordan basis (with respect to $f_{|X}$) that can be extended to a Jordan basis of V. We show that a subspace is hyperinvariant if and only if it is characteristic and marked. If K has more than two elements then each characteristic subspace is hyperinvariant.

1. Introduction

Let V be an n-dimensional vector space over a field K and let $f : V \to V$ be K-linear. We assume that the characteristic polynomial of f splits over K such that all eigenvalues of f are in K. In this paper we deal with three types of f-invariant subspaces, namely with hyperinvariant, characteristic and marked subspaces. To describe these three concepts we use the following notation. Let $\mathrm{Inv}(V)$ be the lattice of f-invariant subspaces of V and let $\mathrm{End}_f(V)$ be the algebra of all endomorphisms of V that commute with f. If a subspace X remains invariant for all $g \in \mathrm{End}_f(V)$ then X is called *hyperinvariant* for f [13, p. 305]. Let $\mathrm{Hinv(V)}$ be the set of hyperinvariant subspaces of V. It is obvious that $\mathrm{Hinv(V)}$ is a lattice. Because of $f \in \mathrm{End}_f(V)$ we have $\mathrm{Hinv(V)} \subseteq \mathrm{Inv}(V)$. We refer to [13], [9], [17], [19] for results on hyperinvariant subspaces. The group of automorphisms of V that commute with f will be denoted by $\mathrm{Aut}_f(V)$. A subspace X of V will be called *characteristic* (with respect to f) if $X \in \mathrm{Inv}(V)$ and $\alpha(X) = X$ for all $\alpha \in \mathrm{Aut}_f(V)$. Let $\mathrm{Chinv(V)}$ be set of characteristic subspaces of V. Obviously, also $\mathrm{Chinv(V)}$ is a lattice, and $\mathrm{Hinv(V)} \subseteq \mathrm{Chinv(V)}$.

Set $\iota = \mathrm{id}_V$ and $f^0 = \iota$. Let $\langle x \rangle_f = \mathrm{span}\{f^i x, i \geqslant 0\}$ be the cyclic subspace generated by $x \in V$. If $B \subseteq V$ we define $\langle B \rangle_f = \sum_{b \in B} \langle b \rangle_f$. Let λ be an eigenvalue of f such that $V_\lambda = \mathrm{Ker}(f - \lambda\iota)^n$ is the corresponding generalized eigenspace. Let $\dim \mathrm{Ker}(f - \lambda\iota) = k$, and let $s^{t_1}, \dots, s^{t_k}$, be the elementary divisors of $f_{|V_\lambda}$. Then there exist vectors $u_1, \dots, u_k$, such that

$$V_\lambda = \langle u_1 \rangle_{f-\lambda\iota} \oplus \cdots \oplus \langle u_k \rangle_{f-\lambda\iota},$$

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and $(f-\lambda\iota)^{t_i-1}u_i \neq 0$, $(f-\lambda\iota)^{t_i}u_i = 0$, $i=1,\dots,k$. We call $U_\lambda = \{u_1,\dots,u_k\}$ a set of *generators* of V_λ. Each U_λ gives rise to a Jordan basis of V_λ, namely

$$\{u_1, (f-\lambda\iota)u_1, \dots, (f-\lambda\iota)^{t_1-1}u_1 \dots, u_k, (f-\lambda\iota)u_k, \dots, (f-\lambda\iota)^{t_k-1}u_k\}.$$

Define $f_\lambda = f_{|V_\lambda}$. Let Y be an f_λ-invariant subspace of V_λ. Then Y is said to be *marked* in V_λ (with respect to f_λ) if there exists a set U_λ of generators of V_λ and corresponding integers r_i, $0 \leqslant r_i \leqslant t_i$, such that

$$Y = \left\langle (f-\lambda\iota)^{r_1} u_1 \right\rangle_{f-\lambda\iota} \oplus \cdots \oplus \left\langle (f-\lambda\iota)^{r_k} u_k \right\rangle_{f-\lambda\iota}.$$

Thus Y has a Jordan basis which can be extended to a Jordan basis of V_λ. Let $\sigma(f) = \{\lambda_1,\dots,\lambda_m\}$ be the spectrum of f. Then

$$V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_m}. \tag{1.1}$$

If $X \in \operatorname{Inv}_f V$ then $X_{\lambda_i} = X \cap V_{\lambda_i}$ is f_{λ_i}-invariant in V_{λ_i}, and

$$X = X_{\lambda_1} \oplus \dots \oplus X_{\lambda_m}. \tag{1.2}$$

We say that X is marked in V if each subspace X_{λ_i} in (1.2) is marked in V_{λ_i}. The set of marked subspaces of V will be denoted by $\operatorname{Mark(V)}$. We assume $0 \in \operatorname{Mark(V)}$. Marked subspaces can be traced back to [13, p. 83]. They have been studied in [4], [8], [1], and [6]. For marked (A,C)-invariant subspaces we refer to [5] and [7]. We mention applications to algebraic Riccati equations [2] and to stability of invariant subspaces of commuting matrices [15].

The following examples show that to a certain extent the three types of invariant subspaces are independent of each other. Suppose f is nilpotent. If $x \in V$ then the smallest nonnegative integer ℓ with $f^\ell x = 0$ is called the *exponent* of x. We write $\mathrm{e}(x) = \ell$. A nonzero vector x is said to have *height* q if $x \in f^q V$ and $x \notin f^{q+1}V$. In this case we write $\mathrm{h}(x) = q$. We set $\mathrm{h}(0) = -\infty$. For $j \geqslant 0$ we define $V[f^j] = \operatorname{Ker} f^j$.

EXAMPLE 1.1. Let $K = \mathbb{Z}_2$. Consider $V = K^4$ and

$$f = \operatorname{diag}(0, N_3),\ N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $e_1,\dots,e_4$, be the unit vectors of K^4. Then $f^3 = 0$ and $V = \langle e_1\rangle_f \oplus \langle e_2\rangle_f$. Define $z = e_1 + e_3$ and $Z = \langle z\rangle_f$. Then

$$Z = \{0, z, z+e_4, e_4\} = \left\langle v;\ \mathrm{e}(v) = 2, \mathrm{h}(v) = 0, \mathrm{h}(fv) = 2 \right\rangle_f.$$

If $\alpha \in \operatorname{Aut}_f(V)$ then $|\alpha(Z)| = |Z|$. Moreover α preserves height and exponent. Hence $\alpha(Z) = Z$, and Z is characteristic. Let $g = \operatorname{diag}(1,0,0,0)$ be the orthogonal projection on Ke_1. Then $g \in \operatorname{End}_f(V)$. We have $gz = e_1 \in g(Z)$, but $e_1 \notin Z$. Therefore Z is not hyperinvariant. The Jordan bases of Z are $J_1 = \{z, e_4\}$ and $J_2 = \{z+e_4, e_4\}$. If $y \in K^4$ then $z \neq fy$ and $z + e_4 \neq fy$. Hence neither J_1 nor J_2 can be extended to a Jordan basis of K^4. Therefore Z is not marked.

EXAMPLE 1.2. Let $V = K^2$ and $f = 0$. Then $K^2 = \langle e_1\rangle_f \oplus \langle e_2\rangle_f$ and the subspace $X = \langle e_1\rangle_f$ is marked. From $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{Aut}_f(V)$ and $\alpha(e_1) = e_1 + e_2$ follows that X is not characteristic.

In contrast to $\mathrm{Hinv}(V)$ or $\mathrm{Chinv}(V)$ the set $\mathrm{Mark}(V)$ in general is not a lattice.

EXAMPLE 1.3. $V = K^6$, $f = \mathrm{diag}(0, N_3, N_2)$. The subspaces $Z_1 = \langle e_5\rangle$ and $Z_2 = \langle e_5 + e_3 + e_1\rangle$ are marked but $Z_1 + Z_2 = \langle e_5\rangle \oplus \langle e_3 + e_1\rangle$ is not marked. Thus the set of marked subspaces is not closed under addition.

In this paper we study the following problems. Under what conditions is a marked subspace characteristic? When is each characteristic subspace hyperinvariant? Because of the Lemma 1.4 below one can deal separately with single components V_{λ_i} in (1.1) and the corresponding restrictions $f_{\lambda_i} = f_{|V_{\lambda_i}}$, $i = 1, \dots, m$.

LEMMA 1.4. *An f-invariant subspace $X \subseteq V$ is hyperinvariant (resp. characteristic, resp. marked) if and only if, with respect to f_{λ_i}, each component X_{λ_i} in* (1.2) *is hyperinvariant (resp. characteristic, resp. marked) in V_{λ_i}.*

Proof. If $\eta \in \mathrm{End}_f(V)$ then it is known ([11, p. 223]) that the subspaces V_{λ_i} in (1.1) are invariant under η, and that $\eta_{|V_{\lambda_i}} \in \mathrm{End}_{f_{\lambda_i}}(V_{\lambda_i})$. Hence, if $X \in \mathrm{Inv}(V)$ then (1.2) implies

$$\eta(X) = \eta_{|V_{\lambda_1}}(X_{\lambda_1}) \oplus \cdots \oplus \eta_{|V_{\lambda_i}}(X_{\lambda_m})$$

Hence if $X_{\lambda_i} \in \mathrm{Hinv}(V_{\lambda_i})$, resp. $X_{\lambda_i} \in \mathrm{Chinv}(V_{\lambda_i})$, $i = 1, \dots, m$, then $X \in \mathrm{Hinv}(V)$, resp. $X \in \mathrm{Chinv}(V)$.

Now suppose now that X is hyperinvariant. Let us show that $X_{\lambda_i} \in \mathrm{Hinv}(V_{\lambda_i})$, $i = 1, \dots, m$. Take $i = 1$. Set $\hat{V} = V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}$ and $\hat{X} = X_{\lambda_2} \oplus \cdots \oplus X_{\lambda_m}$. Let $\beta_1 \in \mathrm{End}_{f_{\lambda_1}}(V_{\lambda_1})$. Define $\beta = \beta_1 + \mathrm{id}_{|\hat{V}}$. Then $\beta \in \mathrm{End}_f(V)$. Hence $\beta(X) \subseteq X = X_{\lambda_1} \oplus \hat{X}$, and $\beta(X) = \beta_1(X_{\lambda_1}) \oplus \hat{X}$. From $X_{\lambda_1} \subseteq V_{\lambda_1}$ and $\beta_1(X_{\lambda_1}) \subseteq V_{\lambda_1}$ we obtain $\beta_1(X_{\lambda_1}) \subseteq X_{\lambda_1}$. Therefore $X_{\lambda_1} \in \mathrm{Hinv}(V_{\lambda_1})$. A similar argument shows that $X \in \mathrm{Chinv}(V)$ implies $X_{\lambda_i} \in \mathrm{Chinv}(V_{\lambda_i})$, $i = 1, \dots, m$. In the case of marked subspaces the assertion is obvious. □

2. Auxiliary results

Because of Lemma 1.4 it suffices to consider an endomorphism f with only one eigenvalue λ. We shall assume $\sigma(f) = \{0\}$ such that $f^n = 0$. Let

$$s^{t_1}, \dots, s^{t_k}, \ 0 < t_1 \leqslant \cdots \leqslant t_k, \tag{2.1}$$

be the elementary divisors of f. We call $U = (u_1, \dots, u_k)$ a *generator tuple* of V if

$$V = \langle u_1\rangle_f \oplus \cdots \oplus \langle u_k\rangle_f \tag{2.2}$$

and if U is ordered according to (2.1) such that

$$\mathrm{e}(u_1) = t_1 \leqslant \cdots \leqslant \mathrm{e}(u_k) = t_k.$$

Let $\mathscr{U}$ be the set of generator tuples of V. In the following we omit the subscript f in (2.2) and we write $\langle u_i \rangle$ instead of $\langle u_i \rangle_f$. We say that a k-tuple $r = (r_1, \ldots, r_k)$ of integers is *admissible* if

$$0 \leqslant r_i \leqslant t_i,\ i = 1, \ldots, k. \tag{2.3}$$

Each $U \in \mathscr{U}$ together with an admissible tuple r gives rise to a subspace

$$W(r,U) = \langle f^{r_1} u_1 \rangle \oplus \cdots \oplus \langle f^{r_k} u_k \rangle, \tag{2.4}$$

which is marked in V. Conversely, a subspace W is marked in V only if $W = W(r,U)$ for some $U \in \mathscr{U}$ and some admissible r. The following example shows that, in general, $W(r,U) \neq W(r,\tilde{U})$ if $U \neq \tilde{U}$.

EXAMPLE 2.1. Let $V = K^5$ and $f = \mathrm{diag}(N_2, N_3)$. Then $V = \langle e_1 \rangle \oplus \langle e_3 \rangle$, and $U = (e_1, e_3)$ and $\tilde{U} = (e_1, e_3 + e_1)$ are generator tuples. Choose $r = (1,0)$. Then the corresponding subspaces $W(r,U) = \langle e_2 \rangle \oplus \langle e_3 \rangle$ and $W(r,\tilde{U}) = \langle e_2 \rangle \oplus \langle e_3 + e_1 \rangle$ are different from each other.

The construction of invariant subspaces of the form $W(r,U)$ is a standard procedure in linear algebra and systems theory. It is used in [16], [12, p.61], [3, p.28], [18]. Hence it is important to know whether for a given r different choices of U will always result in the same subspace. Theorem 3.1 will provide a necessary and sufficient condition for r such that $W(r,U)$ is independent of the choice of U. Let r be admissible and define

$$W(r) = f^{r_1} V \cap V[f^{t_1 - r_1}] + \cdots + f^{r_k} V \cap V[f^{t_k - r_k}]. \tag{2.5}$$

Subspaces of the form $f^{\nu} V$ and $V[f^{\mu}]$ are hyperinvariant, and $\mathrm{Hinv}(V)$ is a lattice. Therefore (see e.g. [9]) we have $W(r) \in \mathrm{Hinv}(V)$.

The following lemma shows that each $\alpha \in \mathrm{Aut}_f(V)$ is uniquely determined by the image of a given generator tuple.

LEMMA 2.2. *Let* $U = (u_1, \ldots, u_k) \in \mathscr{U}$ *be given. For* $\alpha \in \mathrm{Aut}_f(V)$ *define* $\Theta_U(\alpha) = \big(\alpha(u_1), \ldots, \alpha(u_k)\big)$. (i) *Then*

$$\alpha \mapsto \Theta_U(\alpha),\ \Theta_U : \mathrm{Aut}_f(V) \to \mathscr{U},$$

is a bijection. (ii) *If* $\tilde{U} = \Theta(\alpha)$ *then* $W(r,\tilde{U}) = \alpha\big(W(r,U)\big)$.

Proof. (i) It is easy to see that $\Theta_U(\alpha) \in \mathscr{U}$. Hence Θ_U maps $\mathrm{Aut}_f(V)$ into $\mathscr{U}$. Let $x \in V$ and

$$x = \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=0}^{\mathrm{e}(u_i)-1} c_{ij} f^j u_i. \tag{2.6}$$

Suppose $\alpha, \beta \in \mathrm{Aut}_f(V)$ and $\Theta_U(\alpha) = \Theta_U(\beta) = (\hat{u}_1, \ldots, \hat{u}_k)$. Then

$$\alpha(x) = \sum \sum c_{ij} f^j \hat{u}_i = \beta(x).$$

Hence $\alpha=\beta$, and Θ_U is injective. Now consider $\tilde{U}=(\tilde{u}_1,\dots,\tilde{u}_k)\in\mathscr{U}$. Let $x\in V$ be the vector in (2.6). Define $\gamma: x\mapsto \sum_i\sum_j c_{ij}f^j\tilde{u}_i$. Then $\gamma\in \mathrm{Aut}_f(V)$ and $\tilde{U}=\Theta_U(\gamma)$. Hence Θ_U is surjective.
(ii) It is obvious that $\alpha\big(W(r,U)\big)=\langle f^{r_1}\alpha(u_1)\rangle_f\oplus\cdots\oplus\langle f^{r_k}\alpha(u_k)\rangle_f=W(r,\tilde{U})$. $\square$

In group theory fully invariant subgroups play the role of hyperinvariant subspaces. Hence the decomposition (2.8) below is an analog to a distributive law in Lemma 9.3 in [10, p. 47].

LEMMA 2.3. *Suppose*

$$V=V_1\oplus\cdots\oplus V_q,\ V_i\in \mathrm{Inv}(V),\ i=1,\dots,q. \tag{2.7}$$

(i) *If X is a hyperinvariant subspace of V, or*
(ii) *if X characteristic and $|K|>2$, then*

$$X=(X\cap V_1)\oplus\cdots\oplus(X\cap V_q). \tag{2.8}$$

Proof. If $x\in V$ then $x=\sum_{i=1}^q x_i,\ x_i\in V_i$. Set $X_i=X\cap V_i$, and $S=\oplus_{i=1}^q X_i$. Then $S\subseteq X$. To prove the converse inclusion we note that

$$fx=\sum_{i=1}^q f_{|V_i}(x_i). \tag{2.9}$$

(i) Let π_i be the projection on V_i induced by (2.7). Then (2.9) implies $\pi_i\in \mathrm{End}_f(V)$. Hence, if $x\in X$ then and $\pi_i(x)=x_i\in X$. Thus $x_i\in X_i$, and therefore $X\subseteq S$.
(ii) Let $a\in K$ be different from 0 and 1, and define $\gamma_i=\iota-a\pi_i$. Then $\gamma_i\in \mathrm{Aut}_f(V)$. Hence $\gamma_i(x)=x-ax_i\in X$ if $x\in X$. Thus we obtain $x_i\in X_i$. $\square$

EXAMPLE 2.4. In Lemma 2.3 (ii) one can not drop the assumption $|K|>2$. Suppose $|K|=2$, and let V and f be as in Example 1.1. The subspace $Z=\langle e_1+e_3\rangle$ is characteristic. Both $V_1=\langle e_1\rangle$ and $V_2=\langle e_2\rangle$ are in $\mathrm{Inv}(V)$, and we have $V=V_1\oplus V_2$. But $Z\cap V_1=0$ and $Z\cap V_2=\langle e_4\rangle$ imply $Z\supsetneqq(Z\cap V_1)\oplus(Z\cap V_2)$.

The next lemma is an intermediate result.

LEMMA 2.5. *Each hyperinvariant subspace of V is marked, and*

$$\mathrm{Hinv}(V)\subseteq \mathrm{Mark}(V)\cap \mathrm{Chin}(V). \tag{2.10}$$

Proof. Let $U=(u_1,\dots,u_k)\in\mathscr{U}$. If X is invariant then $X\cap\langle u_i\rangle=\langle f^{r_i}u_i\rangle$ for some r_i. Thus, if X is hyperinvariant then (2.8) in Lemma 2.3 implies $X=\oplus_{i=1}^k\langle f^{r_i}u_i\rangle$. Therefore X is marked, and $\mathrm{Hinv}(V)\subseteq \mathrm{Chin}(V)$ yields the inclusion (2.10). $\square$

3. Hyperinvariant = characteristic + marked

We now characterize those marked subspaces which are characteristic. The theorem below includes results from [2] with new proofs.

THEOREM 3.1. *Let* $U \in \mathscr{U}$ *and let* $r = (r_1, \dots, r_k)$ *be admissible. Then the following statements are equivalent.*

(i) *The subspace* $W(r,U)$ *is characteristic.*

(ii) *The subspace* $W(r,U)$ *is independent of the generator tuple* U*, i.e.*

$$W(r,U) = W(r,\tilde{U}) \quad \textit{for all} \quad \tilde{U} \in \mathscr{U}. \tag{3.1}$$

(iii) *The tuples* $t = (t_1, \dots, t_k)$ *and* $r = (r_1, \dots, r_k)$ *satisfy*

$$r_1 \leqslant \cdots \leqslant r_k \tag{3.2}$$

and

$$t_1 - r_1 \leqslant \cdots \leqslant t_k - r_k. \tag{3.3}$$

(iv) *We have* $W(r,U) = W(r)$.

(v) $W(r,U)$ *is the unique marked subspace* W *such that the elementary divisors of* W *and of* V/W *are*

$$s^{t_1 - r_1}, \dots, s^{t_k - r_k}, \quad \text{and} \quad s^{r_1}, \dots, s^{r_k}. \tag{3.4}$$

(vi) *The subspace* $W(r,U)$ *is hyperinvariant.*

Proof. (i) $\Leftrightarrow$ (ii) It follows from Lemma 2.2 that the two statements are equivalent.
(iv) $\Rightarrow$ (vi) This follows from the fact that $W(r)$ is hyperinvariant.
(v) $\Leftrightarrow$ (ii) Let $\tilde{U} \in \mathscr{U}$. Then $W(r,U)$ and the quotient space $V/W(r,U)$, and also $W(r,\tilde{U})$ and $V/W(r,\tilde{U})$, have elementary divisors given by (3.4). (Note that in the right-hand side of (2.4) there may be summands of the form $\langle u_i \rangle$ or $\langle f^{t_i} u_i \rangle = 0$. Thus (3.4) may contain trivial entries of the form $s^0 = 1$.)
(vi) $\Rightarrow$ (i) Obvious, because of $\mathrm{Hinv(V)} \subseteq \mathrm{Chinv(V)}$.
(iii) $\Rightarrow$ (iv) From $\mathrm{e}(u_i) = t_i$ follows

$$\langle f^{r_i} u_i \rangle = \langle u_i \rangle [f^{t_i - r_i}] \subseteq f^{r_i} V \cap V[f^{t_i - r_i}].$$

Hence $W(r,U) \subseteq W(r)$. We have to show that the conditions (3.2) and (3.3) imply the converse inclusion

$$W(r) = f^{r_1} V \cap V[f^{t_1 - r_1}] + \cdots + f^{r_k} V \cap V[f^{t_k - r_k}] \subseteq W(r,U).$$

With regard to the decomposition $V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle$ we define

$$D(\mu, \nu) = f^{r_\nu} \langle u_\mu \rangle \cap \langle u_\mu \rangle [f^{t_\nu - r_\nu}].$$

The subspaces $f^{r_\nu}V \cap V[f^{t_\nu - r_\nu}]$ are hyperinvariant. Therefore Lemma 2.3(i) yields

$$f^{r_\nu}V \cap V[f^{t_\nu - r_\nu}] = \bigoplus_{\mu=1}^{k} \left(f^{r_\nu}V \cap V[f^{t_\nu - r_\nu}] \cap \langle u_\mu \rangle\right) = \bigoplus_{\mu=1}^{k} D(\mu,\nu).$$

Hence

$$W(r) = \sum_{\mu,\nu=1}^{k} D(\mu,\nu). \tag{3.5}$$

Set $q(\mu,\nu) = \max\{r_\nu, t_\mu - (t_\nu - r_\nu)\}$. We have

$$\langle u_\mu \rangle [f^{t_\nu - r_\nu}] = \begin{cases} \langle u_\mu \rangle, & \text{if } t_\nu - r_\nu \geqslant t_\mu, \\ f^{t_\mu - (t_\nu - r_\nu)} \langle u_\mu \rangle, & \text{if } t_\nu - r_\nu \leqslant t_\mu. \end{cases}$$

Hence

$$D(\mu,\nu) = f^{q(\mu,\nu)} \langle u_\mu \rangle.$$

Let us show that $r_\mu \leqslant q(\mu,\nu)$ for all μ. If $\mu \geqslant \nu$, then (3.3) implies

$$q(\mu,\nu) = (t_\mu - t_\nu) + r_\nu = (t_\mu - r_\mu) - (t_\nu - r_\nu) + r_\mu \geqslant r_\mu.$$

If $\mu \leqslant \nu$ then $t_\mu - t_\nu \leqslant 0$, and therefore $q(\mu,\nu) = r_\nu$. Hence (3.2) implies $q(\mu,\nu) \geqslant r_\mu$. It follows that

$$D(\mu,\nu) = f^{q(\mu,\nu)} \langle u_\mu \rangle \subseteq f^{r_\nu} \langle u_\mu \rangle \subseteq W(r,U).$$

for all μ,ν. Thus (3.5) yields $W(r) \subseteq W(r,U)$.
(ii) $\Rightarrow$ (iii) We modify the entries of U and replace u_k by $\tilde{u}_k = u_{k-1} + u_k$. Then $\tilde{U} = (u_1,\dots,u_{k-1},\tilde{u}_k) \in \mathscr{U}$. Set $Y_k = \oplus_{i=1}^{k-1} \langle f^{r_i} u_i \rangle$. Then $W(r,U) = W(r,\tilde{U})$ implies

$$Y_k \oplus \langle f^{r_k} u_k \rangle = Y_k \oplus \langle f^{r_k}(u_{k-1} + u_k) \rangle.$$

From

$$f^{r_k} u_{k-1} + f^{r_k} u_k \ \in\ \langle f^{r_{k-1}} u_{k-1} \rangle \oplus \langle f^{r_k} u_k \rangle$$

follows $r_{k-1} \leqslant r_k$. Proceeding in this manner we obtain the chain of inequalities in (3.2). In order to prove (3.3) we start with the entry of u_1 of $\mathscr{U}$ and replace it by $u_1 + f^{t_2 - t_1} u_2$. Because of $\mathrm{e}(u_1 + f^{t_2 - t_1} u_2) = \mathrm{e}(u_1)$ we have $\hat{U} = (u_1 + f^{t_2 - t_1} u_2, u_2, \dots, u_k) \in \mathscr{U}$. Set $Y_1 = \oplus_{i=2}^{k} \langle f^{r_i} u_i \rangle$. Then $W(r,U) = W(r,\hat{U})$ implies

$$\langle f^{r_1} u_1 \rangle \oplus Y_1 = \langle f^{r_1}(u_1 + f^{t_2 - t_1} u_2) \rangle \oplus Y_1.$$

From

$$f^{r_1} u_1 + f^{r_1 + (t_2 - t_1)} u_2 \ \in\ \langle f^{r_1} u_1 \rangle \oplus \langle f^{r_2} u_2 \rangle$$

follows $r_2 \leqslant r_1 + (t_2 - t_1)$, i.e. $t_1 - r_1 \leqslant t_2 - r_2$, such that we the end up with (3.3). □

Let $[k]$ denote the greatest integer less than or equal to k. If $c \in \mathbb{R}$ and $0 < c < 1$, then $r = ([ct_1],\dots,[ct_m])$ is admissible, and it is not difficult to verify that r satisfies (3.2) and (3.3). We remark that admissible tuples of the form $\hat{r} = ([\frac{1}{2}t_1],\dots,[\frac{1}{2}t_k])$ play a role in the study of maximal invariant neutral subspaces [18]. It follows from Theorem 3.1 that the construction of such subspaces is independent of the choice of the underlying Jordan basis.

THEOREM 3.2. (i) *We have*

$$\operatorname{Hinv}(V) = \operatorname{Chinv}(V) \cap \operatorname{Mark}(V). \tag{3.6}$$

(ii) [9] *A subspace W of V is hyperinvariant if and only if $W = W(r)$ for some r satisfying* (3.2) *and* (3.3)*.*

Proof. (i) From Theorem 3.1 follows $\operatorname{Mark}(V) \cap \operatorname{Chinv}(V) \subseteq \operatorname{Hinv}(V)$. The reverse inclusion is (2.10) in Lemma 2.5. This yields (3.6). Hence a subspace is hyperinvariant if and only if it is both characteristic and marked.
(ii) If W is hyperinvariant then W is marked, that is $W = W(r,U)$. Therefore we can apply Theorem 3.1(iv). It was noted earlier that $W(r) \in \operatorname{Hinv}(V)$. □

We note that hyperinvariant subspaces can be characterized completely by the distributive law in Lemma 2.3.

THEOREM 3.3. *A subspace $X \in \operatorname{Inv}(V)$ is hyperinvariant if and only if X satisfies*

$$X = (X \cap V_1) \oplus \cdots \oplus (X \cap V_q) \tag{3.7}$$

when

$$V = V_1 \oplus \cdots \oplus V_q,\ V_i \in \operatorname{Inv}(V), i = 1, \ldots, q. \tag{3.8}$$

Proof. Because of Lemma 2.3 it remains to prove sufficiency. Let $U = (u_1, \ldots, u_k) \in \mathscr{U}$ and $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_k) \in \mathscr{U}$. Then

$$V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle = \langle \tilde{u}_1 \rangle \oplus \cdots \oplus \langle \tilde{u}_k \rangle. \tag{3.9}$$

Define $X_i = \langle u_i \rangle \cap X$ and $\tilde{X}_i = \langle \tilde{u}_i \rangle \cap X$, $i = 1, \ldots, k$. Then $X_i = \langle f^{r_i} u_i \rangle$ and $\tilde{X}_i = \langle f^{\tilde{r}_i} \tilde{u}_i \rangle$ for some $r_i, \tilde{r}_i$. Set $r = (r_1, \ldots, r_k)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_k)$. In (3.9) we have two direct sums of the form (3.8). Hence the assumption (3.7) implies $X = W(r,U) = W(\tilde{r}, \tilde{U})$. We can pass from U to $\tilde{U}$ in at most k steps, changing a single entry at each step. Suppose we replace u_k in U by $\tilde{u}_k$. Then $\hat{U} = (u_1, \ldots, u_{k-1}, \tilde{u}_k) \in \mathscr{U}$, and $V = \langle u_1 \rangle \oplus \cdots \langle u_{k-1} \rangle \oplus \langle \tilde{u}_k \rangle$. Set $Y_k = \oplus_{i=1}^{k-1} \langle f^{r_i} u_i \rangle$. Then

$$X = Y_k \oplus \langle f^{\tilde{r}_k} \tilde{u}_k \rangle = Y_k \oplus \langle f^{r_k} u_k \rangle.$$

Considering the elementary divisors of V/X we deduce $\tilde{r}_k = r_k$, and at the end we obtain $r = \tilde{r}$, and therefore $W(r,U) = W(r,\tilde{U})$. We conclude that $X = W(r,U)$ is independent of the choice of the generator tuple U. Hence X is hyperinvariant. □

Let us reexamine Example 1.1 and consider a field K of characteristic different from 2.

EXAMPLE 1.1 (CONTINUED). Let $\operatorname{char} K \neq 2$. Then $\gamma : (e_1, e_2) \mapsto (2e_1, e_2)$ determines an f-automorphism. For $Z = \langle e_1 + e_3 \rangle$ we have $\gamma(Z) = \langle 2e_1 + e_3 \rangle \neq Z$. Hence in this case $Z \in \operatorname{Inv}(V)$ is not characteristic.

To identify the characteristic subspaces we screen $\mathrm{Inv}(V)$. Note that

$$\mathrm{Aut}_f(V) = \big\{\alpha : (e_1, e_2) \mapsto (ae_1 + be_4, ce_2 + de_3 + ge_4 + he_1),\\ a,b,c,d,g,h \in K, a \neq 0, c \neq 0\big\}.$$

The nonzero cyclic subspaces are of the form $\langle e_2 + ce_1\rangle$, $\langle e_3 + ce_1\rangle$, and $\langle ae_4 + ce_1\rangle$, $a,c \in K$, $(a,c) \neq (0,0)$. Only $\langle e_3\rangle = fV$ and $\langle e_4\rangle = f^2V$ are characteristic. Moreover, X is a direct sum of two cyclic subspaces if and only if $X \in \{V, \langle e_3\rangle \oplus \langle e_1\rangle = V[f^2], \langle e_4\rangle \oplus \langle e_1\rangle = V[f]\}$. These three subspaces are characteristic. We find $\mathrm{Hinv}(V) = \{0, fV, f^2V, V[f], V[f^2], V\}$. Hence $\mathrm{Hinv}(V) = \mathrm{Chinv}(V)$. The example is a special case of the following general result (see also [14, p. 67]).

THEOREM 3.4. *If* $|K| > 2$ *then each characteristic subspace of* V *is hyperinvariant, i.e.* $\mathrm{Chinv}(V) = \mathrm{Hinv}(V)$.

Proof. Because of Lemma 1.4 it suffices to consider the case where f has only one eigenvalue. We can assume $f^n = 0$. If $|K| > 2$ and X is characteristic then it follows from Lemma 2.3(ii) that (2.7) implies (2.8). Therefore, according to Theorem 3.3, the subspace X is hyperinvariant. □

In the case of vector spaces over $K = \mathbb{Z}_2$ it is an open problem to describe all subspaces that are characteristic without being hyperinvariant.

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