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NUMERICAL RANGES OF RESTRICTED SHIFTS AND UNITARY DILATIONS

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Abstract. This paper considers the restricted shift operator associated with an infinite Blaschke product, expressing the closure of its numerical range as the intersection of the closures of the numerical ranges of a parametrized family of unitary dilations (or, equivalently, unitary perturbations of a modified restricted shift). The techniques used are based on interpolation and Clark measures. The results generalize known theorems for numerical ranges of matrices associated with finite-dimensional Blaschke products, which can be expressed geometrically in terms of the Poncelet property.

1. Introduction

A Blaschke product

$$B(z) = z\left(\frac{z-a}{1-\overline{a}z}\right)\left(\frac{z-b}{1-\overline{b}z}\right),$$

where a and b are distinct nonzero points in the disc, is a three-to-one mapping. If we consider a point λ on the unit circle and the three points mapped by B to λ, that is, $B(z_1) = B(z_2) = B(z_3) = \lambda$, we may connect these points to obtain a triangle. If we do this for each point λ on the unit circle, the intersection of all these triangles is an ellipse with foci at the nonzero zeroes of B [4]. Gau and Wu showed (see [10, Thm. 5.1]) that a similar result holds for Blaschke products of degree n (see also [14]). The key observation in the papers of Gau and Wu as well as that of Mirman is that the region we consider is the numerical range of a dilation of a matrix A with eigenvalues at the nonzero zeroes of a finite Blaschke product. Thus, the numerical range of A is a convex region bounded by a smooth curve C. Considering the points identified by the Blaschke product and connecting successive points, the results in [9, 10] and [14] show that the polygon obtained in this manner circumscribes the curve and each line segment in the polygon is tangent to the curve C at a single point. Such curves are known as Poncelet curves because they have the Poncelet property: beginning at *any* point on the unit circle and drawing the line from the point tangent to the curve, another point on the circle will be obtained. Continuing to draw tangent lines in this

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manner will *always* produce a closed polygon, and it will do so in the same number of steps each time, a number that does not depend on the starting point on the unit circle. Using interpolation results obtained in [11] as well as Blaschke products that do the interpolation, new proofs of these results can be obtained along with some new connections between Blaschke products and these curves (see [5]). This phenomenon is illustrated in Figure 1.

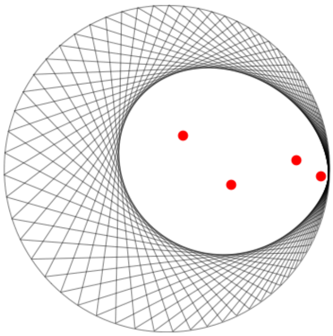

Figure 1. Poncelet curves corresponding to a finite Blaschke product

This paper studies the connections between infinite Blaschke products and numerical ranges. While it may not make sense to study the value of the Blaschke product on the unit circle, if the radial limit exists at a point we will see that these points can be tied to the study of the numerical ranges of the Blaschke products. Thus, given a Blaschke product B of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_j}{|a_j|} \frac{a_j - z}{1 - \overline{a}_j z}$$

(the Blaschke factor being interpreted as z if $a_j = 0$), the corresponding results for such Blaschke products require a study of interpolation by points on the unit circle. In Section 2, by considering compressions and dilations of appropriate operators, along the lines of [10, Thm. 6.3], we obtain relations between the numerical ranges of the restricted shift associated with an infinite Blaschke product and those associated with its partial products. This allows us in Section 3 to obtain the appropriate generalization of the Poncelet property for the numerical range of the restricted shift associated with an infinite Blaschke product.

The main result, Theorem 2.1 of the paper, is the following (all necessary notation is explained below).

THEOREM. *Suppose that $B = \prod_{k=1}^{\infty} b_k$ is an infinite Blaschke product and let U_α^B be the unitary rank-one perturbation of the restricted shift S_{zB} associated with $\alpha \in \mathbb{T}$. Then*

$$\bigcap_{\alpha \in \mathbb{T}} \overline{W(U_\alpha^B)} = \overline{W(S_B)}.$$

1.1. Notation and Background

We let $\theta \in H^\infty(\mathbb{D})$ be an inner function, and $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$, the model space. Write $P_{K_\theta} : H^2(\mathbb{D}) \to K_\theta$ for the orthogonal projection.

Let $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ denote the shift (multiplication by the independent variable) and define $S_\theta \in \mathscr{L}(K_\theta)$ the *restricted shift* by $S_\theta = P_{K_\theta} S_{|K_\theta}$. Note that $S_\theta^* = S^*_{|K_\theta}$. Let H be a Hilbert space. For an operator $T \in \mathscr{L}(H)$, the numerical range $W(T)$ is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\},$$

and is a convex subset of $\mathbb{C}$. If H is finite dimensional, then $W(T)$ is closed. In the case of a normal operator T, the set $\overline{W(T)}$ is the closed convex hull of the spectrum $\operatorname{spec}(T)$.

2. Dilations and numerical ranges

2.1. The link with rank-one perturbations

A *dilation* of a Hilbert-space operator $T \in \mathscr{L}(H)$ (in the sense of Halmos [12]) is an operator $\widetilde{T} \in \mathscr{L}(K)$, where K is a Hilbert space containing H, such that $T = P_H \widetilde{T}_{|H}$. In an obvious matrix notation $\widetilde{T} = \begin{pmatrix} T & * \\ * & * \end{pmatrix}$. We shall not require the more complicated Sz.-Nagy–Foias dilation [16] here, so when we talk of dilations we mean dilations in the above sense.

In [5], a main object of study is an $n \times n$ matrix $A = (a_{ij})_{i,j}$, defined in terms of points $a_1, \ldots, a_n \in \mathbb{D}$ by

$$a_{ij} = \begin{cases} a_j & \text{if } i = j, \\ \left(\prod_{k=i+1}^{j-1} (-\overline{a}_k)\right) \sqrt{1-|a_i|^2}\sqrt{1-|a_j|^2} & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

Note that the matrix A corresponds to the matrix of the operator S_B, where B is a Blaschke product with zeroes $a_1, \ldots, a_n$, with respect to an orthonormal basis constructed by applying the Gram–Schmidt process to the reproducing kernels $k_j(z) = 1/(1-\overline{a}_j z)$ (taken in reverse order). Thus it makes sense to consider the operator S_θ where θ is a general inner function.

Let $\mathscr{D}_1$ denote the class of all *unitary 1-dilations* of S_θ, by which we mean unitary dilations of S_θ defined on spaces of the form $K_\theta \oplus \mathbb{C}$. As we shall see it is natural to formulate the following conjecture.

CONJECTURE 2.1.

$$\overline{W(S_\theta)} = \bigcap_{\widetilde{T} \in \mathscr{D}_1} \overline{W(\widetilde{T})}.$$

Obviously we do have $\subseteq$ in general. We shall prove this conjecture in the special case when θ is a Blaschke product (see Theorem 2.1).

We can characterise all unitary 1-dilations of S_θ, by means of two orthogonal decompositions of K_θ. We refer here to [2], although the formulae can be found in many other places, e.g., [7].

Let

$$\mathscr{M}_1 = \mathbb{C}(S^*\theta) = \{x(\theta(z)-\theta(0))/z : x \in \mathbb{C}\} \qquad \text{and} \qquad \mathscr{N}_1 = K_\theta \ominus \mathscr{M}_1. \quad (1)$$

Also let

$$\mathscr{M}_2 = \mathbb{C}(\theta\overline{\theta(0)}-1) \qquad \text{and} \qquad \mathscr{N}_2 = K_\theta \ominus \mathscr{M}_2. \quad (2)$$

Then

$$S_\theta(xS^*\theta + w) = x((\theta\overline{\theta(0)}-1)\theta(0) + Sw$$

for $x \in \mathbb{C}$ and $w \in \mathscr{N}_1$.

Thus with respect to the two orthogonal decompositions of K_θ, S_θ has the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & S \end{pmatrix}$, where $|\lambda| < 1$ and S is a surjective isometry. Indeed, if $\theta(0) = 0$ we have $\lambda = 0$.

Thus the unitary 1-dilations must look like

$$\widetilde{T} = \begin{pmatrix} \lambda & 0 & \alpha\sqrt{1-|\lambda|^2} \\ 0 & S & 0 \\ \beta\sqrt{1-|\lambda|^2} & 0 & -\alpha\beta\overline{\lambda} \end{pmatrix}, \quad (3)$$

with respect to the two orthogonal decompositions of $K_\theta \oplus \mathbb{C}$, where $\alpha, \beta \in \mathbb{T}$. However, up to unitary equivalence, there is only one free parameter, namely, the value of $\alpha\beta$.

Note that, if θ is a finite Blaschke product of degree n, then $\dim K_\theta = n$ and $\dim K_{z\theta} = n+1$. This makes it plausible that there is a link between the unitary 1-dilations of S_θ and the rank-1 unitary perturbations of $S_{z\theta}$, and this is in fact the case for general inner functions θ. We now explain this in detail for the case $\theta(0) = 0$. The general case can be found in [10, Thm. 6.3].

Such rank-1 perturbations are well-understood, thanks to the theory of Clark measures [3]. In particular, one may decompose $K_{z\theta}$ orthogonally as

$$K_{z\theta} = \mathscr{M}_1 \oplus \mathscr{N}_1 = \mathbb{C}\theta \oplus (\mathbb{C}\theta)^\perp,$$

and

$$K_{z\theta} = \mathscr{M}_2 \oplus \mathscr{N}_2,$$

as in (1) and (2). With respect to these decompositions one has $S_{z\theta} = 0 \oplus S$. It is then possible to define for each $\alpha \in \mathbb{T}$ a unitary operator $U_\alpha \in \mathscr{L}(K_{z\theta})$ by $U_\alpha = \alpha 1 \oplus S$, where 1 denotes the constant function, which lies in $K_{z\theta}$. This is the complete set of rank-1 unitary perturbations of $S_{z\theta}$.

LEMMA 2.1. [10] *In the case $\theta(0) = 0$, all unitary 1-dilations of S_θ are equivalent to rank-1 perturbations of $S_{z\theta}$.*

Proof. It is easily verified that $K_{z\theta} = K_\theta \oplus \mathbb{C}\theta$, and $K_\theta = \mathbb{C}1 \oplus zK_{\theta/z}$, giving us two orthogonal decompositions

$$K_{z\theta} = \mathbb{C}\theta/z \oplus K_{\theta/z} \oplus \mathbb{C}\theta = \mathbb{C}1 \oplus zK_{\theta/z} \oplus \mathbb{C}\theta.$$

With respect to these two decompositions, the operator $S_{z\theta}$ has the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & M_z & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and its rank-1 unitary perturbation U_α is $\begin{pmatrix} 0 & 0 & \alpha \\ 0 & M_z & 0 \\ 1 & 0 & 0 \end{pmatrix}$, so that clearly $P_{K_\theta} U_{\alpha|K_\theta} = S_\theta$, i.e., U_α is unitarily equivalent to a unitary 1-dilation of S_θ. But, by virtue of the expression

$$\widetilde{T} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & M_z & 0 \\ \beta & 0 & 0 \end{pmatrix},$$

which is derived from (3) in the case $\theta(0) = 0$, with respect to the orthogonal decompositions $K_\theta = \mathbb{C}\theta/z \oplus K_{\theta/z}$ and $K_\theta = \mathbb{C}1 \oplus zK_{\theta/z}$, we see that every unitary 1-dilation is equivalent to some U_α. $\square$

Thus we may reformulate Conjecture 2.1 as asserting that

$$\overline{W(S_\theta)} = \bigcap_{\alpha \in \mathbb{T}} \overline{W(U_\alpha)}. \tag{4}$$

A special case of this is proved below, Theorem 2.1.

2.2. Numerical ranges of restricted shifts

From [6, Thm. 9.3.4] we know that if $A_n = P_{X_n} A P_{X_n}$ is the compression of $A \in \mathscr{L}(H)$ to X_n, and if X_n is an increasing sequence of subspaces with dense union in H, then

$$\operatorname{spec}(A_n) \subseteq \overline{W(A_n)} \subseteq \overline{W(A)}, \qquad \overline{W(A)} = \overline{\cup_{n=1}^{\infty} \overline{W(A_n)}}, \tag{5}$$

and moreover $\overline{W(A_n)}$ is an increasing sequence of sets.

This enables us to express the numerical range of S_B when B is an infinite Blaschke product in terms of the more easily-analysed numerical ranges corresponding to finite Blaschke products.

LEMMA 2.2. *Let $B = \prod_{k=1}^{\infty} b_k$ be an infinite Blaschke product with elementary Blaschke factor b_k, and $B_n = \prod_{k=1}^{n} b_k$ the partial product. Then*

$$\overline{W(S_B)} = \overline{\bigcup_{n=1}^{\infty} W(S_{B_n})}.$$

Proof. Note that $BH^2 = \bigcap_{n=1}^{\infty} B_n H^2$, a decreasing intersection, and so $K_B = \overline{\bigcup_{n=1}^{\infty} K_{B_n}}$: for K_B clearly contains the union, and if it is strictly bigger then there is a non-zero function in K_B orthogonal to all the K_{B_n}, hence in $B_n H^2$ for each n, hence in BH^2, which is impossible.

Now S_{B_n} is the compression of S_B to K_{B_n}, so by (5) the result follows, noting that $W(T)$ is closed if T is defined on a finite-dimensional Hilbert space. Alternatively we may use adjoints, since $S_{B_n}^* = S_{B|K_{B_n}}^*$, and $W(T^*) = \{\overline{z} : z \in W(T)\}$ for any operator T. $\square$

Note that the result remains true when the B_n are themselves infinite Blaschke products, in the form $\overline{W(S_B)} = \overline{\bigcup_{n=1}^{\infty} \overline{W(S_{B_n})}}$.

We may apply the above result to describe $W(S_\theta)$ for θ an arbitrary inner function. To do this we use Frostman's theorem [8, p. 79] to approximate θ uniformly by Blaschke products, whose numerical range can be described by Lemma 2.2. Note that the orthogonal projection onto K_θ is given by

$$P_{K_\theta} f = \theta P_-(\overline{\theta} f), \qquad \text{for} \quad f \in H^2,$$

where $P_- : L^2(\mathbb{T}) \to L^2(\mathbb{T}) \ominus H^2$ is the orthogonal projection. Hence if $\|B_n - \theta\|_\infty \to 0$ we also have $\|P_{K_{B_n}} - P_{K_\theta}\| \to 0$.

However, as in [6, Prob. 9.3.3], we know that if $\|A_1 - A_2\| < \varepsilon$, then $W(A_1) \subseteq \{z \in \mathbb{C} : \operatorname{dist}(z, W(A_2)) < \varepsilon\}$.

Thus $W(S_\theta)$ is the limit of the sets $W(S_n)$ where S_n is a restricted shift corresponding to a Blaschke product.

Let B be an infinite Blaschke product. Then the zeroes of B accumulate on a compact subset Z of $\mathbb{T}$. In the case that $Z \neq \mathbb{T}$, for $0 < \eta < \pi/2$ we define the closed set $E_\eta \subset \mathbb{T}$ to consist of all points at an (angular) distance at least η from Z. Observe that $B_n \to B$ uniformly on the set K_η consisting of all points in $\mathbb{C}$ at a distance at most $\eta/2$ from E_η.

We shall require a technical lemma.

LEMMA 2.3. *For all $\varepsilon > 0$ sufficiently small, satisfying also $0 < \varepsilon < \eta/2$, there is an integer n such that for every $\alpha \in \mathbb{T}$, and $z \in E_\eta$ with $B(z) = \alpha$ one can find $w \in K_\eta \cap \mathbb{T}$ with $|w - z| < \varepsilon$ and $B_n(w) = \alpha$.*

Proof. Write

$$\delta(\varepsilon) = \inf_{z \in E_\eta} \inf_{z' \in K_\eta, |z - z'| = \varepsilon} |B(z) - B(z')|,$$

noting that for ε sufficiently small $\delta(\varepsilon) > 0$ since otherwise there exist sequences $z_n \in E_\eta$ and $z_n' \in K_\eta$ with $0 < |z_n - z_n'| \to 0$, such that $B(z_n) - B(z_n') = 0$ (since the infimum is attained in each case). Let ζ be any limit point of (z_n); then $B'(\zeta) = 0$, by the argument principle, which is impossible since the zeroes of B' in $\overline{\mathbb{D}}$ lie in the closed convex hull of the zeroes of B [1, Thm. 2.1].

Now, $B_n \to B$ uniformly on K_η, and so for n sufficiently large we have $|B_n - B| < \delta(\varepsilon)$ on K_η; hence by Rouché's theorem, $B_n - \alpha$ has a zero w in $\{\gamma : |\gamma - z| < \varepsilon\}$ whenever $B(z) = \alpha$. Since $\alpha \in \mathbb{T}$ and B_n is inner, w also lies on $\mathbb{T}$. $\square$

We are now ready to prove a special case of Conjecture 2.1.

THEOREM 2.1. *Suppose that* $B = \prod_{k=1}^{\infty} b_k$ *is an infinite Blaschke product and let* U_α^B *be the unitary rank-one perturbation of the restricted shift* S_{zB} *associated with* $\alpha \in \mathbb{T}$. *Then*

$$\bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^B)} = \overline{W(S_B)}.$$

Proof. By Lemma 2.1, we have $\overline{W(S_B)} \subseteq \bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^B)}$.
It remains to check that

$$\bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^B)} \subseteq \overline{W(S_B)}. \tag{6}$$

We know by Lemma 2.2 that $\overline{W(S_B)} = \overline{\bigcup_{n=1}^{\infty} W(S_{B_n})}$, where $B_n = \prod_{k=1}^{n} b_k$ and $U_\alpha^{B_n}$ is the unitary rank-one perturbation of the restricted shift S_{zB_n}, associated with α.

Moreover, by [5, 10] and Lemma 2.1, $W(S_{B_n}) = \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n})$. Therefore (6) is equivalent to

$$\bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^B)} \subseteq \overline{\bigcup_{n=1}^{\infty} \left(\bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n}) \right)}. \tag{7}$$

We intend to prove first that

$$\bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^B)^\circ \subseteq \bigcup_{n=1}^{\infty} \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n})^\circ. \tag{8}$$

To do this we need the following technical lemma.

LEMMA 2.4. *Let* $z \in W(U_\alpha^B)^\circ$. *Then there exist* $n(\alpha)$ *and* $\varepsilon(n(\alpha)) > 0$ *such that* $z \in W(U_{\alpha'}^{B_n})^\circ$, *whenever* $n \geqslant n(\alpha)$, $\alpha' \in \mathbb{T}$ *and* $|\alpha - \alpha'| < \varepsilon(n(\alpha))$.

Proof. Since $\overline{W(U_\alpha^B)}$ is the closed convex hull of $\operatorname{spec}(U_\alpha^B)$, and since $W(U_\alpha^B)^\circ$ is convex, for $z \in W(U_\alpha^B)^\circ$, there exist $\xi_1, \dots, \xi_k \in \operatorname{spec}(U_\alpha^B)$ such that $z = \sum_{j=1}^{k} \lambda_j \xi_j$, with $\lambda_j \geqslant 0$ for each j, and $\sum_{j=1}^{k} \lambda_j = 1$ and such that z lies in the interior of the convex hull of $\{\xi_1, \dots, \xi_k\}$.

If ξ_j is an accumulation point of the zeroes of B, and hence in $\operatorname{spec}(S_B)$, then $\xi_j \in \overline{W(S_B)}$, and so by Lemma 2.2 we can find n_j independent of α such that for any $n \geqslant n_j$ there is a point $\eta_j^{(n)} \in W(U_\alpha^{B_n})$, such that for $n \geqslant n_j$ we can have $|\eta_j^{(n)} - \xi_j|$ as small as we wish.

If ξ_j is not an accumulation point, then $B(\xi_j) = \alpha$ and, choosing η sufficiently small, and applying Lemma 2.3 we may for $n \geqslant n_j$ find a solution η_j to $B_n(\eta_j) = \alpha$ arbitrarily close to ξ_j.

Since z is in the interior of the convex hull of $\{\xi_1, \dots, \xi_k\}$, it is also in the interior of the convex hull of $\{\eta_1, \dots, \eta_k\}$ if $|\xi_j - \eta_j|$ is sufficiently small for each j. Thus $z \in W(U_\alpha^{B_n})^\circ$ for n sufficiently large, say, $n \geqslant n(\alpha) := \max\{n_1, \dots, n_k\}$.

Clearly there is a neighbourhood of α in $\mathbb{T}$, say $|\alpha' - \alpha| < \varepsilon(n(\alpha))$, where we still have $z \in W(U_{\alpha'}^{B_n})^\circ$. $\square$

Continuing with the proof of Theorem 2.1, we use the compactness of $\mathbb{T}$ to find $\alpha_1, \dots, \alpha_m \in \mathbb{T}$ such that every point of $\mathbb{T}$ is at a distance at most $\varepsilon(n(\alpha_j))$ from one of the points α_j.

We apply Lemma 2.4, taking $N = \max\{n(\alpha_1), \dots, n(\alpha_m)\}$. It follows that (8) holds.

Observe that $\bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^B)^\circ$ is nonempty. For if B has at least three non-collinear zeroes, then these lie in the spectrum of S_B, and thus there is a nontrivial triangle in $W(S_B)$, and hence in every $W(U_\alpha^B)$. Otherwise, the zeroes of B all lie on a line, and accumulate non-tangentially at either one or two points on $\mathbb{T}$. We then have a sequence of arcs $(\xi_0 e^{i\beta_n}, \xi_0 e^{i\beta_{n-1}})_n$ tending towards one of the accumulation points, say $\xi_0 \in \mathbb{T}$, on which B is analytic and attains all the values in $\mathbb{T}$. Note, however that for $\xi \in \mathbb{T}$ one has $\xi \in W(U_\alpha^B)$ whenever $B(\xi) = \alpha$ and B is analytic at ξ. From these considerations and the convexity of $W(U_\alpha^B)$ we see that there are points lying in the interior of every $W(U_\alpha^B)$ simultaneously. Indeed, each numerical range contains a common triangle with one vertex at ξ_0 (see Figure 2).

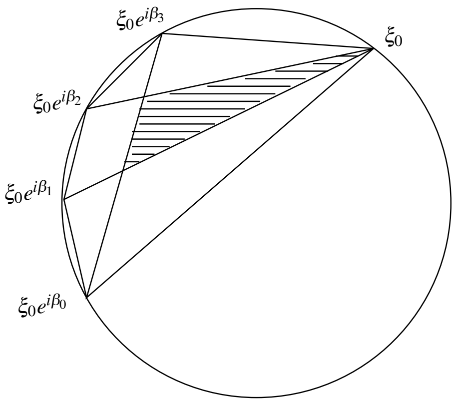

Figure 2. The interior of $W(U_\alpha^B)$ is nonempty

Finally, if $z \in \bigcap_{\alpha\in\mathbb{T}} \overline{W(U_\alpha^B)}$, then taking $w \in \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^B)^\circ$, we see that every point of the line segment joining z and w, with the possible exception of z itself, lies in $\bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^B)^\circ$, and hence in $\bigcup_{n=1}^\infty \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n})^\circ$. We conclude that z lies in $\overline{\bigcup_{n=1}^\infty \left(\bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n})\right)}$, as required. $\square$

3. The Geometric Viewpoint

Suppose now that $B_n(0) = 0$. We have seen that

$$W(S_{B_n}) = \bigcap_{\alpha\in\mathbb{T}} W(U_\alpha^{B_n}).$$

Now the numerical range of $U_\alpha^{B_n}$ is a polygon with n vertices, each at the points B_n maps to α. When we interpolate two sets of points on the unit circle $\{z_1,\dots,z_n\}$ and $\{z_1',\dots,z_n'\}$ to two distinct values, α and α' on the unit circle with a Blaschke product B of degree n satisfying $B(0)=0$, then this Blaschke product is unique. As noted above, for finite Blaschke products we know that $W(S_{B_n})$ is the intersection of the $W(U_\alpha^{B_n})$. Geometrically, this means that if we take the intersection of all the closed sets bounded by the polygons that we get from the $W(U_\alpha^{B_n})$, the numerical range of S_{B_n} will be this intersection. The previous result says that even in the case of an infinite Blaschke product, a similar result holds. Since U_α^B is unitary, $\overline{W(U_\alpha^B)}$ is the closed convex hull of its spectrum, just as in the finite case. If the Blaschke product is discontinuous at a point $\gamma \in \partial\mathbb{D}$, then, an argument based on Rouché's theorem, similar to the one used in Lemma 2.3, implies that for n large, B_n will assume the value α close to γ, and so $W(S_{B_n})$ will contain points close to α. We now consider the geometric picture in the case in which there is an isolated singularity. Geometrically, this is the most interesting case.

Thus, the setting is the following: We consider a Blaschke product B with $B(0)=0$ and a singularity at the point $z=1$ (and only the point $z=1$). Ordering the points where $B^*(z)=\alpha$ in terms of their arguments, we may connect a point where $B^*(z)=\alpha$ to the subsequent one. Consider two points z_1 and z_2 on the unit circle that are successive points with the property that $B^*(z_1)=B^*(z_2)=\alpha$.

Apply Lemma 2.2 to conclude that $\overline{W(S_B)}=\overline{\bigcup_n W(S_{B_n})}$. For n sufficiently large, we know that $B_n(z_{1,n})=B_n(z_{2,n})=\alpha$ for some points $z_{1,n}$ and $z_{2,n}$ on $\mathbb{T}$ with $|z_1-z_{1,n}|+|z_2-z_{2,n}|<\varepsilon(n)$, where $\varepsilon(n)\to 0$. Now by [4] and [13, p. 10], the line joining two points $z_{1,n}$ and $z_{2,n}$ identified by B_n is tangent to $W(S_{B_n})$ at the point $\zeta_n=(m_{1,n}z_{2,n}+m_{2,n}z_{1,n})/(m_{1,n}+m_{2,n})$ where $m_{j,n}=\dfrac{B_n(z_{j,n})}{z_{j,n}B_n'(z_{j,n})}$. It follows from [4] that $0<m_{j,n}<1$.

Thus, $z_{1,n}\to z_1$ and $z_{2,n}\to z_2$. Now, because B has a singularity at the point $z=1$ only and we assume that neither z_1 nor z_2 are the point $z=1$, we know that B is analytic in a neighbourhood of z_1 and in a neighbourhood of z_2. Thus, B_n and its derivative B_n' converge uniformly to B and B', respectively, on these neighbourhoods. In particular, $B_n'(z_{j,n})$ remains bounded as $n\to\infty$ and is bounded away from 0. Thus, the points $\zeta_n\in W(S_{B_n})\subseteq W(S_B)$ converge to a point $\zeta=(m_1z_2+m_2z_1)/(m_1+m_2)$, and using the formula presented above for m_j, we see that $0<m_j<1$ for $j=1,2$. In particular, the line joining z_1 to z_2 intersects $\overline{W(S_B)}$. Note that our assumptions on the sequence allow us to find an explicit formula for m_1 and m_2. In fact,

$$m_j=\frac{B(z_j)}{z_jB'(z_j)} \qquad \text{for} \quad j=1,2.$$

We claim that there are two possibilities for the line segment joining z_1 and z_2; that is, we claim that it can either be a bounding line segment or it will cross $W(S_B)$ at exactly one point. To see why this is, suppose that the line segment joining z_1 and z_2 intersects $W(S_B)$ in more than one point and that the line segment is not contained in the boundary of $\overline{W(S_B)}$. Rotating, we may assume that the line joining z_1 and z_2

is horizontal. Now we assume that this line segment intersects $W(S_B)$ in (at least) two points, w_1 and w_2. But the convexity of $W(S_B)$ implies that the line segment joining these two points is contained in $W(S_B)$. Since it is not a bounding line segment, there must be a point of $W(S_B)$ on either side of this line segment. Let w denote a point above this line segment. Thus, $w \in \overline{W(S_B)} \subseteq \bigcap_{\gamma\in\mathbb{T}} \overline{W(U_\gamma^B)}$. In particular, $w \in \overline{W(U_\alpha^B)}$ and it lies above the line joining z_1 and z_2. Since $W(U_\alpha^B)$ is the closed convex hull of its spectrum, there must be a point of the spectrum of U_α^B lying above z_1 and z_2. Since B is analytic on the arc joining z_1 and z_2, this contradicts our assumption that there is no point between z_1 and z_2 at which B assumes the value α.

REMARK 3.1. As observed by one of the referees, another link between a restricted shift and its unitary perturbations U_α is given by the formula

$$S_\theta = \int_{\mathbb{T}} U_\alpha \frac{|d\alpha|}{2\pi},$$

as in [15]. It is possible that this identity will lead to further results in the same direction.

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