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STRUCTURED DECOMPOSITIONS FOR MATRIX TRIPLES: SVD–LIKE CONCEPTS FOR STRUCTURED MATRICES

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In memory of Ralph Byers
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Abstract. Canonical forms for matrix triples $(A,G,\hat{G})$, where A is arbitrary rectangular and G, $\hat{G}$ are either real symmetric or skew symmetric, or complex Hermitian or skew Hermitian, are derived. These forms generalize classical singular value decompositions. In [1] a similar canonical form has been obtained for the complex case. In this paper, we provide an alternative proof for the complex case which is based on the construction of a staircase-like form with the help of a structured QR-like decomposition. This approach allows generalization to the real case.

1. Introduction

Let $\mathbb{F}$ denote either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. Consider a triple of matrices $(A,G,\hat{G})$ with $A\in\mathbb{F}^{m\times n}$, $G\in\mathbb{F}^{m\times m}$, and $\hat{G}\in\mathbb{F}^{n\times n}$, where G and $\hat{G}$ are nonsingular and either Hermitian or skew-Hermitian (in the complex case) or symmetric or skew-symmetric (in the real case). In this paper we derive canonical forms $(A_{\mathrm{CF}},G_{\mathrm{CF}},\hat{G}_{\mathrm{CF}})$ under the transformation

$$(A_{\mathrm{CF}},G_{\mathrm{CF}},\hat{G}_{\mathrm{CF}}) := (X^*AY, X^*GX, Y^*\hat{G}Y), \tag{1.1}$$

with nonsingular matrices $X\in\mathbb{F}^{m\times m}$ and $Y\in\mathbb{F}^{n\times n}$. (Here A^* denotes the conjugate transpose of a matrix A if $\mathbb{F}=\mathbb{C}$ or the transpose if $\mathbb{F}=\mathbb{R}$.)

The canonical form for the complex case is already known and has appeared in [1], although uniqueness of the canonical form had not been considered there. The real case, however, has only been investigated in [27] so far for the special case that

$$G=\begin{bmatrix}0 & I_m\\ -I_m & 0\end{bmatrix} \quad\text{and}\quad \hat{G}=I_n$$

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and in [28] where a numerical method was derived for this case.

In this paper, we give an alternative proof of the canonical form in the complex case which is based on the construction of a staircase-like form with the help of a structured QR-like decomposition. These stair-case decompositions have analogues in the real case which allow a generalization of the results for the complex case to cover the real case as well.

The difficulties encountered in the treatment of the real case in full generality stem from the fact that one has to distinguish the cases that G and $\hat{G}$ are either both symmetric, or both skew-symmetric, or that one is symmetric and the other skew-symmetric. In the complex case, in contrast, there is no need to distinguish between Hermitian and skew-Hermitian matrices G and $\hat{G}$, because multiplication with the *imaginary unit* $\imath$ easily converts an Hermitian matrix into a skew-Hermitian matrix and vice versa. A corresponding transformation can be performed on the canonical form so that all cases are covered by presenting the canonical form for the case that G and $\hat{G}$ are both Hermitian.

A third case besides the real case and the complex case with Hermitian and skew-Hermitian G and $\hat{G}$ is obtained if one assumes that G and $\hat{G}$ are complex symmetric or complex skew-symmetric and if one replaces the conjugate transpose in (1.1) by the transpose. This case has been investigated in [19]. So together with this paper the complete set of canonical forms for real and complex matrix triples of the form (1.1) is available.

The study of the described canonical forms is motivated by the goal of unifying the solution procedures for eigenvalue problems associated with structured matrices from Lie and Jordan algebras related to indefinite inner products, [3, 8, 21, 22]. Consider for example a *signature matrix*

$$\Sigma_{\pi_1,\nu_1} = \begin{bmatrix} I_{\pi_1} & 0 \\ 0 & -I_{\nu_1} \end{bmatrix}, \qquad \pi_1+\nu_1 = m.$$

A matrix $\mathscr{H} \in \mathbb{C}^{m\times m}$ is called Σ_{π_1,ν_1}*-Hermitian* if $(\Sigma_{\pi_1,\nu_1}\mathscr{H})^* = \Sigma_{\pi_1,\nu_1}\mathscr{H}$, i.e., if $\Sigma_{\pi_1,\nu_1}\mathscr{H}$ is Hermitian. The matrix $\Sigma_{\pi_1,\nu_1}\mathscr{H}$ then possesses a factorization $\Sigma_{\pi_1,\nu_1}\mathscr{H} = A\Sigma_{\pi_2,\nu_2}A^*$, where Σ_{π_2,ν_2} is another signature matrix and $A \in \mathbb{R}^{m\times n}$ with $n = \pi_2+\nu_2$. This means that $\mathscr{H}$ has a factorization

$$\mathscr{H} = \Sigma_{\pi_1,\nu_1}A\Sigma_{\pi_2,\nu_2}A^*.$$

If, for the triple $(A,\Sigma_{\pi_1,\nu_1},\Sigma_{\pi_2,\nu_2})$, we can determine a suitable canonical form

$$(A_{\mathrm{CF}},G_{\mathrm{CF}},\hat{G}_{\mathrm{CF}}) = (X^*AY, X^*\Sigma_{\pi_1,\nu_1}X, Y^*\Sigma_{\pi_2,\nu_2}Y),$$

then this will allow us to determine the eigenstructure of $\mathscr{H}$, because

$$X^{-1}\mathscr{H}X = (X^*\Sigma_{\pi_1,\nu_1}X)^{-1}(X^*AY)(Y^*\Sigma_{\pi_2,\nu_2}Y)^{-1}(Y^*A^*X) = G_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}\hat{G}_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}^*.$$

Simultaneously the eigenstructure for the Σ_{π_2,ν_2}-Hermitian matrix $\hat{\mathscr{H}} = \Sigma_{\pi_2,\nu_2}A^*\Sigma_{\pi_1,\nu_1}A$, is obtained, because $Y^{-1}\Sigma_{\pi_2,\nu_2}A^*\Sigma_{\pi_1,\nu_1}AY = \hat{G}_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}^*G_{\mathrm{CF}}^{-1}A$.

In general, the canonical form (1.1) of the matrix triple $(A, G, \hat{G})$ will allow us to simultaneously determine the eigenstructures of the two structured matrices

$$\mathscr{H} = G^{-1}A\hat{G}^{-1}A^*, \qquad \hat{\mathscr{H}} = \hat{G}^{-1}A^*G^{-1}A. \tag{1.2}$$

Structured matrices with such product representations cover all the structured matrices from the Lie and Jordan algebras (see [3]), associated with the sesquilinear forms

$$\langle x, y\rangle_G = x^*Gy, \qquad \langle x, y\rangle_{\hat{G}} = x^*\hat{G}y. \tag{1.3}$$

Furthermore, the form (1.1) can be interpreted as a generalization of the singular value decomposition (SVD) [9] of a matrix $A \in \mathbb{C}^{m\times n}$, i.e., the decomposition

$$A_{\text{CF}} := U^*AV = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}, \quad \sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$$

with unitary matrices U, V. Indeed, the SVD can be considered as a canonical form for the matrix triple (A, I_m, I_n) under the transformation

$$(A, I_m, I_n) \mapsto (A_{\text{CF}}, I_m, I_n) = (X^*AY, X^*I_mX, Y^*I_nY). \tag{1.4}$$

Here, the equation for the first components of the two matrix triples in (1.4) is the actual singular value decomposition, while the equations for the second and third components just force the transformation matrices to be unitary. The canonical form then displays the eigenstructure of the I_n-selfadjoint matrix A^*A and the I_m-selfadjoint matrix AA^*, because the nonzero singular values $\sigma_1, \ldots, \sigma_r$ are just the square roots of the nonzero eigenvalues of A^*A and AA^*.

When generalizing the concept of the singular value decomposition to analogous factorizations for linear maps $\mathscr{L} : \mathbb{C}^n \to \mathbb{C}^m$, where the spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ are equipped with indefinite inner products given by invertible Hermitian matrices $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$, one may consider to apply a transformation $A \mapsto X^*AY$ to a matrix representation of $\mathscr{L}$, where X and Y are matrices that are unitary with respect to the sesquilinear forms (1.3), i.e., where $X^*GX = G$, $Y^*\hat{G}Y = \hat{G}$. However, if one allows general changes of bases in the spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, i.e., changes that affect the indefinite inner products as well, then this corresponds exactly to the transformation as in (1.1) and the canonical forms will appear to be less complicated.

Generalizations of the singular value decomposition in the sense of this paper have been studied earlier in the literature, probably starting with [10, 11]. The generalized singular value decomposition defined there corresponds to (1.1) for the case that for all three matrices A_{CF}, $\mathscr{H}_{\text{CF}} := G_{\text{CF}}^{-1}A_{\text{CF}}\hat{G}_{\text{CF}}^{-1}A_{\text{CF}}^*$, and $\hat{\mathscr{H}}_{\text{CF}} := \hat{G}_{\text{CF}}^{-1}A_{\text{CF}}^*G_{\text{CF}}^{-1}A_{\text{CF}}$ a diagonal representation can be chosen. The general complex case (allowing also non-diagonal representations) was then discussed in [1].

In [15], the canonical forms of the matrices $X^{[*]}X$ and $XX^{[*]}$ were investigated, where $X^{[*]} = H^{-1}XH$ denotes the adjoint of a matrix $X \in \mathbb{C}^{n\times n}$ with respect to the indefinite inner product induced by the nonsingular Hermitian matrix $H \in \mathbb{C}^{n\times n}$. This question is motivated from the theory of polar decompositions in indefinite inner product spaces. It is said that a matrix $X \in \mathbb{C}^{n\times n}$ allows an H-polar decomposition, if there exists an H-selfadjoint matrix B, i.e., a matrix satisfying $B^*H = HB$, and an H-unitary matrix U, i.e., a matrix satisfying $U^*HU = H$, such that $X = UB$. It was shown in [20] that X allows an H-polar decomposition if and only if the two matrices $X^{[*]}X$ and $XX^{[*]}$ have the same canonical forms as H-selfadjoint matrices. Setting $A = X$, $G = H^{-1}$, and $\hat{G} = H$, we find that

$$X^{[*]}X = \hat{G}^{-1}A^*G^{-1}A = \mathscr{H} \quad \text{and} \quad XX^{[*]} = A\hat{G}^{-1}A^*G^{-1} = G\mathscr{H}G^{-1},$$

and thus, the canonical forms of $X^{[*]}X$ and $XX^{[*]}$ can be read off from the canonical form for the matrix triple $(A,G,\hat{G}) = (X,H^{-1},H)$. Consequently, many of the results from [15] can be recovered from the results in this paper. Recently, the relation of the spectra of $X^{[*]}X$ and $XX^{[*]}$ has been investigated in terms of infinite dimensional indefinite inner product spaces (also known as Krein spaces) in [23].

A canonical form closely related to the one obtained under the transformation (1.1) is the canonical form for pairs of matrices (B,C), $B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}$ under transformations of the form

$$(B,C) \mapsto (X^{-1}BY, Y^{-1}CX). \tag{1.5}$$

This form corresponds to the canonical representation of the quiver $1 \rightleftarrows 2$, see [4, 13] and the discussion in [24]. In particular, this canonical form reveals the Jordan structures of the products BC and CB. In our framework, this corresponds to a canonical form of the pair of matrices $(G^{-1}A, \hat{G}^{-1}A^*)$ rather than for the triple $(A,G,\hat{G})$. When focussing on matrix triples, our approach is more general, because the canonical form for the pair $(G^{-1}A, \hat{G}^{-1}A^*)$ can be easily read off from the canonical form for $(A,G,\hat{G})$, but not vice versa. Moreover, the canonical form for $(A,G,\hat{G})$ allows the construction of structured canonical forms for the structured matrices and matrix pencils mentioned in the previous paragraphs. The approach via (1.5), on the other hand, focusses on different aspects and allows to consider pairs (B,C), where the ranks of B and C are distinct. This case is not covered by the canonical forms obtained in this paper or in [19] as the considered pairs of matrices always have the same rank.

The paper is organized as follows. In Section 2 we review the definitions of matrices having structures with respect to indefinite inner products and provide some auxiliary results. In Section 3 we then derive the canonical forms for the complex case and for the real case when G and $\hat{G}$ are both real symmetric. In Section 4 we study the case that one of $G,\hat{G}$ is real symmetric and the other is real skew-symmetric. In Section 5 we present the canonical forms for the case that both $G,\hat{G}$ are real skew-symmetric.

Throughout the paper we use $\mathbb{F}$ to denote the field of real or complex matrices, i.e., $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. $\mathbb{R}_-$ ($\mathbb{R}_+$) is the set of real negative (positive) numbers, and $\mathbb{C}_-$ ($\mathbb{C}_+$) is the open left (right) half complex plane. The $n\times n$ identity and $n\times n$ zero matrices are denoted by I_n and $\mathscr{O}_n$, respectively. The $m\times n$ zero matrix is denoted by $\mathscr{O}_{m\times n}$ and e_j is the jth column of the identity matrix or, equivalently, the jth standard basis vector of $\mathbb{F}^n$. Moreover, we introduce

$$\Sigma_{\pi,\nu,\delta} = \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & \mathscr{O}_\delta \end{bmatrix}, \quad \Sigma_{\pi,\nu} = \Sigma_{\pi,\nu,0} = \begin{bmatrix} I_\pi & 0 \\ 0 & -I_\nu \end{bmatrix}, \quad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

The transpose and conjugate transpose of a matrix A are denoted by A^T and A^*, respectively. We use $A_1 \oplus \ldots \oplus A_k$ to denote a block diagonal matrix with diagonal blocks $A_1, \ldots, A_k$. If $A = [a_{ij}] \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{\ell\times k}$, then $A \otimes B = [a_{ij}B] \in \mathbb{F}^{n\ell\times mk}$ denotes the Kronecker product of A and B. For a real symmetric or complex Hermitian matrix A we call (π,ν,δ) the *Sylvester inertia index* with π, ν, δ being the number of positive, negative, and zero eigenvalues of A, respectively. For a square matrix A, $\sigma(A)$ denotes the spectrum of A. We use

$$R_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \quad \mathscr{J}_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

to denote the $n\times n$ reverse identity or the $n\times n$ upper triangular Jordan block associated with the eigenvalue λ, respectively, and

$$\mathscr{J}_n(a,b) = I_n \otimes \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \mathscr{J}_n(0) \otimes I_2 = \begin{bmatrix} a & b & 1 & 0 & & & & 0 \\ -b & a & 0 & 1 & & & & \\ & & a & b & \ddots & & & \\ & & -b & a & & & & \\ & & & & \ddots & & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & a & b \\ 0 & & & & & & -b & a \end{bmatrix}$$

for blocks associated with complex conjugate eigenvalues in the real Jordan form of a real matrix.

2. Matrices structured with respect to sesquilinear forms

Our general theory will cover and generalize results for the following classes of matrices.

DEFINITION 2.1. *Let $G \in \mathbb{F}^{n\times n}$ be invertible and let $\mathscr{H}, \mathscr{K} \in \mathbb{F}^{n\times n}$ be such that*

$$(G\mathscr{H})^* = G\mathscr{H} \quad \textit{and} \quad (G\mathscr{K})^* = -G\mathscr{K}.$$

1) *If $\mathbb{F} = \mathbb{C}$ and G is Hermitian or skew-Hermitian, then $\mathscr{H}$ is called* G-Hermitian *and $\mathscr{K}$ is called* G-skew-Hermitian.

2) *If $\mathbb{F}=\mathbb{R}$ and G is symmetric, then $\mathscr{H}$ is called G-*symmetric *and $\mathscr{K}$ is called G-*skew-symmetric.

3) *If $\mathbb{F}=\mathbb{R}$ and G is skew-symmetric, then $\mathscr{H}$ is called G-*Hamiltonian *and $\mathscr{K}$ is called G-*skew-Hamiltonian.

G-Hermitian and G-symmetric matrices are often called *G-selfadjoint matrices* as they are selfadjoint with respect to the indefinite inner product induced by G. In this paper, we prefer the notions G-Hermitian and G-symmetric in order to clearly distinguish between the complex and the real case. Observe that transformations of the form

$$(\mathscr{H},G)\mapsto(P^{-1}\mathscr{H}P,P^*GP),\quad P\in\mathbb{F}^{n\times n}\text{ invertible},$$

preserve the structure of $\mathscr{H}$ with respect to G, i.e., if, for example, $\mathscr{H}$ is G-Hermitian, then $P^{-1}\mathscr{H}P$ is P^*GP-Hermitian as well. Clearly, each complex Hermitian or real symmetric invertible matrix G is congruent to $\Sigma_{\pi,\nu}$ for some π,ν and each real skew-symmetric invertible matrix G is congruent to J_n for some n. Thus, we may always restrict ourselves to the case that either $G=\Sigma_{\pi,\nu}$ or $G=J_n$. In the latter case, we refer to J_n-Hamiltonian or J_n-skew-Hamiltonian matrices simply as *Hamiltonian* or *skew-Hamiltonian* matrices, respectively.

G-(skew-)Hermitian, G-(skew-)symmetric, and G-(skew-)Hamiltonian matrices have been intensively studied in the literature. In particular, canonical forms for such matrices have been derived in many places. We review these well-known canonical forms in the following.

THEOREM 2.2. (Canonical form for G-Hermitian matrices, [8, 17, 25])
Let $G\in\mathbb{C}^{n\times n}$ be Hermitian and invertible and let $\mathscr{H}\in\mathbb{C}^{n\times n}$ be G-Hermitian. Then there exists an invertible matrix $X\in\mathbb{C}^{n\times n}$ such that

$$X^{-1}\mathscr{H}X=\mathscr{H}_c\oplus\mathscr{H}_r,\quad X^*GX=G_c\oplus G_r,$$

where

$$\begin{aligned}\mathscr{H}_c&=\mathscr{H}_{c,1}\oplus\cdots\oplus\mathscr{H}_{c,m_c},\quad G_c=G_{c,1}\oplus\cdots\oplus G_{c,m_c},\\ \mathscr{H}_r&=\mathscr{H}_{r,1}\oplus\cdots\oplus\mathscr{H}_{r,m_r},\quad G_r=G_{r,1}\oplus\cdots\oplus G_{r,m_r},\end{aligned}$$

and where the diagonal blocks have the following forms:

1) *blocks associated with pairs $(\lambda_j,\overline{\lambda}_j)$ of nonreal eigenvalues of $\mathscr{H}$:*

$$\mathscr{H}_{c,j}=\begin{bmatrix}\mathscr{J}_{\xi_j}(\lambda_j)&0\\0&\mathscr{J}_{\xi_j}(\bar{\lambda}_j)\end{bmatrix},\quad G_{c,j}=R_{2\xi_j}=\begin{bmatrix}0&R_{\xi_j}\\R_{\xi_j}&0\end{bmatrix},$$

where $\operatorname{Im}\lambda_j>0$ and $\xi_j\in\mathbb{N}$ for $j=1,\dots,m_c$;

2) *blocks associated with real eigenvalues:*

$$\mathscr{H}_{r,j}=\mathscr{J}_{\eta_j}(\alpha_j),\quad G_{r,j}=s_jR_{\eta_j},$$

where $\alpha_j\in\mathbb{R}$, $s_j\in\{-1,1\}$, and $\eta_j\in\mathbb{N}$ for $j=1,\dots,m_r$.

$\mathscr{H}$ has the (not necessarily pairwise distinct) nonreal eigenvalues $\lambda_1,\dots,\lambda_{m_c},\overline{\lambda}_1,\dots,\overline{\lambda}_{m_c}$ and (not necessarily pairwise distinct) real eigenvalues $\alpha_1,\dots,\alpha_{m_r}$.

REMARK 2.3. Besides the eigenvalues, the signs $s_1,\dots,s_{m_r}$ associated with the real eigenvalues are additional invariants of G-Hermitian matrices. The collection of these signs is called the *sign characteristic* of $\mathscr{H}$, sometimes also called *Krein signature*, [14]. For details on the sign characteristics, we refer to [8] and the references therein.

The real version of Theorem 2.2 is as follows:

THEOREM 2.4. (Canonical form for real G-symmetric matrices, [8, 16, 17, 26]) *Let $G\in\mathbb{R}^{n\times n}$ be symmetric and invertible and let $\mathscr{H}\in\mathbb{R}^{n\times n}$ be G-symmetric. Then there exists an invertible matrix $X\in\mathbb{R}^{n\times n}$ such that*

$$X^{-1}\mathscr{H}X=\mathscr{H}_c\oplus\mathscr{H}_r,\quad X^TGX=G_c\oplus G_r,$$

where

$$\begin{aligned}\mathscr{H}_c&=\mathscr{H}_{c,1}\oplus\cdots\oplus\mathscr{H}_{c,m_c},\quad & G_c&=G_{c,1}\oplus\cdots\oplus G_{c,m_c},\\ \mathscr{H}_r&=\mathscr{H}_{r,1}\oplus\cdots\oplus\mathscr{H}_{r,m_r},\quad & G_r&=G_{r,1}\oplus\cdots\oplus G_{r,m_r},\end{aligned}$$

and where the diagonal blocks have the following forms:

1) *blocks associated with pairs $(\lambda_j,\overline{\lambda}_j)$ of nonreal eigenvalues of $\mathscr{H}$:*

$$\mathscr{H}_{c,j}=\mathscr{J}_{\xi_j}(a_j,b_j),\quad G_{c,j}=R_{2\xi_j},$$

where $b_j=\operatorname{Im}\lambda_j>0$, $a_j=\operatorname{Re}\lambda_j$, and $\xi_j\in\mathbb{N}$ for $j=1,\dots,m_c$;

2) *blocks associated with real eigenvalues:*

$$\mathscr{H}_{r,j}=\mathscr{J}_{\eta_j}(\alpha_j),\quad G_{r,j}=s_jR_{\eta_j},$$

where $\alpha_j\in\mathbb{R}$, $s_j\in\{-1,1\}$, and $\eta_j\in\mathbb{N}$ for $j=1,\dots,m_r$.

$\mathscr{H}$ has the (not necessarily pairwise distinct) nonreal eigenvalues $\lambda_1,\dots,\lambda_{m_c},\overline{\lambda}_1,\dots,\overline{\lambda}_{m_c}$ and (not necessarily pairwise distinct) real eigenvalues $\alpha_1,\dots,\alpha_{m_r}$.

The corresponding canonical form for G-skew-Hermitian matrices immediately follows from Theorem 2.2, because a matrix $\mathscr{K}$ is G-skew-Hermitian if and only if $\mathscr{H}=\imath\mathscr{K}$ is G-Hermitian. In the real case, however, the trick of multiplying by the imaginary unit $\imath$ is not an option and a canonical form has to be derived separately. We also need additional notation. We use the notation

$$\varXi_n=\begin{bmatrix}(-1)^0 & & 0\\ & \ddots & \\ 0 & & (-1)^{n-1}\end{bmatrix},\quad \varGamma_n=\varXi_nR_n=\begin{bmatrix}0 & & (-1)^0\\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{n-1} & & 0\end{bmatrix}.$$

THEOREM 2.5. (Canonical form for G-skew-symmetric matrices, [16, 18, 26]) *Let $G \in \mathbb{R}^{n\times n}$ be symmetric and invertible and let $\mathscr{K} \in \mathbb{R}^{n\times n}$ be G-skew symmetric. Then there exists an invertible matrix $X \in \mathbb{R}^{n\times n}$ such that*

$$X^{-1}\mathscr{K}X = \mathscr{K}_c \oplus \mathscr{K}_r \oplus \mathscr{K}_\imath \oplus \mathscr{K}_z, \quad X^T G X = G_c \oplus G_r \oplus G_\imath \oplus G_z,$$

where

$$\begin{array}{ll} \mathscr{K}_c = \mathscr{K}_{c,1} \oplus \cdots \oplus \mathscr{K}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathscr{K}_r = \mathscr{K}_{r,1} \oplus \cdots \oplus \mathscr{K}_{r,m_r}, & G_r = G_{r,1} \oplus \cdots \oplus G_{r,m_r}, \\ \mathscr{K}_\imath = \mathscr{K}_{\imath,1} \oplus \cdots \oplus \mathscr{K}_{\imath,m_\imath}, & G_\imath = G_{\imath,1} \oplus \cdots \oplus G_{\imath,m_\imath}, \\ \mathscr{K}_z = \mathscr{K}_{z,1} \oplus \cdots \oplus \mathscr{K}_{z,m_o+m_e}, & G_z = G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e}, \end{array}$$

and where the diagonal blocks have the following forms:

1) *blocks associated with quadruples $(\lambda_j, \bar{\lambda}_j, -\lambda_j, -\bar{\lambda}_j)$ of nonreal, non purely imaginary eigenvalues of $\mathscr{K}$:*

$$\mathscr{K}_{c,j} = \begin{bmatrix} \mathscr{J}_{\xi_j}(a_j,b_j) & 0 \\ 0 & -\mathscr{J}_{\xi_j}(a_j,b_j) \end{bmatrix}, \quad G_{c,j} = R_{4\xi_j} = \begin{bmatrix} 0 & R_{2\xi_j} \\ R_{2\xi_j} & 0 \end{bmatrix},$$

where $a_j = \operatorname{Re}\lambda_j > 0$, $b_j = \operatorname{Im}\lambda_j > 0$, and $\xi_j \in \mathbb{N}$ for $j = 1,\ldots,m_c$;

2) *blocks associated with pairs $(\alpha_j, -\alpha_j)$ of real nonzero eigenvalues of $\mathscr{K}$:*

$$\mathscr{K}_{r,j} = \begin{bmatrix} \mathscr{J}_{\eta_j}(\alpha_j) & 0 \\ 0 & -\mathscr{J}_{\eta_j}(\alpha_j) \end{bmatrix}, \quad G_{r,j} = R_{2\eta_j} = \begin{bmatrix} 0 & R_{\eta_j} \\ R_{\eta_j} & 0 \end{bmatrix},$$

where $\alpha_j > 0$ and $\eta_j \in \mathbb{N}$ for $j = 1,\ldots,m_r$;

3) *blocks associated with pairs $(\imath\beta_j, -\imath\beta_j)$ of purely imaginary nonzero eigenvalues of $\mathscr{K}$:*

$$\mathscr{K}_{\imath,j} = \begin{bmatrix} 0 & \mathscr{J}_{\rho_j}(\beta_j) \\ -\mathscr{J}_{\rho_j}(\beta_j) & 0 \end{bmatrix}, \quad G_{\imath,j} = s_j \begin{bmatrix} R_{\rho_j} & 0 \\ 0 & R_{\rho_j} \end{bmatrix},$$

where $\beta_j > 0$, $s_j \in \{-1,1\}$, and $\rho_j \in \mathbb{N}$ for $j = 1,\ldots,m_\imath$;

4) *blocks associated with the eigenvalue $\lambda = 0$ of $\mathscr{K}$:*

$$\mathscr{K}_{z,j} = \mathscr{J}_{\zeta_j}(0), \quad G_{z,j} = t_j \Gamma_{\zeta_j},$$

where $\zeta_j \in \mathbb{N}$ is odd, $t_j \in \{-1,1\}$ for $j = 1,\ldots,m_o$, and

$$\mathscr{K}_{z,j} = \begin{bmatrix} \mathscr{J}_{\zeta_j}(0) & 0 \\ 0 & -\mathscr{J}_{\zeta_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\zeta_j} \\ R_{\zeta_j} & 0 \end{bmatrix},$$

where $\zeta_j \in \mathbb{N}$ is even for $j = m_o+1,\ldots,m_o+m_e$.

$\mathscr{K}$ has the (not necessarily pairwise distinct) eigenvalues $\pm\lambda_1,\ldots,\pm\lambda_{m_c}, \pm\overline{\lambda}_1,\ldots,\pm\overline{\lambda}_{m_c}$, $\pm\alpha_1,\ldots,\pm\alpha_{m_r}$, $\pm\imath\beta_1,\ldots,\pm\imath\beta_{m_\imath}$, and the additional eigenvalue 0, provided that $m_o + m_e > 0$.

If G is skew-Hermitian, then the canonical form for G-Hermitian (G-skew-Hermitian) matrices follows directly from Theorem 2.2, because $\imath G$ is Hermitian and a matrix $\mathscr{H}$ is G-Hermitian (G-skew-Hermitian) if and only if $\mathscr{H}$ is $\imath G$-Hermitian ($\imath G$-skew-Hermitian). The real case, once again, has to be treated separately.

THEOREM 2.6. (Canonical form for G-Hamiltonian matrices, [16, 18, 26]) *Let $G \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and invertible and let $\mathscr{H} \in \mathbb{R}^{2n\times 2n}$ be G-Hamiltonian. Then there exists an invertible matrix $X \in \mathbb{R}^{2n\times 2n}$ such that*

$$X^{-1}\mathscr{H}X = \mathscr{H}_c \oplus \mathscr{H}_r \oplus \mathscr{H}_\imath \oplus \mathscr{H}_z, \quad X^T GX = G_c \oplus G_r \oplus G_\imath \oplus G_z,$$

where

$$\begin{aligned}
\mathscr{H}_c &= \mathscr{H}_{c,1} \oplus \cdots \oplus \mathscr{H}_{c,m_c}, & G_c &= G_{c,1} \oplus \cdots \oplus G_{c,m_c},\\
\mathscr{H}_r &= \mathscr{H}_{r,1} \oplus \cdots \oplus \mathscr{H}_{r,m_r}, & G_r &= G_{r,1} \oplus \cdots \oplus G_{r,m_r},\\
\mathscr{H}_\imath &= \mathscr{H}_{\imath,1} \oplus \cdots \oplus \mathscr{H}_{\imath,m_\imath}, & G_\imath &= G_{\imath,1} \oplus \cdots \oplus G_{\imath,m_\imath},\\
\mathscr{H}_z &= \mathscr{H}_{z,1} \oplus \cdots \oplus \mathscr{H}_{z,m_o+m_e}, & G_z &= G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e},
\end{aligned}$$

and where the diagonal blocks have the following forms:

1) *blocks associated with quadruples $(\lambda_j, \bar{\lambda}_j, -\lambda_j, -\bar{\lambda}_j)$ of nonreal, non purely imaginary eigenvalues of $\mathscr{H}$:*

$$\mathscr{H}_{c,j} = \begin{bmatrix} \mathscr{J}_{\xi_j}(a_j,b_j) & 0 \\ 0 & -\mathscr{J}_{\xi_j}(a_j,b_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{2\xi_j} \\ -R_{2\xi_j} & 0 \end{bmatrix},$$

where $a_j = \operatorname{Re}\lambda_j > 0$, $b_j = \operatorname{Im}\lambda_j > 0$, and $\xi_j \in \mathbb{N}$ for $j = 1,\ldots,m_c$;

2) *blocks associated with pairs $(\alpha_j, -\alpha_j)$ of real, nonzero eigenvalues of $\mathscr{H}$:*

$$\mathscr{H}_{r,j} = \begin{bmatrix} \mathscr{J}_{\eta_j}(\alpha_j) & 0 \\ 0 & -\mathscr{J}_{\eta_j}(\alpha_j) \end{bmatrix}, \quad G_{r,j} = \begin{bmatrix} 0 & R_{\eta_j} \\ -R_{\eta_j} & 0 \end{bmatrix},$$

where $\alpha_j > 0$ and $\eta_j \in \mathbb{N}$ for $j = 1,\ldots,m_r$;

3) *blocks associated with pairs $(\imath\beta_j, -\imath\beta_j)$ of purely imaginary, nonzero eigenvalues of $\mathscr{H}$:*

$$\mathscr{H}_{\imath,j} = \begin{bmatrix} 0 & \mathscr{J}_{\rho_j}(\beta_j) \\ -\mathscr{J}_{\rho_j}(\beta_j) & 0 \end{bmatrix}, \quad G_{\imath,j} = s_j \begin{bmatrix} 0 & R_{\rho_j} \\ -R_{\rho_j} & 0 \end{bmatrix},$$

where $\beta_j > 0$, $s_j \in \{-1,1\}$, and $\rho_j \in \mathbb{N}$ for $j = 1,\ldots,m_\imath$;

4) *blocks associated with the eigenvalue $\lambda = 0$ of $\mathscr{H}$:*

$$\mathscr{H}_{z,j} = \begin{bmatrix} \mathscr{J}_{\zeta_j}(0) & 0 \\ 0 & -\mathscr{J}_{\zeta_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\zeta_j} \\ -R_{\zeta_j} & 0 \end{bmatrix},$$

where $\zeta_j \in \mathbb{N}$ is odd for $j = 1,\ldots,m_o$, and

$$\mathscr{H}_{z,j} = \Xi_{\zeta_j}\mathscr{J}_{\zeta_j}(0), \quad G_{z,j} = t_j \Gamma_{\zeta_j},$$

where $\zeta_j \in \mathbb{N}$ is even and $t_j \in \{-1,1\}$ for $j = m_o+1,\ldots,m_o+m_e$.

$\mathscr{H}$ has the (not necessarily pairwise distinct) eigenvalues $\pm\lambda_1 \dots, \pm\lambda_{m_c}, \pm\overline{\lambda}_1, \dots, \pm\overline{\lambda}_{m_c}$, $\pm\alpha_1, \dots, \pm\alpha_{m_r}$, $\pm\imath\beta_1, \dots, \pm\imath\beta_{m_\imath}$, and the additional eigenvalue 0, provided that $m_o + m_e > 0$.

THEOREM 2.7. (Canonical form for G-skew-Hamiltonian matrices, [5, 18, 26]) *Let $G \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and invertible and let $\mathscr{K} \in \mathbb{R}^{2n\times 2n}$ be G-skew-Hamiltonian. Then there exists an invertible matrix $X \in \mathbb{R}^{2n\times 2n}$ such that*

$$X^{-1}\mathscr{K}X = \mathscr{K}_c \oplus \mathscr{K}_r, \quad X^T G X = G_c \oplus G_r,$$

where

$$\begin{array}{ll} \mathscr{K}_c = \mathscr{K}_{c,1} \oplus \cdots \oplus \mathscr{K}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathscr{K}_r = \mathscr{K}_{r,1} \oplus \cdots \oplus \mathscr{K}_{r,m_r}, & G_r = G_{r,1} \oplus \cdots \oplus G_{r,m_r}, \end{array}$$

and where the diagonal blocks have the following forms:

1) *blocks associated with pairs $(\lambda_j, \bar{\lambda}_j)$ of nonreal eigenvalues of $\mathscr{K}$:*

$$\mathscr{K}_{c,j} = \begin{bmatrix} \mathscr{J}_{\xi_j}(a_j,b_j) & 0 \\ 0 & \mathscr{J}_{\xi_j}(a_j,b_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{2\xi_j} \\ -R_{2\xi_j} & 0 \end{bmatrix},$$

where $a_j = \operatorname{Re}\lambda_j \in \mathbb{R}$, $b_j = \operatorname{Im}\lambda_j > 0$, and $\xi_j \in \mathbb{N}$ for $j = 1, \dots, m_c$;

2) *blocks associated with real eigenvalues α_j of $\mathscr{K}$:*

$$\mathscr{K}_{r,j} = \begin{bmatrix} \mathscr{J}_{\eta_j}(\alpha_j) & 0 \\ 0 & \mathscr{J}_{\eta_j}(\alpha_j) \end{bmatrix}, \quad G_{r,j} = \begin{bmatrix} 0 & R_{\eta_j} \\ -R_{\eta_j} & 0 \end{bmatrix},$$

where $\alpha_j \in \mathbb{R}$ and $\eta_j \in \mathbb{N}$ for $j = 1, \dots, m_r$.

$\mathscr{K}$ has the (not necessarily pairwise distinct) nonreal eigenvalues $a_1 \pm \imath b_1, \dots, a_{m_c} \pm \imath b_{m_c}$, and the (not necessarily pairwise distinct) real eigenvalues $\alpha_1, \dots, \alpha_{m_r}$ (possibly including zero).

In the following we need some results concerning the existence of structured square roots of structured matrices. This question has been deeply investigated in the literature mostly in the context of polar decompositions, and necessary and sufficient conditions for the existence of square roots have been developed, see [1, 2, 5]. We do not quote the results in full generality, but only consider the following special cases.

THEOREM 2.8. *Let $G \in \mathbb{F}^{n\times n}$ be Hermitian and nonsingular and let $\mathscr{H} \in \mathbb{F}^{n\times n}$ be G-Hermitian, nonsingular, and such that $\sigma(\mathscr{H}) \cap \mathbb{R}_- = \emptyset$. Then there exists a square root $\mathscr{S} \in \mathbb{F}^{n\times n}$ of $\mathscr{H}$ that satisfies $\sigma(\mathscr{S}) \subseteq \mathbb{C}_+$. This square root is unique and is a real polynomial in $\mathscr{H}$ (i.e., a polynomial in $\mathscr{H}$ whose coefficients are real). In particular, $\mathscr{S}$ is G-Hermitian.*

Proof. Comparing the canonical forms of G-Hermitian and G-symmetric matrices, it is easily seen that any pair $(G,\mathscr{H}) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$, where G is a nonsingular Hermitian matrix and $\mathscr{H}$ is G-Hermitian, can be transformed into a real pair $(G_r,\mathscr{H}_r) = (P^*GP, P^{-1}\mathscr{H}P)$ by some *complex* nonsingular transformation matrix $P \in \mathbb{C}^{n\times n}$. (This corresponds to the well-known fact that a matrix $\mathscr{H}$ is G-Hermitian for some Hermitian G if and only if $\mathscr{H}$ is similar to a real matrix, see [8].) Therefore, it is sufficient to consider the real case. Then by the discussion in Chapter 6.4 in [12], we obtain that a square root $\mathscr{S}$ of $\mathscr{H}$ with $\sigma(\mathscr{S}) \subseteq \mathbb{C}_+$ exists, is unique, and can be expressed as a polynomial (with real coefficients) in $\mathscr{H}$. Clearly, this polynomial stays invariant under the transformation with the transformation matrix P in the complex case. It is then straightforward to check that a real polynomial in $\mathscr{H}$ is again G-Hermitian. □

In the case of a skew-symmetric real bilinear form, we have a similar result. The proof follows exactly the same line as the proof of the preceding theorem.

THEOREM 2.9. *Let $G \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and nonsingular and let $\mathscr{K} \in \mathbb{R}^{2n\times 2n}$ be G-skew-Hamiltonian, nonsingular, and such that $\sigma(\mathscr{K}) \cap \mathbb{R}_- = \emptyset$. Then there exists a square root $\mathscr{S} \in \mathbb{R}^{2n\times 2n}$ of $\mathscr{K}$ that satisfies $\sigma(\mathscr{S}) \subseteq \mathbb{C}_+$. This square root is unique and is a real polynomial in $\mathscr{K}$. In particular, $\mathscr{S}$ is G-skew-Hamiltonian.*

One might ask whether G-Hamiltonian matrices have G-Hamiltonian or G-skew-Hamiltonian square roots, but this is never the case because squares of such matrices must always be G-skew-Hamiltonian. On the other hand, each real G-skew-Hamiltonian matrix $\mathscr{K}$ will have a G-Hamiltonian square root [5], but this square root cannot be a polynomial in $\mathscr{K}$, because such a polynomial would be G-skew-Hamiltonian again.

3. Canonical form for $G, \hat{G}$ Hermitian

In this section, we investigate the matrix triple $(A, G, \hat{G})$ for the case that both $G, \hat{G}$ are Hermitian and nonsingular. We first consider the simpler case that A is square and nonsingular.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and let $G, \hat{G} \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$ such that*

$$X^*AY = A_c \oplus A_r, \qquad X^*GX = G_c \oplus G_r, \qquad Y^*\hat{G}Y = \hat{G}_c \oplus \hat{G}_r, \tag{3.1}$$

and for the $\hat{G}$-Hermitian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^*G^{-1}A \in \mathbb{C}^{n\times n}$ and for the G-Hermitian matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^* \in \mathbb{C}^{n\times n}$, we have that

$$Y^{-1}\hat{\mathscr{H}}Y = \hat{\mathscr{H}}_c \oplus \hat{\mathscr{H}}_r, \qquad X^{-1}\mathscr{H}X = \mathscr{H}_c \oplus \mathscr{H}_r. \tag{3.2}$$

The diagonal blocks in these decompositions have the following forms:

1) *blocks associated with pairs* $(\mu_j^2, \bar{\mu}_j^2)$ *of nonreal eigenvalues of* $\hat{\mathscr{H}}$ *and* $\mathscr{H}$*:*

$$\begin{aligned}
A_c &= \begin{bmatrix} \mathscr{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathscr{J}_{\xi_1}(\bar{\mu}_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}(\mu_{m_c}) & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}(\bar{\mu}_{m_c}) \end{bmatrix}, \\
G_c &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_{m_c}} \\ R_{\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{G}_c &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_{m_c}} \\ R_{\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_c &= \begin{bmatrix} \mathscr{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathscr{J}_{\xi_1}^2(\bar{\mu}_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}^2(\mu_{m_c}) & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}^2(\bar{\mu}_{m_c}) \end{bmatrix}, \\
\mathscr{H}_c &= \begin{bmatrix} \mathscr{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathscr{J}_{\xi_1}^2(\bar{\mu}_1) \end{bmatrix}^* \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}^2(\mu_{m_c}) & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}^2(\bar{\mu}_{m_c}) \end{bmatrix}^*,
\end{aligned}$$

where $\mu_j \in \mathbb{C}$, $\arg \mu_j \in (0, \pi/2)$, *and* $\xi_j \in \mathbb{N}$ *for* $j = 1, \ldots, m_c$*;*

2) *blocks associated with real eigenvalues* α_j *of* $\mathscr{H}$ *and* $\hat{\mathscr{H}}$*:*

$$\begin{aligned}
A_r &= \mathscr{J}_{\eta_1}(\beta_1) \oplus \cdots \oplus \mathscr{J}_{\eta_{m_r}}(\beta_{m_r}), \\
G_r &= s_1 R_{\eta_1} \oplus \cdots \oplus s_{m_r} R_{\eta_{m_r}}, \\
\hat{G}_r &= \hat{s}_1 R_{\eta_1} \oplus \cdots \oplus \hat{s}_{m_r} R_{\eta_{m_r}}, \\
\hat{\mathscr{H}}_r &= s_1\hat{s}_1 \mathscr{J}_{\eta_1}^2(\beta_1) \oplus \cdots \oplus s_{m_r}\hat{s}_{m_r} \mathscr{J}_{\eta_{m_r}}^2(\beta_{m_r}), \\
\mathscr{H}_r &= s_1\hat{s}_1 \big(\mathscr{J}_{\eta_1}^2(\beta_1)\big)^* \oplus \cdots \oplus s_{m_r}\hat{s}_{m_r} \big(\mathscr{J}_{\eta_{m_r}}^2(\beta_{m_r})\big)^*,
\end{aligned}$$

where $\beta_j > 0$, $s_j, \hat{s}_j \in \{+1, -1\}$, *and* $\eta_j \in \mathbb{N}$ *for* $j = 1, \ldots, m_r$. *Thus,* $\alpha_j = \beta_j^2 > 0$ *if* $s_j = \hat{s}_j$ *and* $\alpha_j = -\beta_j^2 < 0$ *if* $s_j \neq \hat{s}_j$.

Moreover, the form (3.1) *is unique up to the simultaneous permutation of blocks in the right hand side of* (3.1).

Proof. The proof will be performed in two steps.
Step 1) We first show that we may assume without loss of generality that $\hat{\mathscr{H}}$ either has only one pair of conjugate complex nonreal eigenvalues $(\lambda, \bar{\lambda})$ or only one real eigenvalue α.

Indeed, in view of Theorem 2.2, there exists a nonsingular matrix $Y \in \mathbb{C}^{n \times n}$ such that

$$Y^{-1}\hat{\mathscr{H}}Y = \hat{\mathscr{H}}_1 \oplus \hat{\mathscr{H}}_2, \quad Y^*\hat{G}Y = \hat{G}_1 \oplus \hat{G}_2,$$

where $\hat{\mathscr{H}}_1, \hat{G}_1 \in \mathbb{C}^{p \times p}$, $\hat{\mathscr{H}}_2, \hat{G}_2 \in \mathbb{C}^{(n-p)\times(n-p)}$, $\sigma(\hat{\mathscr{H}}_1) \cap \sigma(\hat{\mathscr{H}}_2) = \emptyset$, and $\hat{\mathscr{H}}_1$ either has only one eigenvalue that is real or only two eigenvalues that are conjugate complex. Using

$$G^{-1}A\hat{\mathscr{H}} = \mathscr{H}G^{-1}A$$

and the fact that $G^{-1}A$ is nonsingular, we find that $\mathscr{H}$ and $\hat{\mathscr{H}}$ are similar. Thus, there exists a nonsingular matrix $X \in \mathbb{C}^{m \times m}$ such that

$$X^{-1}\mathscr{H}X = \hat{\mathscr{H}}_1 \oplus \hat{\mathscr{H}}_2 = \begin{bmatrix} \hat{\mathscr{H}}_1 & 0 \\ 0 & \hat{\mathscr{H}}_2 \end{bmatrix}, \quad X^*GX = \begin{bmatrix} G_1 & G_{12} \\ G_{12}^* & G_2 \end{bmatrix},$$

Here, G has been partitioned conformably with $\mathscr{H}$. By assumption, $\sigma(\hat{\mathscr{H}}_1) = \sigma(\hat{\mathscr{H}}_1^*)$ and thus $\sigma(\hat{\mathscr{H}}_1^*) \cap \sigma(\hat{\mathscr{H}}_2) = \emptyset$. Then using that $\mathscr{H}$ is G-Hermitian, i.e.,

$$\begin{bmatrix} \hat{\mathscr{H}}_1^* & 0 \\ 0 & \hat{\mathscr{H}}_2^* \end{bmatrix} \begin{bmatrix} G_1 & G_{12} \\ G_{12}^* & G_2 \end{bmatrix} = X^* \mathscr{H}^* G X = X^* G \mathscr{H} X = \begin{bmatrix} G_1 & G_{12} \\ G_{12}^* & G_2 \end{bmatrix} \begin{bmatrix} \hat{\mathscr{H}}_1 & 0 \\ 0 & \hat{\mathscr{H}}_2 \end{bmatrix},$$

we obtain $G_{12} = 0$, because the Sylvester equation $\hat{\mathscr{H}}_1^* G_{12} - G_{12} \hat{\mathscr{H}}_2 = 0$ only has the trivial solution, given that the spectra of the coefficient matrices $\hat{\mathscr{H}}_1^*$ and $\hat{\mathscr{H}}_2$ do not intersect. Next, we will show that X^*AY decomposes in the same way as $\mathscr{H}, \hat{\mathscr{H}}, G$, and $\hat{G}$. To this end, we partition

$$(X^*AY)^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

conformably with G and $\hat{G}$. Then

$$\hat{\mathscr{H}} A^{-1} = \hat{G}^{-1} A^* G^{-1} = (G^{-1} A \hat{G}^{-1})^* = (\mathscr{H} A^{-*})^* = A^{-1} \mathscr{H}^*$$

implies

$$\begin{bmatrix} \hat{\mathscr{H}}_1 & 0 \\ 0 & \hat{\mathscr{H}}_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathscr{H}}_1^* & 0 \\ 0 & \hat{\mathscr{H}}_2^* \end{bmatrix}$$

Using once again the fact that a Sylvester equation only has the trivial solution if the spectra of the coefficient matrices do not intersect, we finally obtain that

$$(X^*AY)^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

and thus X^*AY is block diagonal as well. Repeating this argument several times, we see that it remains to study triples $(A, G, \hat{G})$ for which $\hat{\mathscr{H}}$ has the restricted spectrum as initially stated.

Step 2) By Step 1), we may assume without loss of generality that $\hat{\mathscr{H}}$ either has only one pair of conjugate complex nonreal eigenvalues $(\lambda, \bar{\lambda})$ or only one real eigenvalue α. We discuss these two cases separately.

Case 1: $\sigma(\hat{\mathscr{H}}) = \{\lambda, \bar{\lambda}\}$ for some $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > 0$.

By Theorem 2.8, $\hat{\mathscr{H}}$ has a unique $\hat{G}$-Hermitian square root $S \in \mathbb{C}^{n\times n}$ satisfying $\sigma(S) \subseteq \mathbb{C}_+$. Then by Theorem 2.2, there exists a nonsingular matrix $\widetilde{Y} \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} S_{\text{CF}} := \widetilde{Y}^{-1} S \widetilde{Y} &= \begin{bmatrix} \mathscr{J}_{\xi_1}(\mu) & 0 \\ 0 & \mathscr{J}_{\xi_1}(\bar{\mu}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_m}(\mu) & 0 \\ 0 & \mathscr{J}_{\xi_m}(\bar{\mu}) \end{bmatrix}, \\ G_{\text{CF}} := \widetilde{Y}^* \hat{G} \widetilde{Y} &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}, \\ \mathscr{H}_{\text{CF}} := \widetilde{Y}^{-1} \hat{\mathscr{H}} \widetilde{Y} &= \begin{bmatrix} \mathscr{J}_{\xi_1}^2(\mu) & 0 \\ 0 & \mathscr{J}_{\xi_1}^2(\bar{\mu}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_m}^2(\mu) & 0 \\ 0 & \mathscr{J}_{\xi_m}^2(\bar{\mu}) \end{bmatrix}, \end{aligned}$$

where $\mu = \sqrt{\lambda} \in \mathbb{C}$, $\arg\mu \in (0, \frac{\pi}{2})$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Here, the third identity immediately follows from $\hat{\mathscr{H}} = S^2$. Since $\mathscr{H}$ and $\hat{\mathscr{H}}$ are similar and since $\hat{\mathscr{H}}$

has only a pair of conjugate complex nonreal eigenvalues, we obtain from Theorem 2.2 that the canonical forms of the pairs $(\mathscr{H},G)$ and $(\hat{\mathscr{H}},\hat{G})$ coincide. In particular, this implies the existence of a nonsingular matrix $\widetilde{X}\in\mathbb{C}^{n\times n}$ such that

$$\begin{aligned}
\mathscr{H}_{\text{CF}}=\widetilde{X}^{-1}\mathscr{H}\widetilde{X} &= \begin{bmatrix}\mathscr{J}^2_{\xi_1}(\mu) & 0\\ 0 & \mathscr{J}^2_{\xi_1}(\bar{\mu})\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}\mathscr{J}^2_{\xi_m}(\mu) & 0\\ 0 & \mathscr{J}^2_{\xi_m}(\bar{\mu})\end{bmatrix},\\
G_{\text{CF}}=\widetilde{X}^{*}G\widetilde{X} &= \begin{bmatrix}0 & R_{\xi_1}\\ R_{\xi_1} & 0\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}0 & R_{\xi_m}\\ R_{\xi_m} & 0\end{bmatrix}.
\end{aligned}$$

Finally, setting $X=G^{-1}\widetilde{X}^{-*}$ and $Y=A^{-1}G\widetilde{X}S_{\text{CF}}$, we obtain

$$\begin{aligned}
X^*AY &= \widetilde{X}^{-1}G^{-1}AA^{-1}G\widetilde{X}S_{\text{CF}}=S_{\text{CF}}\\
X^*GX &= \widetilde{X}^{-1}G^{-1}GG^{-1}\widetilde{X}^{-*}=(\widetilde{X}^*G\widetilde{X})^{-1}=G_{\text{CF}}^{-1}=G_{\text{CF}}\\
Y^*\hat{G}Y &= S_{\text{CF}}^*\widetilde{X}^*GA^{-*}\hat{G}A^{-1}G\widetilde{X}S_{\text{CF}}\\
&= S_{\text{CF}}^*\widetilde{X}^*G\widetilde{X}\widetilde{X}^{-1}\mathscr{H}^{-1}\widetilde{X}S_{\text{CF}}\\
&= S_{\text{CF}}^*G_{\text{CF}}(\mathscr{H}_{\text{CF}})^{-1}S_{\text{CF}}=G_{\text{CF}}S_{\text{CF}}(\mathscr{H}_{\text{CF}})^{-1}S_{\text{CF}}=G_{\text{CF}}
\end{aligned}$$

as desired, where we have used that S_{CF} is G_{CF}-Hermitian and that $S_{\text{CF}}^2=\mathscr{H}_{\text{CF}}$. It is now easy to check that $X^{-1}\mathscr{H}X$ and $Y^{-1}\hat{\mathscr{H}}Y$ have the claimed forms.

Case 2: $\sigma(\hat{\mathscr{H}})=\{\alpha\}$ for some $\alpha\in\mathbb{R}\setminus\{0\}$.
Observe that $\operatorname{sign}(\alpha)\hat{\mathscr{H}}$ has the only positive eigenvalue $|\alpha|$. Thus, we can apply Theorems 2.8 and 2.2 which yield the existence of a square root $S\in\mathbb{C}^{n\times n}$ of $\operatorname{sign}(\alpha)\hat{\mathscr{H}}$ and a nonsingular matrix $\widetilde{Y}\in\mathbb{C}^{n\times n}$ such that

$$\begin{aligned}
S_{\text{CF}}:=\widetilde{Y}^{-1}S\widetilde{Y} &= \mathscr{J}_{\eta_1}(\beta)\oplus\cdots\oplus\mathscr{J}_{\eta_m}(\beta),\\
\widetilde{Y}^*\hat{G}\widetilde{Y} &= \hat{s}_1R_{\eta_1}\oplus\cdots\oplus\hat{s}_mR_{\eta_m},\\
\mathscr{H}_{\text{CF}}:=\widetilde{Y}^{-1}\hat{\mathscr{H}}\widetilde{Y} &= \operatorname{sign}(\alpha)\mathscr{J}^2_{\eta_1}(\beta)\oplus\cdots\oplus\operatorname{sign}(\alpha)\mathscr{J}^2_{\eta_m}(\beta),
\end{aligned}$$

where $\beta=\sqrt{|\alpha|}$, $\eta_j\in\mathbb{N}$ and $\hat{s}_j\in\{+1,-1\}$ for $j=1,\dots,m$. Again using that $\mathscr{H}$ and $\hat{\mathscr{H}}$ are similar, we obtain from Theorem 2.2 the existence of a nonsingular matrix $\widetilde{X}\in\mathbb{C}^{n\times n}$ such that

$$\begin{aligned}
\mathscr{H}_{\text{CF}}=\widetilde{X}^{-1}\mathscr{H}\widetilde{X} &= \operatorname{sign}(\alpha)\mathscr{J}^2_{\eta_1}(\beta)\oplus\cdots\oplus\operatorname{sign}(\alpha)\mathscr{J}^2_{\eta_m}(\beta),\\
G_{\text{CF}}:=\widetilde{X}^*G\widetilde{X} &= s_1R_{\eta_1}\oplus\cdots\oplus s_mR_{\eta_m}
\end{aligned}$$

for some $s_1,\dots,s_m\in\{+1,-1\}$. Setting $X=G^{-1}\widetilde{X}^{-*}$ and $Y=A^{-1}G\widetilde{X}S_{\text{CF}}$, we obtain as in *Case 1* that $X^*AY=S_{\text{CF}}$, $X^*GX=G_{\text{CF}}$, and $Y^*\hat{G}Y=G_{\text{CF}}S_{\text{CF}}(\mathscr{H}_{\text{CF}})^{-1}S_{\text{CF}}=\operatorname{sign}(\alpha)G_{\text{CF}}$.

We mention in passing that it is possible to link the sign characteristic $(\hat{s}_1,\dots,\hat{s}_m)$ to the sign characteristic $(s_1,\dots,s_m)$, but we refrain from doing so, because the explicit knowledge of the parameters $\hat{s}_1,\dots,\hat{s}_m$ is irrelevant for the development of the canonical form for the triple $(A,G,\hat{G})$. It is now straightforward to check that $X^{-1}\mathscr{H}X$ and $Y^{-1}\hat{\mathscr{H}}Y$ have the claimed forms. Concerning uniqueness, we note that the form (3.1) is uniquely determined by the canonical form of $\hat{\mathscr{H}}$ as a $\hat{G}$-Hermitian matrix, and the restrictions $\arg\mu_j\in(0,\pi/2)$ and $\beta>0$. $\square$

In the general situation that A is non square, we have the following result.

THEOREM 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and let $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} X^*AY &= A_{nz} \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^*GX &= G_{nz} \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^*\hat{G}Y &= \hat{G}_{nz} \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4}. \end{aligned} \tag{3.3}$$

Moreover, for the $\hat{G}$-Hermitian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^*G^{-1}A \in \mathbb{C}^{n\times n}$ and for the G-symmetric matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^* \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathscr{H}}Y &= \hat{\mathscr{H}}_{nz} \oplus \hat{\mathscr{H}}_{z,1} \oplus \hat{\mathscr{H}}_{z,2} \oplus \hat{\mathscr{H}}_{z,3} \oplus \hat{\mathscr{H}}_{z,4}, \\ X^{-1}\mathscr{H}X &= \mathscr{H}_{nz} \oplus \mathscr{H}_{z,1} \oplus \mathscr{H}_{z,2} \oplus \mathscr{H}_{z,3} \oplus \mathscr{H}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\hat{\mathscr{H}}$ and $\mathscr{H}$:*
   $A_{nz}, G_{nz}, \hat{G}_{nz}$ have the forms as in (3.1) *and $\hat{\mathscr{H}}_{nz}, \mathscr{H}_{nz}$ have the forms as in* (3.2)*;*

1) *one block corresponding to n_0 Jordan blocks of size 1×1 of $\hat{\mathscr{H}}$ and m_0 Jordan blocks of size 1×1 of $\mathscr{H}$ associated with the eigenvalue zero:*

$$A_{z,1} = \mathscr{O}_{m_0\times n_0},\ G_{z,1} = \Sigma_{\pi_0,\nu_0},\ \hat{G}_{z,1} = \Sigma_{\hat{\pi}_0,\hat{\nu}_0},\ \hat{\mathscr{H}}_{z,1} = \mathscr{O}_{n_0},\ \mathscr{H}_{z,1} = \mathscr{O}_{m_0},$$

   where $m_0, n_0, \pi_0, \nu_0, \hat{\pi}_0, \hat{\nu}_0 \in \mathbb{N}\cup\{0\}$ and $\pi_0+\nu_0 = m_0$, $\hat{\pi}_0+\hat{\nu}_0 = n_0$;

2) *blocks corresponding to a pair of $j\times j$ Jordan blocks of $\hat{\mathscr{H}}$ and $\mathscr{H}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathscr{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathscr{J}_4(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \mathscr{J}_{2\ell}(0), \\ G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} R_{2\ell}, \\ \hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} R_{2\ell}, \\ \hat{\mathscr{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathscr{J}_2^2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathscr{J}_4^2(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \mathscr{J}_{2\ell}^2(0), \\ \mathscr{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathscr{J}_2^2(0)^T \oplus \bigoplus_{i=1}^{\gamma_2} \mathscr{J}_4^2(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \mathscr{J}_{2\ell}^2(0)^T, \end{aligned}$$

   where $\gamma_1,\ldots,\gamma_\ell \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,2}$ and $\mathscr{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j\times j$, where exactly γ_j blocks have sign $+1$ and γ_j blocks have sign -1, for $j = 1,\ldots,\ell$;

3) *blocks corresponding to a $j\times j$ Jordan block of $\hat{\mathscr{H}}$ and a $(j+1)\times(j+1)$ Jordan block of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_1}\begin{bmatrix} I_1 \\ 0 \end{bmatrix}_{2\times 1} \oplus \bigoplus_{i=1}^{m_2}\begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\ell-1}}\begin{bmatrix} I_{\ell-1} \\ 0 \end{bmatrix}_{\ell\times(\ell-1)},\\
G_{z,3} &= \bigoplus_{i=1}^{m_1} s_1^{(i)}R_2 \oplus \bigoplus_{i=1}^{m_2} s_2^{(i)}R_3 \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\ell-1}} s_{\ell-1}^{(i)}R_\ell\,,\\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_1} \hat{s}_1^{(i)}R_1 \oplus \bigoplus_{i=1}^{m_2} \hat{s}_2^{(i)}R_2 \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\ell-1}} \hat{s}_{\ell-1}^{(i)}R_{\ell-1}\,,\\
\hat{\mathscr{H}}_{z,3} &= \bigoplus_{i=1}^{m_1} s_1^{(i)}\hat{s}_1^{(i)}\mathscr{J}_1(0) \oplus \bigoplus_{i=1}^{m_2} s_2^{(i)}\hat{s}_2^{(i)}\mathscr{J}_2(0) \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\ell-1}} s_{\ell-1}^{(i)}\hat{s}_{\ell-1}^{(i)}\mathscr{J}_{\ell-1}(0)\,,\\
\mathscr{H}_{z,3} &= \bigoplus_{i=1}^{m_1} s_1^{(i)}\hat{s}_1^{(i)}\mathscr{J}_2(0)^T \oplus \bigoplus_{i=1}^{m_2} s_2^{(i)}\hat{s}_2^{(i)}\mathscr{J}_3(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\ell-1}} s_{\ell-1}^{(i)}\hat{s}_{\ell-1}^{(i)}\mathscr{J}_\ell(0)^T,
\end{aligned}
$$

where $m_1,\ldots,m_{\ell-1}\in\mathbb{N}\cup\{0\}$, and for $j=1,\ldots,\ell-1$, we have that $s_j^{(i)}=1$ and $\hat{s}_j^{(i)}\in\{+1,-1\}$ if j is odd, and $s_j^{(i)}\in\{+1,-1\}$ and $\hat{s}_j^{(i)}=1$ if j is even; thus $\hat{\mathscr{H}}_{z,3}$ has m_j Jordan blocks of size $j\times j$ with signs $\hat{s}_j^{(i)}$ if j is odd and signs $s_j^{(i)}$ if j is even, and $\mathscr{H}_{z,3}$ has m_j Jordan blocks of size $(j+1)\times(j+1)$ with signs $\hat{s}_j^{(i)}$ if j is odd and signs $s_j^{(i)}$ if j is even for $i=1,\ldots,m_j$ and $j=1,\ldots,\ell-1$;

4) *blocks corresponding to a $(j+1)\times(j+1)$ Jordan blocks of $\hat{\mathscr{H}}$ and a $j\times j$ Jordan block of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{n_1}\begin{bmatrix} 0\, I_1 \end{bmatrix}_{1\times 2} \oplus \bigoplus_{i=1}^{n_2}\begin{bmatrix} 0\, I_2 \end{bmatrix}_{2\times 3} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{\ell-1}}\begin{bmatrix} 0\, I_{\ell-1} \end{bmatrix}_{(\ell-1)\times\ell},\\
G_{z,4} &= \bigoplus_{i=1}^{n_1} s_1^{(i)}R_1 \oplus \bigoplus_{i=1}^{n_2} s_2^{(i)}R_2 \oplus\cdots\oplus \bigoplus_{i=1}^{n_{\ell-1}} s_{\ell-1}^{(i)}R_{\ell-1}\,,\\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{n_1} \hat{s}_1^{(i)}R_2 \oplus \bigoplus_{i=1}^{n_2} \hat{s}_2^{(i)}R_3 \oplus\cdots\oplus \bigoplus_{i=1}^{n_{\ell-1}} \hat{s}_{\ell-1}^{(i)}R_\ell\,,\\
\hat{\mathscr{H}}_{z,4} &= \bigoplus_{i=1}^{n_1} s_1^{(i)}\hat{s}_1^{(i)}\mathscr{J}_2(0) \oplus \bigoplus_{i=1}^{n_2} s_2^{(i)}\hat{s}_2^{(i)}\mathscr{J}_3(0) \oplus\cdots\oplus \bigoplus_{i=1}^{n_{\ell-1}} s_{\ell-1}^{(i)}\hat{s}_{\ell-1}^{(i)}\mathscr{J}_\ell(0),\\
\mathscr{H}_{z,4} &= \bigoplus_{i=1}^{n_1} s_1^{(i)}\hat{s}_1^{(i)}\mathscr{J}_1(0)^T \oplus \bigoplus_{i=1}^{n_2} s_2^{(i)}\hat{s}_2^{(i)}\mathscr{J}_2(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{n_{\ell-1}} s_{\ell-1}^{(i)}\hat{s}_{\ell-1}^{(i)}\mathscr{J}_{\ell-1}(0)^T\,,
\end{aligned}
$$

where $n_1,\ldots,n_{\ell-1}\in\mathbb{N}\cup\{0\}$, and for $j=1,\ldots,\ell-1$, we have that $s_j^{(i)}=1$ and $\hat{s}_j^{(i)}\in\{+1,-1\}$ if j is even, and $s_j^{(i)}\in\{+1,-1\}$ and $\hat{s}_j^{(i)}=1$ if j is odd; thus, $\hat{\mathscr{H}}_{z,4}$ has n_j Jordan blocks of size $(j+1)\times(j+1)$ with signs $s_j^{(i)}$ if j is odd and signs $\hat{s}_j^{(i)}$ if j is even, and $\mathscr{H}_{z,4}$ has n_j Jordan blocks of size $j\times j$ with signs $s_j^{(i)}$ if j is odd and signs $\hat{s}_j^{(i)}$ if j is even for $i=1,\ldots,m_j$ and $j=1,\ldots,\ell-1$;

For the eigenvalue zero, the matrices $\hat{\mathscr{H}}$ and $\mathscr{H}$ have $2\gamma_j + m_j + n_{j-1}$ respectively $2\gamma_j + m_{j-1} + n_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \ell$, where $m_\ell = n_\ell = 0$ and where ℓ is the maximum of the index of $\hat{\mathscr{H}}$ and the index of $\mathscr{H}$. (Here, index *refers to the maximal size of a Jordan block associated with the eigenvalue zero.)*

Furthermore, the form (3.3) *is unique up to simultaneous block permutation of the blocks in the block diagonal of the right hand side of* (3.3).

Proof. Due to its very technical nature, the proof is omitted here and presented in the Appendix. ◻

Since the canonical form of Theorem 3.2 is quite complicated, we present some examples to illustrate this form.

EXAMPLE 3.3. Let A, G, $\hat{G}$ be given by

$$A = \left[\begin{array}{cc|c|cc|ccc} 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ \hline 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&1&0&0&0 \\ \hline 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{array}\right], \quad G = \left[\begin{array}{cc|cc|c|cc} 0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ \hline 0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0 \\ \hline 0&0&0&0&+1&0&0 \\ \hline 0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0 \end{array}\right], \quad \hat{G} = \left[\begin{array}{cc|c|cc|ccc} 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ \hline 0&0&-1&0&0&0&0&0 \\ \hline 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&-1&0&0 \end{array}\right].$$

Then the canonical form consists of one block of type 2) with $j = 2$, one block of type 3) with $j = 1$ and sign $\hat{s}_1^{(1)} = -1$, and two blocks of type 4), one with $j = 1$ and sign $s_1^{(1)} = +1$ and one with $j = 2$ and sign $\hat{s}_2^{(1)} = -1$. Observe that the signs only occur in the blocks of G or $\hat{G}$, respectively, that have odd size. The signs attached to the corresponding even sized blocks are always $+1$. Thus, for example, the signs corresponding to blocks of type 4) will always be found in G if j is odd and they can be read off $\hat{G}$ if j is even.

EXAMPLE 3.4. It is important to note that rectangular matrices with a total number of zero rows or columns are allowed in the canonical form. For example consider the two non-equivalent triples

$$A_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; G_1 = \begin{bmatrix} -1 \end{bmatrix}, \; \hat{G}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{and} \qquad A_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; G_2 = \begin{bmatrix} -1 \end{bmatrix}, \; \hat{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The first example is just one block of type 4) with sign $s_1^{(1)} = -1$. Indeed, forming the products

$$\hat{\mathscr{H}}_1 = \hat{G}_1^{-1}A_1^*G_1^{-1}A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{H}_1 = G_1^{-1}A\hat{G}_1^{-1}A_1^* = \begin{bmatrix} 0 \end{bmatrix},$$

as predicted, $\hat{\mathscr{H}}_1$ has only one Jordan block of size 2 associated with the eigenvalue $\lambda = 0$ and the sign $s = -1$, while $\mathscr{H}_1$ has one Jordan block of size 1 associated with $\lambda = 0$ and the sign $s = -1$. The situation is different in the second case. Here, we obtain

$$\hat{\mathscr{H}}_2 = \hat{G}_2^{-1}A_2^*G_2^{-1}A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathscr{H}_2 = G_2^{-1}A\hat{G}_2^{-1}A_2^* = \begin{bmatrix} 1 \end{bmatrix},$$

i.e., $\hat{\mathscr{H}}_2$ has two Jordan blocks of size 1, one associated with $\lambda = 0$ and sign $s_1 = 1$ and a second one associated with $\lambda = 1$ and sign $s_2 = -1$, while $\mathscr{H}_2$ has one Jordan block of size 1 associated with $\lambda = 1$ and sign $s = -1$. Here, the triple $(A_2, G_2, \hat{G}_2)$ is in canonical form consisting of one block of type 1) and size 0×1 and of one block of type 0):

$$A_2 = \left[\begin{array}{c|c} 0 & 1 \end{array}\right], \; G_2 = \left[\begin{array}{|c} -1 \end{array}\right], \; \hat{G}_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right].$$

We have the following real versions of Theorem 3.1 and Theorem 3.2.

THEOREM 3.5. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let $G, \hat{G} \in \mathbb{R}^{n\times n}$ be symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{R}^{n\times n}$ such that*

$$X^T AY = A_c \oplus A_r, \qquad X^T GX = G_c \oplus G_r, \qquad Y^T \hat{G}Y = \hat{G}_c \oplus \hat{G}_r. \tag{3.4}$$

Moreover, for the $\hat{G}$-symmetric matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^T G^{-1}A$ and for the G-symmetric matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^T$, we have that

$$Y^{-1}\hat{\mathscr{H}}Y = \hat{\mathscr{H}}_c \oplus \hat{\mathscr{H}}_r, \qquad X^{-1}\mathscr{H}X = \mathscr{H}_c \oplus \mathscr{H}_r. \tag{3.5}$$

The diagonal blocks in these decompositions have the following forms:

1) *blocks associated with pairs $(\mu_j^2, \bar{\mu}_j^2)$ of nonreal eigenvalues of $\hat{\mathscr{H}}$ and $\mathscr{H}$:*

$$\begin{aligned}
A_c &= \mathscr{J}_{\xi_1}(a_1,b_1) \oplus\cdots\oplus \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c}),\\
G_c &= R_{2\xi_1} \oplus\cdots\oplus R_{2\xi_{m_c}},\\
\hat{G}_c &= R_{2\xi_1} \oplus\cdots\oplus R_{2\xi_{m_c}},\\
\hat{\mathscr{H}}_c &= \mathscr{J}_{\xi_1}^2(a_1,b_1) \oplus\cdots\oplus \mathscr{J}_{\xi_{m_c}}^2(a_{m_c},b_{m_c}),\\
\mathscr{H}_c &= \mathscr{J}_{\xi_1}^2(a_1,b_1)^T \oplus\cdots\oplus \mathscr{J}_{\xi_{m_c}}^2(a_{m_c},b_{m_c})^T,
\end{aligned}$$

where $a_j, b_j > 0$, $\mu_j = a_j + \imath b_j$, and $\xi_j \in \mathbb{N}$ for $j = 1,\ldots,m_c$;

2) *blocks associated with real eigenvalues α_j of $\mathscr{H}$ and $\hat{\mathscr{H}}$:*

$$\begin{array}{rcccc} A_r &=& \mathscr{J}_{\eta_1}(\beta_1) & \oplus\cdots\oplus & \mathscr{J}_{\eta_{m_r}}(\beta_{m_r}), \\ G_r &=& s_1 R_{\eta_1} & \oplus\cdots\oplus & s_{m_r} R_{\eta_{m_r}}, \\ \hat{G}_r &=& \hat{s}_1 R_{\eta_1} & \oplus\cdots\oplus & \hat{s}_{m_r} R_{\eta_{m_r}}, \\ \hat{\mathscr{H}}_r &=& s_1\hat{s}_1 \mathscr{J}^2_{\eta_1}(\beta_1) & \oplus\cdots\oplus & s_{m_r}\hat{s}_{m_r} \mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r}), \\ \mathscr{H}_r &=& s_1\hat{s}_1 \left(\mathscr{J}^2_{\eta_1}(\beta_1)\right)^T & \oplus\cdots\oplus & s_{m_r}\hat{s}_{m_r} \left(\mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r})\right)^T, \end{array}$$

where $\beta_j > 0$, $s_j, \hat{s}_j \in \{+1,-1\}$, and $\eta_j \in \mathbb{N}$ for $j = 1,\dots,m_r$. Thus, $\alpha_j = \beta_j^2 > 0$ if $s_j = \hat{s}_j$ and $\alpha_j = -\beta_j^2 < 0$ if $s_j \neq \hat{s}_j$.

Furthermore, the form (3.4) *is unique up to the simultaneous permutation of blocks in the right hand side of* (3.4).

Proof. The proof follows exactly the same lines as the proof of Theorem 3.1. (The key point here is that the square roots that are constructed analogously to the proof of Theorem 3.1 are real, see Theorem 2.8.) □

THEOREM 3.6. *Let $A \in \mathbb{R}^{m\times n}$ and let $G \in \mathbb{R}^{m\times m}$ and $\hat{G} \in \mathbb{R}^{n\times n}$ be symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} X^T AY &= A_{nz} \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^T GX &= G_{nz} \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T \hat{G} Y &= \hat{G}_{nz} \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4}. \end{aligned} \tag{3.6}$$

Moreover, for the $\hat{G}$-symmetric matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^T G^{-1} A \in \mathbb{C}^{n\times n}$ and for the G-symmetric matrix $\mathscr{H} = G^{-1} A \hat{G}^{-1} A^T \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathscr{H}} Y &= \hat{\mathscr{H}}_{nz} \oplus \hat{\mathscr{H}}_{z,1} \oplus \hat{\mathscr{H}}_{z,2} \oplus \hat{\mathscr{H}}_{z,3} \oplus \hat{\mathscr{H}}_{z,4}, \\ X^{-1}\mathscr{H} X &= \mathscr{H}_{nz} \oplus \mathscr{H}_{z,1} \oplus \mathscr{H}_{z,2} \oplus \mathscr{H}_{z,3} \oplus \mathscr{H}_{z,4}. \end{aligned}$$

Here, the blocks $A_{nz}, G_{nz}, \hat{G}_{nz}, \hat{\mathscr{H}}_{nz}$, and $\mathscr{H}_{nz}$ have the forms as in (3.4) *and* (3.5)*, while $A_{z,k}$, $G_{z,k}$, $\hat{G}_{z,k}$, $\hat{\mathscr{H}}_{z,k}$, and $\mathscr{H}_{z,k}$ have the forms as in Theorem* 3.2 *for $k = 1,\dots,4$.*

Moreover, the form (3.6) *is unique up to the simultaneous permutation of blocks in the right hand side of* (3.6).

Proof. The proof follows exactly the same lines as the proof of Theorem 3.2. □

In the particular case that one of the Hermitian matrices is positive definite (say G), we obtain the following special case of Theorem 3.2 and Theorem 3.6 that can be interpreted as a generalization of both the Schur form for a Hermitian matrix as well as a generalization of the standard singular value decomposition.

COROLLARY 3.7. *Let $A \in \mathbb{F}^{m\times n}$, let $G \in \mathbb{F}^{m\times m}$ be Hermitian and positive definite, and let $\hat{G} \in \mathbb{F}^{n\times n}$ be Hermitian and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{F}^{m\times m}$ and $Y \in \mathbb{F}^{n\times n}$ such that*

$$X^*AY = \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_{m_r} \end{bmatrix} \oplus \mathscr{O}_{m_0\times n_0} \oplus \begin{bmatrix} \mathscr{O}_{n_1} & I_{n_1} \end{bmatrix},$$

$$X^*GX = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \oplus I_{m_0} \oplus I_{n_1} = I_m,$$

$$Y^*\hat{G}Y = \begin{bmatrix} \hat{s}_1 & & 0 \\ & \ddots & \\ 0 & & \hat{s}_{m_r} \end{bmatrix} \oplus \Sigma_{\hat{\pi}_0,\hat{\nu}_0} \oplus \begin{bmatrix} 0 & I_{n_1} \\ I_{n_1} & 0 \end{bmatrix},$$

where $n_0 = \hat{\pi}_0 + \hat{\nu}_0$ and $\beta_j > 0$, $\hat{s}_j \in \{-1,1\}$ for $j = 1,\dots,m_r$. Moreover,

$$Y^{-1}\hat{G}^{-1}A^*G^{-1}AY = \begin{bmatrix} \hat{s}_1\beta_1^2 & & 0 \\ & \ddots & \\ 0 & & \hat{s}_{m_r}\beta_{m_r}^2 \end{bmatrix} \oplus \mathscr{O}_{n_0} \oplus \begin{bmatrix} 0 & I_{n_1} \\ 0 & 0 \end{bmatrix},$$

$$X^{-1}G^{-1}A\hat{G}^{-1}A^*X = \begin{bmatrix} \hat{s}_1\beta_1^2 & & 0 \\ & \ddots & \\ 0 & & \hat{s}_{m_r}\beta_{m_r}^2 \end{bmatrix} \oplus \mathscr{O}_{m_0+n_1}.$$

Proof. Because G is positive definite, due to the inertia index relation, in the canonical form of Theorem 3.2, G_c, as well as $A_c, \hat{G}_c$ must be void. Furthermore, $\eta_1 = \ldots = \eta_{m_r} = 1$ and $s_1 = \ldots = s_{m_r} = 1$. Concerning the blocks $A_{z,k}$, $G_{z,k}$, $\hat{G}_{z,k}$, the blocks for $k = 1$ may exist, but $G_{z,1}$ has to be the identity matrix I_{m_0}; the blocks for $k = 2$ and $k = 3$ must be void, and the blocks for $k = 4$ may only exist when $j = 1$. In this case $G_{z,4}$ has to be I_{n_1} and applying an appropriate permutation, we can achieve the forms

$$A_{z,4} = \begin{bmatrix} \mathscr{O}_{n_1} & I_{n_1} \end{bmatrix}, \quad G_{z,4} = I_{n_1}, \quad \hat{G}_{z,4} = \begin{bmatrix} 0 & I_{n_1} \\ I_{n_1} & 0 \end{bmatrix}.$$

The proof for the real case is analogous. □

REMARK 3.8. It should be noted that when $G = I_m$, then X is unitary and Corollary 3.7 gives the Schur form of the Hermitian matrix $A\hat{G}^{-1}A^*$. Also, it simultaneously displays the Jordan form of $\hat{G}^{-1}A^*A$. One should observe here the difference in the eigenstructures of $A\hat{G}^{-1}A^*$ and $\hat{G}^{-1}A^*A$ corresponding to the eigenvalue $\lambda = 0$. (Indeed, it is well known that two matrix products AB and BA have identical nonzero eigenvalues including identical algebraic, geometric, and partial multiplicities, but the Jordan structure for the eigenvalue $\lambda = 0$ may be different for both matrices, see [6].) If $G = I_m$ and $\hat{G} = I_n$, then also Y is unitary and Corollary 3.7 becomes the standard singular value decomposition.

4. Canonical form for G symmetric and $\hat{G}$ skew-symmetric

In this section we determine the canonical form for the case that G is symmetric and $\hat{G}$ is skew-symmetric. We only consider the real case, because the corresponding complex case (i.e., G being Hermitian and $\hat{G}$ being skew-Hermitian) can be easily derived from the canonical form in Theorem 3.2 by simply multiplying $\hat{G}$ with $-\imath$. For the real case, the situation is different and the canonical form becomes more complicated. Again, we start with the result for the case that A is square and nonsingular.

THEOREM 4.1. *Let $A \in \mathbb{R}^{2n\times 2n}$ be nonsingular, let $G \in \mathbb{R}^{2n\times 2n}$ be symmetric and nonsingular, and let $\hat{G} \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{R}^{2n\times 2n}$ such that*

$$X^T AY = A_c \oplus A_r \oplus A_\imath, \quad X^T GX = G_c \oplus G_r \oplus G_\imath, \quad Y^T \hat{G} Y = \hat{G}_c \oplus \hat{G}_r \oplus \hat{G}_\imath. \qquad (4.1)$$

Moreover, for the $\hat{G}$-Hamiltonian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^T G^{-1}A$ and for the G-skew-symmetric matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^T$, we have that

$$Y^{-1}\hat{\mathscr{H}}Y = \hat{\mathscr{H}}_c \oplus \hat{\mathscr{H}}_r \oplus \hat{\mathscr{H}}_\imath, \qquad X^{-1}\mathscr{H}X = \mathscr{H}_c \oplus \mathscr{H}_r \oplus \mathscr{H}_\imath. \qquad (4.2)$$

The diagonal blocks in these decompositions have the following forms:

1) *blocks associated with quadruples $((a_j \pm \imath b_j)^2, -(a_j \pm \imath b_j)^2)$ of nonreal and non purely imaginary eigenvalues of $\hat{\mathscr{H}}$ and $\mathscr{H}$:*

$$\begin{aligned}
A_c &= \begin{bmatrix} \mathscr{J}_{\xi_1}(a_1,b_1) & 0 \\ 0 & \mathscr{J}_{\xi_1}(a_1,b_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c}) & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c}) \end{bmatrix}, \\
G_c &= \begin{bmatrix} 0 & R_{2\xi_1} \\ R_{2\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{2\xi_{m_c}} \\ R_{2\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{G}_c &= \begin{bmatrix} 0 & R_{2\xi_1} \\ -R_{2\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{2\xi_{m_c}} \\ -R_{2\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_c &= \begin{bmatrix} -\mathscr{J}_{\xi_1}(a_1,b_1)^2 & 0 \\ 0 & \mathscr{J}_{\xi_1}(a_1,b_1)^2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c})^2 & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c})^2 \end{bmatrix}, \\
\mathscr{H}_c &= \begin{bmatrix} \mathscr{J}_{\xi_1}(a_1,b_1)^2 & 0 \\ 0 & -\mathscr{J}_{\xi_1}(a_1,b_1)^2 \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c})^2 & 0 \\ 0 & -\mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c})^2 \end{bmatrix}^T,
\end{aligned}$$

where $a_j > b_j > 0$ and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m_c$;

2) *blocks associated with pairs of real eigenvalues* $(\alpha_j^2, -\alpha_j^2)$ *of* $\mathscr{H}$ *and* $\hat{\mathscr{H}}$*:*

$$\begin{aligned}
A_r &= \begin{bmatrix} \mathscr{J}_{\eta_1}(\alpha_1) & 0 \\ 0 & \mathscr{J}_{\eta_1}(\alpha_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\eta_{m_r}}(\alpha_{m_r}) & 0 \\ 0 & \mathscr{J}_{\eta_{m_r}}(\alpha_{m_r}) \end{bmatrix}, \\
G_r &= \begin{bmatrix} 0 & R_{\eta_1} \\ R_{\eta_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\eta_{m_r}} \\ R_{\eta_{m_r}} & 0 \end{bmatrix}, \\
\hat{G}_r &= \begin{bmatrix} 0 & R_{\eta_1} \\ -R_{\eta_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\eta_{m_r}} \\ -R_{\eta_{m_r}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_r &= \begin{bmatrix} -\mathscr{J}_{\eta_1}(\alpha_1)^2 & 0 \\ 0 & \mathscr{J}_{\eta_1}(\alpha_1)^2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathscr{J}_{\eta_{m_r}}(\alpha_{m_r})^2 & 0 \\ 0 & \mathscr{J}_{\eta_{m_r}}(\alpha_{m_r})^2 \end{bmatrix}, \\
\mathscr{H}_r &= \begin{bmatrix} \mathscr{J}_{\eta_1}(\alpha_1)^2 & 0 \\ 0 & -\mathscr{J}_{\eta_1}(\alpha_1)^2 \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\eta_{m_r}}(\alpha_{m_r})^2 & 0 \\ 0 & -\mathscr{J}_{\eta_{m_r}}(\alpha_{m_r})^2 \end{bmatrix}^T,
\end{aligned}$$

where $\alpha_j > 0$ *and* $\eta_j \in \mathbb{N}$ *for* $j = 1, \ldots, m_r$*;*

3) *blocks associated with pairs of purely imaginary eigenvalues* $(\imath\beta_j^2, -\imath\beta_j^2)$ *of* $\mathscr{H}$ *and* $\hat{\mathscr{H}}$*:*

$$\begin{aligned}
A_\imath &= \begin{bmatrix} \mathscr{J}_{\rho_1}(\beta_1) & 0 \\ 0 & \mathscr{J}_{\rho_1}(\beta_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath}) & 0 \\ 0 & \mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath}) \end{bmatrix}, \\
G_\imath &= s_1 \begin{bmatrix} R_{\rho_1} & 0 \\ 0 & R_{\rho_1} \end{bmatrix} \oplus \cdots \oplus s_{m_\imath} \begin{bmatrix} R_{\rho_{m_\imath}} & 0 \\ 0 & R_{\rho_{m_\imath}} \end{bmatrix}, \\
\hat{G}_\imath &= s_1 \begin{bmatrix} 0 & R_{\rho_1} \\ -R_{\rho_1} & 0 \end{bmatrix} \oplus \cdots \oplus s_{m_\imath} \begin{bmatrix} 0 & R_{\rho_{m_\imath}} \\ -R_{\rho_{m_\imath}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_\imath &= \begin{bmatrix} 0 & \mathscr{J}_{\rho_1}(\beta_1)^2 \\ -\mathscr{J}_{\rho_1}(\beta_1)^2 & 0 \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} 0 & \mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath})^2 \\ -\mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath})^2 & 0 \end{bmatrix}^T, \\
\mathscr{H}_\imath &= \begin{bmatrix} 0 & -\mathscr{J}_{\rho_1}(\beta_1)^2 \\ \mathscr{J}_{\rho_1}(\beta_1)^2 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -\mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath})^2 \\ \mathscr{J}_{\rho_{m_\imath}}(\beta_{m_\imath})^2 & 0 \end{bmatrix},
\end{aligned}$$

where $\beta_j > 0$, $s_j \in \{+1, -1\}$*, and* $\rho_j \in \mathbb{N}$ *for* $j = 1, \ldots, m_\imath$*;*

Furthermore, the form (4.1) *is unique up to the simultaneous permutation of blocks in the right hand side of* (4.1)*.*

Proof. Analogous to the proof of Theorem 3.1, it can be shown that without loss of generality we may assume that $\sigma(\hat{\mathscr{H}}) = \{\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\}$, where $\lambda \in \mathbb{C} \setminus \{0\}$. We then distinguish the three different cases $\lambda^2 > 0$, $\lambda^2 < 0$, and $\lambda^2 \notin \mathbb{R}$. The proof then proceeds similar to the proof of Theorem 3.1, but instead of constructing a square root of $\hat{\mathscr{H}}$, a square root of a related $\hat{G}$-skew-Hamiltonian matrix $\widetilde{S}$ will be considered. The proof for the cases $\lambda^2 > 0$ and $\lambda^2 \notin \mathbb{R}$ follows exactly the same lines as the proof of Theorem 5.1 in [19] and will not be reproduced here. The proof for the remaining case differs slightly and will therefore be presented here in full detail.

Thus, assume without loss of generality that $\sigma(\hat{\mathscr{H}}) = \{\imath\lambda, -\imath\lambda\}$, where $\lambda > 0$. By Theorem 2.6, there exists a nonsingular matrix $W \in \mathbb{R}^{2n\times 2n}$ such that

$$\begin{aligned} W^{-1}\hat{\mathscr{H}}W &= \begin{bmatrix} 0 & \mathscr{J}_{\rho_1}(\lambda) \\ -\mathscr{J}_{\rho_1}(\lambda) & 0 \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} 0 & \mathscr{J}_{\rho_m}(\lambda) \\ -\mathscr{J}_{\rho_m}(\lambda) & 0 \end{bmatrix}, \\ W^T\hat{G}W &= \hat{s}_1\begin{bmatrix} 0 & R_{\rho_1} \\ -R_{\rho_1} & 0 \end{bmatrix} \oplus\cdots\oplus \hat{s}_m\begin{bmatrix} 0 & R_{\rho_m} \\ -R_{\rho_m} & 0 \end{bmatrix}, \end{aligned}$$

where $\rho_j \in \mathbb{N}$ and $\hat{s}_j \in \{+1,-1\}$ for $j = 1,\ldots,m$. Next, we define the matrix $\widetilde{S}$ to be such that

$$W^{-1}\widetilde{S}W = \begin{bmatrix} \mathscr{J}_{\rho_1}(\lambda) & 0 \\ 0 & \mathscr{J}_{\rho_1}(\lambda) \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} \mathscr{J}_{\rho_m}(\lambda) & 0 \\ 0 & \mathscr{J}_{\rho_m}(\lambda) \end{bmatrix}.$$

Then $\widetilde{S}$ is $\hat{G}$-skew-Hamiltonian and satisfies $\sigma(\widetilde{S}) \subseteq \mathbb{R}_+$. Thus, applying Theorem 2.9, we obtain that $\widetilde{S}$ has unique square root $S \in \mathbb{C}^{n\times n}$ that satisfies $\sigma(S) \subseteq \mathbb{R}_+$ and that is a polynomial in $\widetilde{S}$. Consequently, with $\widetilde{S}$ also its square root S is $\hat{G}$-skew-Hamiltonian. Let $\beta = \sqrt{\lambda}$. Then Theorem 2.7 implies the existence of a nonsingular matrix $\widetilde{Y} \in \mathbb{R}^{2n\times 2n}$ such that

$$\begin{aligned} S_{\mathrm{CF}} := \widetilde{Y}^{-1}S\widetilde{Y} &= \begin{bmatrix} \mathscr{J}_{\rho_1}(\beta) & 0 \\ 0 & \mathscr{J}_{\rho_1}(\beta) \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} \mathscr{J}_{\rho_m}(\beta) & 0 \\ 0 & \mathscr{J}_{\rho_m}(\beta) \end{bmatrix}, \\ \widetilde{Y}^T\hat{G}\widetilde{Y} &= \begin{bmatrix} 0 & R_{\rho_1} \\ -R_{\rho_1} & 0 \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} 0 & R_{\rho_m} \\ -R_{\rho_m} & 0 \end{bmatrix}. \end{aligned}$$

(Note that the Jordan structure of S follows from the fact that S is a polynomial in $\widetilde{S}$.) Moreover, using $G^{-1}A\hat{\mathscr{H}} = \mathscr{H}G^{-1}A$ and the fact that $G^{-1}A$ is nonsingular, we find that $\hat{\mathscr{H}}$ and $\mathscr{H}$ are similar. Thus, by Theorem 2.5, we find that there is a nonsingular matrix $\widetilde{X}_1$ such that

$$\begin{aligned} \widetilde{X}_1^{-1}\mathscr{H}\widetilde{X}_1 &= \begin{bmatrix} 0 & \mathscr{J}_{\rho_1}(\lambda) \\ -\mathscr{J}_{\rho_1}(\lambda) & 0 \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} 0 & \mathscr{J}_{\rho_m}(\lambda) \\ -\mathscr{J}_{\rho_m}(\lambda) & 0 \end{bmatrix}, \\ \widetilde{X}_1^TG\widetilde{X}_1 &= \widetilde{s}_1\begin{bmatrix} R_{\rho_1} & 0 \\ 0 & R_{\rho_1} \end{bmatrix} \oplus\cdots\oplus \widetilde{s}_m\begin{bmatrix} R_{\rho_m} & 0 \\ 0 & R_{\rho_m} \end{bmatrix}, \end{aligned}$$

where $\widetilde{s}_1,\ldots,\widetilde{s}_m \in \{+1,-1\}$. On the other hand, since $\beta^2 = \lambda$, the matrix

$$\mathscr{H}_{\mathrm{CF}} := \begin{bmatrix} 0 & -\mathscr{J}_{\rho_1}^2(\beta) \\ \mathscr{J}_{\rho_1}^2(\beta) & 0 \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} 0 & -\mathscr{J}_{\rho_m}^2(\beta) \\ \mathscr{J}_{\rho_m}^2(\beta) & 0 \end{bmatrix}$$

is similar to $\widetilde{X}_1^{-1}\mathscr{H}\widetilde{X}_1$. It is also obvious that $\mathscr{H}_{\mathrm{CF}}$ is $X_1^TG\widetilde{X}_1$-skew symmetric. Again, by Theorem 2.5 there exists a nonsingular matrix $\widetilde{X}_2$ such that with setting $\widetilde{X} = \widetilde{X}_1\widetilde{X}_2$ we have that

$$\begin{aligned} \mathscr{H}_{\mathrm{CF}} := \widetilde{X}^{-1}\mathscr{H}\widetilde{X} &= \begin{bmatrix} 0 & -\mathscr{J}_{\rho_1}^2(\beta) \\ \mathscr{J}_{\rho_1}^2(\beta) & 0 \end{bmatrix} \oplus\cdots\oplus \begin{bmatrix} 0 & -\mathscr{J}_{\rho_m}^2(\beta) \\ \mathscr{J}_{\rho_m}^2(\beta) & 0 \end{bmatrix}, \\ G_{\mathrm{CF}} := \widetilde{X}^TG\widetilde{X} &= s_1\begin{bmatrix} R_{\rho_1} & 0 \\ 0 & R_{\rho_1} \end{bmatrix} \oplus\cdots\oplus s_m\begin{bmatrix} R_{\rho_m} & 0 \\ 0 & R_{\rho_m} \end{bmatrix}, \end{aligned}$$

for some $s_1,\dots,s_m \in \{+1,-1\}$. (It is actually possible to show that $\widetilde{s}_j = s_j$ for $j = 1,\dots,m$, but we refrain from doing so as it is not necessary for the proof.) Observe that S_{CF} is G_{CF}-symmetric and satisfies

$$S_{\mathrm{CF}}(\mathscr{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} = \begin{bmatrix} 0 & I_{\rho_1} \\ -I_{\rho_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\rho_m} \\ -I_{\rho_m} & 0 \end{bmatrix}.$$

Using this identity and setting $X = G^{-1}\widetilde{X}^{-T}$ and $Y = A^{-1}G\widetilde{X}S_{\mathrm{CF}}$, we obtain

$$\begin{aligned} X^TAY &= \widetilde{X}^{-1}G^{-1}AA^{-1}G\widetilde{X}S_{\mathrm{CF}} = S_{\mathrm{CF}}, \\ X^TGX &= \widetilde{X}^{-1}G^{-1}GG^{-1}\widetilde{X}^{-T} = (\widetilde{X}^TG\widetilde{X})^{-1} = (G_{\mathrm{CF}})^{-1} = G_{\mathrm{CF}}, \\ Y^T\hat{G}Y &= S_{\mathrm{CF}}^T\widetilde{X}^TGA^{-T}\hat{G}A^{-1}G\widetilde{X}S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^T\widetilde{X}^TG\widetilde{X}\widetilde{X}^{-1}\mathscr{H}^{-1}\widetilde{X}S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^TG_{\mathrm{CF}}(\mathscr{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} = G_{\mathrm{CF}}S_{\mathrm{CF}}(\mathscr{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} \\ &= s_1\begin{bmatrix} 0 & R_{\rho_1} \\ -R_{\rho_1} & 0 \end{bmatrix} \oplus \cdots \oplus s_m\begin{bmatrix} 0 & R_{\rho_m} \\ -R_{\rho_m} & 0 \end{bmatrix}. \end{aligned}$$

It is now straightforward to check that $Y^{-1}\mathscr{H}Y$ and $X^{-1}\hat{\mathscr{H}}X$ have the claimed forms. Concerning uniqueness, we note that the form (4.1) is already uniquely determined by the Jordan structure, the sign characteristic of $\mathscr{H}$, and by the restrictions on the parameters. ☐

For the general non square case we have the following result.

THEOREM 4.2. *Let $A \in \mathbb{R}^{m\times 2n}$, let $G \in \mathbb{R}^{m\times m}$ be symmetric nonsingular, and let $\hat{G} \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{2n\times 2n}$ such that*

$$\begin{aligned} X^TAY &= A_{nz}\oplus A_{z,1}\oplus A_{z,2}\oplus A_{z,3}\oplus A_{z,4}\oplus A_{z,5}\oplus A_{z,6}, \\ X^TGX &= G_{nz}\oplus G_{z,1}\oplus G_{z,2}\oplus G_{z,3}\oplus G_{z,4}\oplus G_{z,5}\oplus G_{z,6}, \\ Y^T\hat{G}Y &= \hat{G}_{nz}\oplus \hat{G}_{z,1}\oplus \hat{G}_{z,2}\oplus \hat{G}_{z,3}\oplus \hat{G}_{z,4}\oplus \hat{G}_{z,5}\oplus \hat{G}_{z,6}. \end{aligned} \tag{4.3}$$

Moreover, for the $\hat{G}$-Hamiltonian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^TG^{-1}A \in \mathbb{R}^{2n\times 2n}$ and for the G-skew-symmetric matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^T \in \mathbb{R}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathscr{H}}Y &= \hat{\mathscr{H}}_{nz}\oplus \hat{\mathscr{H}}_{z,1}\oplus \hat{\mathscr{H}}_{z,2}\oplus \hat{\mathscr{H}}_{z,3}\oplus \hat{\mathscr{H}}_{z,4}\oplus \hat{\mathscr{H}}_{z,5}\oplus \hat{\mathscr{H}}_{z,6}, \\ X^{-1}\mathscr{H}X &= \mathscr{H}_{nz}\oplus \mathscr{H}_{z,1}\oplus \mathscr{H}_{z,2}\oplus \mathscr{H}_{z,3}\oplus \mathscr{H}_{z,4}\oplus \mathscr{H}_{z,5}\oplus \mathscr{H}_{z,6}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\hat{\mathscr{H}}$ and $\mathscr{H}$:*
 $A_{nz}, G_{nz}, \hat{G}_{nz}$ have the forms as in (4.1) *and $\hat{\mathscr{H}}_{nz}, \mathscr{H}_{nz}$ have the forms as in* (4.2)*;*

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\hat{\mathscr{H}}$ and m_0 Jordan blocks of size 1×1 of $\mathscr{H}$ associated with the eigenvalue zero:*

$$A_{z,1} = \mathscr{O}_{m_0\times 2n_0},\ G_{z,1} = \Sigma_{\pi_0,\nu_0},\quad \hat{G}_{z,1} = J_{n_0},\quad \hat{\mathscr{H}}_{z,1} = \mathscr{O}_{2n_0},\quad \mathscr{H}_{z,1} = \mathscr{O}_{m_0},$$

 where $m_0, n_0, \pi_0, \nu_0 \in \mathbb{N}\cup\{0\}$ and $m_0 = \pi_0 + \nu_0$;

2) *blocks corresponding to a pair of $j \times j$ Jordan blocks of $\mathscr{H}$ and $\hat{\mathscr{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathscr{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathscr{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \mathscr{J}_{4\ell+2}(0)\,, \\
G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} R_{4\ell+2}\,, \\
\hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \begin{bmatrix} 0 & R_{2\ell+1} \\ -R_{2\ell+1} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} (-\Sigma_{2,2})\mathscr{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} (-\Sigma_{2\ell+1,2\ell+1})\mathscr{J}_{4\ell+2}^2(0), \\
\mathscr{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \Sigma_{3,1}\mathscr{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \Sigma_{2\ell+2,\ell}\mathscr{J}_{4\ell+2}^2(0)^T,
\end{aligned}
$$

where $\gamma_1,\dots,\gamma_\ell \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,2}$ and $\mathscr{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j \times j$ for $j = 1,\dots,2\ell+1$;
moreover, if j is odd, then exactly γ_j Jordan blocks of $\mathscr{H}_{z,2}$ of size $j \times j$ have sign $s = +1$ and exactly γ_j blocks have sign $s = -1$ (even-sized Jordan blocks associated with zero of G-skew-symmetric matrices do not have signs), and if j is even, then exactly γ_j Jordan blocks of $\hat{\mathscr{H}}_{z,2}$ of size $j \times j$ have sign $s = +1$ and exactly γ_j blocks have sign $s = -1$ (odd-sized Jordan blocks associated with zero of $\hat{G}$-Hamiltonian matrices do not have signs);

3) *blocks corresponding to a $2j \times 2j$ Jordan block of $\hat{\mathscr{H}}$ and a $(2j+1)\times(2j+1)$ Jordan block of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_2} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} \begin{bmatrix} I_{2\ell} \\ 0 \end{bmatrix}_{(2\ell+1)\times 2\ell}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_2} s_i^{(2)} R_3 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} s_i^{(2\ell)} R_{2\ell+1}\,, \\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_{z,3} &= \bigoplus_{i=1}^{m_2} (-s_i^{(2)}\Sigma_{1,1})\mathscr{J}_2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} (-s_i^{(2\ell)}\Sigma_{\ell,\ell})\mathscr{J}_{2\ell}(0)\,, \\
\mathscr{H}_{z,3} &= \bigoplus_{i=1}^{m_2} s_i^{(2\ell)}\Sigma_{2,1}\mathscr{J}_3(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} s_i^{(2\ell)}\Sigma_{\ell+1,\ell}\mathscr{J}_{2\ell+1}(0)^T,
\end{aligned}
$$

where $m_2, m_4, \dots, m_{2\ell} \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,3}$ has m_{2j} Jordan blocks of size $2j \times 2j$ with signs $s_i^{(2j)}$, and $\mathscr{H}_{z,3}$ has m_{2j} Jordan blocks of size $(2j+1)\times(2j+1)$ with signs $s_i^{(2j)}$ for $i = 1,\dots,m_{2j}$ and $j = 1,\dots,\ell$;

4) *blocks corresponding to two $(2j-1)\times(2j-1)$ Jordan blocks of $\hat{\mathscr{H}}$ and two $2j\times 2j$ Jordan blocks of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{m_1}\begin{bmatrix}0 & I_1\\ 0 & 0\\ I_1 & 0\\ 0 & 0\end{bmatrix}_{4\times 2} \oplus\cdots\oplus \bigoplus_{i=1}^{m_{2\ell-1}}\begin{bmatrix}0 & I_{2\ell-1}\\ 0 & 0\\ I_{2\ell-1} & 0\\ 0 & 0\end{bmatrix}_{4\ell\times(4\ell-2)},\\
G_{z,4} &= \bigoplus_{i=1}^{m_1} R_4 \oplus\cdots\oplus \bigoplus_{i=1}^{m_{2\ell-1}} R_{4\ell},\\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{m_1}\begin{bmatrix}0 & R_1\\ -R_1 & 0\end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{m_{2\ell-1}}\begin{bmatrix}0 & R_{2\ell-1}\\ -R_{2\ell-1} & 0\end{bmatrix},\\
\hat{\mathscr{H}}_{z,4} &= \bigoplus_{i=1}^{m_1}\begin{bmatrix}-\mathscr{J}_1(0) & 0\\ 0 & \mathscr{J}_1(0)\end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{m_{2\ell-1}}\begin{bmatrix}-\mathscr{J}_{2\ell-1}(0) & 0\\ 0 & \mathscr{J}_{2\ell-1}(0)\end{bmatrix},\\
\mathscr{H}_{z,4} &= \bigoplus_{i=1}^{m_1}\begin{bmatrix}-\mathscr{J}_2(0) & 0\\ 0 & \mathscr{J}_2(0)\end{bmatrix}^T \oplus\cdots\oplus \bigoplus_{i=1}^{m_{2\ell-1}}\begin{bmatrix}-\mathscr{J}_{2\ell}(0) & 0\\ 0 & \mathscr{J}_{2\ell}(0)\end{bmatrix}^T,
\end{aligned}
$$

where $m_1,m_3,\dots,m_{2\ell-1}\in\mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of the size $(2j-1)\times(2j-1)$ and $\mathscr{H}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of size $2j\times 2j$ for $j=1,\dots,\ell$; (no sign-characteristic is involved, because neither even-sized Jordan blocks associated with zero of G-skew-symmetric matrices nor odd-sized Jordan blocks associated with zero of $\hat{G}$-Hamiltonian matrices have signs);

5) *blocks corresponding to a $2j\times 2j$ Jordan block of $\hat{\mathscr{H}}$ and a $(2j-1)\times(2j-1)$ Jordan block of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,5} &= \bigoplus_{i=1}^{n_1}\begin{bmatrix}0 & I_1\end{bmatrix}_{1\times 2} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell-1}}\begin{bmatrix}0 & I_{2\ell-1}\end{bmatrix}_{(2\ell-1)\times 2\ell},\\
G_{z,5} &= \bigoplus_{i=1}^{n_1} s_i^{(1)} R_1 \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell-1}} s_i^{(2\ell-1)} R_{2\ell-1},\\
\hat{G}_{z,5} &= \bigoplus_{i=1}^{n_1}\begin{bmatrix}0 & R_1\\ -R_1 & 0\end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell-1}}\begin{bmatrix}0 & R_\ell\\ -R_\ell & 0\end{bmatrix},\\
\hat{\mathscr{H}}_{z,5} &= \bigoplus_{i=1}^{n_1}(-s_i^{(1)}\Sigma_{1,1})\mathscr{J}_2(0) \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell-1}}(-s_i^{(2\ell-1)}\Sigma_{\ell,\ell})\mathscr{J}_{2\ell}(0),\\
\mathscr{H}_{z,5} &= \bigoplus_{i=1}^{n_1} s_i^{(1)}\Sigma_{1,0}\mathscr{J}_1(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell-1}} s_i^{(2\ell-1)}\Sigma_{\ell,\ell-1}\mathscr{J}_{2\ell-1}(0)^T,
\end{aligned}
$$

where $n_1,n_3\dots,n_{2\ell-1}\in\mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,5}$ has n_{2j-1} Jordan blocks of size $2j\times 2j$ with signs $s_i^{(2j-1)}$, and $\mathscr{H}_{z,5}$ has n_{2j-1} Jordan blocks of size $(2j-1)\times(2j-1)$ with signs $s_i^{(2j-1)}$ for $i=1,\dots,n_{2j-1}$ and $j=1,\dots,\ell$;

6) *blocks corresponding to two* $(2j+1)\times(2j+1)$ *Jordan blocks of* $\hat{\mathscr{H}}$ *and two* $2j\times 2j$ *Jordan blocks of* $\mathscr{H}$ *associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,6} &= \bigoplus_{i=1}^{n_2}\begin{bmatrix} 0 & 0 & 0 & I_2\\ 0 & I_2 & 0 & 0\end{bmatrix}_{4\times 6} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell}}\begin{bmatrix} 0 & 0 & 0 & I_{2\ell}\\ 0 & I_{2\ell} & 0 & 0\end{bmatrix}_{4\ell\times(4\ell+2)},\\
G_{z,6} &= \bigoplus_{i=1}^{n_2} R_4 \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell}} R_{4\ell},\\
\hat{G}_{z,6} &= \bigoplus_{i=1}^{n_2}\begin{bmatrix} 0 & R_3\\ -R_3 & 0\end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell}}\begin{bmatrix} 0 & R_{2\ell+1}\\ -R_{2\ell+1} & 0\end{bmatrix},\\
\hat{\mathscr{H}}_{z,6} &= \bigoplus_{i=1}^{n_2}\begin{bmatrix} -\mathscr{J}_3(0) & 0\\ 0 & \mathscr{J}_3(0)\end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell}}\begin{bmatrix} -\mathscr{J}_{2\ell+1}(0) & 0\\ 0 & \mathscr{J}_{2\ell+1}(0)\end{bmatrix},\\
\mathscr{H}_{z,6} &= \bigoplus_{i=1}^{n_2}\begin{bmatrix} -\mathscr{J}_2(0) & 0\\ 0 & \mathscr{J}_2(0)\end{bmatrix}^T \oplus\cdots\oplus \bigoplus_{i=1}^{n_{2\ell}}\begin{bmatrix} -\mathscr{J}_{2\ell}(0) & 0\\ 0 & \mathscr{J}_{2\ell}(0)\end{bmatrix}^T,
\end{aligned}
$$

where $n_2,n_4,n_6,\ldots,n_{2\ell}\in\mathbb{N}\cup\{0\}$*; thus,* $\hat{\mathscr{H}}_{z,6}$ *has* $2n_{2j}$ *Jordan blocks of size* $(2j+1)\times(2j+1)$ *and* $\mathscr{H}_{z,6}$ *has* $2n_{2j}$ *Jordan blocks of size* $2j\times 2j$ *for* $j=1,\ldots,\ell$*; (no sign-characteristic is involved, because neither even-sized Jordan blocks associated with zero of* G*-skew-symmetric matrices nor odd-sized Jordan blocks associated with zero of* $\hat{G}$*-Hamiltonian matrices have signs);*

For the eigenvalue zero, the matrices $\hat{\mathscr{H}}$ *and* $\mathscr{H}$ *have* $2\gamma_{2j}+m_{2j}+n_{2j-1}$*, respectively* $2\gamma_{2j}+2m_{2j-1}+2n_{2j}$ *Jordan blocks of size* $2j\times 2j$ *for* $j=1,\ldots,\ell$*, and* $2\gamma_{2j+1}+2m_{2j+1}+2n_{2j}$*, respectively* $2\gamma_{2j+1}+m_{2j}+n_{2j+1}$ *Jordan blocks of size* $(2j+1)\times(2j+1)$ *for* $j=0,\ldots,\ell$*. Here* $m_{2\ell+1}=n_{2\ell+1}=0$*, where* $2\ell+1$ *is the smallest odd number that is larger or equal to the maximum of the index of* $\hat{\mathscr{H}}$ *and the index of* $\mathscr{H}$*. (Here,* index *refers to the maximal size of a Jordan block associated with the eigenvalue zero.)*

Furthermore, the form (4.3) *is unique up to the simultaneous permutation of blocks in the right hand side of* (4.3)*.*

Proof. The proof can be found in the Appendix. □

For the special case that G is positive definite, the condensed form simplifies considerably.

COROLLARY 4.3. *Let* $A\in\mathbb{R}^{m\times 2n}$*, let* $G\in\mathbb{R}^{m\times m}$ *be symmetric and positive definite, and let* $\hat{G}\in\mathbb{R}^{2n\times 2n}$ *be skew-symmetric and nonsingular. Then there exist nonsingular matrices* $X\in\mathbb{R}^{m\times m}$ *and* $Y\in\mathbb{R}^{2n\times 2n}$ *such that*

$$
\begin{aligned}
X^TAY &= \begin{bmatrix}\beta_1 & 0\\ 0 & \beta_1\end{bmatrix}\oplus\ldots\oplus\begin{bmatrix}\beta_{m_\iota} & 0\\ 0 & \beta_{m_\iota}\end{bmatrix}\oplus\mathscr{O}_{m_0\times 2n_0}\oplus\begin{bmatrix}\mathscr{O}_{n_1} & I_{n_1}\end{bmatrix},\\
X^TGX &= I_2\oplus\ldots\oplus I_2\oplus I_{m_0}\oplus I_{n_1} = I_m\\
Y^T\hat{G}Y &= J_1\oplus\ldots\oplus J_1\oplus J_{n_0}\oplus J_{n_1},
\end{aligned}
$$

with $\beta_j>0$ *for* $j=1,\ldots,m_\iota$*.*

Moreover,

$$Y^{-1}\hat{G}^{-1}A^TG^{-1}AY = \begin{bmatrix} 0 & -\beta_1^2 \\ \beta_1^2 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & -\beta_{m_i}^2 \\ \beta_{m_\iota}^2 & 0 \end{bmatrix} \oplus \mathscr{O}_{2n_0} \oplus \begin{bmatrix} 0 & -I_{n_1} \\ 0 & 0 \end{bmatrix},$$
$$X^{-1}G^{-1}A\hat{G}^{-1}A^TX == \begin{bmatrix} 0 & \beta_1^2 \\ -\beta_1^2 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & \beta_{m_i}^2 \\ -\beta_{m_\iota}^2 & 0 \end{bmatrix} \oplus \mathscr{O}_{m_0+n_1}.$$

Proof. Because G is positive definite, due to the inertia index relation, in the canonical form of Theorem 4.2, G_c, G_r, as well as $A_c, \hat{G}_c, A_r, \hat{G}_r$ must be void. Furthermore, $\rho_1 = \ldots = \rho_{m_\iota} = 1$ and $s_1 = \ldots = s_{m_\iota} = 1$. Concerning the blocks $A_{z,k}$, $G_{z,k}$, $\hat{G}_{z,k}$, the blocks for $k=1$ may exist, but $G_{z,1}$ has to be the identity matrix I_{m_0}; the blocks for $k = 2,3,4,6$ must be void, and the blocks for $k=5$ may only exist when $j=1$. In this case $G_{z,5}$ has to be I_{n_1} and applying an appropriate permutation, we can achieve the forms $A_{z,4} = \begin{bmatrix} \mathscr{O}_{n_1} & I_{n_1} \end{bmatrix}$, $G_{z,4} = I_{n_1}$, and $\hat{G}_{z,4} = J_{n_1}$. □

The result of Corollary 4.3 first appeared in [27], where an independent proof is given.

5. Canonical form for skew-symmetric G and $\hat{G}$

When $G \in \mathbb{F}^{m\times m}$ and $\hat{G} \in \mathbb{F}^{n\times n}$ are both skew-Hermitian, then in the complex case the canonical form for the triple $(A, G, \hat{G})$, where $A \in \mathbb{F}^{m\times n}$ can easily be derived from the Hermitian case in Section 3, by simply considering the related triple $(A, \imath G, \imath\hat{G})$. The real case, however, is different.

THEOREM 5.1. *Let $A \in \mathbb{R}^{2n\times 2n}$ be nonsingular and let $G, \hat{G} \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{R}^{2n\times 2n}$ such that*

$$X^TAY = A_c \oplus A_r, \qquad X^TGX = G_c \oplus G_r, \qquad Y^T\hat{G}Y = \hat{G}_c \oplus \hat{G}_r. \tag{5.1}$$

Moreover, for the $\hat{G}$-skew-Hamiltonian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^TG^{-1}A$ and for the G-skew-Hamiltonian matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^T$, we have that

$$Y^{-1}\hat{\mathscr{H}}Y = \hat{\mathscr{H}}_c \oplus \hat{\mathscr{H}}_r, \qquad X^{-1}\mathscr{H}X = \mathscr{H}_c \oplus \mathscr{H}_r. \tag{5.2}$$

The diagonal blocks in these decompositions have the following forms:

1) *blocks associated with pairs $(\mu_i^2, \bar{\mu}_i^2)$ of nonreal eigenvalues of $\hat{\mathscr{H}}$ and $\mathscr{H}$:*

$$A_c = \begin{bmatrix} \mathscr{J}_{\xi_1}(a_1,b_1) & 0 \\ 0 & \mathscr{J}_{\xi_1}(a_1,b_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c}) & 0 \\ 0 & \mathscr{J}_{\xi_{m_c}}(a_{m_c},b_{m_c}) \end{bmatrix},$$

$$
\begin{aligned}
G_c &= \begin{bmatrix} 0 & R_{2\xi_1} \\ -R_{2\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{2\xi_{m_c}} \\ -R_{2\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{G}_c &= \begin{bmatrix} 0 & -R_{2\xi_1} \\ R_{2\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -R_{2\xi_{m_c}} \\ R_{2\xi_{m_c}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_c &= \begin{bmatrix} \mathscr{J}^2_{\xi_1}(a_1,b_1) & 0 \\ 0 & \mathscr{J}^2_{\xi_1}(a_1,b_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}^2_{\xi_{m_c}}(a_{m_c},b_{m_c}) & 0 \\ 0 & \mathscr{J}^2_{\xi_{m_c}}(a_{m_c},b_{m_c}) \end{bmatrix}, \\
\mathscr{H}_c &= \begin{bmatrix} \mathscr{J}^2_{\xi_1}(a_1,b_1) & 0 \\ 0 & \mathscr{J}^2_{\xi_1}(a_1,b_1) \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}^2_{\xi_{m_c}}(a_{m_c},b_{m_c}) & 0 \\ 0 & \mathscr{J}^2_{\xi_{m_c}}(a_{m_c},b_{m_c}) \end{bmatrix}^T,
\end{aligned}
$$

where $a_i \in \mathbb{R}$, $b_i > 0$, $\mu_i = a_i + \imath b_i$, and $\xi_i \in \mathbb{N}$ for $i = 1,\ldots,m_c$;

2) *blocks associated with real eigenvalues $\alpha_j = \delta_j \beta_j^2$ of $\mathscr{H}$ and $\hat{\mathscr{H}}$:*

$$
\begin{aligned}
A_r &= \begin{bmatrix} \mathscr{J}_{\eta_1}(\beta_1) & 0 \\ 0 & \mathscr{J}_{\eta_1}(\beta_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathscr{J}_{\eta_{m_r}}(\beta_{m_r}) & 0 \\ 0 & \mathscr{J}_{\eta_{m_r}}(\beta_{m_r}) \end{bmatrix}, \\
G_r &= \begin{bmatrix} 0 & R_{\eta_1} \\ -R_{\eta_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\eta_{m_r}} \\ -R_{\eta_{m_r}} & 0 \end{bmatrix}, \\
\hat{G}_r &= \delta_1 \begin{bmatrix} 0 & -R_{\eta_1} \\ R_{\eta_1} & 0 \end{bmatrix} \oplus \cdots \oplus \delta_{m_r} \begin{bmatrix} 0 & -R_{\eta_{m_r}} \\ R_{\eta_{m_r}} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_r &= \delta_1 \begin{bmatrix} \mathscr{J}^2_{\eta_1}(\beta_1) & 0 \\ 0 & \mathscr{J}^2_{\eta_1}(\beta_1) \end{bmatrix} \oplus \cdots \oplus \delta_{m_r} \begin{bmatrix} \mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r}) & 0 \\ 0 & \mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r}) \end{bmatrix}, \\
\mathscr{H}_r &= \delta_1 \begin{bmatrix} \mathscr{J}^2_{\eta_1}(\beta_1) & 0 \\ 0 & \mathscr{J}^2_{\eta_1}(\beta_1) \end{bmatrix}^T \oplus \cdots \oplus \delta_{m_r} \begin{bmatrix} \mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r}) & 0 \\ 0 & \mathscr{J}^2_{\eta_{m_r}}(\beta_{m_r}) \end{bmatrix}^T,
\end{aligned}
$$

where $\beta_j > 0$, $\delta_j \in \{+1,-1\}$, and $\eta_j \in \mathbb{N}$ for $j = 1,\ldots,m_r$. (Here, δ_j is not a sign in the sense of "sign characteristic", but only depends on $\alpha_j = \delta_j \beta_j^2$ being either positive or negative.)

Furthermore, the form (5.1) *is unique up to the simultaneous permutation of blocks in the right hand side of* (5.1).

Proof. Once again, we can restrict ourselves to the case that either $\sigma(\hat{\mathscr{H}}) = \{\mu^2, \bar{\mu}^2\}$ for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ or $\sigma(\hat{\mathscr{H}}) = \{\alpha\}$, where $\alpha \in \mathbb{R} \setminus \{0\}$. The remainder of the proof then follows exactly the same lines as the proof of Theorem 3.1 by constructing a skew-Hamiltonian square root S of $\hat{\mathscr{H}}$ that is a polynomial in $\hat{\mathscr{H}}$ in the cases $\sigma(\hat{\mathscr{H}}) = \{\mu^2, \bar{\mu}^2\}$ or $\sigma(\hat{\mathscr{H}}) = \{\alpha\}$ and $\alpha > 0$, or by constructing a skew-Hamiltonian square root S of $-\hat{\mathscr{H}}$ otherwise. □

We mention that the choice of the transformation matrices X,Y in Theorem 5.1 so that $X^T G_c X = -Y^T \hat{G}_c Y$ rather than $X^T G_c X = Y^T \hat{G}_c Y$ is just a matter of taste and avoids the occurrence of distracting minus signs in the forms for $\mathscr{H}_c$ and $\hat{\mathscr{H}}_c$.

For the general non square case we have the following result.

THEOREM 5.2. *Let $A \in \mathbb{R}^{2m\times 2n}$ and let $G \in \mathbb{R}^{2m\times 2m}$, $\hat{G} \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and nonsingular. Then there exists nonsingular matrices $X \in \mathbb{R}^{2m\times 2m}$ and $Y \in \mathbb{R}^{2n\times 2n}$ such that*

$$\begin{aligned} X^T AY &= A_{nz} \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^T GX &= G_{nz} \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T \hat{G} Y &= \hat{G}_{nz} \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4}. \end{aligned} \tag{5.3}$$

Moreover, for the $\hat{G}$-skew-Hamiltonian matrix $\hat{\mathscr{H}} = \hat{G}^{-1}A^T G^{-1}A \in \mathbb{R}^{2n\times 2n}$ and for the G-skew-Hamiltonian matrix $\mathscr{H} = G^{-1}A\hat{G}^{-1}A^T \in \mathbb{R}^{2m\times 2m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathscr{H}}Y &= \hat{\mathscr{H}}_{nz} \oplus \hat{\mathscr{H}}_{z,1} \oplus \hat{\mathscr{H}}_{z,2} \oplus \hat{\mathscr{H}}_{z,3} \oplus \hat{\mathscr{H}}_{z,4}, \\ X^{-1}\mathscr{H}X &= \mathscr{H}_{nz} \oplus \mathscr{H}_{z,1} \oplus \mathscr{H}_{z,2} \oplus \mathscr{H}_{z,3} \oplus \mathscr{H}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\mathscr{H}$ and $\hat{\mathscr{H}}$:*
$A_{nz}, G_{nz}, \widetilde{G}_{nz}$ *have the forms as in* (5.1) *and* $\mathscr{H}_{nz}, \hat{\mathscr{H}}_{nz}$ *have the forms as in* (5.2)*;*

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\mathscr{H}$ and $2m_0$ Jordan blocks of size 1×1 of $\hat{\mathscr{H}}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{2m_0\times 2n_0}, \quad G_{z,1} = J_{m_0}, \quad \hat{G}_{z,1} = J_{n_0}, \quad \hat{\mathscr{H}}_{z,1} = 0_{2n_0}, \quad \mathscr{H}_{z,1} = 0_{2m_0};$$

2) *blocks corresponding to a pair of $j\times j$ Jordan blocks of $\hat{\mathscr{H}}$ and $\mathscr{H}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathscr{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathscr{J}_4(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \mathscr{J}_{2\ell}(0), \\ G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\ \hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\ \hat{\mathscr{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \hat{\Gamma}_4 \mathscr{J}_4^2(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \hat{\Gamma}_{2\ell} \mathscr{J}_{2\ell}^2(0), \\ \mathscr{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \Gamma_4 \mathscr{J}_4^2(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{\gamma_\ell} \Gamma_{2\ell} \mathscr{J}_{2\ell}^2(0)^T, \end{aligned}$$

where $\gamma_1,\ldots,\gamma_\ell \in \mathbb{N}\cup\{0\}$, and $\hat{\Gamma}_{2j} = (-I_{j-1}) \oplus I_1 \oplus (-I_j)$ and $\Gamma_{2j} = (-I_j) \oplus I_1 \oplus (-I_{j-1})$ for $j = 2,\ldots,\ell$; thus, $\hat{\mathscr{H}}_{z,2}$ and $\mathscr{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j\times j$ for $j = 1,\ldots,\ell$;

3) *blocks corresponding to two $j\times j$ Jordan blocks of $\hat{\mathscr{H}}$ and two $(j+1)\times(j+1)$*

Jordan blocks of $\mathscr{H}$ associated with the eigenvalue zero:

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & I_1 \\ 0 & 0 \\ I_1 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \oplus \cdots \oplus \bigoplus_{i=1}^{m_\ell} \begin{bmatrix} 0 & I_{\ell-1} \\ 0 & 0 \\ I_{\ell-1} & 0 \\ 0 & 0 \end{bmatrix}_{2\ell\times(2\ell-2)}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} 0 & R_{\ell-1} \\ -R_{\ell-1} & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} \mathscr{J}_1(0) & 0 \\ 0 & \mathscr{J}_1(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} \mathscr{J}_{\ell-1}(0) & 0 \\ 0 & \mathscr{J}_{\ell-1}(0) \end{bmatrix}, \\
\mathscr{H}_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} \mathscr{J}_2(0) & 0 \\ 0 & \mathscr{J}_2(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} \mathscr{J}_\ell(0) & 0 \\ 0 & \mathscr{J}_\ell(0) \end{bmatrix}^T,
\end{aligned}
$$

where $m_1,\ldots,m_{\ell-1} \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,3}$ has $2m_j$ Jordan blocks of size $j\times j$ and $\mathscr{H}_{z,3}$ has $2m_j$ Jordan blocks of size $(j+1)\times(j+1)$ for $j=1,\ldots,\ell-1$;

4) *blocks corresponding to two $(j+1)\times(j+1)$ Jordan blocks of $\hat{\mathscr{H}}$ and two $j\times j$ Jordan blocks of $\mathscr{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & 0 & 0 & I_1 \\ 0 & I_1 & 0 & 0 \end{bmatrix}_{2\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & 0 & 0 & I_{\ell-1} \\ 0 & I_{\ell-1} & 0 & 0 \end{bmatrix}_{(2\ell-2)\times 2\ell}, \\
G_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & R_{\ell-1} \\ -R_{\ell-1} & 0 \end{bmatrix}, \\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{\mathscr{H}}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} \mathscr{J}_2(0) & 0 \\ 0 & \mathscr{J}_2(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} \mathscr{J}_\ell(0) & 0 \\ 0 & \mathscr{J}_\ell(0) \end{bmatrix}, \\
\mathscr{H}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} \mathscr{J}_1(0) & 0 \\ 0 & \mathscr{J}_1(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} \mathscr{J}_{\ell-1}(0) & 0 \\ 0 & \mathscr{J}_{\ell-1}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $n_1,\ldots,n_{\ell-1} \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathscr{H}}_{z,4}$ has $2n_j$ Jordan blocks of size $(j+1)\times(j+1)$ and $\mathscr{H}_{z,4}$ has $2n_j$ Jordan blocks of size $j\times j$ for $j=1,\ldots,\ell-1$;

Then for the eigenvalue zero, the matrices $\hat{\mathscr{H}}$ and $\mathscr{H}$ have $2\gamma_j+2m_j+2n_{j-1}$ respectively $2\gamma_j+2m_{j-1}+2n_j$ Jordan blocks of size $j\times j$ for $j=1,\ldots,\ell$. Here ℓ is the maximum of the indices of $\mathscr{H}$ and $\hat{\mathscr{H}}$. (Here index *refers to the maximal size of a Jordan block associated with the eigenvalue zero.)*

Furthermore, the form (5.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (5.3).

Proof. The proof is presented in the Appendix. □

6. Conclusion

We have presented canonical forms for matrix triples $(A,G,\hat{G})$, where $G,\hat{G}$ are nonsingular and either complex and Hermitian or skew Hermitian or real and symmetric or skew symmetric. These results generalize the canonical forms for matrices that are Hermitian, skew Hermitian or real symmetric, skew symmetric with respect to indefinite scalar products as they are studied in detail in [7, 8, 16, 17, 18].

7. Appendix: Proofs of the main theorems

7.1. Preliminary factorizations

In the following sections, we aim to compute the canonical forms via some type of staircase algorithm. A key factorization needed in the steps of this algorithm is presented in the following lemma.

PROPOSITION 7.1. *Let $B \in \mathbb{F}^{m\times n}$, $m \geqslant n$, and let $\pi,\nu \geqslant 0$ be integers such that $\pi+\nu = m$. Suppose that* $\operatorname{rank} B = n$ *and that the inertia index of the Hermitian matrix $B^*\Sigma_{\pi,\nu}B$ is (π_0,ν_0,δ_0). Then $\pi_0+\nu_0+\delta_0 = n$ and there exists an invertible matrix $X \in \mathbb{F}^{m\times m}$ such that*

$$X^*B = \begin{array}{c} \left[\begin{array}{c} 0 \\ 0 \\ B_0 \end{array}\right] \end{array} \begin{array}{l} \pi_1+\nu_1 \\ \delta_0 \\ n \end{array}, \quad X^*\Sigma_{\pi,\nu}X = \Sigma_{\pi_1,\nu_1} \oplus \begin{bmatrix} & & I_{\delta_0} \\ & \Sigma_{\pi_0,\nu_0} & \\ I_{\delta_0} & & \end{bmatrix},$$

where $B_0 \in \mathbb{F}^{n\times n}$ is nonsingular, $\pi_1 = \pi-\pi_0-\delta_0 \geqslant 0$, and $\nu_1 = \nu-\nu_0-\delta_0 \geqslant 0$.

Proof. By assumption, there exists a nonsingular matrix $Y \in \mathbb{F}^{n\times n}$ such that

$$Y^*B^*\Sigma_{\pi,\nu}BY = \Sigma_{\pi_0,\nu_0,\delta_0}.$$

Let $B_1 \in \mathbb{F}^{m\times\pi_0}$ be the matrix formed by the leading π_0 columns of BY and partition it as

$$B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \quad B_{11} \in \mathbb{F}^{\pi\times\pi_0}, \quad B_{21} \in \mathbb{F}^{\nu\times\pi_0}.$$

Then from $B_1^*\Sigma_{\pi,\nu}B_1 = I_{\pi_0}$ we have that

$$B_{11}^*B_{11} - B_{21}^*B_{21} = I_{\pi_0}. \tag{7.1}$$

Since $B_{11}^*B_{11}$ and $B_{21}^*B_{21}$ are positive semidefinite, it follows that $\operatorname{rank} B_{11} = \operatorname{rank}(I_{\pi_0} + B_{21}^*B_{21}) = \pi_0$ and therefore $\pi \geqslant \pi_0$ by Sylvester's Law of Inertia. Hence, there exists a unitary matrix $U_1 \in \mathbb{F}^{\pi\times\pi}$ such that

$$U_1^*B_{11} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix},$$

where $T_1 \in \mathbb{F}^{\pi_0\times\pi_0}$ is invertible. Since $B_{11}^*B_{11} = T_1^*T_1$, we obtain that (7.1) is equivalent to

$$I_{\pi_0} - (B_{21}T_1^{-1})^*(B_{21}T_1^{-1}) = T_1^{-*}T_1^{-1}.$$

Then the matrix $I_\nu - (B_{21}T_1^{-1})(B_{21}T_1^{-1})^*$ is positive definite, because it easily follows from [6] that it has the same eigenvalues as $I_{\pi_0} - (B_{21}T_1^{-1})^*(B_{21}T_1^{-1})$ with a possible exception for the eigenvalue $\lambda = 1$. Thus, we have the factorization

$$I_\nu - (B_{21}T_1^{-1})(B_{21}T_1^{-1})^* = \widetilde{T}_1\widetilde{T}_1^*, \tag{7.2}$$

for some invertible $\widetilde{T}_1 \in \mathbb{F}^{\nu\times\nu}$. Let

$$X_1 = \begin{bmatrix} U_1 & 0 \\ 0 & I_\nu \end{bmatrix} \begin{bmatrix} T_1 & 0 & -(B_{21}T_1^{-1})^*\widetilde{T}_1^{-*} \\ 0 & I_{\pi-\pi_0} & 0 \\ -B_{21} & 0 & \widetilde{T}_1^{-*} \end{bmatrix}.$$

With (7.1) and (7.2) it is easily verified that

$$X_1^*B_1 = \begin{array}{@{}c@{}l@{}} \begin{bmatrix} I_{\pi_0} \\ 0 \\ 0 \end{bmatrix} & \begin{array}{l} \pi_0 \\ \pi-\pi_0 \\ \nu \end{array} \end{array}, \qquad X_1^*\Sigma_{\pi,\nu}X_1 = \Sigma_{\pi,\nu}.$$

Then, since $\Sigma_{\pi,\nu}^2 = I_m$, the last relation implies that

$$X_1\Sigma_{\pi,\nu}X_1^* = X_1\Sigma_{\pi,\nu}\Sigma_{\pi,\nu}\Sigma_{\pi,\nu}X_1^* = X_1\Sigma_{\pi,\nu}X_1^*\Sigma_{\pi,\nu}X_1\Sigma_{\pi,\nu}X_1^*$$

and thus $\Sigma_{\pi,\nu}X_1\Sigma_{\pi,\nu}X_1^* = I_m$ or, equivalently, $X_1\Sigma_{\pi,\nu}X_1^* = \Sigma_{\pi,\nu}$. Also recall that B_1 consists of the first π_0 columns of BY. Thus, partitioning

$$X_1^*BY = \begin{bmatrix} I_{\pi_0} & B_{12} \\ 0 & \widetilde{B} \end{bmatrix},$$

where $\widetilde{B}$ is $(m-\pi_0)\times(n-\pi_0)$, we obtain from

$$\begin{aligned} \Sigma_{\pi_0,\nu_0,\delta_0} &= Y^*B^*\Sigma_{\pi,\nu}BY = (X_1^*BY)^*\Sigma_{\pi,\nu}(X_1^*BY) \\ &= \begin{bmatrix} I_{\pi_0} & 0 \\ B_{12}^* & \widetilde{B}^* \end{bmatrix} \begin{bmatrix} I_{\pi_0} & 0 \\ 0 & \Sigma_{\pi-\pi_0,\nu} \end{bmatrix} \begin{bmatrix} I_{\pi_0} & B_{12} \\ 0 & \widetilde{B} \end{bmatrix}, \end{aligned}$$

that

$$B_{12} = 0, \qquad \widetilde{B}^*\Sigma_{\pi-\pi_0,\nu}\widetilde{B} = \Sigma_{0,\nu_0,\delta_0} = \begin{bmatrix} -I_{\nu_0} & 0 \\ 0 & \mathscr{O}_{\delta_0} \end{bmatrix}.$$

Letting $B_2 \in \mathbb{F}^{(m-\pi_0)\times\nu_0}$ be the matrix consisting of the leading ν_0 columns of $\widetilde{B}$, we obtain that $B_2^*\Sigma_{\pi-\pi_0,\nu}B_2 = -I_{\nu_0}$. By a procedure analogous to the one used for B_1 above, we can determine a nonsingular matrix $X_2 \in \mathbb{F}^{(m-\pi_0)\times(m-\pi_0)}$ such that

$$X_2^*B_2 = \begin{array}{@{}c@{}l@{}} \begin{bmatrix} 0 \\ I_{\nu_0} \\ 0 \end{bmatrix} & \begin{array}{l} \pi-\pi_0 \\ \nu_0 \\ \nu-\nu_0 \end{array} \end{array}, \qquad X_2^*\Sigma_{\pi-\pi_0,\nu}X_2 = \Sigma_{\pi-\pi_0,\nu},$$

which also shows that $\nu_0 \leqslant \nu$. With $X_3 = X_1(I_{\pi_0} \oplus X_2)$, we then have

$$X_3^* BY = \begin{bmatrix} I_{\pi_0} & 0 & 0 \\ 0 & 0 & B_{13} \\ 0 & I_{\nu_0} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix}, \quad X_3^* \Sigma_{\pi,\nu} X_3 = \Sigma_{\pi,\nu},$$

which also implies $X_3 \Sigma_{\pi,\nu} X_3^* = \Sigma_{\pi,\nu}$ and thus $(X_3^* BY)^* \Sigma_{\pi,\nu} (X_3^* BY) = \Sigma_{\pi_0,\nu_0,\delta_0}$. Then it easily follows that

$$B_{23} = 0 \quad \text{and} \quad 0 = \begin{bmatrix} B_{13} \\ B_{33} \end{bmatrix}^* \Sigma_{\pi-\pi_0,\nu-\nu_0} \begin{bmatrix} B_{13} \\ B_{33} \end{bmatrix} = B_{13}^* B_{13} - B_{33}^* B_{33}. \tag{7.3}$$

Let P_1 be the permutation matrix that interchanges the middle two block-rows of $X_3^* BY$ by pre-multiplication and set $X_4 = X_3 P_1^*$. Then

$$X_4^* BY = \begin{bmatrix} I_{\pi_0} & 0 & 0 \\ 0 & I_{\nu_0} & 0 \\ 0 & 0 & B_{13} \\ 0 & 0 & B_{33} \end{bmatrix}, \quad X_4^* \Sigma_{\pi,\nu} X_4 = \begin{bmatrix} \Sigma_{\pi_0,\nu_0} & 0 \\ 0 & \Sigma_{\pi-\pi_0,\nu-\nu_0} \end{bmatrix}.$$

Now, both B_{13} and B_{33} have full column rank, because otherwise, by (7.3) it is not difficult to show that B_{13} and B_{33} would have a common null space. But this is not possible, because then $X_4^* BY$, as well as B, would have rank less than n, contradicting the assumption. Since $B_{13} \in \mathbb{F}^{(\pi-\pi_0)\times\delta_0}$ and $B_{33} \in \mathbb{F}^{(\nu-\nu_0)\times\delta_0}$, we have that $\pi \geqslant \pi_0 + \delta_0$ and $\nu \geqslant \nu_0 + \delta_0$. Observe that (7.3) implies that the positive definite factors in the polar decompositions of B_{13} and B_{33} coincide, i.e., we have

$$B_{13} = \widetilde{U}_3 W \quad \text{and} \quad B_{33} = \widetilde{U}_4 W$$

for some $\widetilde{U}_3 \in \mathbb{F}^{(\pi-\pi_0)\times\delta_0}$, $\widetilde{U}_4 \in \mathbb{F}^{(\nu-\nu_0)\times\delta_0}$, where $\widetilde{U}_3, \widetilde{U}_4$ have orthonormal columns and $W = (B_{13}^* B_{13})^{1/2} = (B_{33}^* B_{33})^{1/2} \in \mathbb{F}^{\delta_0\times\delta_0}$ is nonsingular. Extending $\widetilde{U}_3$ and $\widetilde{U}_4$ to unitary matrices $U_3 \in \mathbb{F}^{(\pi-\pi_0)\times(\pi-\pi_0)}$, $U_4 \in \mathbb{F}^{(\nu-\nu_0)\times(\nu-\nu_0)}$, we obtain that.

$$U_3 B_{13} = \begin{bmatrix} W \\ 0 \end{bmatrix}, \qquad U_4 B_{33} = \begin{bmatrix} W \\ 0 \end{bmatrix},$$

Setting $X_5 = X_4(I_{\pi_0+\nu_0} \oplus U_3^* \oplus U_4^*)$, we obtain that

$$X_5^* BY = \begin{bmatrix} I_{\pi_0+\nu_0} & 0 \\ 0 & W \\ 0 & 0 \\ 0 & W \\ 0 & 0 \end{bmatrix}, \quad X_5^* \Sigma_{\pi,\nu} X_5 = \Sigma_{\pi_0,\nu_0} \oplus \Sigma_{\pi-\pi_0,\nu-\nu_0}.$$

Let P_2 be the permutation matrix that interchanges the 3rd and 4th block row of $X_5^* BY$ by pre-multiplication, and let $X_6 = X_5 P_2^*$. Then

$$X_6^* BY = \begin{bmatrix} I_{\pi_0+\nu_0} & 0 \\ 0 & W \\ 0 & W \\ 0 & 0 \end{bmatrix}, \quad X_6^* \Sigma_{\pi,\nu} X_6 = \Sigma_{\pi_0,\nu_0} \oplus \Sigma_{\delta_0,\delta_0} \oplus \Sigma_{\pi_1,\nu_1},$$

where $\pi_1 = \pi - \pi_0 - \delta_0$ and $\nu_1 = \nu - \nu_0 - \delta_0$. Then setting

$$Z = \frac{\sqrt{2}}{2}\begin{bmatrix} I_{\delta_0} & I_{\delta_0} \\ I_{\delta_0} & -I_{\delta_0} \end{bmatrix},$$

and $X_7 = X_6(I_{\pi_0+\nu_0} \oplus Z \oplus I_{\pi_1+\nu_1})$, it is easily verified that

$$Z^*\begin{bmatrix} W \\ W \end{bmatrix} = \begin{bmatrix} \sqrt{2}W \\ 0 \end{bmatrix}, \quad Z^*\Sigma_{\delta_0,\delta_0}Z = \begin{bmatrix} 0 & I_{\delta_0} \\ I_{\delta_0} & 0 \end{bmatrix},$$

and thus, we have

$$X_7^*BY = \begin{bmatrix} I_{\pi_0+\nu_0} & 0 \\ 0 & \sqrt{2}W \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_7^*\Sigma_{\pi,\nu}X_7 = \Sigma_{\pi_0,\nu_0} \oplus \begin{bmatrix} 0 & I_{\delta_0} \\ I_{\delta_0} & 0 \end{bmatrix} \oplus \Sigma_{\pi_1,\nu_1}.$$

Let P_3 be the permutation matrix that changes the order the block rows of X_7^*BY to the order $4,3,1,2$ by pre-multiplication and set $X = X_7P_3^*$. Then

$$X^*BY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{\pi_0+\nu_0} & 0 \\ 0 & \sqrt{2}W \end{bmatrix}, \quad X^*\Sigma_{\pi,\nu}X = \Sigma_{\pi_1,\nu_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & \Sigma_{\pi_0,\nu_0} & 0 \\ I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

The desired factorization then follows by multiplying with Y^{-1} from the right and setting $B_0 = (I_{\pi_0+\nu_0} \oplus \sqrt{2}W)Y^{-1}$. □

PROPOSITION 7.2. *Let $B \in \mathbb{R}^{2m\times n}$ and suppose that* $\operatorname{rank} B = n$, $\operatorname{rank} B^TJ_mB = 2n_0$ *(note that the rank of a real skew-symmetric matrix is even), and let $\delta_0 = n - 2n_0$ denote the dimension of the null space of B^TJ_mB. Then there exists an invertible matrix $X \in \mathbb{R}^{2m\times 2m}$ such that*

$$X^TB = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} \end{array}\begin{array}{l} 2n_1 \\ \delta_0 \\ n \end{array}, \quad X^TJ_mX = J_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & J_{n_0} & 0 \\ -I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

where $B_0 \in \mathbb{C}^{n\times n}$ is nonsingular and $n_1 = m - n_0 - \delta_0$.

Proof. The proof follows the same lines as in the complex case (or more precisely, as in the case of a complex skew-symmetric bilinear form induced by J_m), see [19] for details. □

7.2. Proof of Theorem 3.2

We present a constructive and recursive proof in several steps. The proof uses the same strategy as in the case of G and $\hat{G}$ being complex symmetric, see [19]. Although this requires a lot of repetition of the ideas published in [19], we decided to give the full proof of Theorem 3.2 and ideas of proof for the other main theorems, because of two reasons. First, we want this paper to be self-contained, and secondly, the case of complex sesquilinear forms or real bilinear forms is more involved than the case of complex bilinear forms. For example, any complex symmetric matrix is congruent to the identity matrix, but the same is not true for complex Hermitian matrices under congruence or real symmetric matrices under real congruence. This fact results in the existence of the so-called sign characteristic of real eigenvalues of G-Hermitian matrices. It is this point that makes the development of the canonical forms more challenging in the case that G and $\hat{G}$ are complex Hermitian or real symmetric or skew-symmetric.

Step 1) Reduction to a stair-case-like form

Let $(\pi,\nu,0)$ and $(\hat{\pi},\hat{\nu},0)$ be the Sylvester inertia indices of G and $\hat{G}$, respectively. By applying appropriate congruence transformations to G and $\hat{G}$, we may assume that $G=\Sigma_{\pi,\nu}$ and $\hat{G}=\Sigma_{\hat{\pi},\hat{\nu}}$. Let

$$A=B_1C_1^*$$

be a full rank factorization of A, i.e., $B_1\in\mathbb{C}^{m\times r}$, $C_1\in\mathbb{C}^{n\times r}$, $\operatorname{rank}B_1=\operatorname{rank}C_1=r$. Applying Proposition 7.1 to B_1 and C_1, respectively, we can determine nonsingular matrices $X_1\in\mathbb{C}^{m\times m}$ and $Y_1\in\mathbb{C}^{n\times n}$ such that

$$X_1^*B_1=\begin{bmatrix}0\\0\\B_{1,0}\end{bmatrix}\begin{array}{l}\pi_0+\nu_0\\ \delta_1\\ r\end{array},\quad X_1^*\Sigma_{\pi,\nu}X_1=\Sigma_{\pi_0,\nu_0}\oplus\begin{bmatrix}0&0&I_{\delta_1}\\0&\Sigma_{p_1,q_1}&0\\I_{\delta_1}&0&0\end{bmatrix},$$

$$Y_1^*C_1=\begin{bmatrix}0\\0\\C_{1,0}\end{bmatrix}\begin{array}{l}\hat{\pi}_0+\hat{\nu}_0\\ \hat{\delta}_1\\ r\end{array},\quad Y_1^*\Sigma_{\hat{\pi},\hat{\nu}}Y_1=\Sigma_{\hat{\pi}_0,\hat{\nu}_0}\oplus\begin{bmatrix}0&0&I_{\hat{\delta}_1}\\0&\Sigma_{\hat{p}_1,\hat{q}_1}&0\\I_{\hat{\delta}_1}&0&0\end{bmatrix},$$

where $B_{1,0},C_{1,0}\in\mathbb{C}^{r\times r}$ are both invertible, $p_1,q_1,\delta_1,\hat{p}_1,\hat{q}_1,\hat{\delta}_1\geqslant 0$, and

$$p_1+q_1+\delta_1=\hat{p}_1+\hat{q}_1+\hat{\delta}_1=r.$$

Partition

$$B_{1,0}C_{1,0}^*=\begin{array}{c}\\ p_1+q_1\\ \delta_1\end{array}\begin{array}{c}\begin{array}{cc}\hat{p}_1+\hat{q}_1&\hat{\delta}_1\end{array}\\ \begin{bmatrix}A_{3,3}&A_{3,4}\\A_{4,3}&A_{4,4}\end{bmatrix}\end{array},$$

then

$$X_1^*AY_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&A_{3,3}&A_{3,4}\\0&0&A_{4,3}&A_{4,4}\end{bmatrix},\quad X_1^*\Sigma_{\pi,\nu}X_1=\begin{bmatrix}\Sigma_{\pi_0,\nu_0}&0&0&0\\0&0&0&I_{\delta_1}\\0&0&\Sigma_{p_1,q_1}&0\\0&I_{\delta_1}&0&0\end{bmatrix},$$

$$Y_1^*\Sigma_{\hat{\pi},\hat{\nu}}Y_1=\begin{bmatrix}\Sigma_{\hat{\pi}_0,\hat{\nu}_0}&0&0&0\\0&0&0&I_{\hat{\delta}_1}\\0&0&\Sigma_{\hat{p}_1,\hat{q}_1}&0\\0&I_{\hat{\delta}_1}&0&0\end{bmatrix}.$$

Applying the same procedure to the triple $(A_{3,3},\Sigma_{p_1,q_1},\Sigma_{\hat{p}_1,\hat{q}_1})$, we can construct nonsingular matrices $\widetilde{X}_2,\widetilde{Y}_2$ such that

$$\widetilde{X}_2^* A_{3,3}\widetilde{Y}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{5,5} & A_{5,6} \\ 0 & 0 & A_{6,5} & A_{6,6} \end{bmatrix}, \quad \widetilde{X}_2^*\Sigma_{p_1,q_1}\widetilde{X}_2 = \begin{bmatrix} \Sigma_{\pi_1,\nu_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_2} \\ 0 & 0 & \Sigma_{p_2,q_2} & 0 \\ 0 & I_{\delta_2} & 0 & 0 \end{bmatrix},$$

$$\widetilde{Y}_2^*\Sigma_{\hat{p}_1,\hat{q}_1}\widetilde{Y}_2 = \begin{bmatrix} \Sigma_{\hat{\pi}_1,\hat{\nu}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_2} \\ 0 & 0 & \Sigma_{\hat{p}_2,\hat{q}_2} & 0 \\ 0 & I_{\hat{\delta}_2} & 0 & 0 \end{bmatrix},$$

where $p_2,q_2,\delta_2,\hat{p}_2,\hat{q}_2,\hat{\delta}_2 \geqslant 0$, $A_{6,6}\in\mathbb{F}^{\delta_2\times\hat{\delta}_2}$, $A_{5,6}\in\mathbb{F}^{(p_2+q_2)\times\hat{\delta}_2}$, $A_{6,5}\in\mathbb{F}^{\delta_2\times(\hat{p}_2+\hat{q}_2)}$, $A_{5,5}\in\mathbb{F}^{(p_2+q_2)\times(\hat{p}_2+\hat{q}_2)}$, $p_2+q_2+\delta_2=\hat{p}_2+\hat{q}_2+\hat{\delta}_2=\operatorname{rank}A_{3,3}$, and where the matrix

$$\begin{bmatrix} A_{5,5} & A_{5,6} \\ A_{6,5} & A_{6,6} \end{bmatrix} \in \mathbb{F}^{(p_2+q_2+\delta_2)\times(p_2+q_2+\delta_2)}$$

is nonsingular. Letting

$$X_2 = X_1(I_{\pi_0+\nu_0+\delta_1}\oplus\widetilde{X}_2\oplus I_{\delta_1}), \quad Y_2 = Y_1(I_{\hat{\pi}_0+\hat{\nu}_0+\hat{\delta}_1}\oplus\widetilde{Y}_2\oplus I_{\hat{\delta}_1}),$$

we then have

$$X_2^*AY_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{3,7} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{4,7} \\ 0 & 0 & 0 & 0 & A_{5,5} & A_{5,6} & A_{5,7} \\ 0 & 0 & 0 & 0 & A_{6,5} & A_{6,6} & A_{6,7} \\ 0 & 0 & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} \end{bmatrix},$$

$$X_2^*\Sigma_{\pi,\nu}X_2 = \begin{bmatrix} \Sigma_{\pi_0,\nu_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_1} \\ 0 & 0 & \Sigma_{\pi_1,\nu_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\delta_2} & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{p_2,q_2} & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_2} & 0 & 0 & 0 \\ 0 & I_{\delta_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_2^*\Sigma_{\hat{\pi},\hat{\nu}}Y_2 = \begin{bmatrix} \Sigma_{\hat{\pi}_0,\hat{\nu}_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_1} \\ 0 & 0 & \Sigma_{\hat{\pi}_1,\hat{\nu}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{\hat{p}_2,\hat{q}_2} & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 & 0 & 0 \\ 0 & I_{\hat{\delta}_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the matrix $X_2^* A Y_2$ has been partitioned conformably with $X_2^* \Sigma_{\pi,\nu} X_2$ (row-wise) and $Y_2^* \Sigma_{\hat\pi,\hat\nu} Y_2$ (column-wise). The submatrix of $X_2^* A Y_2$ that is obtained by deleting the leading two block rows and block columns is then nonsingular, because it is equivalent to $B_{1,0} C_{1,0}^*$. Thus, $\left[\begin{smallmatrix} A_{3,7} \\ A_{4,7} \end{smallmatrix}\right]$ has full row rank and $[A_{7,3}\ A_{7,4}]$ has full column rank.

We can repeat the procedure for the triple $(A_{5,5}, \Sigma_{p_2,q_2}, \Sigma_{\hat p_2,\hat q_2})$ which finally yields nonsingular matrices X_3 and Y_3 such that (after renaming some blocks in A and using the canonical notation corresponding to the notation in the previous step), we have

$$X_3^* A Y_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&A_{3,10}\\ 0&0&0&0&0&0&0&0&0&A_{4,10}\\ 0&0&0&0&0&0&0&0&A_{5,9}&A_{5,10}\\ 0&0&0&0&0&0&0&0&A_{6,9}&A_{6,10}\\ 0&0&0&0&0&0&A_{7,7}&A_{7,8}&A_{7,9}&A_{7,10}\\ 0&0&0&0&0&0&A_{8,7}&A_{8,8}&A_{8,9}&A_{8,10}\\ 0&0&0&0&A_{9,5}&A_{9,6}&A_{9,7}&A_{9,8}&A_{9,9}&A_{9,10}\\ 0&0&A_{10,3}&A_{10,4}&A_{10,5}&A_{10,6}&A_{10,7}&A_{10,8}&A_{10,9}&A_{10,10} \end{bmatrix},$$

$$X_3^* \Sigma_{\pi,\nu} X_3 = \begin{bmatrix} \Sigma_{\pi_0,\nu_0}&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&I_{\delta_1}\\ 0&0&\Sigma_{\pi_1,\nu_1}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&I_{\delta_2}&0\\ 0&0&0&0&\Sigma_{\pi_2,\nu_2}&0&0&0&0&0\\ 0&0&0&0&0&0&0&I_{\delta_3}&0&0\\ 0&0&0&0&0&0&\Sigma_{p_3,q_3}&0&0&0\\ 0&0&0&0&0&I_{\delta_3}&0&0&0&0\\ 0&0&0&I_{\delta_2}&0&0&0&0&0&0\\ 0&I_{\delta_1}&0&0&0&0&0&0&0&0 \end{bmatrix}, \tag{7.4}$$

$$Y_3^* \Sigma_{\hat\pi,\hat\nu} Y_3 = \begin{bmatrix} \Sigma_{\hat\pi_0,\hat\nu_0}&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&I_{\hat\delta_1}\\ 0&0&\Sigma_{\hat\pi_1,\hat\nu_1}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&I_{\hat\delta_2}&0\\ 0&0&0&0&\Sigma_{\hat\pi_2,\hat\nu_2}&0&0&0&0&0\\ 0&0&0&0&0&0&0&I_{\hat\delta_3}&0&0\\ 0&0&0&0&0&0&\Sigma_{\hat p_3,\hat q_3}&0&0&0\\ 0&0&0&0&0&I_{\hat\delta_3}&0&0&0&0\\ 0&0&0&I_{\hat\delta_2}&0&0&0&0&0&0\\ 0&I_{\hat\delta_1}&0&0&0&0&0&0&0&0 \end{bmatrix},$$

where $[A_{10,3}\ A_{10,4}]$ and $[A_{9,5}\ A_{9,6}]$ have full column rank,

$$\begin{bmatrix} A_{3,10} \\ A_{4,10} \end{bmatrix} \text{ and } \begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} \text{ have full row rank, and } \begin{bmatrix} A_{7,7} & A_{7,8} \\ A_{8,7} & A_{8,8} \end{bmatrix} \text{ is nonsingular.}$$

Continuing recursively, the process clearly has to stagnate after finitely many steps. Using the canonical notation corresponding to the notation in the first two steps of the process, we find that stagnation occurs after the ℓth step either when $A_{2\ell+1,2\ell+1}$ is nonsingular or when $p_\ell = q_\ell = \hat{p}_\ell = \hat{q}_\ell = 0$. In both cases we obviously have that $p_\ell + q_\ell = \hat{p}_\ell + \hat{q}_\ell$, and we end up with a nonsingular matrix

$$\begin{bmatrix} A_{2\ell+1,2\ell+1} & A_{2\ell+1,2\ell+2} \\ A_{2\ell+2,2\ell+1} & A_{2\ell+2,2\ell+2} \end{bmatrix} \in \mathbb{F}^{(p_\ell+q_\ell+\delta_\ell)\times(\hat{p}_\ell+\hat{q}_\ell+\hat{\delta}_\ell)},$$

full row rank matrices

$$\begin{bmatrix} A_{2k+1,3\ell+2-k} \\ A_{2k+2,3\ell+2-k} \end{bmatrix} \in \mathbb{F}^{(\pi_k+\nu_k+\delta_{k+1})\times\hat{\delta}_k}, \quad k = 1,\dots,\ell-1,$$

and full column rank matrices $[A_{3\ell+2-k,2k+1} \; A_{3\ell+2-k,2k+2}] \in \mathbb{F}^{\delta_k\times(\hat{\pi}_k+\hat{\nu}_k+\hat{\delta}_{k+1})}$ for $k = 1,\dots,\ell-1$. Also, we have

$$\delta_\ell = \hat{\delta}_\ell, \tag{7.5}$$

because $p_\ell + q_\ell + \delta_\ell = \hat{p}_\ell + \hat{q}_\ell + \hat{\delta}_\ell$. Finally, we obtain that due to the full rank properties, we have that

$$\delta_{k-1} \geqslant \hat{\pi}_{k-1} + \hat{\nu}_{k-1} + \hat{\delta}_k, \quad \hat{\delta}_{k-1} \geqslant \pi_{k-1} + \nu_{k-1} + \delta_k \tag{7.6}$$

for $k = 2,\dots,\ell$. On the other hand from the reduction process we have

$$p_k + q_k + \delta_k = \hat{p}_k + \hat{q}_k + \hat{\delta}_k, \tag{7.7}$$

for $k = 1,2,\dots,\ell$, and

$$\begin{aligned} p_{k-1} + q_{k-1} &= \pi_{k-1} + \nu_{k-1} + 2\delta_k + p_k + q_k, \\ \hat{p}_{k-1} + \hat{q}_{k-1} &= \hat{\pi}_{k-1} + \hat{\nu}_{k-1} + 2\hat{\delta}_k + \hat{p}_k + \hat{q}_k, \end{aligned}$$

for $k = 2,\dots,l$. The latter two equations can be rewritten as

$$\begin{aligned} p_{k-1} + q_{k-1} + \delta_{k-1} &= \pi_{k-1} + \nu_{k-1} + \delta_k + \delta_{k-1} + (p_k + q_k + \delta_k), \\ \hat{p}_{k-1} + \hat{q}_{k-1} + \hat{\delta}_{k-1} &= \hat{\pi}_{k-1} + \hat{\nu}_{k-1} + \hat{\delta}_k + \hat{\delta}_{k-1} + (\hat{p}_k + \hat{q}_k + \hat{\delta}_k). \end{aligned}$$

By using (7.7) we then obtain

$$\pi_{k-1} + \nu_{k-1} + \delta_k + \delta_{k-1} = \hat{\pi}_{k-1} + \hat{\nu}_{k-1} + \hat{\delta}_k + \hat{\delta}_{k-1},$$

or, equivalently,

$$\hat{\delta}_{k-1} - \pi_{k-1} - \nu_{k-1} - \delta_k = \delta_{k-1} - \hat{\pi}_{k-1} - \hat{\nu}_{k-1} - \hat{\delta}_k \geqslant 0 \tag{7.8}$$

for $k = 2,\dots,\ell$, where the nonnegativity follows from (7.6).

Step 2) Further reduction of the staircase form

We now isolate the nonsingular block $A_{2\ell+1,2\ell+1}$ from the other blocks and compress the remaining part of $X_\ell^* A Y_\ell$ to a more condensed form. We set $\pi_\ell = p_\ell, \nu_\ell = q_\ell, \hat{\pi}_\ell = \hat{p}_\ell, \hat{\nu}_\ell = \hat{q}_\ell$ and

$$m_k := \begin{cases} \pi_k + \nu_k & \text{if } k \text{ is even} \\ \hat{\pi}_k + \hat{\nu}_k & \text{if } k \text{ is odd} \end{cases}, \quad n_k := \begin{cases} \pi_k + \nu_k & \text{if } k \text{ is odd} \\ \hat{\pi}_k + \hat{\nu}_k & \text{if } k \text{ is even} \end{cases}$$

for $k=0,\dots,\ell$. Moreover, (using (7.5) and (7.8)), we define $\gamma_\ell := \delta_\ell = \hat{\delta}_\ell$ and

$$\gamma_k := \hat{\delta}_k - \pi_k - \nu_k - \delta_{k+1} = \delta_k - \hat{\pi}_k - \hat{\nu}_k - \hat{\delta}_{k+1}, \quad k=1,\dots,\ell-1.$$

For the sake of readability of the paper, we will not carry out the proof for the general case, but we will illustrate the procedure for the special case that $\ell = 3$, where we have the matrices as in (7.4). The general case proceeds in a completely analogous way.

If not void, then $A_{7,7}$ in $X_3^* A Y_3$ in (7.4) is nonsingular, and hence, we can annihilate $A_{7,8}$ by post-multiplying $X_3^* A Y_3$ with the matrix

$$Z_1 := I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus I_{\hat{\delta}_3} \oplus \begin{bmatrix} I & -A_{7,7}^{-1}A_{7,8} \\ 0 & I \end{bmatrix} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1}.$$

Correspondingly updating $Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3$ this leads to a fill-in in the $(7,8)$ and $(8,7)$ block positions in $Z_1^* Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3 Z_1$ given by $-\Sigma_{\hat{p}_3,\hat{q}_3} A_{7,7}^{-1} A_{7,8}$ and $-A_{7,8}^* A_{7,7}^{-*} \Sigma_{\hat{p}_3,\hat{q}_3}$, respectively. We can annihilate these two fill-ins by using the $(8,6)$ block entry $I_{\hat{\delta}_3}$ as a pivot, i.e., by applying a congruence transformation to $Z_1^* Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3 Z_1$ with

$$Z_2 = I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus \begin{bmatrix} I & A_{7,8}^* A_{7,7}^{-*} \Sigma_{\hat{p}_3,\hat{q}_3} \\ 0 & I \end{bmatrix} \oplus I_{\hat{\delta}_3} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1}.$$

It is then easy to check that $Z_2^* Z_1^* Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3 Z_1 Z_2 = Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3$ and that the correspondingly updated matrix $X_3^* A Y_3 Z_1 Z_2$ has no further fill-ins. Finally, we update $Y_3 \leftarrow Y_3 Z_1 Z_2$.

Similarly, we can annihilate $A_{8,7}$ by working on the rows of $X_3^* A Y_3$ and applying congruence transformations to $X_3^* \Sigma_{p_1,q_1} X_3$. Then, we can proceed and annihilate the blocks $A_{7,9}$, $A_{9,7}$, $A_{7,10}$, and $A_{10,7}$ in $X_3^* A Y_3$. Since originally the matrix

$$\begin{bmatrix} A_{7,7} & A_{7,8} \\ A_{8,7} & A_{8,8} \end{bmatrix}$$

is nonsingular, we find that after the above reductions the updated block $A_{8,8}$ is nonsingular (or even void). With $A_{8,8}$ as the pivot, we can then annihilate $A_{8,9}$, $A_{9,8}$, $A_{8,10}$, $A_{10,8}$ and recover $X_3^* \Sigma_{\pi,\nu} X_3$ and $Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3$. Observe that this does not change the zero blocks in $X_3^* A Y_3$. Finally post-multiplying $X_3^* A Y_3$ with the matrix

$$Z_3 = I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus A_{8,8}^* \oplus I_{\hat{p}_3+\hat{q}_3} \oplus A_{8,8}^{-1} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1},$$

we then obtain

$$X_3^* A Y_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{3,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{4,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{5,9} & A_{5,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{6,9} & A_{6,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{9,5} & A_{9,6} & 0 & 0 & A_{9,9} & A_{9,10} \\ 0 & 0 & A_{10,3} & A_{10,4} & A_{10,5} & A_{10,6} & 0 & 0 & A_{10,9} & A_{10,10} \end{bmatrix},$$

while $X_3^*\Sigma_{\pi,\nu}X_3$ and $Y_3^*\Sigma_{\hat{\pi},\hat{\nu}}Y_3$ are as in (7.4). (Indeed, observe that the congruence transformation with Z_3 leaves $Y_3^*\Sigma_{\hat{\pi},\hat{\nu}}Y_3$ invariant.) Since the original block $[A_{9,5}\ A_{9,6}]$ has full column rank, it easily follows that the corresponding updated entry

$$\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} \leftarrow \begin{bmatrix} A_{9,5} & A_{9,6}A_{8,8}^* \end{bmatrix}$$

has full column rank as well. Then there exists a nonsingular matrix W_1 such that

$$\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} \leftarrow W_1^* \begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} = \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{\hat{\delta}_3} \\ 0 & 0 \end{bmatrix}. \tag{7.9}$$

Transforming then $X_3^*AY_3$ and $X_3^*\Sigma_{\pi,\nu}X_3$ with a multiplication from the left and congruence transformation, respectively, with a block diagonal matrix having W_1^{-1} in the $(4,4)$-block position and W_1^* in the $(9,9)$-block position, we obtain the desired update in the block $[A_{9,5}\ A_{9,6}]$ while $X_3^*\Sigma_{\pi,\nu}X_3$ and the zero pattern of $X_3^*AY_3$ are invariant under that transformation. We then continue by taking this updated block $[A_{9,5}\ A_{9,6}]$ as a pivot to annihilate $[A_{10,5}\ A_{10,6}]$. Again, this can be done without changing $X_3^*\Sigma_{\pi,\nu}X_3$.

Similarly, due to a full row rank argument, there exists a nonsingular matrix W_2 such that

$$\begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} := \begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} W_2 = \begin{bmatrix} I_{m_2} & 0 & 0 \\ 0 & I_{\delta_3} & 0 \end{bmatrix}. \tag{7.10}$$

and applying appropriate transformation matrices, the corresponding change in $X_3^*AY_3$ can be made without changing $Y_3^*\Sigma_{\hat{\pi},\hat{\nu}}Y_3$. Then, $A_{5,10}$ and $A_{6,10}$ can be annihilated.

Also, we use the pivots $\left[\begin{smallmatrix} A_{5,9} \\ A_{6,9} \end{smallmatrix}\right]$ and $\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix}$, respectively, to annihilate the leading $m_2+\delta_3$ columns of $A_{9,9}$ and $A_{10,9}$, and the leading $n_2+\hat{\delta}_3$ rows of $A_{9,9}$ and $A_{9,10}$. So these three blocks become

$$A_{9,9} \leftarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{9,9} \end{bmatrix}, \quad A_{9,10} \leftarrow \begin{bmatrix} 0 \\ 0 \\ \widetilde{A}_{9,10} \end{bmatrix}, \quad A_{10,9} \leftarrow \begin{bmatrix} 0 & 0 & \widetilde{A}_{10,9} \end{bmatrix},$$

where $\widetilde{A}_{9,9} \in \mathbb{F}^{\gamma_2\times\gamma_2}$, $\widetilde{A}_{9,10} \in \mathbb{F}^{\gamma_2\times\hat{\delta}_1}$, $\widetilde{A}_{10,9} \in \mathbb{F}^{\delta_1\times\gamma_2}$. Since originally the submatrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & A_{5,9} \\ 0 & 0 & 0 & 0 & A_{6,9} \\ 0 & 0 & A_{7,7} & A_{7,8} & A_{7,9} \\ 0 & 0 & A_{8,7} & A_{8,8} & A_{8,9} \\ A_{9,5} & A_{9,6} & A_{9,7} & A_{9,8} & A_{9,9} \end{bmatrix}$$

was nonsingular, we have that $\widetilde{A}_{9,9}$ is nonsingular. We then use $\widetilde{A}_{9,9}$ as pivot block to annihilate $\widetilde{A}_{9,10}$ and $\widetilde{A}_{10,9}$, and transform $\widetilde{A}_{9,9}$ to I_{γ_2}.

In a similar way we can perform the reductions

$$\begin{bmatrix} A_{3,10} \\ A_{4,10} \end{bmatrix} \leftarrow \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{\delta_2} & 0 \end{bmatrix}, \quad \begin{bmatrix} A_{10,3} & A_{10,4} \end{bmatrix} \leftarrow \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{\hat{\delta}_2} \\ 0 & 0 \end{bmatrix},$$

and use them as pivots to reduce $A_{10,10}$ to

$$A_{10,10} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{10,10} \end{bmatrix},$$

where $\widetilde{A}_{10,10} \in \mathbb{F}^{\gamma_1 \times \gamma_1}$, and finally transform $\widetilde{A}_{10,10}$ to I_{γ_1}. After all this, the matrix $X_3^* A Y_3$ has the form

$$X_3^* A Y_3 = \left[\begin{array}{cccccccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{n_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{m_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{n_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_2} & 0 & 0 & 0 \\
\hline
0 & 0 & I_{m_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{\delta}_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_1}
\end{array}\right],$$

while $X_3^* \Sigma_{\pi,\nu} X_3$ and $Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3$ are still as in (7.4). We partition

$$\begin{aligned} I_{\delta_1} &= I_{m_1} \oplus I_{m_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2} \oplus I_{\gamma_1}, & I_{\delta_2} &= I_{n_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2}, \\ I_{\hat{\delta}_1} &= I_{n_1} \oplus I_{n_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2} \oplus I_{\gamma_1}, & I_{\hat{\delta}_2} &= I_{m_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2}, \end{aligned}$$

and replace I_{δ_1}, I_{δ_2}, $I_{\hat{\delta}_1}$, and $I_{\hat{\delta}_2}$ in the matrix triple with these partitions. We then get $X_3^* A Y_3$, $X_3^* \Sigma_{\pi,\nu} X_3$, and $Y_3^* \Sigma_{\hat{\pi},\hat{\nu}} Y_3$ partitioned in 22 block rows and columns. Let P_R be the block permutation that re-arranges the block columns of $X_3^* A Y_3$ in the order

$$13, 1, 6, 22, 5, 10, 17, 21, 4, 9, 12, 14, 16, 20, 2, 7, 18, 3, 8, 11, 15, 19.$$

Let P_L be another block permutation such that P_L^* re-arranges the block rows of $X_3^* A Y_3$ in the same order. Set

$$\widetilde{X} := X_3 P_L, \quad \widetilde{Y} := Y_3 P_R.$$

Then we obtain that

$$\begin{aligned} \widetilde{X}^* A \widetilde{Y} &= \mathscr{A}_{ns} \oplus \mathscr{A}_0 \oplus (\mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3) \oplus (\mathscr{A}_{1,2} \oplus \mathscr{A}_{2,3}), \\ \widetilde{X}^* \Sigma_{\pi,\nu} \widetilde{X} &= \mathscr{G}_{ns} \oplus \mathscr{G}_0 \oplus (\mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \mathscr{G}_3) \oplus (\mathscr{G}_{1,2} \oplus \mathscr{G}_{2,3}), \\ \widetilde{Y}^* \Sigma_{\hat{\pi},\hat{\nu}} \widetilde{Y} &= \hat{\mathscr{G}}_{ns} \oplus \hat{\mathscr{G}}_0 \oplus (\hat{\mathscr{G}}_1 \oplus \hat{\mathscr{G}}_2 \oplus \hat{\mathscr{G}}_3) \oplus (\hat{\mathscr{G}}_{1,2} \oplus \hat{\mathscr{G}}_{2,3}), \end{aligned}$$

where

$$\mathscr{A}_{ns} = A_{2\ell+1,2\ell+1}, \quad \mathscr{G}_{ns} = \Sigma_{\pi_\ell,\nu_\ell}, \quad \hat{\mathscr{G}}_{ns} = \Sigma_{\hat{\pi}_\ell,\hat{\nu}_\ell}, \quad \ell = 3, \tag{7.11}$$

$$\mathscr{A}_0 = 0_{m_0 \times n_0}, \quad \mathscr{G}_0 = \Sigma_{\pi_0,\nu_0}, \quad \hat{\mathscr{G}}_0 = \Sigma_{\hat{\pi}_0,\hat{\nu}_0}, \tag{7.12}$$

$$\mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3 = \begin{bmatrix} 0 & 0 \\ 0 & I_{\gamma_1} \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\gamma_2} \\ 0 & 0 & I_{\gamma_2} & 0 \\ 0 & I_{\gamma_2} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} \\ 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 \\ 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 \\ 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 \\ 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} &\mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \mathscr{G}_3 = \hat{\mathscr{G}}_1 \oplus \hat{\mathscr{G}}_2 \oplus \hat{\mathscr{G}}_3 \\ &= \begin{bmatrix} 0 & I_{\gamma_1} \\ I_{\gamma_1} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & I_{\gamma_2} \\ 0 & 0 & I_{\gamma_2} & 0 \\ 0 & I_{\gamma_2} & 0 & 0 \\ I_{\gamma_2} & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} \\ 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 \\ 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 \\ 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 \\ 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \\ I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\mathscr{A}_{1,2} \oplus \mathscr{A}_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{n_1} \\ 0 & I_{m_1} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 \\ 0 & I_{m_2} & 0 & 0 & 0 \end{bmatrix},$$

$$\mathscr{G}_{1,2} \oplus \mathscr{G}_{2,3} = \begin{bmatrix} 0 & 0 & I_{m_1} \\ 0 & \Sigma_{\pi_1,\nu_1} & 0 \\ I_{m_1} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & I_{m_2} \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & \Sigma_{\pi_2,\nu_2} & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 \\ I_{m_2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\mathscr{G}}_{1,2} \oplus \hat{\mathscr{G}}_{2,3} = \begin{bmatrix} 0 & 0 & I_{n_1} \\ 0 & \Sigma_{\hat{\pi}_1,\hat{\nu}_1} & 0 \\ I_{n_1} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & \Sigma_{\hat{\pi}_2,\hat{\nu}_2} & 0 & 0 \\ 0 & I_{m_2} & 0 & 0 & 0 \\ I_{n_2} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 3) Extraction of Jordan blocks from the staircase-like-form

Completely analogous to the case $\ell = 3$, we proceed in the case $\ell \neq 3$ and obtain the staircase-like-form as

$$\begin{aligned} \widetilde{X}^* A \widetilde{Y} &= \mathscr{A}_{ns} \oplus \mathscr{A}_0 \oplus \bigoplus_{j=1}^{\ell} \mathscr{A}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathscr{A}_{j,j+1}, \\ \widetilde{X}^* \Sigma_{\pi,\nu} \widetilde{X} &= \mathscr{G}_{ns} \oplus \mathscr{G}_0 \oplus \bigoplus_{j=1}^{\ell} \mathscr{G}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathscr{G}_{j,j+1}, \\ \widetilde{Y}^* \Sigma_{\hat{\pi},\hat{\nu}} \widetilde{Y} &= \hat{\mathscr{G}}_{ns} \oplus \hat{\mathscr{G}}_0 \oplus \bigoplus_{j=1}^{\ell} \hat{\mathscr{G}}_j \oplus \bigoplus_{j=1}^{\ell-1} \hat{\mathscr{G}}_{j,j+1}, \end{aligned}$$

where $\mathscr{A}_{ns},\mathscr{G}_{ns},\hat{\mathscr{G}}_{ns}$ are as in (7.11), $\mathscr{A}_0,\mathscr{G}_0,\hat{\mathscr{G}}_0$ are as in (7.12),

$$\mathscr{A}_j = \left(R_{2j}\mathscr{J}_{2j}(0)\right)\otimes I_{\gamma_j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\gamma_j} \\ 0 & 0 & \iddots & 0 \\ 0 & I_{\gamma_j} & 0 & 0 \end{bmatrix}_{(2j)\times(2j)\text{ blocks}}, \tag{7.13}$$

$$\mathscr{G}_j = \hat{\mathscr{G}}_j = R_{2j}\otimes I_{\gamma_j} = \begin{bmatrix} 0 & 0 & I_{\gamma_j} \\ 0 & \iddots & 0 \\ I_{\gamma_j} & 0 & 0 \end{bmatrix}_{(2j)\times(2j)\text{ blocks}}, \tag{7.14}$$

$$\mathscr{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{n_j} \\ & & & \iddots & I_{m_j} & \\ & & \iddots & \iddots & & \\ & 0 & I_{n_j} & & & \\ 0 & I_{m_j} & & & & 0 \end{bmatrix}_{(2j+1)\times(2j+1)\text{ blocks}}, \tag{7.15}$$

$$\mathscr{G}_{j,j+1} = \begin{bmatrix} 0 & & & & & I_{m_j} \\ & & & & I_{n_j} & \\ & & & \iddots & & \\ & & \Sigma_{\pi_j,\nu_j} & & & \\ & \iddots & & & & \\ & I_{n_j} & & & & \\ I_{m_j} & & & & & 0 \end{bmatrix}_{(2j+1)\times(2j+1)\text{ blocks}}, \tag{7.16}$$

$$\hat{\mathscr{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & & I_{n_j} \\ & & & & I_{m_j} & \\ & & & \iddots & & \\ & & \Sigma_{\hat{\pi}_j,\hat{\nu}_j} & & & \\ & \iddots & & & & \\ & I_{m_j} & & & & \\ I_{n_j} & & & & & 0 \end{bmatrix}_{(2j+1)\times(2j+1)\text{ blocks}}. \tag{7.17}$$

The blocks $\mathscr{A}_{ns}$, $\mathscr{G}_{ns}$, $\hat{\mathscr{G}}_{ns}$, $\mathscr{A}_0$, $\mathscr{G}_0$, and $\hat{\mathscr{G}}_0$ are already in the form as indicated in Theorem 3.2. Next, let us investigate in detail the blocks of the form (7.13)–(7.14). Let P_j be the permutation such that premultiplication with P_j^* reorders the rows of $\mathscr{A}_j$ in the order

$$\begin{array}{cccc} 2j\gamma_j, & (2j-1)\gamma_j, & \dots, & \gamma_j, \\ 2j\gamma_j-1, & (2j-1)\gamma_j-1, & \dots, & \gamma_j-1, \\ \vdots & \vdots & \ddots & \vdots \\ 2j\gamma_j-\gamma_j+1, & (2j-1)\gamma_j-\gamma_j+1, & \dots, & 1; \end{array}$$

and let $\widetilde{P}_j$ be the permutation such that postmultiplication with $\widetilde{P}_j$ reorders the columns of $\mathscr{A}_j$ in the order

$$\begin{array}{cccc} \gamma_j, & \ldots, & (2j-1)\gamma_j, & 2j\gamma_j, \\ \gamma_j-1, & \ldots, & (2j-1)\gamma_j-1, & 2j\gamma_j-1, \\ \vdots & \iddots & \vdots & \vdots \\ 1, & \ldots, & (2j-1)\gamma_j-\gamma_j+1, & 2j\gamma_j-\gamma_j+1. \end{array}$$

Then it is easily verified that

$$P_j^*\mathscr{A}_j\widetilde{P}_j = \bigoplus_{i=1}^{\gamma_j} \mathscr{J}_{2j}(0), \quad P_j^*\mathscr{G}_jP_j = \widetilde{P}_j^*\hat{\mathscr{G}}_j\widetilde{P}_j = \bigoplus_{i=1}^{\gamma_j} R_{2j}.$$

Finally, let us return to the blocks of the forms (7.15)–(7.17). Let Z_j be the permutation such that premultiplication with Z_j^* reorders the rows of $\mathscr{A}_{j,j+1}$ in the order

$$\begin{array}{ccccc} (j+1)m_j+jn_j, & jm_j+(j-1)n_j, & \ldots, & 2m_j+n_j, & m_j, \\ (j+1)m_j-1+jn_j, & jm_j-1+(j-1)n_j, & \ldots, & 2m_j-1+n_j, & m_j-1, \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ jm_j+1+jn_j, & (j-1)m_j+1+(j-1)n_j, & \ldots, & m_j+1+n_j, & 1, \\ jm_j+jn_j, & (j-1)m_j+(j-1)n_j, & \ldots, & m_j+n_j, & \\ jm_j+jn_j-1, & (j-1)m_j+(j-1)n_j-1, & \ldots, & m_j+n_j-1, & \\ \vdots & \vdots & \ddots & \vdots & \\ jm_j+(j-1)n_j+1, & (j-1)m_j+(j-2)n_j+1, & \ldots, & m_j+1, & \end{array}$$

and let $\widetilde{Z}_{j+1}$ be the permutation such that postmultiplication with $\widetilde{Z}_{j+1}$ reorders the columns of $\mathscr{A}_{j,j+1}$ in the order

$$\begin{array}{ccccc} m_j+n_j, & 2m_j+n_j, & \ldots, & jm_j+jn_j, & \\ m_j-1+n_j, & 2m_j-1+n_j, & \ldots, & jm_j-1+jn_j, & \\ \vdots & \vdots & \ddots & \vdots & \\ 1+n_j, & m_j+1+n_j, & \ldots, & (j-1)m_j+1+jn_j, & \\ n_j, & m_j+2n_j, & \ldots, & (j-1)m_j+jn_j, & jm_j+(j+1)n_j, \\ n_j-1, & m_j+2n_j-1, & \ldots, & (j-1)m_j+jn_j-1, & jm_j+(j+1)n_j-1, \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1, & m_j+n_j+1, & \ldots, & (j-1)m_j+(j-1)n_j+1, & jm_j+jn_j+1. \end{array}$$

Then it is easily verified that

$$\begin{aligned}
Z_j^*\mathscr{A}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j}\begin{bmatrix} I_j \\ 0 \end{bmatrix}_{(j+1)\times j} \oplus \bigoplus_{i=1}^{n_j}\begin{bmatrix} 0 & I_j \end{bmatrix}_{j\times(j+1)}, \\
Z_j^*\mathscr{G}_{j,j+1}Z_j &= \bigoplus_{i=1}^{\nu_j}\widetilde{R}_{j+1} \oplus \bigoplus_{i=\nu_j+1}^{m_j} R_{j+1} \oplus \bigoplus_{i=1}^{n_j} R_j, \\
\widetilde{Z}_{j+1}^*\hat{\mathscr{G}}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} R_j \oplus \bigoplus_{i=1}^{\hat{\nu}_j}\widetilde{R}_{j+1} \oplus \bigoplus_{i=\hat{\nu}_j+1}^{n_j} R_{j+1},
\end{aligned} \tag{7.18}$$

if j is even and

$$\begin{aligned}
Z_j^*\mathscr{A}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j}\begin{bmatrix} I_j \\ 0 \end{bmatrix}_{(j+1)\times j} \oplus \bigoplus_{i=1}^{n_j}\begin{bmatrix} 0 & I_j \end{bmatrix}_{j\times(j+1)}, \\
Z_j^*\mathscr{G}_{j,j+1}Z_j &= \bigoplus_{i=1}^{m_j} R_{j+1} \oplus \bigoplus_{i=1}^{\nu_j}\widetilde{R}_j \oplus \bigoplus_{i=\nu_j+1}^{n_j} R_j, \\
\widetilde{Z}_{j+1}^*\hat{\mathscr{G}}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{\hat{\nu}_j}\widetilde{R}_j \oplus \bigoplus_{i=\hat{\nu}_j+1}^{m_j} R_j \oplus \bigoplus_{i=1}^{n_j} R_{j+1},
\end{aligned} \tag{7.19}$$

if j is odd, where

$$\widetilde{R}_{2q+1} = \begin{bmatrix} 0 & 0 & R_q \\ 0 & -1 & 0 \\ R_q & 0 & 0 \end{bmatrix}. \tag{7.20}$$

The matrices in (7.18) and (7.19) are block diagonal (with rectangular diagonal blocks in $Z_j^*\mathscr{A}_{j,j+1}\widetilde{Z}_{j+1}$) and it is straightforward to check that with appropriate transformation matrices it is possible to simultaneously transform, say, the kth block in all three matrices without changing the other blocks. We use this observation to finally show that the form (7.18) or (7.19) is equivalent to the corresponding form in Theorem 3.2. It only remains to show that the odd-sized blocks $\widetilde{R}_j$ and $\widetilde{R}_{j+1}$ in (7.18) and (7.19) can be replaced by $-R_j$ and $-R_{j+1}$, respectively, without changing the other blocks. We show this by an induction argument for the triple $([0\ I_j],\widetilde{R}_j,R_{j+1})$ and j odd, the proof in the other cases is similar. For $j=1$ there is nothing to show, so let $j=3$, i.e.,

$$\mathscr{A} = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \quad \mathscr{G} = \begin{bmatrix} 0&0&1 \\ 0&-1&0 \\ 1&0&0 \end{bmatrix}, \quad \hat{\mathscr{G}} = \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}.$$

Then $\mathscr{G}$ can be transformed to $-R_3$ by the congruence transformation with the transformation matrix $\operatorname{diag}(1,1,-1)$. Updating $\mathscr{A}$ accordingly (i.e., by premultiplying $\mathscr{A}$

with the transformation matrix), we obtain

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathscr{G} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathscr{G}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The negative entry in $\mathscr{A}$ can then be reset to $+1$ by postmultiplication with the matrix $\operatorname{diag}(-1,1,1,-1)$. Observe that the congruence transformation with this matrix leaves $\hat{\mathscr{G}}$ invariant. Next, consider the case $j=5$, i.e.,

$$\mathscr{A} = \left[\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & I_3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \mathscr{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \widetilde{R}_3 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathscr{G}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & R_4 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Applying the transformations of the previous step (embedded in slightly larger transformation matrices), we obtain that

$$\mathscr{A} = \left[\begin{array}{c|cc|c} 0 & -1 & 0 & 0 \\ \hline 0 & 0 & I_3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \mathscr{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -R_3 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathscr{G}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & R_4 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Premultiplying $\mathscr{A}$ with $\operatorname{diag}(-1,I_4)$ and applying the corresponding congruence transformation on $\mathscr{G}$ yields

$$\mathscr{A} = \left[\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & I_3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \mathscr{G} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -R_3 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathscr{G}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & R_4 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The remainder of the proof then follows by induction using alternately the arguments as in the cases $j=3$ and $j=5$.

Step 4) Getting the canonical form for $\mathscr{H}$ and $\hat{\mathscr{H}}$

Up to this point, we have proved the existence of the canonical form for the triple $(A,G,\hat{G})$. The corresponding forms for $\hat{\mathscr{H}}$ and $\mathscr{H}$ then immediately follow by forming the products $\hat{G}^{-1}A^*G^{-1}A$ and $G^{-1}A\hat{G}^{-1}A^*$. These forms are already very close to the actual canonical forms of Theorem 2.2, and further reducing them to that canonical form leads to the statements on the eigenvalues and attached signs of $\hat{\mathscr{H}}$ and $\mathscr{H}$.

Step 5) Uniqueness of the form

We highlight that once uniqueness of the parameters γ_j, m_j, n_j has been proved, then all other parameters are already uniquely defined by the unique canonical forms of $\mathscr{H}$ and $\hat{\mathscr{H}}$ as G-Hermitian, respectively $\hat{G}$-Hermitian matrices. (Indeed, the signs $s_j^{(i)}$ and $\hat{s}_j^{(i)}$ can be immediately reconstructed from the sign characteristics of the eigenvalue 0 of $\mathscr{H}$ and $\hat{\mathscr{H}}$.) The proof of uniqueness of γ_j, m_j, n_j follows the same lines as the proof for the corresponding case of complex symmetric G and $\hat{G}$ given in [19]. For the sake of making the paper self-contained, we reproduce this proof here.

Note that there exists a unique sequence of subspaces

$$\operatorname{Eig}_\ell(\mathscr{H},0) \subseteq \operatorname{Eig}_{\ell-1}(\mathscr{H},0) \subseteq \cdots \subseteq \operatorname{Eig}_1(\mathscr{H},0) = \ker \mathscr{H}$$

where $\operatorname{Eig}_j(\mathscr{H},0)$ consists of the zero vector and all eigenvectors of $\mathscr{H}$ associated with zero that can be extended to a Jordan chain of length at least j. Define $\kappa_\ell = \dim\big(\operatorname{Eig}_\ell(\mathscr{H},0)\cap \ker A\big)$ and

$$\kappa_j = \dim\big(\operatorname{Eig}_j(\mathscr{H},0)\cap \ker A\big) - \dim\big(\operatorname{Eig}_{j+1}(\mathscr{H},0)\cap \ker A\big), \quad j = 1,\ldots,\ell-1.$$

Then any eigenvector of $\mathscr{H}$ that is associated with a Jordan block of size $j\times j$ in the canonical form and that is also in the kernel of A contributes to κ_j. Similarly, we define $\hat{\kappa}_\ell = \dim\big(\operatorname{Eig}_\ell(\hat{\mathscr{H}},0)\cap \ker A^*\big)$ and

$$\hat{\kappa}_j = \dim\big(\operatorname{Eig}_j(\hat{\mathscr{H}},0)\cap \ker A^*\big) - \dim\big(\operatorname{Eig}_{j+1}(\hat{\mathscr{H}},0)\cap \ker A^*\big), \quad j = 1,\ldots,\ell-1.$$

Then elementary counting yields

$$\kappa_j = \gamma_j + n_{j-1} \quad \text{and} \quad \hat{\kappa}_j = \gamma_j + m_{j-1}, \quad j = 1,\ldots,\ell.$$

If τ_j, respectively $\hat{\tau}_j$ denotes the number of Jordan blocks of size $j\times j$ in the canonical form of $\mathscr{H}$ and $\hat{\mathscr{H}}$, respectively, we also have that

$$\tau_j = 2\gamma_j + m_j + n_{j-1} \quad \text{and} \quad \hat{\tau}_j = 2\gamma_j + m_{j-1} + n_j, \quad j = 1,\ldots,\ell.$$

Hence, we obtain

$$\tau_j - \kappa_j - \hat{\kappa}_j = m_j - m_{j-1}, \quad \text{and} \quad \hat{\tau}_j - \kappa_j - \hat{\kappa}_j = n_j - n_{j-1}, \quad j = 1,\ldots,\ell,$$

from which we can successively compute m_j, n_j, $j = \ell-1,\ldots,0$ using $m_\ell = n_\ell = 0$. We furthermore obtain that

$$\gamma_j = \frac{1}{2}(\tau_j - m_j - n_{j-1})$$

for $j = 1,\ldots,\ell$. Thus, the numbers γ_j, m_j, n_j are uniquely determined by the invariant numbers $\tau_j, \hat{\tau}_j, \kappa_j, \hat{\kappa}_j$, $j = 1,\ldots,\ell$.

This concludes the proof of Theorem 3.2. □

7.3. Proof of Theorem 4.2

Applying appropriate congruence transformations to G and $\hat{G}$ otherwise, we may assume that $G = \Sigma_{\pi,\nu}$ and $\hat{G} = J_n$. Let

$$A = B_1 C_1^T$$

be a full rank factorization of A, i.e., $B_1 \in \mathbb{R}^{m\times r}$, $C_1 \in \mathbb{R}^{2n\times r}$, $\operatorname{rank} B_1 = \operatorname{rank} C_1 = r$. Repeatedly applying Proposition 7.1 to B_1 and Proposition 7.2 to C_1, respectively, we can determine a staircase-like form that can be further reduced to canonical form. The proof follows the same lines as in the steps 1) and 2) of the proof of Theorem 3.2 and yields the reduced staircase-like form

$$\begin{aligned}
\widetilde{X}^T A\widetilde{Y} &= \mathscr{A}_{ns}\oplus\mathscr{A}_0\oplus\bigoplus_{j=1}^{\ell}\mathscr{A}_j\oplus\bigoplus_{j=1}^{\ell-1}\mathscr{A}_{j,j+1},\\
\widetilde{X}^T\Sigma_{\pi,\nu}\widetilde{X} &= \mathscr{G}_{ns}\oplus\mathscr{G}_0\oplus\bigoplus_{j=1}^{\ell}\mathscr{G}_j\oplus\bigoplus_{j=1}^{\ell-1}\mathscr{G}_{j,j+1},\\
\widetilde{Y}^T J_n\widetilde{Y} &= \hat{\mathscr{G}}_{ns}\oplus\hat{\mathscr{G}}_0\oplus\bigoplus_{j=1}^{\ell}\hat{\mathscr{G}}_j\oplus\bigoplus_{j=1}^{\ell-1}\hat{\mathscr{G}}_{j,j+1},
\end{aligned}$$

where

$$\mathscr{A}_{ns} = A_{2\ell+1,2\ell+1},\quad \mathscr{G}_{ns} = \Sigma_{\pi_\ell,\nu_\ell},\quad \hat{\mathscr{G}}_{ns} = J_{\hat{\pi}_\ell},$$

with $\pi_\ell+\nu_\ell = 2\hat{\pi}_\ell$ and $A_{2\ell+1,2\ell+1}\in\mathbb{C}^{2\hat{\pi}_\ell\times 2\hat{\pi}_\ell}$ being nonsingular,

$$\mathscr{A}_0 = \mathscr{O}_{m_0\times 2n_0},\quad \mathscr{G}_0 = \Sigma_{\pi_0,\nu_0},\quad \hat{\mathscr{G}}_0 = J_{n_0},$$

$$\mathscr{A}_j = \Big(R_{2j}\mathscr{J}_{2j}(0)\Big)\otimes I_{\gamma_j},\quad \mathscr{G}_j = R_{2j}\otimes I_{\gamma_j}\quad ,\hat{\mathscr{G}}_j = \begin{bmatrix} 0 & R_j\\ -R_j & 0\end{bmatrix}\otimes I_{\gamma_j},$$

and $\mathscr{A}_{j,j+1}$, $\hat{\mathscr{G}}_{j,j+1}$, and $\hat{\mathscr{G}}_{j,j+1}$ are $(2j+1)\times(2j+1)$ block matrices, where, if j is odd, the block rows have alternating sizes $n_j, 2m_j$ and the forms

$$\mathscr{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0\\ & & & & 0 & I_{n_j}\\ & & & \iddots & I_{2m_j} & \\ & & \iddots & \iddots & & \\ & 0 & I_{n_j} & & & \\ 0 & I_{2m_j} & & & & 0\end{bmatrix}, \mathscr{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2m_j}\\ & & & I_{n_j} & \\ & & \Sigma_{\pi_j,\nu_j} & & \\ & I_{n_j} & & & \\ I_{2m_j} & & & & 0\end{bmatrix}, \tag{7.21}$$

$$\hat{\mathscr{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{n_j}\\ & & & I_{2m_j} & \\ & & J_{m_j} & & \\ & -I_{2m_j} & & & \\ -I_{n_j} & & & & 0\end{bmatrix}, \tag{7.22}$$

or, if j is even, then the block rows have alternating sizes $2n_j, m_j$ and the forms

$$\mathscr{A}_{j,j+1} = \begin{bmatrix} 0 & & & & 0 \\ & & & 0 & I_{2n_j} \\ & & \iddots & I_{m_j} & \\ & \iddots & \iddots & & \\ & 0 & I_{2n_j} & & \\ 0 & I_{m_j} & & & 0 \end{bmatrix}, \mathscr{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{m_j} \\ & & & I_{2n_j} & \\ & & \Sigma_{\pi_j,\nu_j} & & \\ & \iddots & & & \\ & I_{2n_j} & & & \\ I_{m_j} & & & & 0 \end{bmatrix}, \quad (7.23)$$

$$\hat{\mathscr{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2n_j} \\ & & & I_{m_j} & \\ & & J_{n_j} & & \\ & -I_{m_j} & & & \\ -I_{2n_j} & & & & 0 \end{bmatrix}, \quad (7.24)$$

The blocks $\mathscr{A}_0$, $\mathscr{G}_0$, and $\hat{\mathscr{G}}_0$ are already in the form as indicated in Theorem 3.2, for the blocks $\mathscr{A}_{ns}$, $\mathscr{G}_{ns}$, $\hat{\mathscr{G}}_{ns}$, we can apply Theorem 4.1, and for the blocks $\mathscr{A}_j$, $\hat{\mathscr{G}}_j$, and $\hat{\mathscr{G}}_j$ we can apply an analogous permutation as it has been done for the corresponding blocks in the proof of Theorem 3.2. Moreover, if j is odd, then let Z_j be the permutation such that premultiplication with Z_j^T reorders the rows of $\mathscr{A}_{j,j+1}$ in the order

$$\begin{array}{ccccc}
2(j+1)m_j+jn_j, & 2jm_j+(j-1)n_j, & \dots, & 4m_j+n_j, & 2m_j, \\
2jm_j+m_j+jn_j, & 2(j-1)m_j+m_j+(j-1)n_j, & \dots, & 2m_j+m_j+n_j, & m_j, \\
2(j+1)m_j-1+jn_j, & 2jm_j-1+(j-1)n_j, & \dots, & 4m_j-1+n_j, & 2m_j-1, \\
2jm_j+m_j-1+jn_j, & 2(j-1)m_j+m_j-1+(j-1)n_j, & \dots, & 2m_j+m_j-1+n_j, & m_j-1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2jm_j+m_j+1+jn_j, & 2(j-1)m_j+m_j+1+(j-1)n_j, & \dots, & 2m_j+m_j+1+n_j, & m_j+1, \\
2jm_j+1+jn_j, & 2(j-1)m_j+1+(j-1)n_j, & \dots, & 2m_j+1+n_j, & 1, \\
2jm_j+jn_j, & 2(j-1)m_j+(j-1)n_j, & \dots, & 2m_j+n_j, & \\
2jm_j+jn_j-1, & 2(j-1)m_j+(j-1)n_j-1, & \dots, & 2m_j+n_j-1, & \\
\vdots & \vdots & \ddots & \vdots & \\
2jm_j+(j-1)n_j+1, & 2(j-1)m_j+(j-2)n_j+1, & \dots, & 2m_j+1, &
\end{array}$$

and let $\widetilde{Z}_{j+1}$ be the permutation such that postmultiplication with $\widetilde{Z}_{j+1}$ reorders the columns of $\mathscr{A}_{j,j+1}$ in the order

$$\begin{array}{cccc}
m_j+n_j, & 2m_j+m_j+2n_j, & \dots, & 2(j-1)m_j+m_j+jn_j, \\
2m_j+n_j, & 4m_j+n_j, & \dots, & 2jm_j+jn_j, \\
m_j-1+n_j, & 2m_j+m_j-1+2n_j, & \dots, & 2(j-1)m_j+m_j-1+jn_j, \\
2m_j-1+n_j, & 4m_j-1+n_j, & \dots, & 2jm_j-1+jn_j, \\
\vdots & \vdots & \ddots & \vdots
\end{array}$$

$$\begin{array}{ccccc}
1+n_j, & 2m_j+1+n_j, & \ldots, & 2(j-1)m_j+1+jn_j, & \\
m_j+1+n_j, & 2m_j+m_j+1+n_j, & \ldots, & 2(j-1)m_j+m_j+1+jn_j, & \\
n_j, & 2m_j+2n_j, & \ldots, & 2(j-1)m_j+jn_j, & 2jm_j+(j+1)n_j, \\
n_j-1, & 2m_j+2n_j-1, & \ldots, & 2(j-1)m_j+jn_j-1, & 2jm_j+(j+1)n_j-1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1, & 2m_j+n_j+1, & \ldots, & 2(j-1)m_j+(j-1)n_j+1, & 2jm_j+jn_j+1.
\end{array}$$

Then it is easily verified that

$$\begin{aligned}
Z_j^T \mathscr{A}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & I_j \\ 0 & 0 \\ I_j & 0 \\ 0 & 0 \end{bmatrix}_{2(j+1)\times 2j} \oplus \bigoplus_{i=1}^{n_j} \begin{bmatrix} 0 & I_j \end{bmatrix}_{j\times(j+1)}, \\
Z_j^T \mathscr{G}_{j,j+1} Z_j &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & R_{j+1} \\ R_{j+1} & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\nu_j} \widetilde{R}_j \oplus \bigoplus_{i=\nu_j+1}^{n_j} R_j, \\
\widetilde{Z}_{j+1}^T \hat{\mathscr{G}}_{j,j+1}\widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & R_j \\ -R_j & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_j} \begin{bmatrix} 0 & R_{\frac{j+1}{2}} \\ -R_{\frac{j+1}{2}} & 0 \end{bmatrix},
\end{aligned} \tag{7.25}$$

where $\widetilde{R}_j$ is as in (7.20). Then analogously as in the proof of Theorem 3.2, we can transform $\widetilde{R}_j$ to $-R_j$ without changing any of the other blocks. Thus, we finally obtain blocks as in 4) and 5) in Theorem 4.2. Similarly, an analogous permutation extracts blocks as in 3) and 6) in Theorem 4.2 for the case that j is even, i.e., if we consider the blocks (7.23)–(7.24).

Concerning uniqueness, as in the proof of Theorem 3.2 it remains to show uniqueness of the numbers ℓ_j, $2m_j$, and n_j. This is done exactly in the same way as in the proof of Theorem 3.2. Note that the paired blocks in 4) and 6) in Theorem 4.2 cannot be decomposed into two smaller blocks of equal size, because of the fact that nonsingular skew-symmetric matrices must have even size. □

7.4. Proof of Theorem 5.2

Applying appropriate congruence transformations to G and $\hat{G}$ otherwise, we may assume that $G = J_m$ and $\hat{G} = J_n$. Again, we then compute a staircase-like form for A by considering the full rank factorization

$$A = B_1 C_1^T$$

of A, i.e., $B_1 \in \mathbb{R}^{2m\times r}$, $C_1 \in \mathbb{R}^{2n\times r}$, $\operatorname{rank} B_1 = \operatorname{rank} C_1 = r$, and repeatedly applying Proposition 7.2 to B_1 and C_1. Then continuing as in step 2) of the proof of Theorem 3.2

yields the reduced staircase-like form

$$
\begin{aligned}
\widetilde{X}^T A \widetilde{Y} &= \mathscr{A}_{ns} \oplus \mathscr{A}_0 \oplus \bigoplus_{j=1}^{\ell} \mathscr{A}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathscr{A}_{j,j+1}, \\
\widetilde{X}^T J_m \widetilde{X} &= \mathscr{G}_{ns} \oplus \mathscr{G}_0 \oplus \bigoplus_{j=1}^{\ell} \mathscr{G}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathscr{G}_{j,j+1}, \\
\widetilde{Y}^T J_n \widetilde{Y} &= \hat{\mathscr{G}}_{ns} \oplus \hat{\mathscr{G}}_0 \oplus \bigoplus_{j=1}^{\ell} \hat{\mathscr{G}}_j \oplus \bigoplus_{j=1}^{\ell-1} \hat{\mathscr{G}}_{j,j+1},
\end{aligned}
$$

where

$$\mathscr{A}_{ns} = A_{2\ell+1,2\ell+1}, \quad \mathscr{G}_{ns} = J_{\pi_\ell}, \quad \hat{\mathscr{G}}_{ns} = J_{\hat{\pi}_\ell} = J_{\pi_\ell},$$

with $A_{2\ell+1,2\ell+1} \in \mathbb{R}^{2\pi_\ell \times 2\pi_\ell}$ being nonsingular,

$$\mathscr{A}_0 = 0_{2m_0 \times 2n_0}, \quad \mathscr{G}_0 = J_{m_0}, \quad \hat{\mathscr{G}}_0 = J_{n_0},$$

$$\mathscr{A}_j = \Big(R_{2j} \mathscr{J}_{2j}(0)\Big) \otimes I_{\gamma_j}, \quad \mathscr{G}_j = \hat{\mathscr{G}}_j = \begin{bmatrix} 0 & R_j \\ -R_j & 0 \end{bmatrix} \otimes I_{\gamma_j},$$

and $\mathscr{A}_{j,j+1}$, $\hat{\mathscr{G}}_{j,j+1}$, and $\hat{\mathscr{G}}_{j,j+1}$ are $(2j+1) \times (2j+1)$ block matrices, where the block rows have alternating sizes $2n_j, 2m_j$ and the forms

$$\mathscr{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{2n_j} \\ & & & \cdot^{\cdot^{\cdot}} & I_{2m_j} & \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ & 0 & I_{2n_j} & & & \\ 0 & I_{2m_j} & & & & 0 \end{bmatrix}, \mathscr{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2m_j} \\ & & & I_{2n_j} & \\ & & J_{\pi_j} \cdot^{\cdot^{\cdot}} & & \\ & -I_{2n_j} \cdot^{\cdot^{\cdot}} & & & \\ -I_{2m_j} & & & & 0 \end{bmatrix}, \quad (7.26)$$

$$\hat{\mathscr{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2n_j} \\ & & & I_{2m_j} & \\ & & J_{\hat{\pi}_j} \cdot^{\cdot^{\cdot}} & & \\ & -I_{2m_j} \cdot^{\cdot^{\cdot}} & & & \\ -I_{2n_j} & & & & 0 \end{bmatrix}, \quad (7.27)$$

The remainder of the proof then proceed as the proof of Theorem 4.2 by adapting the permutation used on the blocks of the forms (7.26)–(7.27) similarly as in the proof of Theorem 4.2 in order to allow to group together paired blocks.

Concerning uniqueness, as in the proof of Theorem 3.2 it remains to show uniqueness of the numbers ℓ_j, $2m_j$, and $2n_j$. This is done exactly in the same way as in the proof of Theorem 3.2. □

REFERENCES

[1] Y. BOLSCHAKOV AND B. REICHSTEIN, *Unitary equivalence in an indefinite scalar product: an analogue of singular-value decomposition*, Linear Algebra Appl., **222** (1995), 155–226.

[2] Y. BOLSCHAKOV, C.V.M. VAN DER MEE, A.C.M. RAN, B. REICHSTEIN, AND L. RODMAN, *Polar decompositions in finite-dimensional indefinite scalar product spaces: general theory*, Linear Algebra Appl., **261** (1997), 91–141.

[3] D.Z. DJOKOVIC, J. PATERA, P. WINTERNITZ, AND H. ZASSENHAUS, *Normal forms of elements of classical real and complex Lie and Jordan algebras*, J. Math. Phys., **24** (1983), 1363–1374.

[4] N.M. DOBROVOL'SKAYA AND V.A. PONOMAREV, *A pair of counter-operators* (in Russian), Uspehi Mat. Nauk, **20** (1965), 80–86.

[5] H. FASSBENDER, D. S. MACKEY, N. MACKEY, AND H. XU, *Hamiltonian square roots of skew–Hamiltonian matrices*, Linear Algebra Appl., **287** (1999), 125–159.

[6] H. FLANDERS, *Elementary divisors of AB and BA*, Proc. Amer. Math. Soc., **2** (1951), 871–874.

[7] I. GOHBERG, P. LANCASTER, AND L. RODMAN, *Matrices and Indefinite Scalar Products*, Birkhäuser, Basel, 1983.

[8] I. GOHBERG, P. LANCASTER, AND L. RODMAN, *Indefinite Linear Algebra and Applications*, Birkhäuser, Basel, 2005.

[9] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, Johns Hopkins University Press, 3rd edition, Baltimore, 1996.

[10] S. HASSI, *Matriisin yleisen singulaariarvohajoitelman ominaisuuksia ja tilastollisia sovelluksia*, Master's thesis, University of Helsinki, 1984.

[11] S. HASSI, *A singular value decomposition of matrices in a space with an indefinite scalar product*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, **79**, 1990.

[12] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[13] R. A. HORN AND D. MERINO, *Contragredient equivalence: a canonical form and some applications*, Linear Algebra Appl., **214** (1995), 43–92.

[14] T. KAPITULA, P.G. KEVREKIDIS, AND B. SANDSTEDE, *Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems*, Physica D: Nonlinear Phenomena, **195** (2004), 263–282.

[15] J. KES AND A.C.M. RAN, *On the relation between $XX^{[*]}$ and $X^{[*]}X$ in an indefinite inner product space*, Oper. Matrices, **1** (2007), 181–197.

[16] P. LANCASTER AND L. RODMAN, *The Algebraic Riccati Equation*, Oxford University Press, Oxford, 1995.

[17] P. LANCASTER AND L. RODMAN, *Canonical forms for Hermitian matrix pairs under strict equivalence and congruence*, SIAM Rev., **47**, (2005), 407–443.

[18] P. LANCASTER AND L. RODMAN, *Canonical forms for symmetric/skew symmetric real pairs under strict equivalence and congruence*, Linear Algebra Appl., **406** (2005), 1–76.

[19] C. MEHL, V. MEHRMANN, AND H. XU, *Singular-value-like decompositions for complex matrix triples*, J. Comput. Appl. Math., to appear, 2009.
`http://dx.doi.org/10.1016/j.cam.2008.02.017`.

[20] C. MEHL, A.C.M. RAN, AND L. RODMAN, *Polar decompositions of normal operators in indefinite inner product spaces*, Oper. Theory Adv. Appl., **162** (2006), 277–292.

[21] V. MEHRMANN AND H. XU, *Structured Jordan canonical forms for structured matrices that are Hermitian, skew Hermitian or unitary with respect to indefinite inner products*, Electron J. Linear Algebra, **5** (1999), 67–103.

[22] J. PATERA AND C. ROUSSEAU, *Versal deformations of elements of classical Jordan algebras*, J. Math. Phys., **24** (1983), 1375–1379.

[23] A. C. M. RAN AND M. WOJTYLAK, *Analysis of spectral points of the operators $T^{[*]}T$ and $TT^{[*]}$ in a Krein space*, submitted, 2008.

[24] V.V. SERGEICHUK, *Computation of canonical matrices for chains and cycles of linear mappings*, Linear Algebra Appl., **376** (2004), 235–263.

[25] R. C. THOMPSON, *The characteristic polynomial of a principal submatrix of a Hermitian pencil*, Linear Algebra Appl., **14** (1976), 135–177.

[26] R. C. THOMPSON, *Pencils of complex and real symmetric and skew matrices*, Linear Algebra Appl., **147** (1991), 323–371.

[27] H. XU, *An SVD-like matrix decomposition and its applications*, Linear Algebra Appl., **368** (2003), 1–24.

[28] H. XU, *A numerical method for computing an SVD-like decomposition*, SIAM J. Matrix Anal. Appl., **26** (2005), 1058–1082.

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