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ON BOUNDS FOR DISCRETE SEMIGROUPS

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Abstract. The main result of this note is extension on the infinite dimension of the following known result for finite matrices: while the spectral radius $\rho(T)$ gives only asymptotic decay estimates, the solution X of the *discrete Lyapunov equation* $X - T^*XT = BB^*$ yields rigorous bounds. We also present a new upper bound for the norm of the solution X in the matrix case which depends on the structure of the right hand side. The new bound shows that the structure of B can greatly influence $\|X\|$.

1. Introduction

In this note we consider the exponential decay of the powers T^k of a Hilbert space operator T. There are two main measures of the decay of this sequence: (i) the spectral radius $\rho(T)$ and the solution X of the *discrete Lyapunov equation*[1]

$$X - T^*XT = BB^*.$$

While the spectral radius gives only asymptotic decay estimates, the Lyapunov equation yields rigorous bounds as was shown e.g. in Godunov [3] for finite matrices. Our aim is to further elaborate on the results presented by Godunov, to extend them to the infinite dimensional case and give a new upper bound for the norm of the solution X in the matrix case which depends on the structure of the right hand side of the *discrete Lyapunov equation*. The last result was inspired by the ideas used in [6].

Here we will observe some additional interesting structure yet without rigorous explanation. We hope that our observations will incite further theoretical investigation in this field.

2. The Main Result

In the following $\mathscr{H}$ will denote a real or complex Hilbert space.[2] The techniques of our proofs are close to those used in [3], with slight adaptations they will be seen to hold in the infinite dimensional case as well.[3]. We give full proofs for the sake of the completeness.

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[1]Sometimes also called *the Stein equation*.

[2]Whenever not specified otherwise, we follow the notation and the terminology of [5].

[3]For some general facts on the matrix Stein equation see also e.g. [2] and [4, Sec. IV.2] for the unique solvability of the operator Stein equation

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THEOREM 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{H}_1, \mathscr{H})$ hold. The relation*

$$\sum_{n=0}^{\infty} \|B^* T^k \psi\|^2 < \infty \text{ for all } \psi. \tag{1}$$

is equivalent to the existence of the strong limit

$$\sum_{n=0}^{\infty} T^{*k} B B^* T^k \tag{2}$$

which then satisfies the equation

$$X - T^* X T = B B^*. \tag{3}$$

Conversely, if (3) holds with a non-negative[4] *selfadjoint $X \in \mathscr{B}(\mathscr{H})$ then (2) converges strongly to a solution of (3). This is the smallest of all non-negative selfadjoint solutions of (3).*

Proof. Any strongly convergent sum (2) obviously solves (3). Also obviously the strong convergence of (2) is equivalent to (1). Conversely, (3) implies

$$X = BB^* + T^* X T = \cdots = \sum_{k=0}^{n-1} T^{*k} B B^* T^k + T^{*n} X T^n \tag{4}$$

for any $n = 0, 1, 2, \ldots$. Since all terms on the right hand side of (4) are non-negative, the series in (2) converges strongly to some X_0 which then solves (3). By the same reason $T^{*n} X T^n$ converges strongly to some non-negative selfadjoint Z so X_0 is minimal as stated. $\square$

For the sake of convenience we denote the right-hand side of (3) in factored form, although X only depends on T and BB^*.

THEOREM 2.2. *Let (3) holds and let, in addition,*

$$\gamma = \gamma(T, B) = \sup_{B^* \psi \neq 0} \frac{(X\psi, \psi)}{\|B^* \psi\|^2} < \infty. \tag{5}$$

Then $\gamma \geqslant 1$ and (3) has the minimal solution X from (2) which satisfies

$$\sum_{k=n}^{\infty} T^{*k} B B^* T^k \leqslant \left(1 - \frac{1}{\gamma}\right)^n X. \tag{6}$$

In particular, the series (2) converges in norm. If $\gamma = 1$ then $B^* T = 0$.

[4]The order relation is understood in the sense of quadratic forms.

Proof. The relation $\gamma \geqslant 1$ is obvious. We have

$$X - \sum_{k=0}^{n-1} T^{*k} BB^* T^k = T^{*n} X T^n = \sum_{k=n}^{\infty} T^{*k} BB^* T^k \tag{7}$$

and (cf.[3])

$$T^{*k-1} X T^{k-1} - T^{*k} X T^k = T^{*k-1} BB^* T^{k-1} \geqslant \frac{T^{*k-1} X T^{k-1}}{\gamma}. \tag{8}$$

Thus,

$$T^{*k} X T^k \leqslant \left(1 - \frac{1}{\gamma}\right) T^{*k-1} X T^{k-1} \leqslant \cdots \leqslant \left(1 - \frac{1}{\gamma}\right)^k X. \tag{9}$$

This, together with (7) gives (6) the norm convergence of which is now obvious. The last assertion is obvious, too. □

Clearly, for the minimal solution X above we have

$$\mathscr{N}(B^*) = \mathscr{N}(BB^*) \supseteq \mathscr{N}(X)$$

and X has a non-trivial null space if and only if T maps some non-vanishing vector from $\mathscr{N}(B^*)$ into $\mathscr{N}(B^*)$. Moreover,

$$\mathscr{N}(X) = \cap_{k=0}^{\infty} \mathscr{N}(B^* T^k) \tag{10}$$

As was mentioned in [3] for finite matrices the quantity γ is the greatest root of the equation $\det(X - \lambda BB^*) = 0$.

COROLLARY 2.1. *If BB^* is positive definite then $\mathscr{H}_1$ can be chosen so that γ from (5) is finite and equals $\|B^{-1} X B^{-*}\|$ (in this case $\mathscr{H}_1$ can be chosen so that both B and B^* are bijective). Moreover,*

$$\rho(T) \leqslant \sqrt{1 - \frac{1}{\|B^{-1} X B^{-*}\|}} < 1 \tag{11}$$

and X is positive definite. Conversely, if $\rho(T) < 1$ then (2) converges in norm for any B and X is the unique solution of (3).

Proof. (6) implies

$$\|(B^* T B^{-*})^n\|^2 \leqslant \left(1 - \frac{1}{\gamma}\right)^n \|B^{-1} X B^{-*}\| \tag{12}$$

and this implies (11). The uniqueness follows from

$$Z = T^* Z T \Rightarrow Z = T^{*n} Z T^n$$

for arbitrary n whereas the positive definiteness of X follows from that of BB^*. The last assertion follows if we rewrite (9) as

$$\|T^k\psi\|_X^2 \leqslant \left(1-\frac{1}{\gamma}\right)^k \|\psi\|_X$$

where $\|\psi\|_X = \|X^{1/2}\psi\|$ is a norm equivalent to the original one. $\square$

The second part of the above Corollary can also be derived by renormalising the Hilbert space such that in the new norm $\|T\| < 1$ holds.

PROPOSITION 2.1. *Let $\rho(T) < 1$, let $B^*B \in \mathscr{B}(\mathscr{H}_1)$ be positive definite and let*

$$B^*T = \tau B^* \tag{13}$$

for some $\tau \in \mathscr{B}(\mathscr{H}_1)$. Then γ from (5) is finite. Conversely, if the dimension of $\mathscr{H}_1$ is finite then $\gamma < \infty$ implies (13).

Proof. (13) implies

$$B^*T^n = \tau^n B^* \tag{14}$$

and by (9)

$$\|\tau^n\|^2 \leqslant \|B\|^2\|(B^*B)^{-1}\| \left(1-\frac{1}{\|X_0\|}\right)^k \|X_0\|$$

where

$$T^*X_0T - X_0 = -I.$$

Thus, $\rho(\tau) < 1$ and

$$\begin{aligned} X &= \sum_{k=0}^{\infty} T^{*k}BB^*T^k = B\sum_{k=0}^{\infty}\tau^{*k}\tau^k B^* \\ &= B(I-\tau^*\tau)^{-1}B^* \end{aligned}$$

and

$$\gamma \leqslant \|(1-\tau^*\tau)^{-1}\|. \tag{15}$$

The proof of the converse is straightforward. $\square$

With finite matrices it is always possible to choose B with full column rank, or, equivalently, B^*B positive definite.

COROLLARY 2.2. *Let $\rho(T) < 1$ and let*

$$X - T^*XT = BB^*, \quad Y - TYT^* = CC^*.$$

Then

$$Tr(C^*XC) = Tr(B^*YB).$$

Furthermore, for $B = I$,

$$(X\psi, \psi) = TrY_\psi$$

and

$$\|X\| = \sup_{\psi \neq 0} \frac{TrY_\psi}{\|\psi\|},$$

where

$$Y_\psi - TY_\psi T^* = B_\psi B_\psi^* \quad B_\psi \phi = (\psi, \phi)\psi.$$

For the proof we just mention

$$\begin{aligned} Tr(C^*XC) &= Tr(CC^*X) = Tr(YX) - Tr(TYT^*X) \\ &= Tr(XY) - Tr(T^*XTY) = Tr(BB^*Y) = Tr(B^*YB) \end{aligned}$$

(other statements are straightforward).

3. Solution bounds

We consider the discrete algebraic Lyapunov equation (DALE):

$$X - T^*XT = BB^* \tag{16}$$

where $T \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ and $\rho(T) < 1$, where $\rho(T)$ denotes the spectral radius of the matrix T.

Due to our assumption it follows that a function

$$f(X) = T^*XT + BB^*$$

has the solution X of (16) as the fixed point. Since X is positive semi-definite we can write

$$X = L_X L_X^*.$$

Thus, if we set $L_0 = B$ then the following simple loop

$$\begin{aligned} &for \quad i = 1 : k \\ &\qquad L(:,i) = [T' * L(:,i-1), B]; \\ &end \end{aligned} \tag{17}$$

will converge (since T is a contraction by the assumption) to L_X, that is

$$L_X = \lim_{k\to\infty} L_k.$$

This further means that $X \approx X_k = L_k L_k^*$.

In the following we will present two bounds, one for the norm of the solution $\|X\|$ and the other which will bound the error in our simple approximation, that is the bound for $\|X - X_k\|$.

We will assume that matrix T from (16) has the following simple Jordan structure

$$T^* = SJS^{-1}; \quad S \in \mathbb{C}^{m\times m}, \quad J = J_1 \oplus \ldots \oplus J_{k_0}, \tag{18}$$

where $J_i \oplus J_k$ stands for a direct sum of J_i and J_k and each J_i, $i = 1,\ldots,k_0$ corresponds to subspaces associated with the eigenvalue λ_i, with the following structure

$$\begin{aligned} J_i &= [\lambda_i] && \text{for} \quad i = 1,\ldots,n_0, \\ J_i &= \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \quad \text{or} \quad J_i = \lambda_i I_2 + N, && \text{for} \quad i = n_0+1,\ldots,k_0, \end{aligned}$$

where I_2 is 2×2 identity matrix and

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is nilpotent of order 2. Let the matrix

$$\widehat{B} = S^{-1}B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1s} \\ b_{21} & b_{22} & \ldots & b_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k_0 1} & b_{k_0 2} & \ldots & b_{k_0 s} \end{bmatrix} = \begin{bmatrix} \widehat{b}_1 \\ \widehat{b}_2 \\ \vdots \\ \widehat{b}_{k_0} \end{bmatrix} \tag{19}$$

be partitioned according to the Jordan structure of the matrix T, that is for $i = 1,\ldots,n_0$, $\widehat{b}_i$ denotes the i-th $1\times s$, and for $i = n_0+1,\ldots,k_0$, the i-th $2\times s$, submatrix of the matrix $\widehat{B}$, respectively.

The following theorem contains bound for the norm of the solution of (16).

THEOREM 3.1. *Let X be the solution of (16). Then the following bound holds:*

$$\|X\| \leqslant \|S\|^2 \left(\sum_{p=1}^{n_0} \frac{\|\widehat{b}_p\|}{1-|\lambda_p|} + \sum_{p=1}^{k_0-n_0} \frac{\|\widehat{b}_{n_0+(2p-1)}\| + \|\widehat{b}_{n_0+2p}\|}{1-|\lambda_{n_0+p}|} + \sum_{p=1}^{k_0-n_0} \frac{\|\widehat{b}_{n_0+2p}\|}{(1-|\lambda_{n_0+p}|)^2} \right)^2. \tag{20}$$

Proof. The solution X of (16) can be written as:

$$X = \sum_{j=0}^{\infty} (T^*)^j BB^* T^j.$$

Using (18) and the above equality we can write

$$\|X\| \leqslant \sum_{j=0}^{\infty} \|(T^*)^j B\|^2 \leqslant \|S\|^2 \sum_{j=0}^{\infty} \|J^j \widehat{B}\|^2,$$

where $\widehat{B} = S^{-1}B$ and

$$J = \lambda_1 \oplus \ldots \oplus \lambda_{n_0} + \left(\lambda_{n_0+1} I_2 + N\right) \oplus \ldots \oplus \left(\lambda_{k_0} I_2 + N\right).$$

Note that $2k_0 - n_0 = m$.

We will proceed with bounding the term $\|J^j\widehat{B}\|$. For that purpose we will write $\widehat{B}$ in the form which corresponds to the structure of J. Thus let

$$\widehat{B} = \left[\widehat{b}_1, \ldots, \widehat{b}_{n_0}, Q_b(1)^T, \ldots, Q_b(k_0-n_0)^T\right]^T$$

where $Q_b(p)$ is given by where

$$Q_b(p) = \begin{bmatrix} \widehat{b}_{n_0+(2p-1)} \\ \widehat{b}_{n_0+2p} \end{bmatrix}.$$

Now it is easy to show that

$$\begin{aligned}\|X\| &\leqslant \|S\|^2 \sum_{j=0}^{\infty}\left(\sum_{p=1}^{n_0}\|(\lambda_p)^j\widehat{b}_p\| + \sum_{p=1}^{k_0-n_0}\|\left(\lambda_{n_0+p}I_2+N\right)^j Q_b(p)\|\right)^2 \\ &\leqslant \|S\|^2 \sum_{j=0}^{\infty}\left(\sum_{p=1}^{n_0}|\lambda_p|^j\|\widehat{b}_p\| + \sum_{p=1}^{k_0-n_0}|\lambda_{n_0+p}|^j\|Q_b(p)\| + j|\lambda_{n_0+p}|^{j-1}\|NQ_b(p)\|\right)^2\end{aligned}$$

Now the bound (20) follows simply by summation of infinite series from the above inequality. $\square$

Let L_k be obtained after k steps of (17). The approximate solution of (16) then can be written as

$$\widetilde{X} = L_k L_k^* = \sum_{j=0}^{k}(T^*)^j BB^* T^j. \tag{21}$$

Now from the Theorem 3.1 it is easy to derive the upper bound for $\|\widetilde{X}-X\|$.

COROLLARY 3.1. *Let $\widetilde{X}$ be k-th approximation of the solution X of DALE (16) defined by (21). Then the following bound holds:*

$$\|X-\widetilde{X}\| \leqslant \|S\|^2\left(\sum_{p=1}^{n_0}\frac{|\lambda_p|^{k+1}\,\|\widehat{b}_p\|}{1-|\lambda_p|} + \sum_{p=1}^{k_0-n_0}|\lambda_p|^{k+1}\,\frac{\|\widehat{b}_{n_0+(2p-1)}\|+\|\widehat{b}_{n_0+2p}\|}{1-|\lambda_{n_0+p}|}\right. \\ \left. + \sum_{p=1}^{k_0-n_0}\frac{|\lambda_p|^k\,\|\widehat{b}_{n_0+2p}\|}{(1-|\lambda_{n_0+p}|)^2}\right)^2. \tag{22}$$

Proof. Using the same arguments as in the proof of the Theorem 3.1, bound (22) follows from

$$\|X-\widetilde{X}\| = \|\sum_{j=k+1}^{\infty}(T^*)^j BB^* T^j\|,$$

and the facts that

$$\sum_{j=k+1}^{\infty}|\lambda_p|^j q_p = \frac{|\lambda_p|^{k+1} q_p}{1-|\lambda_p|}, \qquad \sum_{j=k+1}^{\infty} j|\lambda_p|^{j-1} q_p = \frac{|\lambda_p|^k q_p}{(1-|\lambda_p|)^2} \qquad \square$$

The next section illustrates the influence of the structure of the right hand side of the discrete Lyapunov equation (16) on its solution. The all calculations are performed on PC computer using standard Matlab package `dlyap.m` for solving discrete Lyapunov equations.

3.1. Numerical illustration

As an illustration of the bound (20) we will compare it with the standard bound for discrete Lyapunov equation which can be obtained using the following results.

As it has been described in [1, Section 8.3.6], the discrete Lyapunov equation (16) is equivalent to the linear system:

$$Rx = b, \qquad \text{where} \qquad R = I_{n^2} - A^* \otimes A^*, \tag{23}$$

and b is n^2 vector which is obtained by stacking the columns of the matrix BB^* on top of one another. Now, form (23) follows the standard bound:

$$\|X\| \leqslant \|x\| \leqslant \|R^{-1}\|\|b\|. \tag{24}$$

Further, consider the (16), where T is the 6×6 matrix with the following Jordan structure $T^* = SJS^{-1}$

$$S = \begin{bmatrix} -0.59753 & -0.46706 & 0.55739 & 0.11502 & -0.31063 & -0.055246 \\ -0.042629 & -0.15136 & -0.51721 & 0.77593 & -0.31361 & -0.0818 \\ 0.46846 & 0.040048 & 0.55230 & 0.47043 & 0.27916 & -0.41902 \\ -0.42022 & 0.85471 & 0.17322 & 0.22797 & -0.090132 & 0.038395 \\ 0.24984 & 0.15879 & 1.8387\cdot 10^{-3} & -0.30653 & -0.73166 & -0.53218 \\ 0.42742 & 0.039832 & 0.29456 & 0.13054 & -0.42868 & 0.72731 \end{bmatrix}$$

and $J = J_1 \oplus J_2 \oplus J_3$, where

$$J_1 = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 0.99 & 1 \\ 0 & 0.99 \end{bmatrix} \qquad J_3 = \begin{bmatrix} 0.02 & 1 \\ 0 & 0.02 \end{bmatrix}.$$

For the right-hand side in (16) we choose the matrix B, such that $\widehat{B}$ from (19) has the form:

$$\widehat{B} = \begin{bmatrix} 0.9998 & 0.0179 \\ -0.0198 & 0.9998 \\ 0.0002 & 0.0004 \\ -0.0028 & -0.0008 \\ -0.0042 & -0.0015 \\ 0.0011 & 0.0031 \end{bmatrix}$$

Since the spectrum of the matrix T is $\sigma(T) = \{0.1, 0.1, 0.99, 0.99, 0.02, 0.02\}$ and the row norms of the matrix $\widehat{B}$ are

$$\|\widehat{B}(:,1)\| = 1.0000, \|\widehat{B}(:,2)\| = 9.9999e-001, \|\widehat{B}(:,3)\| = 4.1873e-004,$$
$$\|\widehat{B}(:,4)\| = 2.8791e-003, \|\widehat{B}(:,5)\| = 4.4761e-003, \|\widehat{B}(:,6)\| = 3.2520e-003,$$

the bound (20) gives:

$$\|X\| \leqslant 11.879\,.$$

It is important to emphasize that by $\|X\| = 2.0793$ the upper bound looks pessimistic, but this bound is much sharper then any other bound, such as (24), which will ignore the influence of the right-hand side of the (16). In fact from (24) it follows that

$$\|X\| \leqslant 251417.19\,.$$

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REFERENCES

[1] B.N. DATTA, *Numerical Methods for Linear Control Systems*, Elsevier Academic Press, London, UK, 2004.

[2] H. DYM, *Linear Algebra in Action*, Graduate Studies in Mathematics 78, Amer. Math. Soc., Providence RI, 2006.

[3] S. K. GODUNOV, *Modern Aspects of Linear Algebra*, Translations of Mathematical Monographs, Amer. Math. Soc. Providence, RI, Vol. **175**, 1998.

[4] I. GOHBERG, S. GOHBERG AND M. A. KAASHOEK, *Classes of Linear Operators*, Vol.**I**, Birkhäuser OT 49, Basel and Boston, 1990.

[5] T. KATO, *Perturbation Theory for Linear Operators*, 2nd ed., Berlin Heidelberg New York 1980, 1995.

[6] N. TRUHAR, K. VESELIĆ, *Bounds on the trace of solution to the Lyapunov equation with a general stable matrix*, Systems and Control Letters, **56** (2007), 493–503.

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