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ESSENTIALLY HERMITIAN MATRICES AND INCLUSION RELATIONS OF C–NUMERICAL RANGES

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Abstract. Let $\mathbf{M}$ denote the set of all $n\times n$ complex matrices and $\mathbf{M}_n^0$ denote the set of $n\times n$ matrices with trace 0. For any $C\in\mathbf{M}_n^0$, there exists a maximal $\nu(C)\geqslant 0$ such that

$$\nu(C)W_D(A)\subseteq\|D\|_F W_C(A)$$

whenever $D\in\mathbf{M}_n^0$ and $A\in\mathbf{M}_n$. Here $W_C(A)$ denotes the C-numerical range of A and $\|D\|_F$ denotes the Frobenius norm of D. Moreover $\nu(C)=0$ if and only if C is essentially hermitian.

To prove the above result, we have obtained a new characterisation of essentially hermitian matrices.

1. Introduction

Let $\mathbf{M}_n$ denote the set of all $n\times n$ complex matrices over $\mathbb{C}$ and $\mathbf{M}_n^0$ denote the set of $n\times n$ matrices with trace 0. Let $C\in\mathbf{M}_n$, the C-numerical range of A and the C-numerical radius of A for $A\in\mathbf{M}_n$ are defined respectively by

$$W_C(A)=\{\operatorname{tr}(CU^*AU):U\text{ is unitary}\}$$

and

$$r_C(A)=\max\{|a|\ :\ a\in W_C(A)\}.$$

When $C=E_{11}$, the matrix with a 1 at the $(1,1)$-entry and 0 elsewhere, they become the classical numerical range $W(A)$ and the classical numerical radius $r(A)$.

While $W(A)$ is always convex for all A, it is not true for general $W_C(A)$ [1]. There are only three known cases that $W_C(A)$ is convex for all A: C is essentially hermitian (i.e. a linear combination of the scalar matrix and a hermitian matrix) [8, 10]; C is of rank one or $C\in\mathbf{M}_2$ [9]; C is a block-shift matrix (i.e. C is unitarily similar to $e^{i\theta}C$ for any $\theta\in\mathbf{R}$) [7].

First introduced in [4], a survey on C-numerical range could be found in [6]. Some properties of $W_C(A)$ are listed below:

(i) $W_C(A)=W_A(C)$.

(ii) $W_C(aA+bI)=aW_C(A)+b\operatorname{tr}C$.

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(iii) $W_C(A)$ has empty interior only if both A and C are essentially hermitian or one of A and C is a scalar matrix.

(iv) If C is not a scalar matrix and $\operatorname{tr} C \neq 0$ then r_C is a norm on $\mathbf{M}_n$. If $\operatorname{tr} C = 0$ then r_C is not a norm as $r_C(I) = 0$.

Althrough $W_C(A)$ fails to be convex in general, [3] confirms that $W_C(A)$ is always star-shaped. A key in [3] is the following set:

$$S(C) := \{D \in \mathbf{C}^{n\times n} : W_D(A) \subseteq W_C(A) \text{ for all } A \in \mathbf{M}_n\}.$$

A study of the set $S(C)$ could be found in [2]. Indeed [2] uses $S(C)$ to construct an alternative proof of Property (iii).

If $\operatorname{tr} C = 0$ then r_C fails to be a norm on $\mathbf{M}_n$. However, if $C \neq 0$ then r_C is a norm on $\mathbf{M}_n^0$. Let $0 \neq D \in \mathbf{M}_n^0$, then r_D is another norm on $\mathbf{M}_n^0$. Thus there exists a $\nu > 0$ such that

$$\nu\, r_D(A) \leqslant r_C(A)$$

for all $A \in \mathbf{M}_n^0$. If C is not essentially hermitian then we have a much stronger property, which is related to Property (iii). We will prove in this article that

THEOREM 1.1. *If $C \in M_n^0$ is not essentially hermitian, then there exists $\nu > 0$ such that*

$$\nu W_D(A) \subseteq \|D\|_F W_C(A)$$

for all $A \in \mathbf{M}_n^0$, where ν depends on C only and $\|D\|_F$ is the Frobenius norm of D.

By Property (iii) alone, we can deduce a similar result, except that $\nu > 0$ may depend on A and D also. The set $S(C)$ is again a key to prove Theorem 1.1. Before we prove Theorem 1.1, we obtain a characterisation of essentially hermitian matrices in the next section.

2. A characterisation of essentially hermitian matrices

We have the following characterisation of essentially hermitian matrices.

THEOREM 2.1. *Let $A \in \mathbf{M}_n$. Suppose*
(P): *for any orthonormal vectors x, y satisfying $x^*Ax = y^*Ay = \frac{1}{n}\operatorname{tr} A$,*
we have $|x^*Ay| = |y^*Ax|$,
then A is essentially hermitian.

To prove the statement, it suffices to consider the case when $\operatorname{tr} A = 0$, i.e. $A \in \mathbf{M}_n^0$. We need to use the following trivial fact about essentially hermitian matrix:

LEMMA 2.2. *Let $A = (a_{ij}e^{i\theta_{ij}}) \in \mathbf{M}_n^0$ with zero diagonal, where $a_{ij} \geqslant 0$ and $-\pi < \theta_{ij} \leqslant \pi$. If*

(1) $a_{12} \neq 0$;

(2) $a_{ij} = a_{ji}$ *for all* i, j*;*

(3) $\theta_{ij} + \theta_{ji} = \theta_{12} + \theta_{21} + 2m\pi$ *for some integers* m*, whenever* $a_{ij} \neq 0$.

then $A = e^{i(\theta_{12}+\theta_{21})/2}H$ *where* H *is an essentially hermitian matrix.*

Note that a matrix is always unitarily similar to a matrix of equal diagonal entries [5, Theorem 1.3.4]. We prove Theorem 2.1 in four steps.

Case $n = 2$.

If $A \in \mathbf{M}_2^0$ satisfies (P), then A is unitarily similar to a matrix of the form

$$\begin{pmatrix} 0 & ae^{i\theta_{12}} \\ ae^{i\theta_{21}} & 0 \end{pmatrix} = e^{i(\theta_{12}+\theta_{21})/2} \begin{pmatrix} 0 & ae^{i(\theta_{12}-\theta_{21})/2} \\ ae^{i(\theta_{21}-\theta_{12})/2} & 0 \end{pmatrix}.$$

Case $n = 3$.

LEMMA 2.3. *Let* A *satisfy* (P) *and* $\operatorname{tr} A = 0$. *If* A *is singular, then every eigenvector corresponding to* 0 *is a normal eigenvector.*

Proof. Let v be a unit eigenvector of A corresponding to 0. Construct an unitary matrix $U = [v, v_2, \ldots, v_n]$ such that v as the first column and that U^*AU has zero diagonal. $Av = 0$ implies $v_j^*Av = 0$ and, as A satisfies (P), $v^*Av_j = 0$ for all j. Thus $v^*A = 0$. ☐

Let $A \in \mathbf{M}_3^0$ satisfy (P). Without loss of generality,

$$A = \begin{pmatrix} 0 & a_{12}e^{i\theta_{12}} & a_{13}e^{i\theta_{13}} \\ a_{12}e^{i\theta_{21}} & 0 & a_{23}e^{i\theta_{23}} \\ a_{13}e^{i\theta_{31}} & a_{23}e^{i\theta_{32}} & 0 \end{pmatrix}$$

for some $a_{12} > 0, a_{13}, a_{23} \geqslant 0, -\pi < \theta_{ij} \leqslant \pi$.

Suppose A is singular. By Lemma 2.3, A is unitarily similar to $0 \oplus A_1$ where A_1 is a 2×2 matrix satisfying (P). A_1 is essential hermitian, and so is A.

Suppose A is nonsingular. In this case,

(i) a_{12}, a_{13}, a_{23} are all nonzero and

(ii) $\theta_{12} - \theta_{13} - \theta_{21} + \theta_{23} + \theta_{31} - \theta_{32}$ is not a odd multiple of π.

Let $x = (0, \cos t, e^{i(\theta_{23}-\theta_{32}+\pi)/2}\sin t)^*$, $y = (1,0,0)^*$. $x^*Ax = y^*Ay = 0$ and $x^*y = 0$. By (P), $|x^*Ay|^2 = |y^*Ax|^2$ and thus

$$\begin{aligned} &|a_{12}e^{i\theta_{21}}\cos t + a_{13}e^{i(\theta_{31}+\theta_{23}/2-\theta_{32}/2+\pi/2)}\sin t|^2 \\ &= |a_{12}e^{i\theta_{12}}\cos t + a_{13}e^{i(\theta_{13}-\theta_{32}/2+\theta_{23}/2-\pi/2)}\sin t|^2. \end{aligned}$$

Expand both sides and cancel like terms, we get

$$\cos(\theta_{21} - \theta_{31} - \theta_{23}/2 + \theta_{32}/2 - \pi/2) = \cos(\theta_{12} - \theta_{13} + \theta_{32}/2 - \theta_{23}/2 + \pi/2).$$

Hence

$$\theta_{21}-\theta_{31}-\theta_{23}/2+\theta_{32}/2-\pi/2=\theta_{12}-\theta_{13}+\theta_{32}/2-\theta_{23}/2+\pi/2+2k\pi$$

or

$$\theta_{21}-\theta_{31}-\theta_{23}/2+\theta_{32}/2-\pi/2=2k\pi-(\theta_{12}-\theta_{13}+\theta_{32}/2-\theta_{23}/2+\pi/2)$$

for some integers k. The first equality reduces to

$$\theta_{12}-\theta_{13}-\theta_{21}+\theta_{23}+\theta_{31}-\theta_{32}=(2k+1)\pi$$

contradicting (ii). Hence the second equality holds and it is equivalent to

$$\theta_{13}+\theta_{31}=\theta_{12}+\theta_{21}+2k\pi.$$

Similarly

$$\theta_{23}+\theta_{32}=\theta_{12}+\theta_{21}+2m\pi$$

for some integers m. By Lemma 2.2, A is essentially hermitian.

Case $n=4$.

Let $0\neq A\in\mathbf{M}_4^0$ satisfy (P). Then A is unitarily similar to a matrix A' with zero diagonals and the $(1,2)$- and the $(2,1)$-entries are nonzero. If the $(1,2)$- and the $(2,1)$-entries of A' are the only nonzero entries, then we are done. Otherwise, it is unitarily similar to a matrix A'' with zero diagonals, the $(1,2)$- and $(2,1)$- entries are nonzero and that at least one of $(1,3)$-, $(1,4)$-, $(2,3)$-, $(2,4)$- entries is nonzero. We assume that $A=A''$.

Write $A=(a_{ij}e^{i\theta_{ij}})\in\mathbf{M}_n^0$ with zero diagonal, where $a_{ij}=a_{ji}\geqslant 0$ and $-\pi<\theta_{ij}\leqslant\pi$.

Suppose $a_{13}\neq 0$. Consider the submatrix $A(1,2,3)$ which satisfies (P) and thus it is essentially hermitian. By Lemma 2.2, $\theta_{12}+\theta_{21}=\theta_{13}+\theta_{31}+2k\pi$ for some integers k. Similarly for the $(1,4)$-, $(2,3)$- and $(2,4)$-entries.

Suppose $a_{34}\neq 0$. Note that at least one of $a_{13},a_{23},a_{14},a_{24}$ is nonzero. Say, $a_{13}\neq 0$. By considering the submatrices $A(1,2,3)$ and $A(1,3,4)$, we have $\theta_{12}+\theta_{21}=\theta_{13}+\theta_{31}+2k\pi=\theta_{34}+\theta_{43}+2m\pi$ for some integers k,m.

By Lemma 2.2, A is essentially hermitian.

Case $n>4$.

Let $A\in\mathbf{M}_n^0$ satisfy (P). Without loss of generality, assume that the diagonal entries of A are zero and that the $(1,2)$-entry is nonzero. Write $A=(a_{ij}e^{i\theta_{ij}})\in\mathbf{M}_n^0$ with zero diagonal, where $a_{ij}=a_{ji}\geqslant 0$ and $-\pi<\theta_{ij}\leqslant\pi$.

If $a_{ij}\neq 0$, then consider a 4×4-submatrix $A(\alpha)$, where $1,2,i,j\in\alpha$. $A(\alpha)$ is essentially hermitian and thus $\theta_{12}+\theta_{21}=\theta_{ij}+\theta_{ji}+2k\pi$ for some integers k.

By Lemma 2.2, A is essentially hermitian.

3. Inclusion Relation of Numerical Ranges

We start with some old results.

LEMMA 3.1. [2, Theorem 3.1.1] *Let* $C \in \mathbf{M}_n$ *and* $D \in \mathbf{M}_p$ *then* $S(C) \oplus S(D) \subseteq S(C \oplus D)$ *and* $S(C) \otimes S(D) \subseteq S(C \otimes D)$, *where the operations on sets are element-wise.*

LEMMA 3.2. [9] *Let* $C \in \mathbf{M}_2$, *then* $W_C(A)$ *is convex for all* $A \in \mathbf{M}_2$, *equivalently* $S(C) = \operatorname{conv}(\mathbf{U}(C))$, *i.e. the convex hull of the unitarily orbit of* A.

LEMMA 3.3. [3] *Suppose* $D = (b_{ij}) \in S(C)$. *Let* k *be such that* $1 \leqslant k \leqslant n$, $\varepsilon \in [0,1]$, *and* $D' = \left(d'_{ij}\right)$ *be defined by*

$$d'_{ij} = \begin{cases} \varepsilon d_{ij}, & \text{if exactly one of } i, j \text{ equals } k, \\ d_{ij}, & \text{otherwise.} \end{cases}$$

(In other words, D' *is obtained from* D *by multiplying* ε *to the entries on the* k*-th row and on the* k*-th column, except for the* (k,k)*th entry, of* D.*) Then* $D' \in S(C)$.

Apply Lemmas 3.1, 3.2 and 3.3, we have

LEMMA 3.4. *Let* $C = (c_{ij}) \in \mathbf{M}_n^0$ *with zero diagonal, then*

$$\operatorname{conv}\left(\mathbf{U}\left(\begin{pmatrix} 0 & c_{12} \\ c_{21} & 0 \end{pmatrix}\right)\right) \oplus 0_{n-2} \subseteq S(C).$$

If C is a block-shift matrix, then $W_C(A)$ is a circular disc for any C [7]. In particular, it is true for $A = E_{12}$, the matrix with a 1 at the $(1,2)$-entry and 0 elsewhere. Indeed, we have the following result.

LEMMA 3.5. [2, Corollary 3.2.6] *If* $D \in \mathbf{M}_n^0$ *then* $S(D) \subseteq \beta(n)\|D\|_F S(E_{12})$ *where* $\beta(n) = \begin{cases} \frac{2(n-1)\sqrt{2n}}{n} & \text{if } n \text{ is even,} \\ \frac{2(n-1)\sqrt{2n-1}}{n} & \text{if } n \text{ is odd.} \end{cases}$

Let's restate Theorem 1.1.

THEOREM 3.6. *Suppose* $C \in \mathbf{M}_n^0$ *is not essentially hermitian. Then there exists* $\nu > 0$ *such that for any* $D \in \mathbf{M}_n^0$,

$$\nu S(D) \subseteq \|D\|_F S(C),$$

equivalently $\nu W_D(A) \subseteq \|D\|_F W_C(A)$ *for any* $A \in \mathbf{M}_n$.

Proof. By Theorem 2.1, there exists unit vectors x and y such that $x^*Cx = y^*Cy = 0$ and $x^*y = 0$ but $|x^*Cy| \neq |y^*Cx|$. Therefore we can assume that $C = (c_{ij})$ where $c_{ii} = 0$ for all i and $|c_{12}| \neq |c_{21}|$.

By Lemma 3.4, $\operatorname{conv}\left(\mathbf{U}\left(\begin{pmatrix} 0 & c_{12} \\ c_{21} & 0 \end{pmatrix}\right)\right) \oplus 0_{n-2} \subseteq S(C)$. Since $|c_{12}| \neq |c_{21}|$, there exists $\tau > 0$ such that $\begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} = \tau E_{12} \in S(C)$.

By Lemma 3.5, we have for any $D \in \mathbf{M}_n^0$, $\frac{\tau}{\beta(n)} D \in \|D\|_F \tau S(E_{12}) \subseteq \|D\|_F S(C)$. □

For any $C \in \mathbf{M}_n^0$, define

$$\nu(C) := \max\{\nu \geqslant 0 : \nu S(D) \subseteq \|D\|_F S(C) \text{ for all } D \in \mathbf{M}_n^0\}.$$

COROLLARY 3.7. *Let $C \in \mathbf{M}_n^0$. Then $\nu(C) \geqslant 0$ and that $\nu(C) = 0$ if and only if C is essentially hermitian.*

Proof. By Theorem 3.6, if C is not essentially hermitian then $\nu(C) > 0$. If $C \neq 0$ is essentially hermitian, then $W(C)$ and $W(iC)$ are two line segments intersect at 0 only, hence $\nu iC \in \|iC\|_F S(C)$ only if $\nu = 0$, and thus $\nu(C) = 0$. □

REFERENCES

[1] Y. H. AU-YEUNG AND N. K. TSING, *A conjecture of Marcus on the generalized numerical range*, Linear and Multilinear Algebra, **14** (1983), 235–239.

[2] W. S. CHEUNG, *Some Geometrical Aspects Of and Inclusion Relations For Generalized Numerical Ranges*, M. Phil Thesis, The University of Hong Kong, 1996.

[3] W. S. CHEUNG AND N. K. TSING, *The C-numerical range of matrices is star-shaped*, Linear and Multilinear Algebra, **41**, 3 (1996), 245–250.

[4] M. GOLDBERG AND E. G. STRAUS, *Elementary inclusion relations for generalized numerical ranges*, Lin. Alg. Appl., **18** (1977), 1–18.

[5] R. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[6] C.-K. LI, *C-numerical ranges and C-numerical radii*, Linear and Multilinear Algebra, **37** (1994), 51–82.

[7] C.-K. LI AND N. K. TSING, *Matrices with circular symmetry on their unitary similarity orbits and C-numerical ranges*, Proc. Amer. Math. Soc., **111** (1991), 19–28.

[8] Y. T. POON, *Another proof of a result of Westwick*, Linear and Multilinear Algebra, **9** (1980), 35–37.

[9] N. K. TSING, *The constrained bilinear form and the C-numerical range*, Linear Algebra and Its Applications, **56** (1984), 195–206.

[10] R. WESTWICK, *A theorem on numerical ranges*, Linear and Multilinear Algebra, **2** (1975), 311–315.

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