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and
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BANACH SPACES OF FUNCTIONS TAKING VALUES IN A C^*–ALGEBRA

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Abstract. Let $\mathscr{A}$ be a C^*-algebra with identity 1; and let $s(\mathscr{A})$ denote the set of all states on $\mathscr{A}$. The state space $s(\mathscr{A})$ (with the weak* topology) is used to construct classes of Banach spaces of functions defined on a fixed set S taking values in $\mathscr{A}$. The inter-relationship between spaces are considered. Special classes of operators on these spaces are also considered. When $\mathscr{A}$ is taken to be $\mathbb{C}$ and S to be $\mathbb{N}$, all spaces are just the classical spaces.

1. Introduction

Many well-known and beautiful theorems in analysis involve Banach spaces of functions taking values in the real or complex field. In fact, from our more naive and simplistic point of view, the entire theory of operators and operator algebras is built on the sequence (function) space ℓ^2 and the Lebesgue function space L^2 (and operator algebras are themselves Banach spaces of functions taking values in a Banach space). Many of these theorems have been extended to Banach spaces of functions taking values in a fixed Banach space, with the norm on the Banach space replacing the absolute value function on the real or complex field [3]. Since a C^*-algebra has richer structure and resembles the complex field in more ways than a general Banach space, when the complex field is replaced by a C^*-algebra, there are more natural ways to study function spaces taking values in a C^*-algebra. In this paper we use the state space on a fixed C^*-algebra to study Banach spaces of functions that take values in the C^*-algebra.

Let S be a fixed set; and let $\mathscr{A}$ be a C^*-algebra with identity 1 and state space $s(\mathscr{A})$ (with the weak* topology, whenever a topology is invoked). We consider spaces of functions from S to $\mathscr{A}$ that are finite under various norms determined by $s(\mathscr{A})$. The main purpose of this paper is to extend results on classical Banach sequence spaces to these spaces. When the C^*-algebra is taken to be the complex field $\mathbb{C}$, since the state space of $\mathbb{C}$ contains only one element, namely the identity map, all the norms are the same as the classical ones determined by the absolute value function and all spaces are just the classical spaces.

We begin by gathering some fairly well known results about elements in $\mathscr{A}$ and $s(\mathscr{A})$ that will be used in later sections of the paper in section 2. The commonalities of spaces to be studied are gathered together to prove a theorem in section 3 which, in turn,

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is used to prove that certain collections of functions on $S \times S$ act as bounded operators on the space, and these functions form a closed subspace of the space of all bounded operators. We extract the essence of the proofs of the existence and completeness of norms into theorems in section 4. These general results are then applied to show all spaces being considered are Banach spaces in sections 5 and 6. For some of these spaces $\mathscr{X}$, we also have a description of the elements of the space $\mathscr{B}(\mathscr{X},\mathscr{A})$, of bounded linear maps from the space $\mathscr{X}$ into $\mathscr{A}$; an analogue of the dual space.

2. Definitions and Preliminaries

Let $\mathscr{A}$ be a unital C^*-algebra over the complex field $\mathbb{C}$ with state space $s(\mathscr{A})$ (the set of all positive linear functionals of norm 1 on $\mathscr{A}$, i.e., taking the value 1 at the identity, with the weak* topology). Then $\|a\| = \sup_{\varphi \in s(\mathscr{A})} |\varphi(a)| = \max_{\varphi \in s(\mathscr{A})} |\varphi(a)|$ for all self-adjoint elements $a \in \mathscr{A}$ (in fact, it is true for all normal elements, [4], Theorem 4.3.4 (iv), p. 258). By convention $|x| = \sqrt{x^*x}$ for every $x \in \mathscr{A}$.

For a positive linear functional φ on $\mathscr{A}$, the following inequality is just the triangle inequality for the semi-norm $\|a\|_\varphi = [\varphi(a^*a)]^{\frac{1}{2}}$ $(a \in \mathscr{A})$ induced by the semi-inner product defined by $\langle a,b\rangle_\varphi = \varphi(b^*a)$ for all $a,b \in \mathscr{A}$ ([4] p. 256).

LEMMA 2.1. (Minkowski's inequality) *Let $a,b \in \mathscr{A}$; and $\varphi \in s(\mathscr{A})$. Then*

$$\left[\varphi\left(|a+b|^2\right)\right]^{\frac{1}{2}} \leqslant \left[\varphi\left(|a|^2\right)\right]^{\frac{1}{2}} + \left[\varphi\left(|b|^2\right)\right]^{\frac{1}{2}} \qquad \text{i.e.;}$$
$$\|a+b\|_\varphi \leqslant \|a\|_\varphi + \|b\|_\varphi .$$

LEMMA 2.2. *Let $x,y \in \mathscr{A}$ and $\varphi \in s(\mathscr{A})$. Then*

$$\varphi\left(|xy|^2\right) \leqslant \|x\|^2 \varphi\left(|y|^2\right) \qquad \text{i.e.;} \quad \|xy\|_\varphi \leqslant \|x\|\,\|y\|_\varphi .$$

For any $x \in \mathscr{A}$, define the *state norm* of x by $\|x\|_s = \sup_{\varphi \in s(\mathscr{A})} |\varphi(x)|$. It is not hard to see that $\|\cdot\|_s$ is indeed a norm on $\mathscr{A}$. The following proposition, which may be well known to the experts, establishes the equivalence of the state norm and the norm. This is useful in later sections.

PROPOSITION 2.3. *For any $x \in \mathscr{A}$, $\|x\|_s \leqslant \|x\| \leqslant 2\|x\|_s$.*

Proof. Since $s(\mathscr{A}) \subseteq \left(\mathscr{A}^\#\right)_1$ (the closed unit ball of $\mathscr{A}^\#$),

$$\|x\|_s = \sup_{\varphi \in s(\mathscr{A})} |\varphi(x)| \leqslant \sup_{f \in \left(\mathscr{A}^\#\right)_1} = \|x\| .$$

Decompose $x \in \mathscr{A}$ as the sum of its real and imaginary parts $x = \Re x + i\Im x$. Since $|\varphi(\Re x)| \leqslant |\varphi(x)|$ and similarly for the imaginary part, and since the state norm of a self-adjoint element is equal to its norm ([4] p. 258),

$$\|x\| \leqslant \|\Re x\| + \|\Im x\| = \|\Re x\|_s + \|\Im x\|_s \leqslant 2\|x\|_s .$$

3. Bounded "Matrices"

Throughout this paper S denotes a fixed nonempty set, and $\mathscr{F}$ denotes the collection of all finite subsets of S directed by set inclusion: $F \succeq G \Leftrightarrow F \supseteq G$ for $F,G \in \mathscr{F}$. For a set Λ, Λ^S denotes the collection of all functions from S to Λ. Let X be a Banach space; and let $\mathbf{x} \in X^S$ so that $\{\mathbf{x}(s) : s \in S\}$ is a collection of vectors in X. The (unordered) sum $\sum_{s\in S}\mathbf{x}(s)$ is said to be *convergent with sum* $x \in X$ if the net $\left\{\sum_{s\in F}\mathbf{x}(s)\right\}_{F\in\mathscr{F}}$ converges to x, written

$$\sum_{s\in S}\mathbf{x}(s) = \lim_{F\in\mathscr{F}}\left(\sum_{s\in F}\mathbf{x}(s)\right) = x,$$

otherwise, $\sum_{s\in S}\mathbf{x}(s)$ is said to *diverge* or to be *divergent*. By convention, the sum over the empty set is 0. A convergent sequence is always bounded, but a convergent net may not be; e.g., the positive reals directed by the reverse ordering converges to 0 (in the usual topology), but it is not bounded. However, for a convergent sum, the net of "partial sums" is always bounded.

PROPOSITION 3.1. *For* $\mathbf{x} \in X^S$, *if* $\sum_{s\in S}\mathbf{x}(s) = x$ *converges in* X, *then there is an* M *such that* $\left\|\sum_{s\in F}\mathbf{x}(s)\right\| \leqslant M$ *for all* $F \in \mathscr{F}$, *i.e., the net* $\left\{\sum_{s\in F}\mathbf{x}(s)\right\}_{F\in\mathscr{F}}$ *of "partial sums" is bounded.*

Proof. Since this is true in any topological vector space, we give a proof for this more general setting. Assume X is a topological vector space. Let $\mathscr{U}$ be an open set containing 0. Then there is an open, balanced and absorbing set $\mathscr{V}$ containing 0 such that $\mathscr{V} + \mathscr{V} \subseteq \mathscr{U}$. By the convergence of the sum, there is an $F_0 \in \mathscr{F}$ such that for all $F \in \mathscr{F}$ with $F \supseteq F_0$, $x - \sum_{s\in F}\mathbf{x}(s) \in \mathscr{V}$. Since $\mathscr{V}$ is absorbing, for each $G \subseteq F_0$ there is an $\varepsilon_G > 0$ such that $\lambda\left[x - \sum_{s\in G}\mathbf{x}(s)\right] \in \mathscr{V}$ for all $\lambda \in [0, 2\varepsilon_G)$. Let $\varepsilon = \min\{\varepsilon_G : G \subseteq F_0\}$. Then if $\lambda \in [0, 2\varepsilon)$ and $H \subseteq F_0$ we have $\lambda\left[x - \sum_{s\in H}\mathbf{x}(s)\right] \in \mathscr{V}$. Let $t = \max\left\{\frac{1}{\varepsilon}, 1\right\}$, and let $H \in \mathscr{F}$. Since $\mathscr{V}$ is balanced and absorbing, we have

$$\begin{aligned}\sum_{s\in H} x(s) &= \sum_{H\cup F_0}\mathbf{x}(s) - \sum_{s\in F_0\setminus H}\mathbf{x}(s)\\ &= \left[\sum_{H\cup F_0}\mathbf{x}(s) - x\right] - \frac{1}{\varepsilon}\left(\varepsilon\left[\sum_{s\in F_0\setminus H}\mathbf{x}(s) - x\right]\right) \in [-\mathscr{V}] - \frac{1}{\varepsilon}[-\mathscr{V}]\end{aligned}$$

$$\subseteq t\mathscr{V} + t\mathscr{V} = t\,[\mathscr{V} + \mathscr{V}] \subseteq t\mathscr{U}.$$

For $b \in X$ and $s \in S$ define $\mathbf{e}_s(b) \in X^S$ (analogue of the standard basis elements in ℓ^p space) by

$$[\mathbf{e}_s(b)]\,(t) = \begin{cases} b & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \qquad \text{for all } \ t \in S.$$

For $\mathbf{x} \in X^S$ and $F \subseteq S$, denote by $\mathbf{x}_F$ the function

$$\mathbf{x}_F(s) = \begin{cases} \mathbf{x}(s) & \text{if } s \in F \\ 0 & \text{if } s \notin F \end{cases}.$$

From this point on, $\mathscr{X}$ will denote a Banach space of functions from S to $\mathscr{A}$ that satisfies the following conditions:

(i) $\mathbf{e}_s(b) \in \mathscr{X} \ \ \forall\, b \in \mathscr{A}, \ \forall\, s \in S$;

(ii) there exists a $c > 0$ such that $\|\mathbf{x}(s)\|_{\mathscr{A}} \leqslant c\,\|\mathbf{x}\|_{\mathscr{X}} \ \ \forall\, s \in S, \ \forall\, \mathbf{x} \in \mathscr{X}$;

(iii) there exists a $\xi > 0$ such that $\forall\, b \in \mathscr{A}$, $\forall\, s \in S$, $\|\mathbf{e}_s(b)\|_{\mathscr{X}} \leqslant \xi\,\|b\|_{\mathscr{A}}$;

(iv) $\sup\limits_{F \in \mathscr{F}} \|\mathbf{x}_F\|_{\mathscr{X}} = \|\mathbf{x}\|_{\mathscr{X}} \ \ \forall\, \mathbf{x} \in \mathscr{X}$.

THEOREM 3.2. *If* $\mathbf{a} \in \mathscr{A}^S$ *and if* $\sum\limits_{s \in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges in* $\mathscr{A}$ *for all* $\mathbf{x} \in \mathscr{X}$, *then* $T_{\mathbf{a}}(\mathbf{x}) = \sum\limits_{s \in S} \mathbf{a}(s)\mathbf{x}(s)$ *defines a bounded linear transformation from* $\mathscr{X}$ *to* $\mathscr{A}$.

Proof. Let $F \in \mathscr{F}$. Define $T_F : \mathscr{X} \to \mathscr{A}$ by $T_F(\mathbf{x}) = \sum\limits_{s \in F} \mathbf{a}(s)\mathbf{x}(s) \ \ \forall\, \mathbf{x} \in \mathscr{X}$. Then T_F is readily seen to be linear, and

$$\|T_F(\mathbf{x})\|_{\mathscr{A}} = \left\| \sum_{s \in F} \mathbf{a}(s)\mathbf{x}(s) \right\|_{\mathscr{A}} \leqslant \sum_{s \in F} \|\mathbf{a}(s)\|_{\mathscr{A}}\, \|\mathbf{x}(s)\|_{\mathscr{A}} \leqslant \sum_{s \in F} \|\mathbf{a}(s)\|_{\mathscr{A}} \cdot c\, \|\mathbf{x}\|_{\mathscr{X}},$$

which implies that T_F is bounded for each $F \in \mathscr{F}$. Fix $\mathbf{x} \in \mathscr{X}$; since $\sum\limits_{s \in S} \mathbf{a}(s)\mathbf{x}(s)$ converges, and $T_F(\mathbf{x}) \xrightarrow{F \in \mathscr{F}} \sum\limits_{s \in S} \mathbf{a}(s)\mathbf{x}(s)$, there exists, by Proposition 3.1, a constant $M_{\mathbf{x}}$ such that

$$\|T_F(\mathbf{x})\|_{\mathscr{A}} = \left\| \sum_{s \in F} \mathbf{a}(s)\mathbf{x}(s) \right\|_{\mathscr{A}} \leqslant M_{\mathbf{x}} \qquad \text{for all } F \in \mathscr{F}.$$

From the uniform boundedness principle, we infer the existence of a constant M such that $\|T_F\| \leqslant M$ for all $F \in \mathscr{F}$. For every $\mathbf{x} \in \mathscr{X}$, we obtain

$$\|T_{\mathbf{a}}(\mathbf{x})\|_{\mathscr{A}} = \left\| \sum_{s \in S} \mathbf{a}(s)\mathbf{x}(s) \right\|_{\mathscr{A}} = \left\| \lim_{F \in \mathscr{F}} \sum_{s \in F} \mathbf{a}(s)\mathbf{x}(s) \right\|_{\mathscr{A}} = \lim_{F \in \mathscr{F}} \left\| \sum_{s \in F} \mathbf{a}(s)\mathbf{x}(s) \right\|_{\mathscr{A}}$$

$$= \lim_{F\in\mathscr{F}} \|T_F(\mathbf{x})\|_{\mathscr{A}} \leqslant \limsup_{F\in\mathscr{F}} \|T_F\| \, \|\mathbf{x}\|_{\mathscr{X}} \leqslant M\|\mathbf{x}\|_{\mathscr{X}}.$$

Therefore $T_{\mathbf{a}}$ is a bounded linear transformation from $\mathscr{X}$ to $\mathscr{A}$.

Let $\mathscr{B}(\mathscr{X})$ be the set of all bounded linear operators on $\mathscr{X}$. It is well-known that $\mathscr{B}(\mathscr{X})$ with the operator norm is a Banach algebra.

Let $\mathbf{A}$ be a function from $S\times S$ to $\mathscr{A}$. We say that $\mathbf{A}$ *defines a linear operator, $T_{\mathbf{A}}$, on $\mathscr{X}$*, if for each $\mathbf{x}\in\mathscr{X}$

(i) $(T_{\mathbf{A}}\mathbf{x})(s) := \sum_{t\in S} \mathbf{A}(s,t)\mathbf{x}(t)$ converges in $\mathscr{A}$ $\quad \forall\, s\in S$,

(ii) $T_{\mathbf{A}}\mathbf{x}\in\mathscr{X}$.

THEOREM 3.3. *Let $\mathbf{A}\in\mathscr{A}^{S\times S}$. Suppose that $\mathbf{A}$ defines a linear operator, $T_{\mathbf{A}}$, on $\mathscr{X}$. Then $T_{\mathbf{A}}$ is bounded.*

Proof. For a fixed $s\in S$,

$$(T_{\mathbf{A}}\mathbf{x})_{\{s\}}(u) = \delta_s(u)\sum_{t\in S}\mathbf{A}(u,t)\mathbf{x}(t) \quad \forall\,\mathbf{x}\in\mathscr{X} \ \ \forall\, u\in S$$

where $\delta_s(u) = \begin{cases} 1 & ; u = s \\ 0 & ; u\neq s \end{cases}$. Then by condition (iii) on $\mathscr{X}$ and the preceding theorem, there is a constant M_s such that for all $\mathbf{x}\in\mathscr{X}$,

$$\left\|(T_{\mathbf{A}}\mathbf{x})_{\{s\}}\right\|_{\mathscr{X}} \leqslant \xi \left\|\sum_{t\in S}\mathbf{A}(s,t)\mathbf{x}(t)\right\|_{\mathscr{A}} \leqslant \xi M_s\|\mathbf{x}\|_{\mathscr{X}}.$$

Let $F\in\mathscr{F}$. The map $T_{\mathbf{A},F} : \mathbf{x}\mapsto (T_{\mathbf{A}}\mathbf{x})_F$ is a bounded linear operator on $\mathscr{X}$, because

$$\begin{aligned} \left\|T_{\mathbf{A},F}(\mathbf{x})\right\|_{\mathscr{X}} = \|(T_{\mathbf{A}}\mathbf{x})_F\|_{\mathscr{X}} &= \left\|\sum_{s\in F}(T_{\mathbf{A}}\mathbf{x})_{\{s\}}\right\|_{\mathscr{X}} \leqslant \sum_{s\in F}\left\|(T_{\mathbf{A}}\mathbf{x})_{\{s\}}\right\|_{\mathscr{X}} \\ &\leqslant \xi\left(\sum_{s\in F}M_s\right)\|\mathbf{x}\|_{\mathscr{X}}. \end{aligned}$$

Fix $\mathbf{x}\in\mathscr{X}$; since $\left\|T_{\mathbf{A},F}(\mathbf{x})\right\|_{\mathscr{X}} = \|(T_{\mathbf{A}}\mathbf{x})_F\|_{\mathscr{X}} \leqslant \|T_{\mathbf{A}}\mathbf{x}\|_{\mathscr{X}}$ for all $F\in\mathscr{F}$ by the condition (iv) on $\mathscr{X}$, it follows from this and the uniform boundedness principle that there is a constant M such that $\left\|T_{\mathbf{A},F}\right\| \leqslant M$ for all $F\in\mathscr{F}$. Thus $\|(T_{\mathbf{A}}\mathbf{x})_F\|_{\mathscr{X}} = \left\|T_{\mathbf{A},F}(\mathbf{x})\right\|_{\mathscr{X}} \leqslant \left\|T_{\mathbf{A},F}\right\|\|\mathbf{x}\| \leqslant M\|\mathbf{x}\|_{\mathscr{X}}$ $\forall\, F\in\mathscr{F}$. By condition (iv) again, $\|T_{\mathbf{A}}\mathbf{x}\|_{\mathscr{X}} = \sup_{F\in\mathscr{F}}\|(T_{\mathbf{A}}\mathbf{x})_F\|_{\mathscr{X}} \leqslant M\|\mathbf{x}\|_{\mathscr{X}}$. Therefore $T_{\mathbf{A}}$ is bounded with $\|T_{\mathbf{A}}\|\leqslant M$.

Let $\mathscr{M}(\mathscr{X})$ be the set of all $\mathbf{A}$ in the space of functions from $S\times S$ to $\mathscr{A}$ such that $\mathbf{A}$ defines a linear operator on $\mathscr{X}$. Then $\mathscr{M}(\mathscr{X})$ can be regarded as a subspace of $\mathscr{B}(\mathscr{X})$, by the preceding theorem. For simplicity of notation write $\mathbf{A}\mathbf{x} = T_{\mathbf{A}}\mathbf{x}$ for $\mathbf{A}\in\mathscr{M}(\mathscr{X})$, $\mathbf{x}\in\mathscr{X}$.

THEOREM 3.4. *Let $\mathscr{X}$ and $\mathscr{M}(\mathscr{X})$ be as defined above. Then $\mathscr{M}(\mathscr{X})$ with the operator norm is a Banach space. If $\mathscr{X}$ has the property that $\|\mathbf{x}-\mathbf{x}_F\| \overset{F\in\mathscr{F}}{\longrightarrow} 0$ for all $\mathbf{x}\in\mathscr{X}$, then $\mathscr{M}(\mathscr{X})$ is a Banach algebra.*

Proof. For $\mathbf{A}\in\mathscr{M}(\mathscr{X})$ and $(s,t)\in S\times S$, we claim that $\|\mathbf{A}(s,t)\| \leqslant c\xi\|\mathbf{A}\|$; where c and ξ are as in the definition of $\mathscr{X}$. Indeed, by conditions (ii) and (iii) on $\mathscr{X}$,

$$\begin{aligned}\|\mathbf{A}(s,t)\|_{\mathscr{A}} &= \|\mathbf{A}(s,t)\cdot 1\|_{\mathscr{A}} = \|\{\mathbf{A}[\mathbf{e}_t(1)]\}(s)\|_{\mathscr{A}} \\ &\leqslant c\|\mathbf{A}[\mathbf{e}_t(1)]\|_{\mathscr{X}} \leqslant c\|\mathbf{A}\|\,\|\mathbf{e}_t(1)\|_{\mathscr{X}} \\ &\leqslant c\|\mathbf{A}\|\,\xi\,\|1\|_{\mathscr{A}} = c\xi\|\mathbf{A}\|.\end{aligned}$$

To see that $\mathscr{M}(\mathscr{X})$ is a Banach space, we show that $\mathscr{M}(\mathscr{X})$ is closed in $\mathscr{B}(\mathscr{X})$. To that end, let $\{\mathbf{A}_n\}$ be a sequence in $\mathscr{M}(\mathscr{X})$ such that $\mathbf{A}_n\to T$ in $\mathscr{B}(\mathscr{X})$. Then $\{\mathbf{A}_n\}$ is a Cauchy sequence in $\mathscr{M}(\mathscr{X})$. For each fixed $(s,t)\in S\times S$,

$$\|\mathbf{A}_n(s,t)-\mathbf{A}_m(s,t)\|_{\mathscr{A}} = \|(\mathbf{A}_n-\mathbf{A}_m)(s,t)\|_{\mathscr{A}} \leqslant c\xi\|\mathbf{A}_n-\mathbf{A}_m\| \to 0$$

as $n,m\to\infty$. We see that the sequence $\{\mathbf{A}_n(s,t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{A}$. Since $\mathscr{A}$ is complete, there is an element $a_{st}\in\mathscr{A}$ such that $\mathbf{A}_n(s,t)\to a_{st}$ as $n\to\infty$. Define $\mathbf{A}(s,t)=a_{st}$. Let $\mathbf{x}$ be an arbitrary vector in $\mathscr{X}$. For each $s\in S$ and $F\in\mathscr{F}$, we have $[\mathbf{A}_n(\mathbf{x}_F)](s)=\sum_{t\in F}\mathbf{A}_n(s,t)\mathbf{x}(t)$ and hence

$$\begin{aligned}(T(\mathbf{x}_F))(s) &= \lim_{n\to\infty}(\mathbf{A}_n(\mathbf{x}_F))(s) = \lim_{n\to\infty}\sum_{t\in F}\mathbf{A}_n(s,t)\mathbf{x}(t) \\ &= \sum_{t\in F}\left(\lim_{n\to\infty}[\mathbf{A}_n(s,t)\mathbf{x}(t)]\right) = \sum_{t\in F}\mathbf{A}(s,t)\mathbf{x}(t).\end{aligned}$$

Thus $[T(\mathbf{x}_F)](s)=\sum_{t\in F}\mathbf{A}(s,t)\mathbf{x}(t)$ for all $s\in S$. Fix $s\in S$, let $\varepsilon>0$ be given. Since $\mathbf{A}_n\to T$, there exists $N\in\mathbb{N}$ such that for all $n\geqslant N$,

$$\|\mathbf{A}_n-T\| < \frac{\varepsilon}{4c\|\mathbf{x}\|_{\mathscr{X}}+1}.$$

Since $\mathbf{A}_N\in\mathscr{M}(\mathscr{X})$, $[\mathbf{A}_N(\mathbf{x})](s)=\sum_{t\in S}\mathbf{A}_N(s,t)\mathbf{x}(t)$ converges in $\mathscr{A}$. That is

$$[\mathbf{A}_N(\mathbf{x}_F)](s)=\sum_{t\in F}\mathbf{A}_N(s,t)\mathbf{x}(t) \overset{F\in\mathscr{F}}{\longrightarrow} \sum_{t\in S}\mathbf{A}_N(s,t)\mathbf{x}(t)=[\mathbf{A}_N(\mathbf{x})](s).$$

So there is an $F_0\in\mathscr{F}$ such that for all $F\in\mathscr{F}$, with $F\supseteq F_0$,

$$\|[\mathbf{A}_N(\mathbf{x}_F)](s)-[\mathbf{A}_N(\mathbf{x})](s)\|_{\mathscr{A}} < \frac{\varepsilon}{4}.$$

For $F\in\mathscr{F}$ with $F\supseteq F_0$,

$$\left\|[T(\mathbf{x})](s)-\sum_{t\in F}\mathbf{A}(s,t)\mathbf{x}(t)\right\|_{\mathscr{A}} = \|[T(\mathbf{x})](s)-[T(\mathbf{x}_F)](s)\|_{\mathscr{A}}$$

$$
\begin{aligned}
&\leqslant \|[T(\mathbf{x})](s)-[\mathbf{A}_N(\mathbf{x})](s)\|_{\mathscr{A}} + \|[\mathbf{A}_N(\mathbf{x})](s)-[\mathbf{A}_N(\mathbf{x}_F)](s)\|_{\mathscr{A}} \\
&\qquad\qquad\qquad\qquad + \|[\mathbf{A}_N(\mathbf{x}_F)](s)-[T(\mathbf{x}_F)](s)\|_{\mathscr{A}} \\
&= \|[T(\mathbf{x})-\mathbf{A}_N(\mathbf{x})](s)\|_{\mathscr{A}} + \|[\mathbf{A}_N(\mathbf{x})](s)-[\mathbf{A}_N(\mathbf{x}_F)](s)\|_{\mathscr{A}} \\
&\qquad\qquad\qquad\qquad + \|[\mathbf{A}_N(\mathbf{x}_F)-T(\mathbf{x}_F)](s)\|_{\mathscr{A}} \\
&\leqslant c\,\|(T-\mathbf{A}_N)(\mathbf{x})\|_{\mathscr{X}} + \frac{\varepsilon}{4} + c\,\|(\mathbf{A}_N-T)(\mathbf{x}_F)\|_{\mathscr{X}} \\
&\leqslant c\,\|(T-\mathbf{A}_N)\|\,\|\mathbf{x}\|_{\mathscr{X}} + \frac{\varepsilon}{4} + c\,\|\mathbf{A}_N-T\|\,\|\mathbf{x}\|_{\mathscr{X}} \\
&\leqslant c\left[\frac{\varepsilon}{4c\,\|\mathbf{x}\|_{\mathscr{X}}+1}\right]\|\mathbf{x}\|_{\mathscr{X}} + \frac{\varepsilon}{4} + c\left[\frac{\varepsilon}{4c\,\|\mathbf{x}\|_{\mathscr{X}}+1}\right]\|\mathbf{x}\|_{\mathscr{X}} < \varepsilon.
\end{aligned}
$$

Hence $\lim_{F\in\mathscr{F}}\left(\sum_{t\in F}\mathbf{A}(s,t)\mathbf{x}(t)\right)=[T(\mathbf{x})](s)$, i.e., $\sum_{t\in S}\mathbf{A}(s,t)\mathbf{x}(t)=[T(\mathbf{x})](s)$. Since this is true for all $s\in S$, we have that $\mathbf{A}\in\mathscr{M}(\mathscr{X})$ and $\mathbf{A}\mathbf{x}=T\mathbf{x}$ for all vector $\mathbf{x}\in\mathscr{X}$. This implies $T=\mathbf{A}\in\mathscr{M}(\mathscr{X})$.

To see that $\mathscr{M}(\mathscr{X})$ is a Banach algebra when $\mathscr{X}$ has the "finite approximation property", we need only show that it is closed under multiplication. Let $\mathbf{A},\mathbf{B}\in\mathscr{M}(\mathscr{X})$. For each $u\in S$, since $\mathbf{e}_u(1)\in\mathscr{X}$, and hence $\mathbf{y}(t)=\mathbf{B}(t,u)=\mathbf{B}[\mathbf{e}_u(1)](t)$, $t\in S$ defines a function in $\mathscr{X}$, thus $\mathbf{B}(\cdot,u)\in\mathscr{X}$ for every $u\in S$. It follows that, for each fixed $s,u\in S$, the sum $\sum_{t\in S}\mathbf{A}(s,t)\mathbf{B}(t,u)=\lim_{G\in\mathscr{F}}\left[\sum_{t\in G}\mathbf{A}(s,t)\mathbf{B}(t,u)\right]$ converges. Set $T=T_{\mathbf{A}}T_{\mathbf{B}}$. Let $\mathbf{x}\in\mathscr{X}$ and $F\in\mathscr{F}$. Then, for each $s\in S$,

$$
\begin{aligned}
(T\mathbf{x}_F)(s) &= [(T_{\mathbf{A}}T_{\mathbf{B}})(\mathbf{x}_F)](s)=\sum_{t\in S}\mathbf{A}(s,t)(T_{\mathbf{B}}\mathbf{x}_F)(t) \\
&= \sum_{t\in S}\mathbf{A}(s,t)\left[\sum_{u\in S}\mathbf{B}(t,u)\mathbf{x}_F(u)\right] \\
&= \sum_{t\in S}\mathbf{A}(s,t)\left[\sum_{u\in F}\mathbf{B}(t,u)\mathbf{x}_F(u)\right] \\
&= \lim_{G\in\mathscr{F}}\left(\sum_{t\in G}\mathbf{A}(s,t)\left[\sum_{u\in F}\mathbf{B}(t,u)\mathbf{x}_F(u)\right]\right) \\
&= \lim_{G\in\mathscr{F}}\left(\sum_{u\in F}\left[\sum_{t\in G}\mathbf{A}(s,t)\mathbf{B}(t,u)\right]\mathbf{x}_F(u)\right) \\
&= \sum_{u\in F}\left(\left[\lim_{G\in\mathscr{F}}\left(\sum_{t\in G}\mathbf{A}(s,t)\mathbf{B}(t,u)\right)\right]\mathbf{x}_F(u)\right) \\
&= \sum_{u\in F}(\mathbf{AB})(s,u)\mathbf{x}_F(u)=([\mathbf{AB}]\mathbf{x}_F)(s)
\end{aligned}
$$

Since $s\in S$ is arbitrary, $T\mathbf{x}_F=(\mathbf{AB})\mathbf{x}_F$ for all $F\in\mathscr{F}$. Let $s_0\in S$ be fixed. We claim that $\lim_{F\in\mathscr{F}}[(\mathbf{AB})\mathbf{x}_F](s_0)=(T\mathbf{x})(s_0)$. Let $\varepsilon>0$. Since T is bounded, and

$\mathbf{x}_F \to \mathbf{x}$, there is an $F_0 \in \mathscr{F}$ such that for all $F \in \mathscr{F}$ satisfying $F \supseteq F_0$ we have $\|(\mathbf{AB})\mathbf{x}_F - T\mathbf{x}\|_{\mathscr{X}} = \|T\mathbf{x}_F - T\mathbf{x}\|_{\mathscr{X}} < \dfrac{\varepsilon}{c+1}$. Thus

$$\begin{aligned}\|[(\mathbf{AB})\mathbf{x}_F](s_0) - (T\mathbf{x})(s_0)\|_{\mathscr{A}} &= \|[(\mathbf{AB})\mathbf{x}_F - T\mathbf{x}](s_0)\|_{\mathscr{A}} \\ &\leqslant c\,\|(\mathbf{AB})\mathbf{x}_F - T\mathbf{x}\|_{\mathscr{X}} < c\left(\frac{\varepsilon}{c+1}\right) < \varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}(T\mathbf{x})(s_0) &= \lim_{F\in\mathscr{F}} [([(\mathbf{AB})\mathbf{x}_F])(s_0)] = \lim_{F\in\mathscr{F}}\left[\sum_{t\in F}(\mathbf{AB})(s_0,t)\mathbf{x}_F(t)\right] \\ &= \lim_{F\in\mathscr{F}}\left[\sum_{t\in F}(\mathbf{AB})(s_0,t)\mathbf{x}(t)\right] = (\mathbf{AB})\mathbf{x}(s_0)\end{aligned}$$

This shows that for each $s_0 \in S$, $[(\mathbf{AB})\mathbf{x}](s_0)$ exists and is equal to $(T\mathbf{x})(s_0)$. Since $T\mathbf{x} \in \mathscr{X}$, $(\mathbf{AB})\mathbf{x} \in \mathscr{X}$. Thus $\mathbf{AB} \in \mathscr{M}(\mathscr{X})$ and $T = T_{(\mathbf{AB})}$ from the arbitrariness of $\mathbf{x} \in \mathscr{X}$. The submultiplicativity of the norm follows from the observation: by definition $\|\mathbf{AB}\| = \|T_{\mathbf{AB}}\| = \|T_{\mathbf{A}}T_{\mathbf{B}}\| \leqslant \|T_{\mathbf{A}}\|\,\|T_{\mathbf{B}}\| = \|\mathbf{A}\|\,\|\mathbf{B}\|$. The proof is complete.

4. The $\ell^p_*(S,\mathscr{A})$ and $\ell^p_{*u}(S,\mathscr{A})$ spaces

We begin this section with some general results that will be used for all spaces to be considered.

PROPOSITION 4.1. *Let $\mathfrak{B}$ be a Banach space of functions from S to $\mathbb{C}$; and let $\eta \in \mathscr{A}^S$. Then $\varphi\circ\eta \in \mathfrak{B}\ \forall\, \varphi \in s(\mathscr{A})$ iff $\sup\limits_{\varphi\in s(\mathscr{A})} \|\varphi\circ\eta\| < \infty$.*

Proof. $[\Leftarrow]$ This is clear by definition.

$[\Rightarrow]$ Fix $\eta \in \mathscr{A}^S$ such that $\varphi\circ\eta \in \mathfrak{B}\ \ \forall\, \varphi \in s(\mathscr{A})$. Let $\mathscr{A}^{\#}$ be the dual space of $\mathscr{A}$ and $f \in \mathscr{A}^{\#}$. By Corollary 4.3.7 of [4] (p. 260), $f = \sum\limits_{\nu=1}^{4} \alpha_\nu \varphi_\nu$, where $\alpha_\nu \in \mathbb{C}$ and $\varphi_\nu \in s(\mathscr{A})$; $\nu = 1,2,3,4$. Since each $\varphi_\nu\circ\eta \in \mathfrak{B}$, $f\circ\eta = \sum\limits_{\nu=1}^{4} \alpha_\nu(\varphi_\nu\circ\eta) \in \mathfrak{B}$. Therefore $T_\eta(f) = f\circ\eta$ maps $\mathscr{A}^{\#}$ to $\mathfrak{B}$. We show that T_η is bounded by showing that the graph of T_η is closed in $\mathscr{A}^{\#}\oplus\mathfrak{B}$. Let $f_n \to f$ in $\mathscr{A}^{\#}$ and $T_\eta(f_n) \to y$ in $\mathfrak{B}$. We must show that $T_\eta(f) = y$. From $f_n \to f$, we get that, for each $s \in S$, $(f_n\circ\eta)(s) \to (f\circ\eta)(s)$ as $n\to\infty$. From $T_\eta(f_n) \to y$, we see that, for each $s \in S$, $(f_n\circ\eta)(s) \to y(s)$ as $n\to\infty$. So $(f\circ\eta)(s) = y(s)$ for all s. Therefore $T_\eta(f) = y$. This shows that the graph of T_η is closed. Since $\mathscr{A}^{\#}$ and $\mathfrak{B}$ are Banach spaces, by closed graph theorem, T_η is a bounded linear mapping from $\mathscr{A}^{\#}$ to $\mathfrak{B}$. Then $\sup\limits_{\varphi\in s(\mathscr{A})}\|\varphi\circ\eta\| = \sup\limits_{\varphi\in s(\mathscr{A})}\|T_\eta(\varphi)\| \leqslant \sup\limits_{\varphi\in s(\mathscr{A})}\|T_\eta\|\,\|\varphi\| = \|T_\eta\| < \infty$.

THEOREM 4.2. *Let X be a Banach space, and let $\|\cdot\| : X^S \to [0,\infty]$ be a function satisfying the following conditions:*

1. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \ \forall\, \mathbf{x} \in X^S$;
2. $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\| \ \ \forall\, \mathbf{x},\mathbf{y} \in X^S$;
3. $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\| \ (0\cdot\infty = 0) \ \forall\, \mathbf{x} \in X^S \ \forall\, \alpha \in \mathbb{C}$;
4. $\exists\, c > 0$ *such that* $\|\mathbf{x}(s)\|_X \leqslant c\,\|\mathbf{x}\| \ \ \forall\, \mathbf{x} \in X^S \ \forall\, s \in S$;
5. $\exists\, \xi > 0$ *such that* $\|\mathbf{e}_s(a)\| \leqslant \xi\,\|a\|_X \ \ \forall\, a \in X \ \forall\, s \in S$;
6. $\sup\limits_{F\in\mathscr{F}} \|\mathbf{x}_F\| = \|\mathbf{x}\| \ \ \forall\, \mathbf{x} \in X^S$.

Then $Y = \left\{\mathbf{x} \in X^S : \|\mathbf{x}\| < \infty\right\}$ is a Banach space with norm $\|\cdot\|$.

Proof. Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in Y. Then for each $s \in S$,

$$\|\mathbf{x}_n(s) - \mathbf{x}_m(s)\|_X = \|(\mathbf{x}_n - \mathbf{x}_m)(s)\|_X \leqslant c\,\|\mathbf{x}_n - \mathbf{x}_m\| \to 0 \ \text{ as } \ n,m \to \infty.$$

Hence $\{\mathbf{x}_n(s)\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists $\mathbf{x}(s) \in X$ such that $\mathbf{x}_n(s) \to \mathbf{x}(s)$ as $n \to \infty$. For a fixed $F \in \mathscr{F}$,

$$\begin{aligned}\|(\mathbf{x}_n)_F - \mathbf{x}_F\| = \|(\mathbf{x}_n - \mathbf{x})_F\| &= \left\|\sum_{s\in F} \mathbf{e}_s\left((\mathbf{x}_n - \mathbf{x})(s)\right)\right\| \\ &= \left\|\sum_{s\in F} \mathbf{e}_s\left(\mathbf{x}_n(s) - \mathbf{x}(s)\right)\right\| \leqslant \xi \sum_{s\in F} \|\mathbf{x}_n(s) - \mathbf{x}(s)\| \to 0\end{aligned}$$

as $n \to \infty$. Let $\varepsilon > 0$. Since $\{\mathbf{x}_n\}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for all $n,m \geqslant N$, $\|\mathbf{x}_n - \mathbf{x}_m\| < \frac{\varepsilon}{2}$. Then, for $n,m \geqslant N$, $F \in \mathscr{F}$

$$\|(\mathbf{x}_n)_F - (\mathbf{x}_m)_F\| = \|(\mathbf{x}_n - \mathbf{x}_m)_F\| \leqslant \|\mathbf{x}_n - \mathbf{x}_m\| < \frac{\varepsilon}{2}.$$

Holding n and F fixed and letting $m \to \infty$ in the left most and right most expressions above, we obtain

$$\|(\mathbf{x}_n - \mathbf{x})_F\| = \|(\mathbf{x}_n)_F - \mathbf{x}_F\| \leqslant \frac{\varepsilon}{2} \qquad \forall\ n \geqslant N \ \ \forall\ F \in \mathscr{F}.$$

Thus $\|\mathbf{x}_n - \mathbf{x}\| = \sup\limits_{F\in\mathscr{F}} \|(\mathbf{x}_n - \mathbf{x})_F\| = \sup\limits_{F\in\mathscr{F}} \|(\mathbf{x}_n)_F - \mathbf{x}_F\| \leqslant \dfrac{\varepsilon}{2} \ \forall\, n \geqslant N$. So $\mathbf{x}_N - \mathbf{x} \in Y$ and hence $\mathbf{x} = \mathbf{x}_N - (\mathbf{x}_N - \mathbf{x}) \in Y$. In addition, we also have $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$.

For $\mathbf{a} \in \mathscr{A}^S$, define $\mathbf{a}^*$ by $\mathbf{a}^*(s) = [\mathbf{a}(s)]^* \ \forall\ s \in S$. For $\mathbf{x},\mathbf{y} \in \mathscr{A}^S$, define $\mathbf{x}\mathbf{y} \in \mathscr{A}^S$ by $(\mathbf{x}\mathbf{y})(s) = \mathbf{x}(s)\mathbf{y}(s)\ \forall\, s \in S$. For $\varphi \in s(\mathscr{A})$, $\mathbf{x} \in \mathscr{A}^S$ define $\widetilde{\varphi}(\mathbf{x}) \in \mathbb{C}^S$ by $\left(\widetilde{\varphi}(\mathbf{x})\right)(s) = \varphi(\mathbf{x}(s))$ for $s \in S$.

THEOREM 4.3. *Let* $\mathbf{y} \in \mathscr{A}^S$, *and* $p \in [2,\infty)$.

(a) The function $\|\cdot\| : \mathscr{A}^S \to [0,\infty]$ *defined, for each* $\mathbf{y} \in \mathscr{A}^S$, *by*

$$\|\mathbf{y}\| = \sup_{\varphi \in s(\mathscr{A})} \left(\sum_{s \in S} [\varphi(\mathbf{y}(s)^* \mathbf{y}(s))]^{\frac{p}{2}} \right)^{\frac{1}{p}} = \sup_{\varphi \in s(\mathscr{A})} \left[\sum_{s \in S} \|\mathbf{y}(s)\|_\varphi^p \right]^{\frac{1}{p}}$$

satisfies all six conditions in Theorem 4.2 *(with* $c = \xi = 1$*).*

(b) The condition $\sum_{s \in S} [\varphi(\mathbf{y}(s)^* \mathbf{y}(s))]^{\frac{p}{2}} < \infty \quad \forall\ \varphi \in s(\mathscr{A})$ *is necessary and sufficient for* $\sup_{\varphi \in s(\mathscr{A})} \sum_{s \in S} [\varphi(\mathbf{y}(s)^* \mathbf{y}(s))]^{\frac{p}{2}} < \infty$.

Proof. Part (a) has six subparts. Since they all have straightforward verifications, we prove only condition (6). Let $\mathbf{x} \in \mathscr{A}^S$. If $\|\mathbf{x}\| = \infty$, for each $M > 0$, there is a $\varphi \in s(\mathscr{A})$ such that $\sum_{s \in S} \|\mathbf{x}(s)\|_\varphi^p > M^p$. Thus there is a $G \in \mathscr{F}$ such that $\sum_{s \in G} \|\mathbf{x}(s)\|_\varphi^p > M^p$. That is $\|\mathbf{x}_G\|^p > M^p$. Therefore $\sup_{F \in \mathscr{F}} \|\mathbf{x}_F\| = \infty$. It is easy to check that $\|\mathbf{x}_F\| \leqslant \|\mathbf{x}\|$ for each $F \in \mathscr{F}$. Thus $\sup_{F \in \mathscr{F}} \|\mathbf{x}_F\| \leqslant \|\mathbf{x}\|$.

If $\|\mathbf{x}\| < \infty$, then for each $\varepsilon > 0$, there is a $\varphi \in s(\mathscr{A})$ such that

$$\sum_{s \in S} \|\mathbf{x}(s)\|_\varphi^p > \|\mathbf{x}\|^p - \frac{\varepsilon}{2}.$$

Thus there is a $G \in \mathscr{F}$ such that $\sum_{s \in G} \|\mathbf{x}(s)\|_\varphi^p > \|\mathbf{x}\|^p - \varepsilon$. That is $\|\mathbf{x}_G\|^p > \|\mathbf{x}\|^p - \varepsilon$. Since ε is arbitrary, $\sup_{G \in \mathscr{F}} \|\mathbf{x}_G\| \geqslant \|\mathbf{x}\|$.

(b) $[\Rightarrow]$ Fix $\mathbf{x} \in \mathscr{A}^S$. Define $\eta_{\mathbf{x}}(s) = \mathbf{x}(s)^* \mathbf{x}(s)$, $s \in S$. Then $\eta_{\mathbf{x}}$ maps S to $\mathscr{A}$ and $(\varphi \circ \eta_{\mathbf{x}})(s) = \varphi(\mathbf{x}(s)^* \mathbf{x}(s)) \quad \forall\ \varphi \in s(\mathscr{A})$. By assumption $\varphi \circ \eta_{\mathbf{x}} \in \ell^{\frac{p}{2}}(S, \mathbb{C}) \quad \forall\ \varphi \in s(\mathscr{A})$. Then by Theorem 4.1,

$$\sup_{\varphi \in s(\mathscr{A})} \left(\sum_{s \in S} [\varphi(\mathbf{x}(s)^* \mathbf{x}(s))]^{\frac{p}{2}} \right)^{\frac{2}{p}} = \sup_{\varphi \in s(\mathscr{A})} \|\varphi \circ \eta_{\mathbf{x}}\| < \infty.$$

Hence $\sup_{\varphi \in s(\mathscr{A})} \left(\sum_{s \in S} [\varphi(\mathbf{x}(s)^* \mathbf{x}(s))]^{\frac{p}{2}} \right)^{\frac{1}{p}} < \infty$. The converse is obvious.

With $\mathscr{A}$ and $s(\mathscr{A})$ as above, for $p \in [2,\infty)$, we consider the following spaces:

$$\ell_*^p(S, \mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sum_{s \in S} [\varphi(\mathbf{x}(s)^* \mathbf{x}(s))]^{\frac{p}{2}} < \infty \ \forall\ \varphi \in s(\mathscr{A}) \right\};$$

and

$$\ell^p_{*u}(S,\mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sum_{s\in S} \left[\varphi\left(\mathbf{x}(s)^*\mathbf{x}(s)\right)\right]^{\frac{p}{2}} \right.$$

$$\left. \text{converges uniformly for } \varphi \in s(\mathscr{A}) \right\}.$$

It is clear from the definitions that $\ell^p_{*u}(S,\mathscr{A}) \subseteq \ell^p_*(S,\mathscr{A})$, but the following example shows that the converse is not true in general.

EXAMPLE 4.4. *For $p \in [2,\infty)$, $\ell^p_{*u}(S,\mathscr{A}) \subsetneq \ell^p_*(S,\mathscr{A})$ for any C^*-algebra $\mathscr{A}$ (of operators) with infinite family of operators having mutually orthogonal ranges and $S = \mathbb{N}$.*

Proof. Let $\{x_n^*\}$ be a family of norm 1 contractions in $\mathscr{A}$ with mutually orthogonal ranges. Then the ranges of $x_n^*x_n = y_ny_n^*$ are orthogonal, and hence $\left\|\sum_{n=1}^k x_n^*x_n\right\| = \max_{1\leqslant n\leqslant k} \|x_n^*x_n\|$. Since $\|x_n\|_\varphi \leqslant \|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$, for each $\varphi \in s(\mathscr{A})$,

$$\sum_{n=1}^k \|\mathbf{x}_n\|_\varphi^p \leqslant \sum_{n=1}^k \|x_n\|_\varphi^2 = \sum_{n=1}^k \varphi(x_n^*x_n) = \varphi\left(\sum_{n=1}^k x_n^*x_n\right)$$
$$\leqslant \left\|\sum_{n=1}^k x_n^*x_n\right\| = \max_{1\leqslant n\leqslant k} \|x_n\|^2 \leqslant 1.$$

So $\mathbf{x}(n) = x_n$, $n \in \mathbb{N}$, defines an element of $\ell^p_*(\mathbb{N},\mathscr{A})$. On the other hand, for each $n \in \mathbb{N}$, there exists $\varphi_n \in s(\mathscr{A})$ such that $\varphi_n(x_n^*x_n) = \|x_n\|^2 = 1$ ([4] Theorem 4.3.4, p, 258). Thus

$$\sum_{j=n}^\infty \left[\varphi_n\left(x_j^*x_j\right)\right]^{\frac{p}{2}} \geqslant \left[\varphi_n\left(x_n^*x_n\right)\right]^{\frac{p}{2}} = 1 > \frac{1}{2}.$$

That is $\mathbf{x} \notin \ell^p_{*u}(\mathbb{N},\mathscr{A})$ (the convergence is not uniform over $s(\mathscr{A})$).

An interesting question arises naturally here. Is there an infinite dimensional C^*-algebra $\mathscr{A}$ such that these two spaces are equal?

THEOREM 4.5. *The function $\|\cdot\|$ defined in Theorem 4.3 is a norm on the space $\ell^p_*(S,\mathscr{A})$ $\left[\text{resp. } \ell^p_{*u}(S,\mathscr{A})\right]$, and $\ell^p_*(S,\mathscr{A})$ $\left[\text{resp. } \ell^p_{*u}(S,\mathscr{A})\right]$ is a Banach algebra under this norm and the point-wise product.*

Proof. That $\|\cdot\|$ is indeed a norm follows directly from Theorem 4.3. From the definition, we see that $\ell^p_*(S,\mathscr{A}) = \left\{\mathbf{y} \in \mathscr{A}^S : \|\mathbf{y}\| < \infty\right\}$, and hence it is a Banach space by Theorem 4.2. To see that $\ell_*^p(S,\mathscr{A})$ is closed under point-wise multiplication and the submultiplicativity of $\|\cdot\|$, let $\mathbf{x}$, $\mathbf{y} \in \ell_*^p(S,\mathscr{A})$ and $\varphi \in s(\mathscr{A})$. We have

$$\left(\sum_{s\in S} \|\mathbf{x}(s)\mathbf{y}(s)\|_\varphi^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{s\in S} \|\mathbf{x}(s)\|^p \|\mathbf{y}(s)\|_\varphi^p\right)^{\frac{1}{p}}$$
$$\leqslant \|\mathbf{x}\| \left(\sum_{s\in S} \|\mathbf{y}(s)\|_\varphi^p\right)^{\frac{1}{p}} \leqslant \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Thus $\mathbf{xy} \in \ell_*^p(S,\mathscr{A})$. Since $\varphi \in s(\mathscr{A})$ was arbitrary, $\|\mathbf{xy}\| \leqslant \|\mathbf{x}\|\,\|\mathbf{y}\|$.

For the space $\ell_{*u}^p(S,\mathscr{A})$, it suffices to show that $\ell_{*u}^p(S,\mathscr{A})$ is a closed subspace of $\ell_*^p(S,\mathscr{A})$ for its completeness, and that it is closed under point-wise product. For the former, let $\{\mathbf{x}_n\}$ be a sequence in $\ell_{*u}^p(S,\mathscr{A})$ such that $\mathbf{x}_n \to \mathbf{x}$ in $\ell_*^p(S,\mathscr{A})$. Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that $\|\mathbf{x}_n - \mathbf{x}\| < \frac{\varepsilon}{2}$ for all $n \geqslant N$. Since $\mathbf{x}_N \in \ell_{*u}^p(S,\mathscr{A})$, there exists $F_0 \in \mathscr{F}$ such that for all $G \in \mathscr{F}$ with $G \subseteq S \setminus F_0$ and for all $\varphi \in s(\mathscr{A})$, $\sum\limits_{s\in G} \|\mathbf{x}_N(s)\|_\varphi^p < \left(\frac{\varepsilon}{2}\right)^p$. Let $H \in \mathscr{F}$ be such that $H \subseteq S \setminus F_0$ and let $\varphi \in s(\mathscr{A})$. We obtain

$$\left(\sum_{s\in H} \|\mathbf{x}(s)\|_\varphi^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{s\in H} \left[\|\mathbf{x}(s) - \mathbf{x}_N(s)\|_\varphi + \|\mathbf{x}_N(s)\|_\varphi\right]^p\right)^{\frac{1}{p}}$$
$$\leqslant \left(\sum_{s\in H} \|\mathbf{x}(s) - \mathbf{x}_N(s)\|_\varphi^p\right)^{\frac{1}{p}} + \left(\sum_{s\in H} \|\mathbf{x}_N(s)\|_\varphi^p\right)^{\frac{1}{p}}$$
$$\leqslant \|\mathbf{x} - \mathbf{x}_N\| + \frac{\varepsilon}{2} \quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad = \varepsilon,$$

which implies that $\mathbf{x} \in \ell_{*u}^p(S,\mathscr{A})$. For the closure of $\ell_{*u}^p(S,\mathscr{A})$ under point-wise product, we show that $\ell_{*u}^p(S,\mathscr{A})$ is in fact a left ideal in $\ell_*^p(S,\mathscr{A})$. Let $\mathbf{x} \in \ell_*^p(S,\mathscr{A})$, $\mathbf{y} \in \ell_{*u}^p(S,\mathscr{A})$; and let $\varepsilon > 0$. There is an $F_0 \in \mathscr{F}$ such that for all $G \in \mathscr{F}$ with $G \subseteq S \setminus F_0$ and all $\varphi \in s(\mathscr{A})$, $\sum\limits_{s\in G} \|\mathbf{y}(s)\|_\varphi^p < \left[\dfrac{\varepsilon}{\|\mathbf{x}\|+1}\right]^p$. Thus for such a G, we also have

$$\sum_{s\in G} \|\mathbf{x}(s)\mathbf{y}(s)\|_\varphi^p \leqslant \sum_{s\in G} \|\mathbf{x}(s)\|^p \|\mathbf{y}(s)\|_\varphi^p \leqslant \|\mathbf{x}\|^p \sum_{s\in G} \|\mathbf{y}(s)\|_\varphi^p$$
$$\leqslant \|\mathbf{x}\|^p \left[\frac{\varepsilon}{\|\mathbf{x}\|+1}\right]^p < \varepsilon^p.$$

Thus $\mathbf{xy} \in \ell_{*u}^p(S,\mathscr{A})$. Since the norm on $\ell_{*u}^p(S,\mathscr{A})$ is the same as that of $\ell_*^p(S,\mathscr{A})$, and submultiplicativity is already proved above, the proof is thus complete.

Recall that, for $\mathbf{x} \in \mathscr{A}^S$, and $F \in \mathscr{F}$ $\quad \mathbf{x}_F(s) = \mathbf{x}(s)$ if $s \in F$ and $\mathbf{x}(s) = 0$ otherwise.

PROPOSITION 4.6. *For each* $\mathbf{x} \in \ell_{*u}^p(S,\mathscr{A})$, $\|\mathbf{x} - \mathbf{x}_F\| \to 0$. *This is false in* $\ell_*^p(\mathbb{N},\mathscr{A})$ *for* $\mathscr{A}$ *as in Example* 4.4.

Proof. Let $\varepsilon > 0$ be given. Since $\mathbf{x} \in \ell^p_{*u}(S,\mathscr{A})$, there exists $F_0 \in \mathscr{F}$ such that for all $G \in \mathscr{F}$ with $G \subseteq S \setminus F_0$ $\forall \varphi \in s(\mathscr{A})$, $\sum_{s\in G} \|\mathbf{x}(s)\|^p_\varphi < \left(\frac{\varepsilon}{2}\right)^p$. Hence $\sum_{s\in S\setminus F_0} \|\mathbf{x}(s)\|^p_\varphi < \left(\frac{\varepsilon}{2}\right)^p$, i.e., $\left\|\mathbf{x} - \mathbf{x}_{F_0}\right\|^p \leqslant \left(\frac{\varepsilon}{2}\right)^p$. Thus for all $F \in \mathscr{F}$ with $F \supseteq F_0$,

$$\|\mathbf{x} - \mathbf{x}_F\|^p \leqslant \left\|\mathbf{x} - \mathbf{x}_{F_0}\right\|^p \leqslant \left(\frac{\varepsilon}{2}\right)^p.$$

This means $\mathbf{x}_F \to \mathbf{x}$.

Let x_n be as defined in Example 4.4, and $\mathbf{x}(n) = x_n$ $(n \in \mathbb{N})$. Then $\mathbf{x} \in \ell^p_*(\mathbb{N},\mathscr{A})$. Let $F \in \mathscr{F}(\mathbb{N})$. Choose $m \in \mathbb{N} \setminus F$ and $\varphi \in s(\mathscr{A})$ such that $\varphi(x_m^* x_m) = 1$. Then $\|\mathbf{x} - \mathbf{x}_F\| \geqslant \|\mathbf{x}(m)\|_\varphi = 1$.

COROLLARY 4.7. *The set $\mathscr{M}\left(\ell^p_*(S,\mathscr{A})\right)$ $\left[\text{resp. } \mathscr{M}\left(\ell^p_{*u}(S,\mathscr{A})\right)\right]$, of all $\mathbf{A}$ in the space of functions from $S \times S$ to $\mathscr{A}$ such that $\mathbf{A}$ defines a bounded linear operator on $\ell^p_*(S,\mathscr{A})$ $\left[\text{resp. } \ell^p_{*u}(S,\mathscr{A})\right]$, with the operator norm is a Banach space [resp. Banach algebra].*

Proof. This follows directly from Theorem 3.4 and the preceding proposition.

The following example shows that for $p \in [2,\infty)$, $\ell^p_*(S,\mathscr{A})$ and $\ell^p_{*u}(S,\mathscr{A})$ may not be "self-adjoint" in the sense that if $\mathbf{x} \in \ell^p_*(S,\mathscr{A})$ or $\mathbf{x} \in \ell^p_{*u}(S,\mathscr{A})$ then $\mathbf{x}^*$ may not be in the space.

EXAMPLE 4.8. *Let $\mathscr{A}$ be a von Neumann algebra that contains an infinite family of partial isometries with the same kernel and mutually orthogonal ranges. Then there exists $\mathbf{x} = \{x_k\} \in \ell^p_{*u}(\mathbb{N},\mathscr{A}) \subseteq \ell^p_*(\mathbb{N},\mathscr{A})$ but $\mathbf{x}^* = \left\{x_k^*\right\} \notin \ell^p_*(\mathbb{N},\mathscr{A})$, and hence $\mathbf{x}^* = \left\{x_k^*\right\} \notin \ell^p_{*u}(\mathbb{N},\mathscr{A})$.*

Proof. Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence of partial isometries in $\mathscr{A}$ with the same kernel and mutually orthogonal ranges, and $\mathbf{x}(k) = k^{-1/p} y_k^*$ for $k \in \mathbb{N}$. Then $\mathbf{x}(k)^*\mathbf{x}(k) = k^{-2/p} y_k y_k^*$ have mutually orthogonal ranges. Let $\varepsilon > 0$. Choose N such that $N^{-2/p} < \varepsilon$. Then for each $\varphi \in s(\mathscr{A})$, since $x_k^* x_k \leqslant I$ for all $k \in \mathbb{N}$, we have $\varphi(x_k^* x_k) \leqslant \varphi(I) = 1$. Since $p \geqslant 2$, for each $q \geqslant N$,

$$\begin{aligned}\sum_{k=N}^{q} \left[\varphi\left(x_k^* x_k\right)\right]^{\frac{p}{2}} &\leqslant \sum_{k=N}^{q} \varphi\left(x_k^* x_k\right) = \varphi\left(\sum_{k=N}^{q} x_k^* x_k\right) \\ &\leqslant \left\|\sum_{k=N}^{q} x_k^* x_k\right\| = \max_{N\leqslant k\leqslant q} \|x_k^* x_k\| = \frac{1}{N^{2/p}} < \varepsilon.\end{aligned}$$

Hence $\mathbf{x} = \{x_k\}$ is in $\ell^p_{*u}(\mathbb{N},\mathscr{A})$ [and hence in $\ell^p_*(\mathbb{N},\mathscr{A})$].

To see that $\mathbf{x}^* = \left\{x_k^*\right\}$ is not in $\ell^p_*(\mathbb{N},\mathscr{A})$, let $\psi \in s(\mathscr{A})$ be the vector state defined by a unit vector in the orthogonal complement of the common kernel of y_k, $k \in \mathbb{N}$. Then $x_k x_k^* = k^{-2/p} y_k^* y_k$ is a multiple of the projection onto the orthogonal complement of the common kernel, and hence $\psi(x_k x_k^*) = \dfrac{1}{k^{2/p}}$ for all $k \in \mathbb{N}$. Thus $\sum_{k=1}^{\infty} [\psi(x_k x_k^*)]^{p/2} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$, i.e., $\mathbf{x}^* = \{x_k^*\}$ is not in $\ell_*^p(\mathbb{N},\mathscr{A})$ and hence it is not in $\ell_{*u}^p(\mathscr{A})$.

5. Alternate descriptions of $\ell_*^2(S,\mathscr{A})$ and $\ell_{*u}^p(S,\mathscr{A})$

In this section, we present alternate descriptions of $\ell_*^2(S,\mathscr{A})$ and $\ell_{*u}^p(S,\mathscr{A})$. Define

$$\widetilde{\ell_*^2}(S,\mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sup_{F \in \mathscr{F}} \left\| \sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right\|_{\mathscr{A}} < \infty \right\};$$

$$\|\mathbf{x}\|_{\widetilde{\ell_*^2}} = \left[\sup_{F \in \mathscr{F}} \left\| \sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right\|_{\mathscr{A}} \right]^{\frac{1}{2}} \qquad \mathbf{x} \in \widetilde{\ell_*^2}(S,\mathscr{A}).$$

PROPOSITION 5.1. $\ell_*^2(S,\mathscr{A}) = \widetilde{\ell_*^2}(S,\mathscr{A})$.

Proof. $(\subseteq)$ Suppose that $\mathbf{x} \in \ell_*^2(S,\mathscr{A})$. Let $F \in \mathscr{F}$. Then

$$\begin{aligned} \left\| \sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right\|_{\mathscr{A}} &= \sup_{\varphi \in s(\mathscr{A})} \left[\varphi \left(\sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right) \right] \\ &= \sup_{\varphi \in s(\mathscr{A})} \left[\sum_{s \in F} \varphi\left(\mathbf{x}(s)^* \mathbf{x}(s)\right) \right] \leqslant \|\mathbf{x}\|^2_{\ell_*^2(S,\mathscr{A})} < \infty. \end{aligned}$$

So $\sup_{F \in \mathscr{F}} \left\| \sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right\|_{\mathscr{A}} < \infty$ and hence $\mathbf{x} \in \widetilde{\ell_*^2}(S,\mathscr{A})$. Therefore $\ell_*^2(S,\mathscr{A}) \subseteq \widetilde{\ell_*^2}(S,\mathscr{A})$.

$(\supseteq)$ Suppose that $\widetilde{\mathbf{x}} \in \widetilde{\ell_*^2}(S,\mathscr{A})$. Let $\varphi \in s(\mathscr{A})$. Then

$$\sum_{s \in F} \varphi\left(\widetilde{\mathbf{x}}(s)^* \widetilde{\mathbf{x}}(s)\right) = \varphi\left(\sum_{s \in F} \widetilde{\mathbf{x}}(s)^* \widetilde{\mathbf{x}}(s) \right) \leqslant \left\| \sum_{s \in F} \widetilde{\mathbf{x}}(s)^* \widetilde{\mathbf{x}}(s) \right\|_{\mathscr{A}} \leqslant \|\widetilde{\mathbf{x}}\|^2_{\widetilde{\ell_*^2}}$$

for all $F \in \mathscr{F}$. So $\sum_{s \in S} \varphi\left(\widetilde{\mathbf{x}}(s)^* \widetilde{\mathbf{x}}(s)\right) \leqslant \|\widetilde{\mathbf{x}}\|^2_{\widetilde{\ell_*^2}}$ $\forall\ \varphi \in s(\mathscr{A})$ and hence $\widetilde{\mathbf{x}} \in \ell_*^2(S,\mathscr{A})$. Therefore $\widetilde{\ell_*^2}(S,\mathscr{A}) \subseteq \ell_*^2(S,\mathscr{A})$.

Note that, from the above argument, we also have

$$\sup_{F \in \mathscr{F}} \left\| \sum_{s \in F} \mathbf{x}(s)^* \mathbf{x}(s) \right\|_{\mathscr{A}} = \|\mathbf{x}\|^2_{\widetilde{\ell_*^2}} = \|\mathbf{x}\|^2_{\ell_*^2} = \sup_{\varphi \in s(\mathscr{A})} \left[\sum_{s \in S} \varphi\left(\mathbf{x}(s)^* \mathbf{x}(s)\right) \right].$$

PROPOSITION 5.2. *For* $p \in [2,\infty)$, *let* $\ell^p_{*c}(S,\mathscr{A})$ *denote the space of functions* $\mathbf{x} \in \mathscr{A}^S$ *such that the map* $\varphi \mapsto \widetilde{\varphi}(\mathbf{x}^*\mathbf{x})$ *is weak* to norm continuous from* $s(\mathscr{A})$ *to* $\ell^{\frac{p}{2}}(S,\mathbb{C})$. *Then* $\ell^p_{*c}(S,\mathscr{A}) = \ell^p_{*u}(S,\mathscr{A})$

Proof. $(\subseteq)$ Suppose that $\mathbf{x} \in \ell^p_{*c}(S,\mathscr{A})$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon^{2/p}$. Then for each $\varphi \in s(\mathscr{A})$ there is a weak* neighborhood, $\mathscr{U}_\varphi$, of φ such that for all $\psi \in \mathscr{U}_\varphi$,

$$\|\widetilde{\varphi}(\mathbf{x}^*\mathbf{x}) - \widetilde{\psi}(\mathbf{x}^*\mathbf{x})\| < \frac{\delta}{2}.$$

Since $\bigcup_{\varphi\in s(\mathscr{A})} \mathscr{U}_\varphi \supseteq s(\mathscr{A})$, and $s(\mathscr{A})$ is weak* compact, there are $\varphi_1,\ldots,\varphi_n \in s(\mathscr{A})$ such that $\mathscr{U}_{\varphi_1} \bigcup \mathscr{U}_{\varphi_2} \bigcup \ldots \bigcup \mathscr{U}_{\varphi_n} \supseteq s(\mathscr{A})$. Since each $\widetilde{\varphi}_j(\mathbf{x}^*\mathbf{x}) \in \ell^{\frac{p}{2}}(S,\mathbb{C})$, there is an $F_j \in \mathscr{F}$ such that $\sum_{s\in S\setminus F_j} [\varphi_j(\mathbf{x}(s)^*\mathbf{x}(s))]^{\frac{p}{2}} < \left(\frac{\delta}{2}\right)^{\frac{p}{2}}$. Put $F_0 = \bigcup_{j=1}^n F_j$. Let $\psi \in s(\mathscr{A})$. Then $\psi \in \mathscr{U}_{\varphi_j}$ for some $j = 1,2,\ldots,n$ and hence

$$\begin{aligned}
\left(\sum_{s\in S\setminus F_0} [\psi(\mathbf{x}(s)^*\mathbf{x}(s))]^{\frac{p}{2}}\right)^{\frac{2}{p}} &= \left\|[\widetilde{\psi}(\mathbf{x}^*\mathbf{x})]_{S\setminus F_0}\right\|_{\ell^{p/2}(S,\mathbb{C})} \\
&\leqslant \left\|[\widetilde{\psi}(\mathbf{x}^*\mathbf{x}) - \widetilde{\varphi}_j(\mathbf{x}^*\mathbf{x})]_{S\setminus F_0}\right\|_{\ell^{p/2}(S,\mathbb{C})} + \left\|[\widetilde{\varphi}_j(\mathbf{x}^*\mathbf{x})]_{S\setminus F_0}\right\|_{\ell^{p/2}(S,\mathbb{C})} \\
&\leqslant \left\|\widetilde{\psi}(\mathbf{x}^*\mathbf{x}) - \widetilde{\varphi}_j(\mathbf{x}^*\mathbf{x})\right\|_{\ell^{p/2}(S,\mathbb{C})} + \left\|[\widetilde{\varphi}_j(\mathbf{x}^*\mathbf{x})]_{S\setminus F_j}\right\|_{\ell^{p/2}(S,\mathbb{C})} \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Thus $\sum_{s\in S\setminus F_0} [\psi(\mathbf{x}(s)^*\mathbf{x}(s))]^{\frac{p}{2}} < \delta^{\frac{p}{2}} = \varepsilon$. This implies that $\mathbf{x} \in \ell^p_{*u}(S,\mathscr{A})$.

$(\supseteq)$ Suppose that $\mathbf{x} \in \ell^p_{*u}(S,\mathscr{A})$. Let $\varepsilon > 0$ be given. Then there exists $F_0 \in \mathscr{F}$ such that for all $G \subseteq S \setminus F_0$, $\sum_{s\in G} \|\mathbf{x}(s)\|^p_\varphi < \left(\frac{\varepsilon}{8}\right)^{\frac{p}{2}}$ $\forall\ \varphi \in s(\mathscr{A})$. Fix $\varphi \in s(\mathscr{A})$. Let $n = \operatorname{card} F_0$. Then the set

$$\mathscr{U} = \left\{\psi \in s(\mathscr{A}) : |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}} < \frac{1}{n^p}\left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}},\ s \in F_0\right\}$$

is an open set in the weak* topology. Let $F \in \mathscr{F}$. For $\psi \in \mathscr{U}$, we have

$$\begin{aligned}
&\sum_{s\in F} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}} \\
&= \sum_{s\in F\cap F_0} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}} \\
&\qquad + \sum_{s\in F\setminus F_0} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}}
\end{aligned}$$

$$\leqslant \sum_{s\in F_0} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}} + \sum_{s\in F\setminus F_0} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}}$$

$$< \left[n\cdot\frac{1}{n^p}\left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}}\right] + \left\{\left(\sum_{s\in F\setminus F_0}\|\mathbf{x}(s)\|_\psi^p\right)^{\frac{2}{p}} + \left(\sum_{s\in F\setminus F_0}\|\mathbf{x}(s)\|_\varphi^p\right)^{\frac{2}{p}}\right\}^{\frac{p}{2}}$$

$$< \left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}} + \left[\frac{\varepsilon}{8}+\frac{\varepsilon}{8}\right]^{\frac{p}{2}} = \left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}} + \left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}} = 2\left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}}.$$

Since this is true for any $F \in \mathscr{F}$, we have

$$\sum_{s\in S} |\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}} \leqslant 2\left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}}.$$

Thus

$$\|\widetilde{\psi}(\mathbf{x}^*\mathbf{x}) - \widetilde{\varphi}(\mathbf{x}^*\mathbf{x})\| = \left[\sum_{s\in S}|\psi(\mathbf{x}(s)^*\mathbf{x}(s)) - \varphi(\mathbf{x}(s)^*\mathbf{x}(s))|^{\frac{p}{2}}\right]^{\frac{2}{p}}$$

$$\leqslant \left[2\left(\frac{\varepsilon}{4}\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} = 2^{\frac{2}{p}}\cdot\frac{\varepsilon}{4} \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

This proves the continuity of the map $\varphi \mapsto \widetilde{\varphi}(\mathbf{x}^*\mathbf{x})$. Therefore for $p \in [2,\infty)$, $\ell^p_{*c}(S,\mathscr{A}) = \ell^p_{*u}(S,\mathscr{A})$.

In fact, for $p = 2$, these are also equal to

$$\widetilde{\ell^2_{*u}}(S,\mathscr{A}) = \left\{\mathbf{x}\in\mathscr{A}^S : \sum_{s\in S}\mathbf{x}(s)^*\mathbf{x}(s) \text{ converges in } \mathscr{A}\right\}.$$

To see this, we will show that $\ell^2_{*u}(S,\mathscr{A}) = \widetilde{\ell^2_{*u}}(S,\mathscr{A})$. To that end, suppose that $\mathbf{x} \in \ell^2_{*u}(S,\mathscr{A})$. Let $\varepsilon > 0$. There exists $F_0 \in \mathscr{F}$ such that for all $G \in \mathscr{F}$ with $G \subseteq S\setminus F_0$ and for all $\varphi \in s(\mathscr{A})$ $\varphi\left(\sum_{s\in G}\mathbf{x}(s)^*\mathbf{x}(s)\right) = \sum_{s\in G}[\varphi(\mathbf{x}(s)^*\mathbf{x}(s))] < \frac{\varepsilon}{2}$. Thus,

$$\left\|\sum_{s\in G}\mathbf{x}(s)^*\mathbf{x}(s)\right\|_{\mathscr{A}} = \sup_{\varphi\in s(\mathscr{A})}\varphi\left(\sum_{s\in G}\mathbf{x}(s)^*\mathbf{x}(s)\right) \leqslant \frac{\varepsilon}{2} < \varepsilon,$$

the Cauchy criterion is satisfied, and hence $\sum_{s\in S}\mathbf{x}(s)^*\mathbf{x}(s)$ converges in $\mathscr{A}$. That is $\ell^2_{*u}(S,\mathscr{A}) \subseteq \widetilde{\ell^2_{*u}}(S,\mathscr{A})$.

On the other hand, suppose that $\mathbf{x} \in \widetilde{\ell^2_{*u}}(S,\mathscr{A})$. Let $\varepsilon > 0$. Then there is $F_0 \in \mathscr{F}$ such that for all $G \in \mathscr{F}$ with $G \subseteq S \setminus F_0$, $\left\| \sum_{s\in G} \mathbf{x}(s)^*\mathbf{x}(s) \right\|_{\mathscr{A}} < \varepsilon$. Thus for each $\varphi \in s(\mathscr{A})$,

$$\sum_{s\in G} \varphi\left(\mathbf{x}(s)^*\mathbf{x}(s)\right) = \varphi\left(\sum_{s\in G} \mathbf{x}(s)^*\mathbf{x}(s)\right) \leqslant \sup_{\varphi\in s(\mathscr{A})} \varphi\left(\sum_{s\in G} \mathbf{x}(s)^*\mathbf{x}(s)\right) = \left\| \sum_{s\in G} \mathbf{x}(s)^*\mathbf{x}(s) \right\|_{\mathscr{A}} < \varepsilon.$$

Therefore $\mathbf{x} \in \ell^2_{*u}(S,\mathscr{A})$ and hence $\widetilde{\ell^2_{*u}}(S,\mathscr{A}) \subseteq \ell^p_{*u}(S,\mathscr{A})$. Thus $\ell^2_{*u}(S,\mathscr{A}) = \widetilde{\ell^2_{*u}}(S,\mathscr{A})$.

Since $\ell^2_*(S,\mathbb{C}) = \ell^2_{*u}(S,\mathbb{C}) = \ell^2(S,\mathbb{C})$, the following are analogues of the duality relationship: for each $x \in \mathbb{C}^S$, $x \in \ell^2(S,\mathbb{C})$ iff $xy \in \ell^1(S,\mathbb{C})$ for all $y \in \ell^2(S,\mathbb{C})$.

THEOREM 5.3. *Let* $\mathbf{a} \in \mathscr{A}^S$. *Then*

(1) $\sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges in* $\mathscr{A}$ $\forall\, \mathbf{x} \in \ell^2_*(S,\mathscr{A}) \Leftrightarrow \mathbf{a}^* \in \ell^2_{*u}(S,\mathscr{A})$;

(2) $\sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges in* $\mathscr{A}$ $\forall\, \mathbf{x} \in \ell^2_{*u}(S,\mathscr{A}) \Leftrightarrow \mathbf{a}^* \in \ell^2_*(S,\mathscr{A})$.

Proof. (1) $(\Rightarrow)$ Suppose that $\sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ converges in $\mathscr{A}$ $\forall\, \mathbf{x} \in \ell^2_*(S,\mathscr{A})$. By Theorem 3.2, $T_{\mathbf{a}}(\mathbf{x}) = \sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ defines a bounded linear transformation from $\ell^2_*(S,\mathscr{A})$ to $\mathscr{A}$. For a fixed $F \in \mathscr{F}$, define

$$\mathbf{y}(s) = \begin{cases} \mathbf{a}(s)^* & \text{if } s \in F \\ 0 & \text{if } s \notin F. \end{cases}$$

Then $\mathbf{y} \in \ell^2_*(S,\mathscr{A})$ and

$$\|\mathbf{y}\|^2 = \sup_{\varphi\in s(\mathscr{A})} \left(\sum_{s\in F} \left[\varphi\left(\mathbf{y}(s)^*\mathbf{y}(s)\right)\right]\right) = \left\| \sum_{s\in F} \mathbf{y}(s)^*\mathbf{y}(s) \right\|_{\mathscr{A}} = \left\| \sum_{s\in F} \mathbf{a}(s)\mathbf{a}(s)^* \right\|_{\mathscr{A}}.$$

From

$$\left\| \sum_{s\in F} \mathbf{a}(s)\mathbf{a}(s)^* \right\|_{\mathscr{A}} = \left\| \sum_{s\in F} \mathbf{a}(s)\mathbf{y}(s) \right\|_{\mathscr{A}} = \|T_{\mathbf{a}}(\mathbf{y})\| \leqslant \|T_{\mathbf{a}}\|\,\|\mathbf{y}\| = \|T_{\mathbf{a}}\| \left\| \sum_{s\in F} \mathbf{a}(s)\mathbf{a}(s)^* \right\|_{\mathscr{A}}^{\frac{1}{2}},$$

we have $\left\|\sum_{s\in F}\mathbf{a}(s)\mathbf{a}(s)^*\right\|_{\mathscr{A}}^{\frac{1}{2}} \leqslant \|T_{\mathbf{a}}\|$. Since this is true for any $F\in\mathscr{F}$,

$$\sup_{F\in\mathscr{F}}\left\|\sum_{s\in F}\mathbf{a}(s)\mathbf{a}(s)^*\right\|_{\mathscr{A}} \leqslant \|T_{\mathbf{a}}\|^2 < \infty.$$

This means $\mathbf{a}^*\in\ell^2_*(S,\mathscr{A})$. Put $\mathbf{x}=\mathbf{a}^*\in\ell^2_*(S,\mathscr{A})$ in the assumption, we get that $\sum_{s\in S}\mathbf{a}(s)\mathbf{a}(s)^*$ converges in $\mathscr{A}$. By the previous result, $\mathbf{a}^*\in\widetilde{\ell^2_{*u}}(S,\mathscr{A})=\ell^2_{*u}(S,\mathscr{A})$.

($\Leftarrow$) Let $\mathbf{x}\in\ell^2_*(S,\mathscr{A})$. Suppose that $\mathbf{a}^*\in\ell^2_{*u}(S,\mathscr{A})$. Let $\varepsilon>0$. There exists $F_0\in\mathscr{F}$ such that for all $\varphi\in s(\mathscr{A})$ and all $G\in\mathscr{F}$ with $G\subseteq S\setminus F_0$,

$$\sum_{s\in G}\left[\varphi\left(\mathbf{a}(s)\mathbf{a}(s)^*\right)\right] < \frac{\varepsilon^2}{16\left(\|\mathbf{x}\|^2+1\right)}.$$

Fix $\varphi\in s(\mathscr{A})$. Let $H\in\mathscr{F}$ be such that $H\subseteq S\setminus F_0$. Then

$$\begin{aligned}
\left|\varphi\left(\sum_{s\in H}\mathbf{a}(s)\mathbf{x}(s)\right)\right| &= \left|\sum_{s\in H}\varphi\left(\mathbf{a}(s)\mathbf{x}(s)\right)\right| \leqslant \sum_{s\in H}\left|\varphi\left(\mathbf{a}(s)\mathbf{x}(s)\right)\right| \\
&\leqslant \sum_{s\in H}\left(\left[\varphi\left(\mathbf{a}(s)\mathbf{a}(s)^*\right)\right]^{\frac{1}{2}}\cdot\left[\varphi\left(\mathbf{x}(s)^*\mathbf{x}(s)\right)\right]^{\frac{1}{2}}\right) \\
&\leqslant \left[\sum_{s\in H}\varphi\left(\mathbf{a}(s)\mathbf{a}(s)^*\right)\right]^{\frac{1}{2}}\cdot\left[\sum_{s\in H}\varphi\left(\mathbf{x}(s)^*\mathbf{x}(s)\right)\right]^{\frac{1}{2}} \\
&\leqslant \left[\sum_{s\in H}\varphi\left(\mathbf{a}(s)\mathbf{a}(s)^*\right)\right]^{\frac{1}{2}}\cdot\|\mathbf{x}\| \\
&< \left[\frac{\varepsilon^2}{16\left(\|\mathbf{x}\|^2+1\right)}\right]^{\frac{1}{2}}\cdot\|\mathbf{x}\| < \frac{\varepsilon}{4}.
\end{aligned}$$

Thus $\left\|\sum_{s\in H}\mathbf{a}(s)\mathbf{x}(s)\right\|_s = \sup_{\varphi\in s(\mathscr{A})}\left|\varphi\left(\sum_{s\in H}\mathbf{a}(s)\mathbf{x}(s)\right)\right| \leqslant \frac{\varepsilon}{4}$.

Therefore $\left\|\sum_{s\in H}\mathbf{a}(s)\mathbf{x}(s)\right\|_{\mathscr{A}} \leqslant \frac{\varepsilon}{2}<\varepsilon$. So $\sum_{s\in S}\mathbf{a}(s)\mathbf{x}(s)$ converges in $\mathscr{A}$.

(2) Simple adaptations of the proof of the previous statement provides a proof for this statement and we omit it.

These can also be restated as: $\mathbf{a}^*\in\ell^2_{*u}(S,\mathscr{A})$ iff $T_{\mathbf{a}}\in\mathscr{B}(\ell^2_*(S,\mathscr{A}),\mathscr{A})$; and $\mathbf{a}^*\in\ell^2_*(S,\mathscr{A})$ iff $T_{\mathbf{a}}\in\mathscr{B}(\ell^2_{*u}(S,\mathscr{A}),\mathscr{A})$.

The following corollary is immediate from Theorem 3.2 and Theorem 5.3.

COROLLARY 5.4. *Let* $\mathbf{a} \in \mathscr{A}^S$. *Then the following are equivalent:*

1. $T_{\mathbf{a}}(\mathbf{x}) = \sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges* $\forall\, \mathbf{x} \in \ell^2_{*u}(S,\mathscr{A})$*;*
2. $T_{\mathbf{a}} : \ell^2_{*u}(S,\mathscr{A}) \to \mathscr{A}$ *is bounded;*
3. $\mathbf{a}^* \in \ell^2_*(S,\mathscr{A})$.

So are the following:

4. $T_{\mathbf{a}}(\mathbf{x}) = \sum_{s\in S} \mathbf{a}(s)\mathbf{x}(s)$ *converges* $\forall\, \mathbf{x} \in \ell^2_{*}(S,\mathscr{A})$*;*
5. $T_{\mathbf{a}} : \ell^2_{*}(S,\mathscr{A}) \to \mathscr{A}$ *is bounded;*
6. $\mathbf{a}^* \in \ell^2_{*u}(S,\mathscr{A})$.

6. The $\ell^p(S,\mathscr{A})$ spaces

In this section we consider another class of function spaces. We begin with the following analogue of Theorem 4.3, whose proof, which is very similar, is omitted.

THEOREM 6.1. *Let* $\mathbf{y} \in \mathscr{A}^S$, *and* $p \in [1,\infty)$.

(a) The function $\|\cdot\| : \mathscr{A}^S \to [0,\infty]$ *defined by*

$$\|\mathbf{y}\| = \sup_{\varphi\in s(\mathscr{A})} \left[\sum_{s\in S} |\varphi(\mathbf{y}(s))|^p\right]^{\frac{1}{p}} \qquad (\mathbf{y} \in \mathscr{A}^S)$$

satisfies all six conditions in Theorem 4.2 *with* $c = 2$ *and* $\xi = 1$.

(b) The condition $\sum_{s\in S} |\varphi(\mathbf{y}(s))|^p < \infty\ \forall\ \varphi \in s(\mathscr{A})$ *is necessary and sufficient for* $\sup_{\varphi\in s(\mathscr{A})} \sum_{s\in S} |\varphi(\mathbf{y}(s))|^p < \infty$.

For a finite $p \geqslant 1$, let

$$\ell^p(S,\mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sum_{s\in S} |\varphi(\mathbf{x}(s))|^p < \infty\ \forall\ \varphi \in s(\mathscr{A}) \right\}, \qquad \text{and}$$

$$\ell^p_u(S,\mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sum_{s\in S} |\varphi(\mathbf{x}(s))|^p \text{ converges uniformly for } \varphi \in s(\mathscr{A}) \right\}.$$

We note first that $\ell^p_u(S,\mathscr{A}) \subseteq \ell^p(S,\mathscr{A})$. From the fact that $\varphi(a^*) = \overline{\varphi(a)}$ for each $a \in \mathscr{A}$ and each $\varphi \in s(\mathscr{A})$, we see that both spaces are self-adjoint in the sense that $\mathbf{x}^*$ is in the space whenever $\mathbf{x}$ is.

The following example shows that the inclusion $\ell_u^p(S,\mathscr{A}) \subseteq \ell^p(S,\mathscr{A})$ can be proper.

EXAMPLE 6.2. *Let $\mathscr{A}$ be a C^*-algebra containing an infinite family $\{y_k\}_{k\in\mathbb{N}}$ of contractions of norm 1 with orthogonal ranges. Then $\mathbf{x}(k) = y_k y_k^*$, $k \in \mathbb{N}$ defines a function that is in $\ell^p(\mathbb{N},\mathscr{A})$, but not in $\ell_u^p(\mathbb{N},\mathscr{A})$.*

Proof. For each $\varphi \in s(\mathscr{A})$ and $k \in \mathbb{N}$, since $I \geqslant \mathbf{x}(k) \geqslant 0$, and $\varphi(\mathbf{x}(k)) = |\varphi(\mathbf{x}(k))| \leqslant \varphi(I) = 1$,

$$\sum_{k=1}^{N} |\varphi(\mathbf{x}(k))|^p \leqslant \sum_{k=1}^{N} |\varphi(\mathbf{x}(k))| = \sum_{k=1}^{N} \varphi(\mathbf{x}(k)) = \varphi\left(\sum_{k=1}^{N} \mathbf{x}(k)\right) \leqslant \varphi(I) = 1$$

for all N. Hence $\sum_{k=1}^{\infty} |\varphi(\mathbf{x}(k))|^p \leqslant 1\ \ \forall\, \varphi \in s(\mathscr{A})$, i.e., $\mathbf{x} \in \ell^p(\mathbb{N},\mathscr{A})$.

To see that $\mathbf{x} \notin \ell_u^p(\mathbb{N},\mathscr{A})$, choose $\varepsilon = 1$; and for each $l \in \mathbb{N}$, there is a $\varphi \in s(\mathscr{A})$ such that $\varphi_l(\mathbf{x}(l)) = \|\mathbf{x}(l)\|_{\mathscr{A}} = 1$. Then $\sum_{k=l}^{\infty} |\varphi_l(\mathbf{x}(k))|^p \geqslant 1 = \varepsilon$. Hence $\mathbf{x} \notin \ell_u^p(\mathbb{N},\mathscr{A})$.

THEOREM 6.3. *The set $\ell^p(S,\mathscr{A})$ $[$resp. $\ell_u^p(S,\mathscr{A})]$ is a Banach space under the usual (point-wise) scalar multiplication and addition, and the norm*

$$\|\mathbf{x}\| := \sup_{\varphi\in s(\mathscr{A})} \left(\sum_{s\in S} |\varphi(\mathbf{x}(s))|^p\right)^{\frac{1}{p}}.$$

Proof. That the function as defined is indeed a norm on $\ell^p(S,\mathscr{A})$ follows directly from Theorem 6.1. By Theorem 4.2, $\ell^p(S,\mathscr{A}) = \left\{\mathbf{y} \in \mathscr{A}^S : \|\mathbf{y}\| < \infty\right\}$ is a Banach space. The completeness of $\ell_u^p(S,\mathscr{A})$ follows from its closedness in $\ell^p(S,\mathscr{A})$, which can be proved by an argument similar to that used in Theorem 4.5.

It is not hard to see that $\lim_{F\in\mathscr{F}} \|\mathbf{x} - \mathbf{x}_F\| = 0$ for all $\mathbf{x} \in \ell_u^p(S,\mathscr{A})$ and $\ell^p(S,\mathscr{A})$ does not have this property (see Example 6.2). The following corollary is an immediate consequence of this result and Theorem 3.4.

COROLLARY 6.4. *The set $\mathscr{M}(\ell^p(S,\mathscr{A}))$ $[$resp. $\mathscr{M}(\ell_u^p(S,\mathscr{A}))]$, of all $\mathbf{A}$ in the space of functions from $S \times S$ to $\mathscr{A}$ such that $\mathbf{A}$ defines a bounded linear operator on $\ell^p(S,\mathscr{A})$ $[$resp. $\ell_u^p(S,\mathscr{A})]$ with the operator norm is a Banach space $[$resp. Banach algebra$]$.*

PROPOSITION 6.5. *For $p \in [2,\infty)$, $\ell_*^p(S,\mathscr{A}) \subseteq \ell^p(S,\mathscr{A})$.*

Proof. Let $\mathbf{x}\in\ell_*^p(S,\mathscr{A})$ and let $\varphi\in s(\mathscr{A})$. Then $\sum_{s\in S}[\varphi(x(s)^*x(s))]^{\frac{p}{2}}<\infty$. Since

$$|\varphi(x(s))|^{2p}=|\varphi(1^*\cdot x(s))|^{2p}\leqslant[\varphi(x(s)^*x(s))\cdot\varphi(1^*1)]^p=[\varphi(x(s)^*x(s))]^p,$$
$$\sum_{s\in S}|\varphi(x(s))|^p\leqslant\sum_{s\in S}[\varphi(x(s)^*x(s))]^{\frac{p}{2}}<\infty.$$

This means $\mathbf{x}\in\ell^p(S,\mathscr{A})$. Therefore $\ell_*^p(S,\mathscr{A})\subseteq\ell^p(S,\mathscr{A})$.

For $\mathscr{A}=\mathbb{C}$, it is not hard to see that $\ell_*^p(S,\mathbb{C})=\ell^p(S,\mathbb{C})$. The following example shows that the inclusion in the previous proposition can be proper.

EXAMPLE 6.6. *For $p\in[2,\infty)$, $\ell_*^p(S,\mathscr{A})$ can be a proper subset of $\ell^p(S,\mathscr{A})$.*

Proof. Let $\mathscr{A}=\mathscr{B}(\ell^2)$ and $S=\mathbb{N}$. Let x_n be the matrix having a 1 in the $(n,1)$ entry and 0 in all others. Then $x_n^*x_n$ is the matrix having a 1 in the $(1,1)$ position and 0 in all other entries. Define $\varphi(A)=a_{11}$ for each $A=[a_{jk}]\in\mathscr{B}(\ell^2)$. Then $\varphi(x_n^*x_n)=1$ for all n. Hence $\sum_{n=1}^{\infty}[\varphi(x_n^*x_n)]^{\frac{p}{2}}$ diverges. This means $\mathbf{x}=\{x_n\}_{n=1}^{\infty}\notin\ell_*^p(S,\mathscr{A})$.

Next, we show that $\mathbf{x}=\{x_n\}_{n=1}^{\infty}\in\ell^p(S,\mathscr{A})$. Let $\varphi\in s(\mathscr{A})$. By Dixmier's Theorem ([5] p. 50; [2]), there are unique $A:=A_\varphi=[a_{jk}]\in\mathscr{C}^1$ and $\psi\in[\mathscr{A}^\#]_\sigma$, where $\mathscr{C}^1$ is the space of trace class operators and $[\mathscr{A}^\#]_\sigma$ is the space consisting of the zero functional and functionals on $\mathscr{A}$ which vanish on the compact operators, such that for all $x\in\mathscr{A}$ $\varphi(x)=\operatorname{trace}(Ax)+\psi(x)$. Since each x_n is compact, $\psi(x_n)=0$. Therefore $\varphi(x_n)=\operatorname{trace}(Ax_n)=a_{1n}$. Since $|a_{1n}|\leqslant\|A\|\leqslant\|A\|_{\mathscr{C}^1}\leqslant\|\varphi\|=1$, $\sum_{n=1}^{\infty}|\varphi(x_n)|^p=\sum_{n=1}^{\infty}|a_{1n}|^p\leqslant\sum_{n=1}^{\infty}|a_{1n}|^2\leqslant\|A\|^2<\infty$. Hence $\{x_n\}_{n=1}^{\infty}\in\ell^p(S,\mathscr{A})$.

With a suitable adaptation of the argument used in the proof of Proposition 5.2, we have the following proposition.

PROPOSITION 6.7. *For $p\in[1,\infty)$, set*

$$\ell_c^p(S,\mathscr{A})=\Big\{\mathbf{x}\in\mathscr{A}^S:\varphi\mapsto\widetilde{\varphi}(\mathbf{x})\text{ is weak}^*\text{ to norm continuous from } s(\mathscr{A})\text{ to }\ell^p(S,\mathbb{C})\Big\}.$$

Then

$$\ell_c^p(S,\mathscr{A})=\ell_u^p(S,\mathscr{A})=\Big\{\mathbf{x}\in\mathscr{A}^S:\forall\varepsilon>0\ \exists F_0\in\mathscr{F}\text{ such that } \big\|\widetilde{\varphi}\big(\mathbf{x}_{S\setminus F_0}\big)\big\|<\varepsilon\ \ \forall\,\varphi\in s(\mathscr{A})\Big\}.$$

Recall that, for $p\in[1,\infty)$:

$$\ell^p(S,\mathscr{A})=\left\{\mathbf{x}\in\mathscr{A}^S:\sum_{s\in S}|\varphi(\mathbf{x}(s))|^p<\infty\ \ \forall\,\varphi\in s(\mathscr{A})\right\},\text{ and}$$

$$\ell_*^{2p}(S,\mathscr{A}) = \left\{ \mathbf{x} \in \mathscr{A}^S : \sum_{s\in S} [\varphi(\mathbf{x}(s)^*\mathbf{x}(s))]^p < \infty \ \forall \varphi \in s(\mathscr{A}) \right\}.$$

The following theorem gives a relation between $\ell^p(S,\mathscr{A})$ and $\ell_*^{2p}(S,\mathscr{A})$.

THEOREM 6.8. *Let* $\mathbf{a} \in \mathscr{A}^S$. *Then* $\mathbf{ax} \in \ell^p(S,\mathscr{A})$ *for every* $\mathbf{x} \in \ell_*^{2p}(S,\mathscr{A})$ *if and only if* $\mathbf{a}^* \in \ell_*^{2p}(S,\mathscr{A})$. *If* $\mathbf{a}$ *satisfies either of these conditions, then* $T_{\mathbf{a}}(\mathbf{x}) = \mathbf{ax}$ *defines a bounded linear transformation from* $\ell_*^{2p}(S,\mathscr{A})$ *to* $\ell^p(S,\mathscr{A})$ *and* $\|T_{\mathbf{a}}\| = \|\mathbf{a}^*\|_{\ell_*^{2p}}$.

Proof. Suppose $\mathbf{a}^* \in \ell_*^{2p}(S,\mathscr{A})$. Let $\mathbf{x} \in \ell_*^{2p}(S,\mathscr{A})$ and let $\varphi \in s(\mathscr{A})$. Then

$$\begin{aligned}\sum_{s\in S} |\varphi(\mathbf{a}(s)\mathbf{x}(s))|^p &\leqslant \sum_{s\in S}\left(\|\mathbf{x}(s)\|_\varphi^p \cdot \|\mathbf{a}(s)^*\|_\varphi^p\right) \\ &\leqslant \left(\sum_{s\in S}\|\mathbf{x}(s)\|_\varphi^{2p}\right)^{\frac{1}{2}} \left(\sum_{s\in S}\|\mathbf{a}(s)^*\|_\varphi^{2p}\right)^{\frac{1}{2}} \qquad (\dagger) \\ &\leqslant \|\mathbf{x}\|_{\ell_*^{2p}}^p \|\mathbf{a}^*\|_{\ell_*^{2p}}^p < \infty\end{aligned}$$

which implies that $\mathbf{ax} \in \ell^p(S,\mathscr{A})$.

Conversely, define $T_{\mathbf{a}} : \ell_*^{2p}(S,\mathscr{A}) \to \ell^p(S,\mathscr{A})$ by

$$T_{\mathbf{a}}(\mathbf{x}) = \mathbf{ax} \qquad \forall\, \mathbf{x} \in \ell_*^{2p}(S,\mathscr{A}).$$

Let $F \in \mathscr{F}$. Define

$$T_F(\mathbf{x}) = (\mathbf{ax})_F \qquad \forall\, \mathbf{x} \in \ell_*^{2p}(S,\mathscr{A})$$

i.e.,

$$[T_F(\mathbf{x})](s) = \begin{cases} \mathbf{a}(s)\mathbf{x}(s) & \text{if } s \in F \\ 0 & \text{if } s \notin F \end{cases}.$$

Then it is not hard to show that each T_F is bounded. An application of the uniform boundedness principle shows that there exists $M > 0$ such that $\|T_F\| \leqslant M \ \ \forall\, F \in \mathscr{F}$. Thus, for each $\mathbf{x} \in \ell_*^{2p}(S,\mathscr{A})$,

$$\|T_{\mathbf{a}}\mathbf{x}\|_{\ell^p(S,\mathscr{A})} = \lim_{F\in\mathscr{F}} \|T_F\mathbf{x}\|_{\ell^p(S,\mathscr{A})} \leqslant \limsup_{F\in\mathscr{F}} \|T_F\|\,\|\mathbf{x}\|_{\ell_*^{2p}} \leqslant M\,\|\mathbf{x}\|_{\ell_*^{2p}}$$

that is $T_{\mathbf{a}}$ is a bounded linear transformation from $\ell_*^{2p}(S,\mathscr{A})$ to $\ell^p(S,\mathscr{A})$. Define

$$\mathbf{y}(s) = \begin{cases} \mathbf{a}(s)^* & \text{if } s \in F \\ 0 & \text{if } s \notin F. \end{cases}$$

Then $\mathbf{y} \in \ell_*^{2p}(S,\mathscr{A})$ and $\|\mathbf{y}\|_{\ell_*^{2p}}^{2p} = \|\mathbf{a}_F^*\|_{\ell_*^{2p}}^{2p}$. Given $\varepsilon > 0$, there is a $\psi \in s(\mathscr{A})$ such that

$$\|\mathbf{y}\|_{\ell_*^{2p}}^{2p} - \varepsilon \ < \ \sum_{s\in F} \|\mathbf{a}(s)^*\|_\psi^{2p} \leqslant \|T_F(\mathbf{y})\|_{\ell^p}^p \leqslant \|T_F\|^p\,\|\mathbf{y}\|_{\ell_*^{2p}}^p \leqslant M^p\,\|\mathbf{y}\|_{\ell_*^{2p}}^p.$$

So $\dfrac{\|\mathbf{y}\|_{\ell_*^{2p}}^{2p}-\varepsilon}{\|\mathbf{y}\|_{\ell_*^{2p}}^{p}} < M^p$ for all $\varepsilon > 0$ and hence $\|\mathbf{y}\|_{\ell_*^{2p}}^{p} = \dfrac{\|\mathbf{y}\|_{\ell_*^{2p}}^{2p}}{\|\mathbf{y}\|_{\ell_*^{2p}}^{p}} \leqslant M^p$. Therefore $\sup\limits_{\varphi\in s(\mathscr{A})}\left(\sum\limits_{s\in F}\|\mathbf{a}(s)^*\|_{\varphi}^{2p}\right)^{\frac{1}{2}} = \|\mathbf{a}_F^*\|_{\ell_*^{2p}}^{p} = \|\mathbf{y}\|_{\ell_*^{2p}}^{p} \leqslant M^p$ for all $F\in\mathscr{F}$. Thus for each fixed $\varphi\in s(\mathscr{A})$, $\left(\sum\limits_{s\in F}\|\mathbf{a}(s)^*\|_{\varphi}^{2p}\right)^{\frac{1}{2}} \leqslant M^p$ $\forall\, F\in\mathscr{F}$ and hence $\left(\sum\limits_{s\in S}\|\mathbf{a}(s)^*\|_{\varphi}^{2p}\right)^{\frac{1}{2}} \leqslant M^p$. Since this is true for all $\varphi\in s(\mathscr{A})$,

$$\|\mathbf{a}^*\|_{\ell_*^{2p}}^{p} = \sup_{\varphi\in s(\mathscr{A})}\left(\sum_{s\in S}\|\mathbf{a}(s)^*\|_{\varphi}^{2p}\right)^{\frac{1}{2}} \leqslant M^p$$

which implies that $\mathbf{a}^*\in\ell_*^{2p}(S,\mathscr{A})$.

We now show that $\|T_{\mathbf{a}}\| = \|\mathbf{a}^*\|_{\ell_*^{2p}}$. For any $\mathbf{x}\in\ell_*^{2p}(S,\mathscr{A})$, by the inequality ($\dagger$) above,

$$\begin{aligned}\|T_{\mathbf{a}}(\mathbf{x})\|_{\ell^p}^{p} = \|\mathbf{a}\mathbf{x}\|_{\ell^p}^{p} &\leqslant \sup_{\varphi\in s(\mathscr{A})}\left(\sum_{s\in S}\|\mathbf{a}(s)^*\|_{\varphi}^{2p}\right)^{\frac{1}{2}}\cdot\sup_{\varphi\in s(\mathscr{A})}\left(\sum_{s\in S}\|\mathbf{x}(s)\|_{\varphi}^{2p}\right)^{\frac{1}{2}}\\ &= \|\mathbf{a}^*\|_{\ell_*^{2p}}^{p}\cdot\|\mathbf{x}\|_{\ell_*^{2p}}^{p}.\end{aligned}$$

Thus $\Big(T_{\mathbf{a}}:\ell_*^{2p}(S,\mathscr{A})\to\ell^p(S,\mathscr{A})$ is bounded and$\Big)$ $\|T_{\mathbf{a}}\|\leqslant\|\mathbf{a}^*\|_{\ell_*^{2p}}$. Let $F\in\mathscr{F}$ and $\mathbf{y}$ be as defined above. Then

$$\begin{aligned}\|\mathbf{y}\|_{\ell_*^{2p}}^{2p} = \|\mathbf{a}_F^*\|_{\ell_*^{2p}}^{2p} &= \sup_{\varphi\in s(\mathscr{A})}\left(\sum_{s\in F}\left[\varphi\left(\mathbf{a}(s)\mathbf{a}(s)^*\right)\right]^p\right)\\ &= \sup_{\varphi\in s(\mathscr{A})}\left(\sum_{s\in F}\left|\varphi\left(\mathbf{a}(s)\mathbf{y}(s)\right)\right|^p\right) = \|T_{\mathbf{a}}(\mathbf{y})\|^p \leqslant \|T_{\mathbf{a}}\|^p\,\|\mathbf{y}\|_{\ell_*^{2p}}^{p}\\ &= \|T_{\mathbf{a}}\|^p\,\|\mathbf{a}_F^*\|_{\ell_*^{2p}}^{p}.\end{aligned}$$

Hence $\|\mathbf{a}_F^*\|_{\ell_*^{2p}}^{p}\leqslant\|T_{\mathbf{a}}\|^p$ or $\|\mathbf{a}_F^*\|_{\ell_*^{2p}}\leqslant\|T_{\mathbf{a}}\|$ $\forall\; F\in\mathscr{F}$. Taking supremum over all $F\in\mathscr{F}$, we have $\|\mathbf{a}^*\|_{\ell_*^{2p}}\leqslant\|T_{\mathbf{a}}\|$. Therefore $\|T_{\mathbf{a}}\| = \|\mathbf{a}^*\|_{\ell_*^{2p}}$.

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