Operators
and
Matrices
Volume 3, Number 3 (2009), 439–449

ASYMPTOTIC BEHAVIOR OF GELFAND–NAIMARK DECOMPOSITION

HUAJUN HUANG

(*Communicated by C.-K. Li*)

Abstract. Let $X = L\sigma U$ be the Gelfand-Naimark decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$, where L is unit lower triangular, σ is a permutation matrix, and U is upper triangular. Call $u(X) := \operatorname{diag} U$ the u-component of X. We show that in a Zariski dense open subset of the ω-orbit of certain Bruhat decomposition,

$$\lim_{m\to\infty} |u(X^m)|^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|, \cdots, |\lambda_{\omega(n)}|).$$

The other situations where $\lim_{m\to\infty} |u(X^m)|^{1/m}$ converge to different limits or diverge are also discussed.

1. Introduction

Gelfand-Naimark decomposition asserts that each $X \in \mathrm{GL}_n(\mathbb{C})$ can be decomposed as $X = L\sigma U$, where L is unit lower triangular, σ is a permutation matrix, and U is upper triangular. Though Gelfand-Naimark decomposition is not unique, σ and $\operatorname{diag} U$ are uniquely determined by X. We denote

$$u(X) := (u_1(X), \cdots, u_n(X)) = \operatorname{diag} U \in \mathbb{C}^n$$

the u-component of X.

Suppose that X has eigenvalues

$$\lambda(X) := (\lambda_1(X), \cdots, \lambda_n(X)) \in \mathbb{C}^n$$

with ascending moduli: $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_n|$. Let

$$s(X) := (s_1(X), \cdots, s_n(X)) \in \mathbb{R}^n_+$$

be the singular values of X in ascending order: $s_1(X) \leqslant \cdots \leqslant s_n(X)$. Let $X = QR$ be the QR decomposition of X and denote

$$a(X) := \operatorname{diag} R = (a_1(X), \cdots, a_n(X)) \in \mathbb{R}^n_+.$$

Mathematics subject classification (2000): 15A23, 15A42.
Keywords and phrases: Gelfand-Naimark decomposition, Bruhat decomposition, u-component, eigenvalues.

Paper OaM-03-25

We view $\lambda(X)$, $s(X)$, $a(X)$ and $u(X)$ as diagonal matrices. Relations between those four quantities were very recently studied [5].

Yamamoto proved [9] that

$$\lim_{m\to\infty}[s(X^m)]^{1/m} = \operatorname{diag}(|\lambda_1|,\cdots,|\lambda_n|). \tag{1.1}$$

Huang and Tam proved [7, Theorem 2.7] (also see [4]) that

$$\lim_{m\to\infty}[a(X^m)]^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|,\cdots,|\lambda_{\omega(n)}|). \tag{1.2}$$

Here ω is a permutation obtained as follow: Let $X = C^{-1}TC$ in which $T = \left[t_{ij}\right] \in \mathrm{GL}_n(\mathbb{C})$ is an upper triangular matrix with $t_{ii} = \lambda_i$ (so $|t_{11}| \leqslant \cdots \leqslant |t_{nn}|$). Then ω comes from the permutation matrix (also denoted by ω) in a Bruhat decomposition $C = V'\omega U'$ of C, for certain upper triangular matrices V' and U'. This result can be applied to both the Jordan decomposition and the Schur triangularization of X. Note that the Gelfand-Naimark decomposition is a variation of the Bruhat decomposition [8].

When X has distinct eigenvalue moduli: $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$, consider the QR iteration:

$$X_1 = X = Q_1R_1; \qquad X_m := R_{m-1}Q_{m-1} = Q_mR_m, \qquad m = 2,3,\cdots$$

where $X_m = Q_mR_m$ is the QR decomposition of X_m for $m = 1,2,\cdots$. In [8, Theorem 3.1], Tyrtyshnikov essentially proved that the lower triangular part of $\{X_m\}_{m=1}^{\infty}$ converges to 0 and

$$\lim_{m\to\infty}\operatorname{diag}(X_m) = \operatorname{diag}(\lambda_{\omega(1)},\lambda_{\omega(2)},\cdots,\lambda_{\omega(n)}), \tag{1.3}$$

where ω is the same as in (1.2). Indeed, Tyrtyshnikov proved this result for GR iterations with no shifts. A detailed description on the asymptotic behavior of $\{X_m\}_{m=1}^{\infty}$ for the above QR iteration is given in [6, Theorem 5.1].

The main purpose of this paper is to discuss the asymptotic behavior of $|u(X^m)|^{1/m}$ as $m\to\infty$. Unlike $s(X^m)^{1/m}$ and $a(X^m)^{1/m}$, the sequence $\{|u(X^m)|^{1/m}\}_{m=1}^{\infty}$ may not converge. However, as we will see, the permutation ω obtained from the Bruhat decomposition of C continues to play a significant role for the asymptotic behavior of $|u(X^m)|^{1/m}$. We provide some sufficient conditions for the convergence of the sequence. In particular, when X is positive definite (it has Cholesky decomposition [3]) [7, Theorem 3.1]

$$\lim_{m\to\infty}|u(X^m)|^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|,\cdots,|\lambda_{\omega(n)}|).$$

In Section 2, we give an upper bound on the asymptotic behavior of $|u_1(X^m)\cdots u_k(X^m)|^{1/m}$ in terms of ω. Then we prove that $\lim_{m\to\infty}|u_1(X^m)\cdots u_k(X^m)|^{1/m}$ converges to a product of k eigenvalue moduli of X if the k-compound matrix of X has distinct eigenvalue moduli. Moreover, when C goes through a Zariski dense open subset of the ω-orbit in the Bruhat decomposition $C = V'\omega U'$, the matrix $X^m = C^{-1}\lambda(X)^mC$ has LU-decomposition for sufficiently large m, and

$$\lim_{m\to\infty}|u(X^m)|^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|,\cdots,|\lambda_{\omega(n)}|).$$

In particular, $\lim_{m\to\infty}|u(X^m)|^{1/m} = \operatorname{diag}(|\lambda_n|,|\lambda_{n-1}|,\cdots,|\lambda_1|)$ when X is in a dense open subset (in Euclidean topology) of $\mathrm{GL}_n(\mathbb{C})$.

In Section 3, we use examples to illustrate the theoretical results given in Section 2. In particular, we show that when the eigenvalue moduli of the k-compound of X are not distinct for some k, $\lim_{m\to\infty}|u(X^m)|^{1/m}$ may diverge.

2. The asymptotic behavior of $|u(X^m)|^{1/m}$

Obviously, $u_1(X)$ in the Gelfand-Naimark decomposition $X = L\sigma U$ is the first nonzero entry in the first column of X, and $a_1(X)$ in the QR decomposition $X = QR$ is the norm of the first column of X. Therefore, $|u_1(X)| \leqslant a_1(X)$. Using compound matrix technique, we get the following relationship between $u(X)$ and $a(X)$.

THEOREM 2.1. [5, Theorem 3.1] *The scalars $u(X) \in \mathbb{C}^n$ and $a(X) \in \mathbb{R}^n_+$ satisfy that*

$$\begin{aligned} |u_1(X)\cdots u_k(X)| &\leqslant a_1(X)\cdots a_k(X), \qquad 1 \leqslant k \leqslant n-1,\\ u_1(X)\cdots u_n(X) &= \pm a_1(X)\cdots a_n(X), \end{aligned}$$

where the sign on the last equality depends on whether σ is even ($+$) or σ is odd ($-$).

Suppose that $X = C^{-1}TC \in \mathrm{GL}_n(\mathbb{C})$ where $T = \left[t_{ij}\right]$ is an upper triangular matrix with $t_{ii} = \lambda_i$ (so $|t_{11}| \leqslant |t_{22}| \leqslant \cdots \leqslant |t_{nn}|$), and C has a Bruhat decomposition $C = V'\omega U'$ where V' and U' are upper triangular, and ω is a permutation matrix. By Theorem 2.1 and (1.2), we get the following result:

THEOREM 2.2. *The asymptotic behavior of $|u(X^m)|^{1/m}$ is bounded by*

$$\begin{aligned} \overline{\lim_{m\to\infty}}|u_1(X^m)\cdots u_k(X^m)|^{1/m} &\leqslant |\lambda_{\omega(1)}\cdots\lambda_{\omega(k)}|, \quad 1 \leqslant k \leqslant n-1;\\ |u_1(X^m)\cdots u_n(X^m)|^{1/m} &= |\lambda_{\omega(1)}\cdots\lambda_{\omega(n)}| = |\det(X)|. \end{aligned}$$

Denote

$$C^{-1} = \left[c'_{ij}\right], \quad C = \left[c_{ij}\right], \quad X^m = \left[x_{ij}^{(m)}\right], \quad \text{and } T^m = \left[t_{ij}^{(m)}\right].$$

The coefficients of X^m are related to those of T^m by $X^m = C^{-1}T^mC$. To study X^m, we first investigate the coefficients of T^m.

A subsequence $s := \{s_0, s_1, \cdots, s_k\}$ of $\{1,2,\cdots,n\}$ is called a *T-path* if $t_{s_is_{i+1}} \neq 0$ for $i = 0,\cdots,k-1$. Clearly $s_0 < s_1 < \cdots < s_k$. We call s_0 the *initial point*, s_k the *terminal point*, and $|s| := k$ the *length*, of the T-path s respectively. Let S_{ij} denote the set of all T-paths of with the initial point i and the terminal point j. Then $S_{ij} = \emptyset$ whenever $i > j$, since T is upper triangular.

Let

$$T(s) := (t_{s_0s_0}, \cdots, t_{s_ks_k})$$

be the $(k+1)$-tuple of the diagonal entries of T corresponding to the T-path s. Define the polynomial

$$M_p(T(s)) := M_p(t_{s_0s_0},\cdots,t_{s_ks_k}) := \sum_{\substack{a_0,\cdots,a_k\geqslant 0\\ a_0+\cdots+a_k=p}} {t_{s_0s_0}}^{a_0}\cdots {t_{s_ks_k}}^{a_k} \tag{2.1}$$

the sum of all degree p monomials for the variables $t_{s_0s_0}$, $\cdots$, t_{s_k,s_k}. Then $M_p(T(s)) = 0$ whenever $p<0$. Define the polynomial

$$f_s(T) := \begin{cases} t_{s_0s_1}t_{s_1s_2}\cdots t_{s_{k-1}s_k}, & |s|\geqslant 1;\\ 1, & |s|=0.\end{cases} \tag{2.2}$$

LEMMA 2.3. *The entries of* $T^m = \left[t_{ij}^{(m)}\right]$ *for an upper triangular matrix* $T = \left[t_{ij}\right]$ *are given by*

$$t_{ij}^{(m)} = \sum_{s\in S_{ij}} f_s(T)M_{m-|s|}(T(s)). \tag{2.3}$$

Proof. By direct computation,

$$\begin{aligned} t_{ij}^{(m)} &= \sum_{i=i_0\leqslant i_1\leqslant\cdots\leqslant i_m=j}\left(\prod_{p=0}^{m-1} t_{i_pi_{p+1}}\right)\\ &= \sum_{\substack{i=j_0<j_1<\cdots<j_k=j\\ a_0\geqslant 0,\ \cdots,\ a_k\geqslant 0\\ a_0+\cdots+a_k=m-k}}\left(\prod_{p=0}^{k-1} t_{j_pj_{p+1}}\cdot\prod_{p=0}^{k} t_{j_pj_p}^{a_p}\right)\\ &= \sum_{i=j_0<j_1<\cdots<j_k=j}\left(\prod_{p=0}^{k-1} t_{j_pj_{p+1}}\right)\left[\sum_{\substack{a_0\geqslant 0,\ \cdots,\ a_k\geqslant 0\\ a_0+\cdots+a_k=m-k}}\left(\prod_{p=0}^{k} t_{j_pj_p}^{a_p}\right)\right]\\ &= \sum_{s\in S_{ij}} f_s(T)M_{m-|s|}(T(s)). \end{aligned}$$

This proves (2.3).

Formula (2.3) implies the following asymptotic result when the diagonal entries of T have strictly ascending moduli.

THEOREM 2.4. *Suppose that the upper triangular matrix* $T\in \mathrm{GL}_n(\mathbb{C})$ *has strictly ascending moduli diagonal entries, that is,* $|t_{11}|<|t_{22}|<\cdots<|t_{nn}|$. *Let diag*$(T)$ *be the diagonal matrix of* T. *Then* $\lim_{m\to\infty} T^m diag(T)^{-m}$ *converges to an upper triangular matrix. Precisely,* $t_{jj}^{(m)}/{t_{jj}}^m = 1$ *for* $j=1,2,\cdots,n$, *and*

$$\lim_{m\to\infty}\frac{t_{ij}^{(m)}}{{t_{jj}}^m} = \sum_{(s_0,\cdots,s_k)\in S_{ij}} \frac{\prod_{p=0}^{k-1} t_{s_ps_{p+1}}}{\prod_{p=0}^{k-1}(t_{jj}-t_{s_ps_p})} \quad \textit{for } 1\leqslant i<j\leqslant n. \tag{2.4}$$

REMARK 2.5. Therefore, $\lim_{m\to\infty} \left| t_{ij}^{(m)} \right|^{1/m} = 0$ or $|t_{jj}|$, for $1 \leqslant i \leqslant j \leqslant n$.

Proof. [Proof of Lemma 2.4] According to (2.3), for $i < j$ we have

$$\frac{t_{ij}^{(m)}}{t_{jj}{}^m} = \sum_{s\in S_{ij}} \left[\frac{f_s(T)}{t_{jj}{}^{|s|}} \cdot \frac{M_{m-|s|}(T(s))}{t_{jj}{}^{m-|s|}} \right]. \tag{2.5}$$

Let us discuss the asymptotic behavior of $\frac{M_\ell(T(s))}{t_{jj}{}^\ell}$ when $\ell \to \infty$. Rewrite $T(s) := (y_0, \cdots, y_k)$ where $|y_0| < \cdots < |y_k|$. Then

$$\begin{aligned}
\frac{M_\ell(y_0,\cdots,y_k)}{y_k^\ell} &= \sum_{\substack{a_0\geqslant 0,\ \cdots,\ a_k\geqslant 0\\ a_0+\cdots+a_k=\ell}} \frac{y_0^{a_0}\cdots y_k^{a_k}}{y_k^{a_0+\cdots+a_k}} \\
&= 1 + M_1\left(\frac{y_0}{y_k},\cdots,\frac{y_{k-1}}{y_k}\right) + M_2\left(\frac{y_0}{y_k},\cdots,\frac{y_{k-1}}{y_k}\right) \\
&\qquad + \cdots + M_\ell\left(\frac{y_0}{y_k},\cdots,\frac{y_{k-1}}{y_k}\right) \\
&= \sum_{q=0}^{\ell} M_q\left(\frac{y_0}{y_k},\cdots,\frac{y_{k-1}}{y_k}\right).
\end{aligned} \tag{2.6}$$

Therefore, the following limit converges by $|y_0| < \cdots < |y_k|$,

$$\begin{aligned}
\lim_{\ell\to\infty} \frac{M_\ell(y_0,\cdots,y_k)}{y_k^\ell} &= \sum_{q=0}^{\infty} M_q\left(\frac{y_0}{y_k},\cdots,\frac{y_{k-1}}{y_k}\right) \\
&= \prod_{p=0}^{k-1}\left[1 + \left(\frac{y_p}{y_k}\right) + \left(\frac{y_p}{y_k}\right)^2 + \left(\frac{y_p}{y_k}\right)^3 + \cdots\right] \\
&= \frac{1}{\prod_{p=0}^{k-1}(1-\frac{y_p}{y_k})} \\
&= \frac{y_k^k}{\prod_{p=0}^{k-1}(y_k - y_p)}.
\end{aligned} \tag{2.7}$$

From (2.5) we get

$$\begin{aligned}
\lim_{m\to\infty} \frac{t_{ij}^{(m)}}{t_{jj}{}^m} &= \sum_{s=(s_0,\cdots,s_k)\in S_{ij}} \left[\frac{f_s(T)}{t_{jj}{}^{|s|}} \cdot \frac{t_{jj}{}^{|s|}}{\prod_{p=0}^{|s|-1}(t_{jj} - t_{s_p s_p})} \right] \\
&= \sum_{(s_0,\cdots,s_k)\in S_{ij}} \frac{\prod_{p=0}^{k-1} t_{s_p s_{p+1}}}{\prod_{p=0}^{k-1}(t_{jj} - t_{s_p s_p})}.
\end{aligned}$$

This completes the proof.

The above results can be used to analyze the coefficients and the u-components in the Gelfand-Naimark decomposition of $X^m = C^{-1}T^mC$ where T is an upper triangular matrix with ascending moduli diagonal entries.

Suppose that X has distinct eigenvalue moduli: $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$. There are many possible ways to decompose $X = C^{-1}TC$ where the upper triangular matrix T has ascending moduli diagonal entries. For example, Jordan decomposition and Schur triangularization provide two different such decompositions. However, the permutation ω obtained from the Bruhat decomposition of C is uniquely determined by X. This could be seen from (1.2) and the fact that X has distinct eigenvalue moduli.

Because X has distinct eigenvalue moduli and thus is diagonalizable, it has a decomposition $X = C^{-1}TC$ where $T = \lambda(X)$ is an ascending moduli diagonal matrix. The equality $X^m = C^{-1}\lambda(X)^mC$ implies that

$$x_{ij}^{(m)} = \sum_{p=1}^{n} c'_{ip}c_{pj}\lambda_p^m. \tag{2.8}$$

So $x_{11}^{(m)} = \sum_{p=1}^{n} c'_{1p}c_{p1}\lambda_p^m$. Suppose that C has the Bruhat decomposition $C = V'\omega U'$ for certain upper triangular matrices V' and U', then $\omega(1)$ is the largest q such that $c_{q1} \neq 0$ in C. Thus

$$x_{11}^{(m)} = \sum_{p=1}^{\omega(1)} c'_{1p}c_{p1}\lambda_p^m. \tag{2.9}$$

Since $\sum_{p=1}^{\omega(1)} c'_{1p}c_{p1} = \sum_{p=1}^{n} c'_{1p}c_{p1} = 1$, there exists the largest integer $r \leqslant \omega(1)$ such that $c'_{1r}c_{r1} \neq 0$. Equality (2.9) implies that $x_{11}^{(m)} \neq 0$ when m is sufficiently large. Moreover, the following lemma holds:

LEMMA 2.6. *Suppose that $X = C^{-1}\lambda(X)C$ has distinct eigenvalue moduli. Then*

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = |\lambda_r| \tag{2.10}$$

where $r \leqslant \omega(1)$ is the largest integer such that $c'_{1r}c_{r1} \neq 0$.

As shown below, $r = \omega(1)$ in a Zariski dense open set. The Zariski topology on $\mathrm{GL}_n(\mathbb{C})$ is defined in the way that a Zariski closed set is the zeros of a set of polynomials on the matrix coefficients and $\det^{-1}$ [2, page 7].

LEMMA 2.7. *Suppose that Λ is a diagonal matrix with strictly ascending moduli, and ω is a permutation matrix, both in $\mathrm{GL}_n(\mathbb{C})$. Let $\mathscr{O}_{\Lambda,\omega}$ denote the set of all matrices $X = C^{-1}\Lambda C$ where C has a Bruhat decomposition $C = V'\omega U'$ for some upper triangular matrices V' and U'. Then on a Zariski dense open subset of the ω-orbit in the Bruhat decomposition of C,*

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = \lim_{m\to\infty} |x_{11}^{(m)}|^{1/m} = |\lambda_{\omega(1)}|.$$

Proof. The coefficient $c_{\omega(1)1}$ is always nonzero by the Bruhat decomposition $C = V'\omega U'$. Therefore, (2.9) implies that whenever $c'_{1\omega(1)} \neq 0$,

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = \lim_{m\to\infty} |x_{11}^{(m)}|^{1/m} = |\lambda_{\omega(1)}|.$$

Clearly, $c'_{1\omega(1)} \neq 0$ if and only if the cofactor of the $(\omega(1),1)$ entry of C is nonzero. This forms a Zariski dense open subset of the ω-orbit in the Bruhat decomposition of C.

Let $C_k(X)$ denote the k-compound matrix of $X \in \mathrm{GL}_n(\mathbb{C})$, $k = 1,\cdots,n$. Let $Q_{k,n}$ denote the set of all k-subsequences of $\{1,2,\cdots,n\}$. The entries of $C_k(X)$ are of the form $x_{\alpha,\beta}$ ($\alpha,\beta \in Q_{k,n}$), where $x_{\alpha,\beta}$ is the determinant of the submatrix formed by the α-rows and the β-columns of X. So $C_k(X) \in \mathrm{GL}_{\binom{n}{k}}(\mathbb{C})$. The k-compound of a permutation (resp. diagonal, upper triangular, lower triangular) matrix is still a permutation (resp. diagonal, upper triangular, lower triangular) matrix. Therefore, the k-compound preserves the Gelfand-Naimark decomposition and the Bruhat decomposition.

Assume that the k-compound of X has distinct eigenvalue moduli for every k. In other words, $\prod_{i\in\alpha} |\lambda_i(X)|$ for all $\alpha \in Q_{k,n}$ are mutually distinct. Then Lemma 2.6 and Lemma 2.7 can be extended to a product of $u_i(X)$ by the compound matrix technique.

THEOREM 2.8. *Suppose that the k-compound of $X \in \mathrm{GL}_n(\mathbb{C})$ has distinct eigenvalue moduli for every $k = 1,\cdots,n$. Then X^m has LU decomposition when m is sufficiently large, and*

$$\lim_{m\to\infty} |u_1(X^m)\cdots u_k(X^m)|^{1/m} = \prod_{i\in\alpha_k} |\lambda_i| \text{ for some } \alpha_k \in Q_{k,n}. \tag{2.11}$$

In particular, $\lim_{m\to\infty} |u_k(X^m)|^{1/m}$ *exists for* $k = 1,\cdots,n$.

Proof. Suppose that $X = C^{-1}\lambda(X)C$. Let $X(k|k)$ be the submatrix formed by the first k rows and the first k columns of X. By direct computation on $X^m = C^{-1}\lambda(X)^m C$, the $(1,1)$ entry in $C_k(X)^m = C_k(X^m)$ is

$$\det X^m(k|k) = \sum_{\alpha\in Q_{k,n}} \left(m_\alpha \prod_{i\in\alpha} \lambda_i^m \right) = \sum_{\alpha\in Q_{k,n}} \left[m_\alpha \left(\prod_{i\in\alpha} \lambda_i \right)^m \right] \tag{2.12}$$

where m_α are constants related to the first k-rows of C^{-1} and the first k-columns of C. Clearly $\sum_{\alpha\in Q_{k,n}} m_\alpha = 1$ by setting all $\lambda_i = 1$ in (2.12). By assumption, $\{\prod_{i\in\alpha} |\lambda_i| \mid \alpha \in Q_{k,n}\}$ are mutually distinct. There is one $\alpha_k \in Q_{k,n}$ such that $m_{\alpha_k} \neq 0$ and $\prod_{i\in\alpha_k} |\lambda_i|$ is maximal. Formula (2.12) implies that $\det X^m(k|k) \neq 0$ when m is sufficiently large. Moreover,

$$\begin{aligned}\lim_{m\to\infty} |u_1(X^m)\cdots u_k(X^m)|^{1/m} &= \lim_{m\to\infty} |u_1(C_k(X^m))|^{1/m} \\ &= \lim_{m\to\infty} |\det X^m(k|k)|^{1/m} = \prod_{i\in\alpha_k} |\lambda_i|.\end{aligned}$$

This completes the proof.

REMARK 2.9. It is not necessarily true that $\alpha_{k-1} \subseteq \alpha_k$. So the limit $\lim_{m\to\infty} |u_k(X^m)|^{1/m}$ in Theorem 2.8 may not equal to an eigenvalue modulus of X. An example is presented in Example 3.1 (3).

THEOREM 2.10. *Suppose that $\Lambda = diag\,(\lambda_1, \cdots, \lambda_n) \in \mathrm{GL}_n(\mathbb{C})$ is a diagonal matrix with strictly ascending moduli, and the elements in $\{\prod_{i\in\alpha} |\lambda_i| \mid \alpha \in Q_{k,n}\}$ are mutually distinct for every $k = 1, 2, \cdots, n$. Suppose that $\omega \in \mathrm{GL}_n(\mathbb{C})$ is a permutation matrix. Let $\mathscr{O}_{\Lambda,\omega}$ denote the set of all matrices $X = C^{-1}\Lambda C$ where C has a Bruhat decomposition $C = V'\omega U'$ for some upper triangular matrices V' and U'. Then on a Zariski dense open subset of the ω-orbit in the Bruhat decomposition of C,*

$$\lim_{m\to\infty} |u_k(X^m)|^{1/m} = |\lambda_{\omega(k)}| \qquad \textit{for} \qquad k = 1, \cdots, n.$$

Proof. The proof is done by applying Lemma 2.7 to the k-compound matrices of X for $k = 1, \cdots, n$, and using the fact that the intersection of finitely many Zariski dense open subsets is still a Zariski dense open subset. Note that the diagonal entry moduli of $C_k(\Lambda)$ may not be in strictly ascending order. However, [7, Lemma 2.10] shows that: there exists a permutation matrix $P \in \mathrm{GL}_{\binom{n}{k}}(\mathbb{C})$ such that the diagonal of $P^{-1}C_k(\Lambda)P$ is in ascending moduli order, and $P^{-1}C_k(V')P$ is still upper triangular for every upper triangular matrix $V' \in \mathrm{GL}_n(\mathbb{C})$. Then

$$\begin{aligned} C_k(X) &= C_k(C)^{-1}\, C_k(\Lambda)\, C_k(C) \\ &= (P^{-1}C_k(C))^{-1}(P^{-1}C_k(\Lambda)P)(P^{-1}C_k(C)) \end{aligned}$$

where $P^{-1}C_k(C)$ has a Bruhat decomposition

$$P^{-1}C_k(C) = (P^{-1}C_k(V')P)\,(P^{-1}C_k(\omega))\,C_k(U').$$

This leads to the proof of Theorem 2.10.

3. Examples

Let $X \in \mathrm{GL}_n(\mathbb{C})$ such that the k-compound of X has distinct eigenvalue moduli for $k = 1, \cdots, n$. By Theorem 2.8, $\lim_{m\to\infty} |u_k(X^m)|^{1/m}$ exists for $k = 1, \cdots, n$. Example 3.1 indicates that $\lim_{m\to\infty} |u_k(X^m)|^{1/m}$ may or may not equal to an eigenvalue modulus of X.

EXAMPLE 3.1. Consider the following two situations:

1. Suppose $X = C^{-1}TC$ for $C := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \in \mathrm{SU}_2(\mathbb{C})$ and $T := \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then $\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for a Bruhat decomposition $C = V'\omega U'$ of C. We have

$$X^m = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^m \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2^m & 2^m - 1 \\ 0 & 1 \end{bmatrix}.$$

It turns out that

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = 2 = |\lambda_{\omega(1)}|,$$
$$\lim_{m\to\infty} |u_2(X^m)|^{1/m} = 1 = |\lambda_{\omega(2)}|.$$

2. Suppose $X = C^{-1}TC$ for $C := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \in \mathrm{SU}_2(\mathbb{C})$ and $T := \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.
Then $\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for a Bruhat decomposition $C = V'\omega U'$ of C. We have

$$X^m = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^m \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-2^m & 2^m \end{bmatrix}.$$

Clearly

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = 1 \neq |\lambda_{\omega(1)}| = 2,$$
$$\lim_{m\to\infty} |u_2(X^m)|^{1/m} = 2 \neq |\lambda_{\omega(2)}| = 1.$$

3. Suppose $X = C^{-1}TC$ for

$$C := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad T := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then $C^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, and

$$\begin{aligned} X^m &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1^m & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 3^m \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1+2^m & -1+2^m & 1-2^m \\ -1+3^m & 1+3^m & -1+3^m \\ -2^m+3^m & -2^m+3^m & 2^m+3^m \end{bmatrix}. \end{aligned}$$

It is clear that X^m has LU decomposition and

$$\lim_{m\to\infty} |u_1(X^m)|^{1/m} = \lim_{m\to\infty} \left| \frac{1+2^m}{2} \right|^{1/m} = 2,$$
$$\lim_{m\to\infty} |u_1(X^m)u_2(X^m)|^{1/m} = \lim_{m\to\infty} \begin{vmatrix} \frac{1+2^m}{2} & \frac{-1+2^m}{2} \\ \frac{-1+3^m}{2} & \frac{1+3^m}{2} \end{vmatrix}^{1/m} = 3.$$

Therefore, $\lim_{m\to\infty} |u_2(X^m)|^{1/m} = \frac{3}{2}$ is not an eigenvalue of X.

The next example shows that: if the k-compound of X has no distinct eigenvalue moduli for certain k, then $\lim_{m\to\infty}|u_i(X^m)|^{1/m}$ may not exist. A lemma is needed to illustrate the example.

LEMMA 3.2. *For every $t\in(0,1)$, there exists an irrational number $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and two positive integer sequences $\{p_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$ and $\{m_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$, such that for $k=1,2,\cdots$,*

1. *m_k is divisible by k, and*
2. *$\alpha\in I_k:=\left(\frac{p_k+1/4}{m_k},\frac{p_k+1/4+t^{m_k}}{m_k}\right)$.*

Proof. First we use induction to construct $\alpha\in\mathbb{R}$, $\{p_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$ and $\{m_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$ that satisfy (1) and (2). Then we show that α must be irrational.

Denote $m_1=p_1=1$ and $I_1:=(1+1/4,1+1/4+t)$. Suppose that $m_k,p_k\in\mathbb{Z}^+$, and

$$I_k:=\left(\frac{p_k+1/4}{m_k},\frac{p_k+1/4+t^{m_k}}{m_k}\right)\subseteq\mathbb{R}^+$$

are well-defined. Then there exists a sufficiently large $m_{k+1}\in\mathbb{Z}^+$ and a suitable $p_{k+1}\in\mathbb{Z}^+$ such that $k+1$ divides m_{k+1} and the closure of

$$I_{k+1}:=\left(\frac{p_{k+1}+1/4}{m_{k+1}},\frac{p_{k+1}+1/4+t^{m_{k+1}}}{m_{k+1}}\right)$$

is contained in I_k. By induction we obtain $\{p_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$, $\{m_k\}_{k=1}^{\infty}\subseteq\mathbb{Z}^+$, and the open interval sequence $\{I_k\}_{k=1}^{\infty}$ such that

$$\overline{I}_1\supset I_1\supset\overline{I}_2\supset I_2\supset\overline{I}_3\supset I_3\supset\overline{I}_4\supset\cdots.$$

Because the lengths of closed intervals in $\{\overline{I}_k\}_{k=1}^{\infty}$ are decreasing to 0, by the Nested Interval Theorem [1], $\bigcap_{i=1}^{\infty}\overline{I}_k=\bigcap_{i=1}^{\infty}I_k$ contains exactly one number $\alpha\in\mathbb{R}^+$.

We show that $\alpha\notin\mathbb{Q}$. Suppose on the contrary, $\alpha=\frac{a}{b}$ where $a,b\in\mathbb{Z}^+$. By simultaneously multiplying a positive integer c on a and b, we may assume that b is large enough so that $4t^{m_b}<4t^b<1$. Then

$$\alpha\in I_b=\left(\frac{p_b+1/4}{m_b},\frac{p_b+1/4+t^{m_b}}{m_b}\right).$$

Therefore,

$$4m_b\alpha\in(4p_b+1,4p_b+1+4t^{m_b})\subset(4p_b+1,4p_b+2). \tag{3.1}$$

However, $4m_b\alpha\in\mathbb{Z}^+$ since $\alpha=\frac{a}{b}$ and m_b is an integer multiple of b. This contradicts to (3.1). Thus $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and we are done.

EXAMPLE 3.3. Fix a number $t \in (0,1)$. Let $\theta := 2\pi\alpha$ where α is given in Lemma 3.2. Denote $X := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Then $X^m = \begin{bmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{bmatrix}$. Obviously $\cos m\theta \neq 0$ for all $m \in \mathbb{Z}^+$ and $\overline{\{\cos m\theta \mid m \in \mathbb{Z}^+\}} = [-1,1]$ since $\alpha = \dfrac{\theta}{2\pi}$ is irrational. So every X^m has *LU* decomposition and $|u_1(X^m)|^{1/m} = |\cos m\theta|^{1/m}$. We claim that $\lim_{m\to\infty} |u_1(X^m)|^{1/m} = \lim_{m\to\infty} |\cos m\theta|^{1/m}$ does not exist. On one hand, it is easy to find a subsequence $\{n_1, n_2, \cdots\} \subset \{1, 2, \cdots\}$ such that $\lim_{i\to\infty} |\cos n_i\theta|^{1/n_i} = 1$. On the other hand, let $\{m_k\}_{k=1}^{\infty}$ and $\{p_k\}_{k=1}^{\infty}$ be given as in Lemma 3.2. Then for $k = 1, 2, \cdots$,

$$m_k\theta = 2m_k\pi\alpha \in \left(2p_k\pi + \frac{\pi}{2}, 2p_k\pi + \frac{\pi}{2} + 2\pi t^{m_k}\right). \tag{3.2}$$

Hence

$$|\cos m_k\theta|^{1/m_k} = |-\sin(m_k\theta - 2p_k\pi - \frac{\pi}{2})|^{1/m_k} \leqslant |2\pi t^{m_k}|^{1/m_k} \to t.$$

Therefore, $\lim_{m\to\infty} |u_1(X^m)|^{1/m}$ does not exist.

REFERENCES

[1] R. G. BARTLE AND D. R. SHERBERT, *Introduction to Real Analysis*, Wiley, 1999.

[2] R. GOODMAN AND N. R. WALLACH, *Representations and Invariants of the Classical Groups*, Cambridge University Press, 2000.

[3] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1991.

[4] H. HUANG AND T. Y. TAM, *An asymptotic behavior of QR decomposition*, Linear Algebra and Its Applications, **424** (2007) 96–107.

[5] H. HUANG AND T. Y. TAM, *On Gelfand-Naimark decomposition of a nonsingular matrix*, Linear and Multilinear Algebra (to appear).

[6] H. HUANG AND T. Y. TAM, *On the QR iterations of real matrices*, Linear Algebra and Its Applications, **408** (2005) 161–176.

[7] H. HUANG AND T. Y. TAM, *Some asymptotic behaviors associated with matrix decomposition*, International J. of Information & Systems Sciences, (2008) 148–159.

[8] E. TYRTYSHNIKOV, *Matrix Bruhat decompositions with a remark on the QR (GR) algorithm*, Linear Algebra and Its Applications, **250** (1997) 61–68.

[9] T. YAMAMOTO, Extreme values of the roots of matrices, J. Math. Soc. Japan, **19** (1967) 171–178.

(Received February 28, 2009)

Huajun Huang
Department of Mathematics and Statistics
Auburn University
AL 36849–5310, USA
e-mail: huanghu@auburn.edu