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ESTIMATING MATCHING DISTANCE BETWEEN SPECTRA

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Abstract. We show that if a, b are elements of an unital Banach algebra such that almost all convex combinations of a and b have a finite spectrum of cardinality n, then the optimal matching distance between their spectra satisfies

$$D(\sigma(a),\sigma(b)) \leqslant c_n\,(\|a\|+\|b\|)^{1-1/n}\,\|a-b\|^{1/n},$$

where $c_n \leqslant 8(1+1/n)(n/2)^{1/n}$.

1. Introduction

Let $K=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ and $L=\{\mu_1,\mu_2,\dots,\mu_n\}$ be two n-tuples of (not necessarly distinct) complex numbers. The optimal matching distance between these sets is given by:

$$D(K,L)=\min_{\pi\in S_n}\max_{1\leqslant i\leqslant n}|\lambda_i-\mu_{\pi(i)}|,$$

where S_n denotes the group of all permutations of $\{1,2,\dots,n\}$.

The distance D was introduced by Dufresnoy to resolve some combinatorial problems, D is also computed as:

$$D(K,L)=\max_{\substack{I,J\subset\{1,2,..,n\}\\ \#(I)+\#(J)=n+1}}\ \min_{i\in I,j\in J}|\lambda_i-\mu_i|.$$

Let Δ denote the Hausdorff distance given by:

$$\Delta(K,L)=\max\{\max_{\lambda\in K} dist(\lambda,L),\max_{\lambda\in L} dist(\lambda,K)\}.$$

The optimal matching distance is always greater than the Hausdorff one and they coincide in the case of sets of two elements.

D,Δ and several other distances are used in the study of variation eigenvalues of matrices. In the literature we can find a lot of spectral variation results for hermitian,

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normal and general matrices (see [1], [2], [3], [4]). When dealing with nonnormal matrices the known perturbation bounds are of the form

$$D(\sigma(a),\sigma(b)) \leqslant c_n(\|a\|+\|b\|)^{1-\frac{1}{n}}\|a-b\|^{\frac{1}{n}},$$

where c_n is a constant depending on the size of matrices and $\sigma(a)$ denotes the spectrum of a.

By results of Elsner and Bhatia-Elsner-Krause (see [3], [7]), if a,b are $n\times n$ matrices, then:

$$\Delta(\sigma(a),\sigma(b)) \leqslant (\|a\|+\|b\|)^{1-\frac{1}{n}}\|a-b\|^{\frac{1}{n}},$$
$$D(\sigma(a),\sigma(b)) \leqslant 4(\|a\|+\|b\|)^{1-\frac{1}{n}}\|a-b\|^{\frac{1}{n}}.$$

The results of this type have been generalized for the Hausdorff distance in [6], to algebraic elements of an arbitrary unital Banach algebra: if a, b are algebraic of degree at most n, we have:

$$\Delta(\sigma(a),\sigma(b)) \leqslant c_n(\|a\|+\|b\|)^{1-\frac{1}{n}}\|a-b\|^{\frac{1}{n}},$$

where $c_n \leqslant \left(\frac{2}{3}n+\frac{1}{3}\right)^{\frac{1}{n}}$.

Here an analogous question for the matching distance D is treated, namely, the perturbation bounds for elements of a general unital Banach algebra with finite spectra (in fact, a stronger condition that almost all convex combinations of a and b have a finite spectrum is considered). The main result is:

Let a,b elements of an unital Banach algebra A. Suppose that the cardinality of $\sigma(a+t(b-a))$ is n for almost all $t\in[0,\ 1]$. Then

$$D(\sigma(a),\sigma(b)) \leqslant c_n(\|a\|+\|b\|)^{1-1/n}\|a-b\|^{1/n}$$

where $c_n \leqslant 8(1+1/n)(n/2)^{\frac{1}{n}}$.

2. Algebroid multifunctions

DEFINITION 1. A mapping K from an open subset Ω of $\mathbb{C}$ into the set of non empty compact subsets of $\mathbb{C}$ is called an *algebroid multifunction* if it is of the form

$$K(z)=\{w:\ w^n+a_1(z)w^{n-1}+\dots+a_{n-1}(z)w+a_n(z)=0\},$$

where $a_1,a_2,\dots,a_n$ are holomorphic functions on Ω. The degree of K is the smallest integer n such that K has such a representation.

Algebroid multifunctions are extension of holomorphic functions and characterize the finite analytic multifunctions (see [1], 7.2.4, p. 155).

The following lemma is a version of Schwarz lemma for the matching distance. It can be deduced from [9], Theorem 1.3. We prove the result by the technique used in [2] Theorem VIII 2.4.

LEMMA 1. *Let K be an algebroid multifunction of degree n, on the unit disk U into the set of non empty compacts sets of U. Then*

$$D(K(z),K(0)) \leqslant 8\left|\frac{z}{2}\right|^{\frac{1}{n}} \quad (z \in U).$$

Proof As K is an algebroid multifunction of degree n, the set $K(z)$ contains $w_1(z), w_2(z), \cdots$ and $w_n(z)$, n distinct points for all z outside some closed discrete subset E of U [[1] Theorem 3.4.25].

Let γ be a curve joining 0 to z, staying within the disk of center 0 and radius $|z|$, and avoiding the set E except perhaps for its endpoints. There are n curves $\Gamma_1, \Gamma_2, \cdots$ and Γ_n which trace $K(\gamma)$, setting up a bijection between $K(0)$ and $K(z)$ (counting multiplicities). It follows that:

$$D(K(z),K(0)) \leqslant \max_{1 \leqslant j \leqslant n} \operatorname{diam} \Gamma_j.$$

To get the result, it suffice to estimate the diameters of the curves $\Gamma_k, k = 1,2,\cdots,n$. For this we need a version of Schwarz lemma of algebroid multifunctions presented in [9].

Let b the Blaschke product of degree n associated to $K(0)$:

$$b(w) = \prod_{1 \leqslant j \leqslant n} \left(\frac{w - w_j(0)}{1 - \overline{w_j(0)}w} \right).$$

By [[9] lemma 2.1], we have for all $z \in U$ and $w \in K(z)$:

$$|b(w)| \leqslant |z|.$$

Then

$$\begin{aligned} \prod_{1 \leqslant j \leqslant n} \left|w - w_j(0)\right| &\leqslant |z| \prod_{1 \leqslant j \leqslant n} \left|1 - \overline{w_j(0)}w\right| \\ &\leqslant 2^n |z|. \end{aligned}$$

It is shown in [[2] lemma VIII 1.4, p:228] that if Γ is a compact connected subset of $\mathbb{C}$ and p is a monic polynomial of degree n, then

$$\operatorname{diam} \Gamma \leqslant 4.2^{-\frac{1}{n}} \max_{w \in \Gamma} |p(w)|^{\frac{1}{n}}.$$

In particular for $k \in \{1,2,\cdots,n\}$, we have

$$\begin{aligned}\operatorname{diam} \Gamma_k &\leqslant 4.2^{-\frac{1}{n}} \max_{w\in\Gamma_k} \left(\prod_{1\leqslant j\leqslant n} \left|w - w_j(0)\right| \right)^{\frac{1}{n}} \\ &\leqslant 8.2^{-\frac{1}{n}} |z|^{\frac{1}{n}}.\end{aligned}$$

Finaly

$$\begin{aligned}D(K(z),K(0)) &\leqslant \max_{1\leqslant k\leqslant n} \operatorname{diam} \Gamma_k \\ &\leqslant 8.2^{-\frac{1}{n}} |z|^{\frac{1}{n}}.\end{aligned}$$

3. Matching estimates for the spectrum

Let A,B two $n\times n$ complex matrices, it is known that:

$$D(\sigma(A),\sigma(B)) \leqslant \frac{16}{3\sqrt{3}}(2M)^{1-1/n}\|A-B\|^{1/n},$$

where $\| \, . \, \|$ is the operator norm, and $M = \max(\|A\|,\|B\|)$ ([8], theorem1, p. 78). Similar estimates are given for other norms in matrix algebras (see [3], p. 197).

Using analytic arguments we give here estimation in a general unital Banach algebra.

THEOREM 1. *Let* $(A,\| \ \|)$ *be an unital Banach algebra, and let* $a,b\in A$. *Suppose that the cardinality of* $\sigma(a+t(b-a))$ *is* n *for almost all* $t\in[0,\ 1]$. *Then*

$$D(\sigma(a),\sigma(b)) \leqslant c_n(\|a\|+\|b\|)^{1-1/n}\|a-b\|^{1/n} \tag{3.1}$$

where $c_n \leqslant 8(1+1/n)(n/2)^{\frac{1}{n}}$.

Proof Let f be the analytic function from the complex unit disk U into the unital Banach algebra A defined by

$$f(z) = (\|a\|+r\|b-a\|)^{-1}(a+rz(b-a)) \qquad (z\in U,\ r>1).$$

Note that for all $z\in U$, the spectrum of $f(z)$ is a subset of the unit disk U. Consequently the multifunction defined on U by:

$$K(z) = \sigma(f(z))$$

is analytic on U into the set of non-empty compacts of U.
The condition $\#(\sigma(a+t(b-a))) = n$ for almost all $t\in[0,1]$ ensures that K is finite,

and by [[1] theorem 7.1.7, p. 147], K is an algebroid multifunction of degree n. From the preceding lemma, we deduce that

$$D(K(1/r),K(0)) \leqslant 8.(2r)^{-\frac{1}{n}}.$$

Therefore

$$\begin{aligned} D(\sigma(a),\sigma(b)) &\leqslant 8(2r)^{-\frac{1}{n}}(\|a\|+r\|b-a\|) \\ &\leqslant 8(2r)^{-\frac{1}{n}}(\|a\|+r\|b-a\|+\|b\|). \end{aligned}$$

In the case that $\|a\|+\|b\|>n\|b-a\|$, we take $r=\frac{\|a\|+\|b\|}{n\|b-a\|}$ and conclude

$$D(\sigma(a),\sigma(b)) \leqslant 8(1+1/n)(n/2)^{\frac{1}{n}}(\|a\|+\|b\|)^{1-\frac{1}{n}}\|a-b\|^{\frac{1}{n}}$$

If $\|a\|+\|b\| \leqslant n\|b-a\|$, we obtain

$$\begin{aligned} D(\sigma(a),\sigma(b)) &\leqslant (\|a\|+\|b\|) \\ &\leqslant (\|a\|+\|b\|)^{1-\frac{1}{n}} n^{\frac{1}{n}} \|b-a\|^{\frac{1}{n}}, \end{aligned}$$

and final result follows.

For the Hausdorff distance, the constant can be improved to $2(1+1/n)n^{\frac{1}{n}}$ by replacing the result of lemma by an immediate consequence of Theorem 1.1 of [9], and following the same arguments in the proof of theorem.

4. Remarks

4.1. Polarity assumption

It is stated in Aupetit's book [1] (scarcity theorem), that if $\sigma(a+\lambda(b-a))$ is finite for all λ in some non-polar subset of $\mathbb{C}$, then there exists an integer p such that $\#\sigma(a+\lambda(b-a)) \leqslant p$ for all $\lambda \in \mathbb{C}$, with equality for all λ outside a closed, discrete set. Thus, in Theorem 1, it suffices to assume that $\#\sigma(a+\lambda(b-a))=n$ for λ in some non polar set E in $\mathbb{C}$. Polar sets are sets of capacity zero, they are negligible sets for potential theory (for more details see [10], 3.2, p.55).

4.2. Example of application

Let $\mathscr{B}(X)$ be the algebra of linear bounded operators on a Banach space X. Let A be a finite rank operator and B any algebraic element of $\mathscr{B}(X)$. By [[5], corollary 4.5] for all λ in the unit disk U, the operator $(1-\lambda)A+\lambda B$ is algebraic. And by Aupetit's result, there is an integer n such that n is the degree of almost all algebraic elements in the segment $[A,B]$. Then we have estimation like (3.1) for the operators A and B.

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