Operators
and
Matrices
Volume 3, Number 4 (2009), 509–515

ABSOLUTE CONTINUITY OF MINIMAL UNITARY DILATIONS

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(*Communicated by H. Bercovici*)

Abstract. The paper considers the question of absolute continuity of minimal unitary dilation of an absolutely continuous pair of commuting Hilbert space contractions. More general result covering commuting N-tuples, operator representations and their minimal dilations is given.

1. Introduction and Preliminaries.

The notion of absolute continuity of operator representations and N-tuples of contractions has been studied intensively in [6], [8], [9], [10]. The present paper is based on the mentioned results. The considered problem is also related to the problem of the weak-star continuity of minimal dilations of representations (see [3], [7]). The problem mentioned in the abstract has been communicated to the author by Mark Malamud.

Let X be a compact Hausdorff space. By $C(X)$ we will denote the Banach algebra of all complex continuous functions on X equipped with the supnorm $\|\cdot\|_\infty$, and by $M(X)$ for the set of all complex Borel measures on X. Considered with the total variation norm, $M(X)$ is a Banach space. For a measure $\mu \in M(X)$ denote by $|\mu|$ its variation measure. Let A be a function algebra on X, i.e. a closed subalgebra of $C(X)$ which contains constants and separates the points of X.

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A representation Φ of a function algebra $A \subset C(X)$ in $\mathcal{H}$ is a continuous, unit preserving homomorphism

$$\Phi : A \ni u \to \Phi(u) \in \mathcal{L}(\mathcal{H})$$

of Banach algebras. In other words, Φ is linear, multiplicative, $\Phi(1)$ is the identity operator I and there exists a positive constant K such that

$$\|\Phi(u)\| \leqslant K\|u\|, \qquad u \in A. \tag{1}$$

When $K = 1$, we say that Φ is contractive. If moreover X is a compact subset of $\mathbb{C}^N$ and A contains the coordinate functions z_j, $j \leqslant N$, such a representation is often referred to as an A – functional calculus for $T_1,\ldots,T_N$, where $T_j = \Phi(z_j)$, i.e. a functional calculus for the operator N-tuple $(T_1,\ldots,T_N)$ based on A. We write then $u(T_1,\ldots,T_N)$ rather than $\Phi(u)$.

Mathematics subject classification (2000): Primary: 47A20; Secondary: 47A60, 46J10, 46E25.
Keywords and phrases: contractions, dilations, representations, absolute continuity, decompositions.
This work has been supported by the MNiSzW grant N201 026 32/1350.

Paper OaM-03-28

2. Minimal dilations.

Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{K}$ be a Hilbert space such that $\mathcal{H} \subset \mathcal{K}$. A linear bounded operator W on $\mathcal{K}$ is a dilation of T if $T^n x = P_{\mathcal{H}} W^n x$ for $x \in \mathcal{H}$ and $n \in \mathbb{N}$. ($P_{\mathcal{H}}$ will always denote the orthogonal projection on $\mathcal{H}$.) The operator W is called a minimal dilation if there is no proper closed subspace $\mathcal{M}$ of $\mathcal{K}$ reducing W and such that $\mathcal{H} \subset \mathcal{M}$. A minimal dilation may be non-unique. If W is unitary then it is called a unitary dilation. It is obvious that if W is unitary and has a reducing subspace then its restriction to this subspace is also unitary. So a minimal unitary dilation in the sense that there is no proper subspace containing $\mathcal{H}$ and reducing W to a unitary operator is also a minimal dilation in the general sense.

Let $\Phi : A \ni u \to \Phi(u) \in \mathcal{L}(\mathcal{H})$ be a representation of a function algebra A such that $\|\Phi(u)\| \leqslant K\|u\|$ for some constant K. Similarly like for an operator, a representation $\Psi : A \ni u \to \Psi(u) \in \mathcal{L}(\mathcal{K})$ is a dilation of Φ if $\mathcal{H} \subset \mathcal{K}$, $\Phi(u)x = P_{\mathcal{H}}\Psi(u)x$ for $u \in A$, $x \in \mathcal{H}$, and $\|\Psi(u)\| \leqslant K\|u\|$. Consequently if Ψ is a dilation of Φ then for each $u \in A$ the operator $\Psi(u)$ is a dilation of $\Phi(u)$. We say that a subspace $\mathcal{M}$ reduce a representation Ψ if $\mathcal{M}$ reduce all the operators $\Psi(u)$ for $u \in A$. The representation Ψ is called a minimal dilation if there is no proper closed subspace $\mathcal{M}$ of $\mathcal{K}$ reducing Ψ and such that $\mathcal{H} \subset \mathcal{M}$.

3. Reducing bands of measures.

A measure $\mu \in M(X)$ is said to be *orthogonal to a function algebra A* or *annihilating A* if

$$\int u \, d\mu = 0 \quad \text{for } u \in A.$$

The set of all such measures is denoted by $A^{\perp}$ and called the *annihilator* of A.

A norm closed subspace $\mathcal{M} \subset M(X)$ is called a *band* if it is "closed" also with respect to the absolute continuity i.e.

$$\nu \ll |\mu|,\ \mu \in \mathcal{M} \implies \nu \in \mathcal{M}.$$

If $\mathcal{M}$ is a band then any measure $\mu \in M(X)$ has a Lebesgue-type decomposition of the form

$$\mu = \mu^{\mathcal{M}} + \mu^{\mathcal{M}^s}, \tag{1}$$

where $\mu^{\mathcal{M}} \in \mathcal{M}$ and $\mu^{\mathcal{M}^s} \in \mathcal{M}^s$ (see [5], Thm. V.17.4). As a consequence we get

PROPOSITION 3.1. *If $\mathcal{M} \subset M(X)$ is a band then*

$$(f\mu)^{\mathcal{M}} = f\mu^{\mathcal{M}} \quad \textit{for } f \in C(X),$$

where $f\mu$ denotes the measure defined by $\int g \, d(f\mu) = \int gf \, d\mu$ for $g \in C(X)$.

In case when the only one band is considered we will write μ^s instead of $\mu^{\mathcal{M}^s}$ for the singular part of μ.

Now we give some properties and terminology concerning bands which will be useful later. For proofs and details we refer the reader to [4], sec. 20 and to [5], V.17. For any subset $E \subset M(X)$ we denote by E^s the set of all measures singular to each measure in E. Observe that E^s is always a band. The smallest band containing E is called the band generated by E and will be denoted by $\langle E \rangle$.

PROPOSITION 3.2.

1. *For $E \subset M(X)$ the set $E^{ss} \overset{df}{=} (E^s)^s$ is equal to the band generated by E.*
2. *If $\mathcal{M}$ is a band then $\mathcal{M}^{ss} = \mathcal{M}$.*

A band $\mathcal{M}$ is said to be *reducing with respect to A* if for every measure $\mu \in A^{\perp}$, we have $\mu_{\mathcal{M}} \in A^{\perp}$ in the decomposition (1). The reducing bands have an important application for operator representations of function algebras. Mlak in [13] used the reducibility property of Gleason parts. Szafraniec (cf. [14]) introduced and used in the general context projections in $M(X)$ having some special property. Their ranges were exactly the reducing bands.

4. Band decomposition of representations.

Let us consider a representation $\Phi : A \to \mathcal{L}(\mathcal{H})$ of a function algebra $A \subset C(X)$ and vectors $x, y \in \mathcal{H}$. It is well known that the linear functional

$$A \ni u \to \langle \Phi(u)x, y \rangle \in \mathbb{C}$$

(after extension to $C(X)$) is represented by a Borel measure $\mu_{xy} \in M(X)$ of total variation $\|\mu_{xy}\|$ bounded by $K\|x\|\|y\|$, so that for any $u \in A$ one has

$$\langle \Phi(u)x, y \rangle = \int_X u \, d\mu_{xy}. \tag{1}$$

By a *system of elementary measures for a representation* Φ we mean any system $\{\mu_{xy}\}_{x,y \in \mathcal{H}}$ in $M(X)$ satisfying (1). Such systems are not unique, unless A is a Dirichlet algebra on X. Using them we can introduce the notions of absolute continuity and of singularity for our representations.

DEFINITION 4.1. The representation Φ is absolutely continuous with respect to a band of measures $\mathcal{M} \subset M(X)$, if Φ has a system of elementary measures $\{\mu_{xy}\}_{x,y \in \mathcal{H}} \subset \mathcal{M}$. The singularity of Φ with respect to $\mathcal{M}$ means $\mu_{xy} \in \mathcal{M}^s$ for a certain system of its elementary measures $\{\mu_{xy}\}_{x,y \in \mathcal{H}}$.

If $\langle \nu_0 \rangle = \{\nu \in M(X) : \ \nu \ll \nu_0\}$ is the band generated by ν_0, the relations of absolute continuity and of singularity of Φ with respect to ν_0 defined in the natural way by requiring that for any $x, y \in \mathcal{H}$

$$\mu_{xy} \ll \nu_0, \qquad (\text{resp. } \mu_{xy} \perp \nu_0) \qquad \text{for some } \mu_{xy}$$

coincide with the corresponding absolute continuity (resp. singularity) with respect to the band $\mathcal{M} = \langle \nu_0 \rangle$.

The pioneering work of Mlak [13], without using the notion of bands, was in fact concerned with the bands generated by Gleason parts of A. Many of the results extend easily to the general case of reducing bands implicit in the study done by Szafraniec [14]. Among these we find following useful observation.

REMARK 4.2. For the reducing bands the property of the absolute continuity (resp. singularity) established in Definition 4.1 is equivalent to the fact that the equality

$$\langle \Phi(u)x, y\rangle = \int_X u\, d\mu_{xy}^{\mathcal{M}} \qquad (\text{resp. } \langle \Phi(u)x, y\rangle = \int_X u\, d\mu_{xy}^{s}) \tag{2}$$

holds for any system of elementary measures $\{\mu_{xy}\}_{x,y\in\mathcal{H}}$ of Φ. Consequently it does not depend on the choice of a particular system of elementary measures. (The notion of $\mu_{xy}^{\mathcal{M}}, \mu_{xy}^{s}$ is that of (1).) We may use therefore a suggestive notation

$$\Phi \ll \mathcal{M} \qquad (\text{resp. } \Phi \ll \mathcal{M}^s)$$

for absolute continuity (resp. singularity) of Φ with respect to $\mathcal{M}$.

Basic to further considerations is the possibility of decomposing into a direct sum of an absolutely continuous and a singular representations, established for a special case in [13] and in the general setting (but differently formulated) in [14].

THEOREM 4.3. *If $\mathcal{M} \subset M(X)$ is a reducing band for a function algebra A on X, then any representation $\Phi : A \to \mathcal{L}(\mathcal{H})$ decomposes into the sum*

$$\Phi = \Phi_a + \Phi_s \tag{3}$$

of two representations Φ_a, Φ_s denoted also $\Phi_a = \Phi_{\mathcal{M}} \ll \mathcal{M}$ and $\Phi_s = \Phi_{\mathcal{M}^s} \ll \mathcal{M}^s$. This decomposition corresponds to the direct sum decomposition

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_s \tag{4}$$

onto (invariant for Φ_a, Φ_s) subspaces determined as the ranges of the projections $\Phi_a(1), \Phi_s(1)$. These projections (as well as the decomposition (3)) are orthogonal if the estimate (1) holds with $K = 1$.

To be precise, if for $x, y \in \mathcal{H}$ we perform the Lebesgue-type decomposition of the elementary measures $\mu_{xy} = \mu_{xy}^a + \mu_{xy}^s$ so that $\mu_{xy}^a \in \mathcal{M}$, $\mu_{xy}^s \in \mathcal{M}^s$, then $\Phi_a(u)x$ is defined as the only vector in $\mathcal{H}$ satisfying

$$(\Phi_a(u)x, y) = \int u\, d\mu_{xy}^a, \qquad y \in \mathcal{H}.$$

Using the assumption that $\mathcal{M}$ is reducing we obtain linearity and multiplicativity of Φ_a (and of analogously defined Φ_s), together with the (1)-type estimates with the same bound K as for Φ. If $\mathcal{H}_a, \mathcal{H}_s$ are the ranges of $\Phi_a(1), \Phi_s(1)$, then

$$\mathcal{H}_a \cap \mathcal{H}_s = \{0\}, \qquad \mathcal{H}_a + \mathcal{H}_s = \mathcal{H}$$

and the restrictions $\Phi_a(u)|_{\mathcal{H}_a}$, $\Phi_s(u)|_{\mathcal{H}_s}$ (rather than Φ_a, Φ_s themselves) give representations.

5. Minimal dilations of contractive representations.

Let A be a function algebra on a compact set X, let $\Phi : A \ni u \to \Phi(u) \in \mathcal{L}(\mathcal{H})$ be a contractive representation of A i.e. $\|\Phi(u)\| \leqslant \|u\|$ for $u \in A$. Let $\Psi : A \ni u \to \Psi(u) \in \mathcal{L}(\mathcal{K})$ be a minimal dilation of Φ, and let $\mathcal{M}$ be a reducing band of measures in $M(X)$.

THEOREM 5.1. *If* $\Phi \ll \mathcal{M}$ *then* $\Psi \ll \mathcal{M}$.

Proof. By Theorem 4.3, we have orthogonal decompositions

$$\mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s, \quad \Psi = \Psi_a \oplus \Psi_s,$$

where $\Psi_a \ll \mathcal{M}$, $\Psi_s \ll \mathcal{M}^s$, the subspaces $\mathcal{K}_a$, $\mathcal{K}_s$ reduce Ψ, and

$$\Psi|_{\mathcal{K}_a} = \Psi_a, \quad \Psi|_{\mathcal{K}_s} = \Psi_s.$$

We assumed $\Phi \ll \mathcal{M}$. Then, by (2), the representation Φ has a system of elementary measures $\{\mu_{xy}\}_{x,y\in\mathcal{H}} \subset \mathcal{M}$. So for $u \in A$, $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \int_X u\, d\mu_{xy} &= \langle \Phi(u)x, y\rangle = \langle \Psi(u)x, y\rangle \\ &= \langle \Psi_a(u)x, y\rangle + \langle \Psi_s(u)x, y\rangle = \int_X u\, d\nu_{xy}^{\mathcal{M}} + \int_X u\, d\nu_{xy}^{s} \end{aligned} \tag{1}$$

for any system of elementary measures $\{\nu_{xy}\}_{x,y\in\mathcal{K}}$ of Ψ. Moreover, by (1), for any fixed $x, y \in \mathcal{H}$ we get

$$\int_X u\, d\mu_{xy} = \int_X u\, d\nu_{xy}^{\mathcal{M}} + \int_X u\, d\nu_{xy}^{s} \quad \text{for } u \in A,$$

which means that the measure $\mu_{xy} - \nu_{xy}^{\mathcal{M}} - \nu_{xy}^{s}$ is orthogonal to A. Since $\mu_{xy} \subset \mathcal{M}$ for $x, y \in \mathcal{H}$ and $\mathcal{M}$ is a reducing band then also the measures $\mu_{xy} - \nu_{xy}^{\mathcal{M}}$ and ν_{xy}^{s} must be orthogonal to A. Hence in particular

$$\langle \Psi_a(u)x, y\rangle = \int_X u\, d\nu_{xy}^{\mathcal{M}} = \int_X u\, d\mu_{xy} = \langle \Phi(u)x, y\rangle$$

for all $u \in A$ and $x, y \in \mathcal{H}$. So $P_{\mathcal{H}}\Psi_a|_{\mathcal{H}} = \Phi$, which means that Ψ_a is a dilation of Φ. Since Ψ was a minimal dilation, we have $\Psi = \Psi_a \ll \mathcal{M}$.

6. Minimal dilations of N-tuples of contractions.

The notion of absolute continuity has appeared in the preceding section in the context of representations. Now we would recall its meaning for N-tuples of commuting contractions i.e. operators in $\mathcal{L}(\mathcal{H})$ with the norm less or equal one. We consider here only such N-tuples $(T_1, \ldots, T_N)$ on $\mathcal{H}$ which have unitary dilations i.e. for which there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an N-tuple $(U_1, \ldots, U_N)$ of commuting unitary operators on $\mathcal{K}$ such that $T_1^{n_1}\dots T_N^{n_N}x = P_{\mathcal{H}}U_1^{n_1}\dots U_N^{n_N}x$ for $x\in\mathcal{H}$ and all the indices $n_1,\dots,n_N$. Single contractions and pairs of commuting contractions always have such dilations (see [1]).

Denote by $\mathbb{D}^N$ the closed unit polydisc in $\mathbb{D}^N$, and by $A(\mathbb{D}^N)$ the algebra of all continuous functions on $\mathbb{D}^N$ which are analytic on its interior. By Spectral Theorem for unitary operators, the N-tuple $(U_1,\dots,U_N)$ generates a contractive representation $\Psi : A(\mathbb{D}^N)\ni u\to\Psi(u)\in\mathcal{L}(\mathcal{K})$ such that $U_j=\Psi(z_j)$ for the coordinate functions z_j, $j=1,\dots,N$. Hence if we define $\Phi(u)x \overset{\text{df}}{=} P_{\mathcal{H}}\Psi(u)x$ for $u\in A(\mathbb{D}^N)$ and $x\in\mathcal{H}$, then the mapping $\Phi : A(\mathbb{D}^N)\ni u\to\Phi(u)\in\mathcal{L}(\mathcal{H})$ will be a representation generated by $(T_1,\dots,T_N)$ i.e. satisfying $T_j=\Phi(z_j)$ for $j=1,\dots,N$.

We say that Φ is *absolutely continuous* if it is absolutely continuous with respect to the band generated by all measures representing points in the interior of $\mathbb{D}^N$ (see [10] for details). By the absolute continuity of the N-tuple we mean the absolute continuity of the related representation. So, by Theorem 4.3 we get

THEOREM 6.1. *Let $(T_1,\dots,T_N)$ be an absolutely continuous N-tuple of commuting contractions on a Hilbert space $\mathcal{H}$, which has a unitary dilation $(U_1,\dots,U_N)$. If $(U_1,\dots,U_N)$ is minimal, then it is also absolutely continuous.*

REMARK 6.2. It is shown ([10], Main Theorem) that the absolute continuity of $(T_1,\dots,T_N)$ is equivalent to the following condition formulated by Apostol in [2]:

$$\lim_{n\to\infty}\sup_{\|p\|_\infty\leqslant 1}|\langle p(T_1,\dots,T_N)T_j^n x,y\rangle| = 0 \quad \text{for } x,y\in\mathcal{H},$$

where the supremum is taken over all complex polynomials $p\in\mathbb{C}[z_1,\dots,z_N]$ with $\|p\|_\infty\leqslant 1$.

REMARK 6.3. The above condition can be established also for N-tuples which do not generate any representation. In such a case it implies so called *total non singularity* of $(T_1,\dots,T_N)$ (see [11]).

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(Received March 16, 2009)

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