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BISHOP'S PROPERTY (β) FOR PARANORMAL OPERATORS

ATSUSHI UCHIYAMA AND KOTARO TANAHASHI

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Abstract. For an operator T on a separable complex Hilbert space $\mathscr{H}$, we say that T has Bishop's property (β) if for any open subset $\mathscr{D} \subset \mathbb{C}$ and any sequence of analytic functions $f_n : \mathscr{D} \to \mathscr{H}$ such as $\|(T-z)f_n(z)\| \to 0$ as $n \to \infty$ uniformly on every compact subset $\mathscr{K} \subset \mathscr{D}$, then $f_n \to 0$ uniformly on $\mathscr{K}$. It is a very important property in spectral theory. It is well-known that every normal operator ($T^*T = TT^*$) has Bishop's property (β). Now, many mathematicians attempt to extend this result to non-normal operators.

In this paper, we shall show that every paranormal operator ($\|T^2x\|\|x\| \geqslant \|Tx\|^2$ for all $x \in \mathscr{H}$) has Bishop's property (β).

1. Introduction.

Studying non-normal operators is a very important subject in operator theory, Bishop's property (β) is one of important and interesting topics in this subject and has been studied by many mathematicians. Several important Hilbert space operator classes are defined as follows: T belongs to

(1) the class of hyponormal operators if and only if $T^*T \geqslant TT^*$,
(2) the class of p-hyponormal operators for a $p > 0$ if and only if $(T^*T)^p \geqslant (TT^*)^p$,
(3) the class of w-hyponormal operators if and only if $|\widetilde{T}| \geqslant |T| \geqslant |\widetilde{T}^*|$,
(4) class A if and only if $|T^2| \geqslant |T|^2$,
(5) the class of paranormal operators if and only if $\|T^2x\|\|x\| \geqslant \|Tx\|^2$ for all $x \in \mathscr{H}$,

where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transform [1] of T with the polar decomposition $T = U|T|$.

The class of p-hyponormal operators was defined by Xia [12] and Aluthge [1], the class of w-hyponormal operators was defined by Aluthge and Wang [2], class A was defined by Furuta, Ito and Yamazaki [6] and the class of paranormal operators was defined by Istrăţescu, Saitō and Yoshino [7] as class (N) and Furuta renamed this class from class (N) to paranormal [5]. Inclusion relations among these operator classes are well known as follows:

$$\{\text{normal}\} \subset \{\text{hyponormal}\} \subset \{p\text{-hyponormal}\,(0<p<1)\} \subset \{w\text{-hyponormal}\} \subset \{\text{class } A\} \subset \{\text{paranormal}\}.$$

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To study a class of non-normal operators, it is important and interesting to verify that every operator which belongs to the class has Bishop's property (β) or not. Recently Kimura [8] has shown that every w-hyponormal operator has Bishop's property (β). Moreover, Chō and Yamazaki [4] extend this result to class A operators. These are subclasses of the class of paranormal operators. We extend their results to the class of paranormal operators. Laursen [9] proved that if T is totally paranormal, i.e., $T-\lambda$ is paranormal for every $\lambda \in \mathbb{C}$, then T has the single valued extension property (SVEP) and the property (C). Here, we say that an operator $T \in \mathscr{B}(\mathscr{H})$ has the (SVEP) if for every open set $\mathscr{D} \subset \mathbb{C}$ the zero function is the only analytic solution $f : \mathscr{D} \to \mathscr{H}$ of the equation

$$(T-z)f(z) = 0,$$

and T has the property (C) if the analytic spectral manifold

$$X_T(F) = \{x \in \mathscr{H} \mid {}^{\exists} f : F^c \to \mathscr{H}; \text{analytic s.t. } (z-T)f(z) \equiv x\}.$$

is closed for every closed set $F \subset \mathbb{C}$. We remark that Bishop's property (β) implies the property (C) and the property (C) implies the (SVEP). Thus our result is also a further extension of Lausen's results.

For an operator T, we denote the approximate point spectrum of T by $\sigma_a(T)$. We define the spectral properties (I), (I') and (II) as follows: T has the property

(I) if $\lambda \in \sigma_a(T)$ and $\{x_n\}$ is a sequence of bounded vectors of $\mathscr{H}$ satisfying $\|(T-\lambda)x_n\| \to 0$ (as $n \to \infty$), then $\|(T-\lambda)^* x_n\| \to 0$ (as $n \to \infty$),

(I′) if $\lambda \in \sigma_a(T) \setminus \{0\}$ and $\{x_n\}$ is a sequence of bounded vectors of $\mathscr{H}$ satisfying $\|(T-\lambda)x_n\| \to 0$ (as $n \to \infty$), then $\|(T-\lambda)^* x_n\| \to 0$ (as $n \to \infty$),

(II) if $\lambda,\ \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and sequences of bounded vectors $\{x_n\}$ and $\{y_n\}$ of $\mathscr{H}$ satisfy $\|(T-\lambda)x_n\| \to 0$ and $\|(T-\mu)y_n\| \to 0$ (as $n \to \infty$), then $\langle x_n, y_n\rangle \to 0$ (as $n \to \infty$),

where $\langle \cdot,\cdot \rangle$ is the inner product on $\mathscr{H}$.

It is well known that every hyponormal or p-hyponormal operator has the property (I) (see [3], [12]), and every w-hyponormal or class A operator has the property (I') (see [10], [11]). Clearly the property (I) implies the property (I').

In this paper, we shall show the following:

(i) The property (I') implies the property (II).

(ii) Every paranormal operator has the property (II).

(iii) The property (II) implies Bishop's property (β).

Thus every paranormal operator on a complex Hilbert space has Bishop's property (β).

2. Preliminaries.

LEMMA 2.1. *An operator T with the property* (I') *has the property* (II).

Proof.. Let $\lambda,\ \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and sequences $\{x_n\}, \{y_n\}$ of bounded vectors in $\mathscr{H}$ satisfy $\|(T-\lambda)x_n\| \to 0$ and $\|(T-\mu)y_n\| \to 0$ (as $n \to \infty$). We may assume that $\mu \neq 0$. Since T has the property (I'), we have $\|(T-\mu)^* y_n\| \to 0$ (as $n \to \infty$). Hence,

$$(\lambda-\mu)\langle x_n, y_n\rangle = \langle (\lambda - T)x_n, y_n\rangle + \langle x_n, (T-\mu)^* y_n\rangle \to 0 \ (n \to \infty).$$

This implies that $\langle x_n, y_n\rangle \to 0$ and the proof is completed. $\square$

LEMMA 2.2. *Let* a, b, c_n $(n = 1,2,3,\cdots) \in \mathbb{C}$ *be* $a \neq 0, a \neq b$, $\sup|c_n| < \infty$ *and* $T_n = \begin{pmatrix} a & c_n \\ 0 & b \end{pmatrix}$ *satisfy*

$$\liminf_{n\to\infty} \left\langle \left((T_n)^{2*}(T_n)^2 - 2k(T_n)^* T_n + k^2\right) v, v\right\rangle \geqslant 0$$

for each $k > 0$ *and* $v = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2$. *Then* $\lim_{n\to\infty} c_n = 0$.

Proof. Without loss of generality, we may assume $a = 1$. Then

$$(T_n)^{2*}(T_n)^2 - 2k(T_n)^* T_n + k^2 = \left(\begin{array}{c|c} (1-k)^2 & (1+b-2k)c_n \\ \hline (1+b-2k)c_n & (|1+b|^2-2k)|c_n|^2 + (|b|^2-k)^2 \end{array}\right).$$

Put $k = 1$. Then

$$S_n := (T_n)^{2*}(T_n)^2 - 2(T_n)^*(T_n) + 1 = \left(\begin{array}{c|c} 0 & (b-1)c_n \\ \hline (b-1)c_n & (|1+b|^2-2)|c_n|^2 + (|b|^2-1)^2 \end{array}\right)$$
$$= \left(\begin{array}{c|c} 0 & (b-1)c_n \\ \hline (b-1)c_n & d_n \end{array}\right).$$

Since $\{c_n\}$ is bounded, above $\{d_n\}$ is also a bounded sequence of real numbers. Put $M = \sup|d_n| < \infty$ and let $\varepsilon > 0$, $\theta \in \mathbb{R}$ be arbitrary and $v = \begin{pmatrix} \frac{1}{\varepsilon}e^{i\theta} \\ \varepsilon \end{pmatrix}$.

Then $\langle S_n v, v\rangle = 2\,\mathrm{Re}\left((b-1)c_n e^{-i\theta}\right) + \varepsilon^2 d_n \leqslant 2\,\mathrm{Re}\left((b-1)c_n e^{-i\theta}\right) + M\varepsilon^2$. By the assumption,

$$0 \leqslant \liminf_{n\to\infty} \langle S_n v, v\rangle \leqslant \liminf_{n\to\infty} 2\,\mathrm{Re}\left((b-1)c_n e^{-i\theta}\right) + M\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, above inequality implies that

$$\liminf_{n\to\infty} \mathrm{Re}\left((b-1)c_n e^{-i\theta}\right) \geqslant 0 \text{ for all } \theta,$$

which is equivalent to

$$\liminf_{n\to\infty} \mathrm{Re}\left(c_n e^{-i\theta}\right) \geqslant 0 \text{ for all } \theta. \tag{1}$$

Let $\{c_{n_k}\}$ be an arbitrary subsequence of $\{c_n\}$. Since $\{c_n\}$ is bounded, $\{c_{n_k}\}$ has a convergent subsequence $\{c_{n_{k_l}}\}$. Let $c = \lim_{l\to\infty} c_{n_{k_l}}$. Then the inequality (1) implies that

$$\mathrm{Re}\left(ce^{-i\theta}\right) \geqslant 0 \text{ for all } \theta,$$

hence $c = 0$. Any subsequence of $\{c_n\}$ has a convergent subsequence which converges to 0, it follows that $\{c_n\}$ converges to 0 as $n \to \infty$. $\square$

LEMMA 2.3. *Every paranormal operator T has the property* (II).

Proof. It suffices to show that if $\|(T-1)x_n\| \to 0$, $\|(T-\mu)y_n\| \to 0$ $(n \to \infty)$, $\|x_n\| = \|y_n\| = 1$ for all n and $\mu \neq 1$, then $\langle x_n, y_n \rangle \to 0$ as $n \to \infty$. Put $y_n = a_n x_n + b_n z_n$, where $a_n, b_n \in \mathbb{C}$ and z_n with $z_n \perp x_n$ and $\|z_n\| = 1$. We shall show that $a_n (= \langle y_n, x_n \rangle)$ converges to 0 as $n \to \infty$. Since $\|(T-1)x_n\|$ and $\|(T-\mu)y_n\|$ converge to 0, we have

$$\|b_n T z_n - \{(\mu-1)a_n x_n + \mu b_n z_n\}\| = \|(T-\mu)y_n - a_n(T-1)x_n\| \to 0. \tag{2}$$

If there exists a subsequence $\{b_{n_k}\}$ which converges to 0, then $|a_{n_k}| \to 1$ and $\mu - 1 = 0$ follows from (2), a contradiction, so there exists $\varepsilon > 0$ such that $|b_n| > \varepsilon$ for all n. Hence,

$$\|Tz_n - (\mu-1)\frac{a_n}{b_n}x_n + \mu z_n\| \to 0.$$

So,

$$\begin{aligned} &\|T(px_n + qz_n) - \{px_n + qc_n(x_n \otimes z_n)z_n + \mu q z_n\}\| \\ &\qquad = \|T(px_n + qz_n) - (px_n + qc_n x_n + \mu q z_n)\| \to 0, \end{aligned}$$

where $c_n = (\mu-1)\dfrac{a_n}{b_n}$, $p, q \in \mathbb{C}$ and $x_n \otimes z_n$ is a rank one operator defined by

$$(x_n \otimes z_n)u = \langle u, z_n \rangle x_n.$$

Also, since $\|(T^2-1)x_n\|$ and $\|(T^2-\mu^2)y_n\|$ converge to 0, we have

$$\|T^2 z_n - \{(1+\mu)c_n x_n + \mu^2 z_n\}\| = \left\| T^2\left(\frac{1}{b_n}y_n - \frac{a_n}{b_n}x_n\right) - \left(\frac{\mu^2}{b_n}y_n - \frac{a_n}{b_n}x_n\right)\right\| \to 0.$$

So,

$$\begin{aligned} &\|T^2(px_n + qz_n) - \{px_n + q(1+\mu)c_n(x_n \otimes z_n)z_n + \mu^2 q z_n\}\| \\ &\qquad = \|T^2(px_n + qz_n) - \{px_n + q(1+\mu)c_n x_n + \mu^2 q z_n\}\| \to 0. \end{aligned}$$

Since $x_n \perp z_n$ and $\|x_n\| = \|y_n\| = 1$, we have

$$\begin{aligned} \|px_n + qc_n(x_n \otimes z_n)z_n + \mu q z_n\|^2 &= \|px_n + qc_n x_n + \mu q z_n\|^2 \\ &= |p + qc_n|^2 + |\mu q|^2 \\ &= \left\| \begin{pmatrix} 1 & c_n \\ 0 & \mu \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \right\|^2, \end{aligned}$$

and

$$\begin{aligned} \|px_n + q(1+\mu)c_n(x_n \otimes z_n)z_n + \mu^2 q z_n\|^2 &= \left\| \begin{pmatrix} 1 & (1+\mu)c_n \\ 0 & \mu^2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} 1 & c_n \\ 0 & \mu \end{pmatrix}^2 \begin{pmatrix} p \\ q \end{pmatrix} \right\|^2. \end{aligned}$$

Hence

$$\|T^2(px_n+qz_n)\|^2 - \left\|\begin{pmatrix}1 & c_n\\ 0 & \mu\end{pmatrix}^2\begin{pmatrix}p\\ q\end{pmatrix}\right\|^2$$
$$= \|T^2(px_n+qz_n)\|^2 - \|px_n+q(1+\mu)c_n(x_n\otimes z_n)z_n+\mu^2 qz_n\|^2 \to 0,$$
$$\|T(px_n+qz_n)\|^2 - \left\|\begin{pmatrix}1 & c_n\\ 0 & \mu\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\right\|^2$$
$$= \|T(px_n+qz_n)\|^2 - \|px_n+qc_n(x_n\otimes z_n)z_n+\mu qz_n\|^2 \to 0$$

for each $\begin{pmatrix}p\\ q\end{pmatrix} \in \mathbb{C}^2$. Put $v_n = px_n+qz_n \in \mathscr{H}$, $v = \begin{pmatrix}p\\ q\end{pmatrix} \in \mathbb{C}^2$ and $T_n = \begin{pmatrix}1 & c_n\\ 0 & \mu\end{pmatrix}$. Then

$$\langle T^{2*}T^2 v_n, v_n\rangle - \langle (T_n)^{2*}(T_n)^2 v, v\rangle = \|T^2 v_n\|^2 - \|(T_n)^2 v\|^2 \to 0,$$
$$\langle T^*T v_n, v_n\rangle - \langle (T_n)^*(T_n) v, v\rangle = \|T v_n\|^2 - \|T_n v\|^2 \to 0.$$

It follows that

$$\langle (T^{2*}T^2 - 2kT^*T + k^2)v_n, v_n\rangle - \langle ((T_n)^{2*}(T_n)^2 - 2k(T_n)^*(T_n) + k^2)v, v\rangle \to 0 \qquad (3)$$

for each $k>0$ and $v = \begin{pmatrix}p\\ q\end{pmatrix} \in \mathbb{C}^2$.

Paranormality of T implies that $\langle (T^{2*}T^2 - 2kT^*T + k^2)v_n, v_n\rangle \geqslant 0$, so (3) implies that

$$\liminf_{n\to\infty} \left\langle ((T_n)^{2*}(T_n)^2 - 2k(T_n)^*(T_n) + k^2)v, v\right\rangle \geqslant 0$$

for each $k>0$ and $v = \begin{pmatrix}p\\ q\end{pmatrix} \in \mathbb{C}^2$. Since $\sup|c_n| \leqslant \dfrac{|\mu-1|}{\varepsilon} < \infty$ we have $\lim_{n\to\infty} c_n = 0$ by Lemma 2.2 and $|a_n| = \dfrac{|c_n||b_n|}{|\mu-1|} \leqslant \dfrac{|c_n|}{|\mu-1|} \to 0$. This completes the proof. $\square$

3. Main theorem.

The single valued extension property (SVEP) is a famous property which is weaker than Bishop's property (β).

We can easily prove that if an operator T has the property (II) then T has the (SVEP). We give a short proof of the assertion.

PROPOSITION 3.1. *If T has the property* (II) *then T also has the (SVEP).*

Proof. Let f be an analytic function such that $(T-z)f(z) = 0$ on $\mathscr{D}$. Since T has the property (II),

$$\ker(T-z) \perp \ker(T-w) \ (z \neq w).$$

The assumption implies that $f(z) \in \ker(T-z)$ for every $z \in \mathscr{D}$, hence,

$$\|f(z)\|^2 = \lim_{w\to z}\langle f(z), f(w)\rangle = 0. \quad \square$$

COROLLARY 3.2. *Every paranormal operator has the (SVEP).*

For $R > 0$ and $z \in \mathbb{C}$, we denote the open ball with center z and radius R by $B(z;R)$. Let $\mathscr{D} \subset \mathbb{C}$ be an open set, $f : \mathscr{D} \to \mathscr{H}$ an analytic function and

$$f(z) = \sum_{l=0}^{\infty} (z - z_0)^l a_l \ (|z - z_0| < R) \tag{4}$$

the Taylor expansion of f. Here $z_0 \in \mathscr{D}$, $\overline{B(z_0;R)} \subset \mathscr{D}$ and $a_l \in \mathscr{H}$. For each compact set $\mathscr{K}$, define the norm $\| \quad \|_{\mathscr{K}}$ by

$$\|f\|_{\mathscr{K}} := \sup_{z \in \mathscr{K}} |f(z)|.$$

LEMMA 3.3. *Let $\mathscr{D}$, $z_0 \in \mathscr{D}$, $R > 0$ and $f(z) = \sum_{l=0}^{\infty} (z - z_0)^l a_l \ (|z - z_0| < R)$ be as above. If f is bounded (i.e., $M = \sup_{z \in \mathscr{D}} |f(z)| < \infty$), then*

$$\|a_l\| \leqslant \frac{M}{R^l}.$$

LEMMA 3.4. *Let $\mathscr{D}$ be an open set of $\mathbb{C}$, $z_0 \in \mathscr{D}$, $R > 0$ such as $\overline{B(z_0;R)} \subset \mathscr{D}$, $f_n : \mathscr{D} \to \mathscr{H}$ a sequence of analytic functions and $f_n(z) = \sum_{l=0}^{\infty} (z - z_0)^l a_l^{(n)} \ (|z - z_0| < R)$ be the Taylor expansion of f_n. If f_n is uniformly bounded on $\overline{B(z_0;R)}$ (i.e., $M = \sup_{n \geqslant 1} \|f_n\|_{\overline{B(z_0;R)}} < \infty$), then*

$$\|f_n(z) - f_n(z_0)\| \leqslant \frac{Mr}{R - r} \text{ for all } z \in \overline{B(z_0;r)}, 0 < r < R. \tag{5}$$

Proof.

$$\begin{aligned} \|f_n(z) - f_n(z_0)\| &\leqslant \sum_{l=1}^{\infty} |z - z_0|^l \|a_l^{(n)}\| \leqslant M \sum_{l=1}^{\infty} \left(\frac{r}{R}\right)^l \\ &= M \cdot \frac{\frac{r}{R}}{1 - \frac{r}{R}} = \frac{Mr}{R - r}. \end{aligned}$$

A sequence of analytic functions $f_n : \mathscr{D} \to \mathscr{H}$, where $\mathscr{D}$ is an open subset of $\mathbb{C}$, converges uniformly to 0 on every compact subset $\mathscr{K}$ of $\mathscr{D}$ if and only if for any $\varepsilon > 0$ and any $z_0 \in \mathscr{D}$ there exist $r > 0$ and $N \in \mathbb{N}$ such that $\overline{B(z_0;r)} \subset \mathscr{D}$ and $\|f_n\|_{\overline{B(z_0;r)}} \leqslant \varepsilon$ for all $n > N$.

THEOREM 3.5. *If an operator T has the property* (II)*, then T also has Bishop's property* (β).

Proof. Let $\mathscr{D} \subset \mathbb{C}$ be an open subset and $f_n : \mathscr{D} \to \mathscr{H}$ be a sequence of analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 on every compact subset $\mathscr{K}$ of $\mathscr{D}$. We shall show that f_n converges uniformly to 0 on every compact subset $\mathscr{K}$ of

$\mathscr{D}$. If necessary, let $g_n = \dfrac{f_n}{1+\|f_n\|_{\mathscr{K}}}$ instead of f_n, we may assume $\sup_n \|f_n\|_{\mathscr{K}} < \infty$ for every compact subset $\mathscr{K}$ of $\mathscr{D}$ without loss of generality.

Let $\varepsilon > 0$ be arbitrary, $z_0 \in \mathscr{D}$ any point and $R > 0$ such as $\overline{B(z_0;R)} \subset \mathscr{D}$. Put $M = \sup \|f_n\|_{\overline{B(z_0;R)}} < \infty$. Then

$$\|f_n(z) - f_n(z_0)\| \leqslant \frac{Mr}{R-r} \text{ for all } z \in \overline{B(z_0;r)},\ 0 < r < R,$$

by Lemma 3.4. Choose a sufficiently small $r > 0$ such that $\dfrac{M^2 r}{R-r} < \dfrac{\varepsilon^2}{8}$, $\dfrac{Mr}{R-r} < \dfrac{\varepsilon}{2}$. Then for all n and for all $z \in \overline{B(z_0;r)}$

$$\|f_n(z_0)\|^2 \leqslant |\langle f_n(z), f(z_0)\rangle| + \frac{M^2 r}{R-r} \leqslant |\langle f_n(z), f_n(z_0)\rangle| + \frac{\varepsilon^2}{8}, \tag{6}$$

$$\|f_n(z)\| \leqslant \|f_n(z_0)\| + \frac{Mr}{R-r} \leqslant \|f_n(z_0)\| + \frac{\varepsilon}{2}. \tag{7}$$

Let $z_1 \in B(z_0;r) \setminus \{z_0\}$ be arbitrary. Then, by the assumption

$$\|(T - z_0)f_n(z_0)\| \to 0 \text{ and } \|(T - z_1)f_n(z_1)\| \to 0.$$

Since T has the property (II)

$$\langle f_n(z_1), f_n(z_0)\rangle \to 0.$$

Hence there exists a natural number N such that $|\langle f_n(z_1), f_n(z_0)\rangle| \leqslant \dfrac{\varepsilon^2}{8}$ for all $n \geqslant N$. Thus $\|f_n(z_0)\|^2 \leqslant |\langle f_n(z_1), f_n(z_0)\rangle| + \frac{\varepsilon^2}{8} < \frac{\varepsilon^2}{8} + \frac{\varepsilon^2}{8} = \frac{\varepsilon^2}{4}$ by (6) and

$$\|f_n(z)\| \leqslant \|f_n(z_0)\| + \frac{\varepsilon}{2} \leqslant \varepsilon \quad \text{for all } z \in B(z_0;r),$$

by (7) for all $n > N$. This completes the proof. □

COROLLARY 3.6. *Every paranormal operator on a complex Hilbert space has Bishop's property* (β).

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Atsushi Uchiyama
Department of Mathematical Science, Faculty of Science
Yamagata University
Yamagata, 990-8560
Japan
e-mail: uchiyama@sci.kj.yamagata-u.ac.jp

Kotaro Tanahashi
Department of Mathematics
Tohoku Pharmaceutical University
Sendai, 981-8558
Japan
e-mail: tanahasi@tohoku-pharm.ac.jp