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STUDY OF A DIFFERENTIAL OPERATOR OF HEUN TYPE ARISING IN FLUID DYNAMICS

MARINA CHUGUNOVA AND HANS VOLKMER

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Abstract. The paper studies the non-selfadjoint linear differential operator

$$Ly = \frac{d}{dt}\left((1-a\cos t)y + b\sin t\,\frac{dy}{dt}\right)$$

acting in the Hilbert space $L^2(-\pi,\pi)$ that originated from a steady state stability problem in fluid dynamics. The operator L is of Heun type and involves two parameters a,b related to the hydrostatic pressure and capillary properties of the fluid. The results concern (1) the properties of functions in the domain of definition of L, (2) conditions on a,b for the linear span of the Fourier basis $\{e^{int}\}$ to be core of L, and (3) the matrix representation of the reduced resolvent of L in the Fourier basis. In particular, it is shown that the reduced resolvent is compact and of trace class $\mathscr{S}_1$.

1. Introduction

The flow dynamic model of a viscous incompressible capillary fluid on the surface of a horizontal circular cylinder rotating around its axis under gravity for the case when surface tension effect is neglected due to the small ratio of the thickness of the fluid film and the radius of the cylinder has been investigated a lot since the pioneer work of Moffatt [13]. In this paper we study the two parametric non-selfadjoint linear differential operator

$$Ly := \frac{d}{dt}\left((1-a\cos t)y + b\sin t\,\frac{dy}{dt}\right), \quad y = y(t), -\pi < t < \pi \tag{1.1}$$

that originated from the stability analysis of steady flows in this model [1, 2]. We will assume throughout that the parameters a,b satisfy

$$a \in (-1,1), \quad b > 0. \tag{1.2}$$

The substitution $x = e^{it}$ shows that L is of Heun type; see [7]. The operator L has the special feature that $t = -\pi, 0, \pi$ are regular singularities of the differential equation $Ly = 0$. It is common to encounter differential operators with singularities at the end

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points of the underlying interval. However, operators involving internal singularities are far less well understood.

We consider L as a linear operator $L: D(L) \to L^2(-\pi,\pi)$, where the domain $D(L)$ is the linear subspace of the Hilbert space $H := L^2(-\pi,\pi)$ with inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$$

defined as follows. The domain $D(L)$ consists of all function y such that

(1) $y \in C[-\pi,\pi]$;

(2) $y(-\pi) = y(\pi)$;

(3) y and y' are locally absolutely continuous on $(-\pi,0)\cup(0,\pi)$;

(4) $\frac{d}{dt}((1-a\cos t)y(t) + b\sin t\, y'(t)) \in H$.

In [7] we used a different definition of $D(L)$ with (1) replaced by $y \in AC[-\pi,\pi]$. However, we prove in Theorem 2.6 that the definitions are equivalent. Under the additional assumption that $a \geqslant 0$ the authors proved in [7] that all eigenvalues of L are purely imaginary or 0 and found their asymptotic behavior. It should be mentioned that the results of [7] are not used in this paper except once in Section 3 where we assume $a \geqslant 0$.

In Section 2 of this paper we show that L is a closed operator with range

$$H_0 := \{f \in H : \int_{-\pi}^{\pi} f(t)\,dt = 0\}.$$

We also show that the restriction L_0 of L to the domain $D(L_0) := D(L)\cap H_0$ is bijective from $D(L_0)$ onto H_0 and its inverse $L_0^{-1} : H_0 \to H_0$ is compact. The proof is based on the fact that the embedding of the Sobolev space

$$H^1(-\pi,\pi) = \{f \in AC[-\pi,\pi] : f' \in L^2(-\pi,\pi)\}$$

into $C[-\pi,\pi]$ (equipped with the maximum-norm) is compact.

Note that the functions $e_n(t) = e^{int}$, $n \in \mathbb{Z}$, lie in $D(L)$. In Section 3 we answer the question whether the linear span E of the sequence $\{e_n\}$ forms a core of $D(L)$, that is, whether the closure of the restriction of L to E is equal to L. This question arises naturally when we represent L in the Fourier basis $\{e_n\}$ by an infinite tridiagonal matrix.

In Section 4 we represent the reduced resolvent L_0^{-1} by an infinite matrix in the basis $\{e_n\}$. We find quite explicit formulas for the matrix entries. As an application we determine $p > 0$ such that L_0^{-1} belongs to Schatten class $\mathscr{S}_p$. In particular, we show that the reduced resolvent is nuclear and as a result the set of eigenfunctions of L is complete in $L^2(-\pi,\pi)$ for $a = 0$ and $b > 0$.

Some results of this paper are known under additional assumptions on the range of the parameters a,b. It should be noted that our methods of proof are different from those used in the papers mentioned below.

For the case when $a = 0,\ 0 < b < 2$, the tridiagonal matrix representation of the operator L_0 with respect to the Fourier basis was analyzed in [8] and it was shown that it has a compact inverse of the Hilbert-Schmidt type, i.e., L_0^{-1} is of Schatten class $\mathscr{S}_p$ for $p \geqslant 2$.

Under the same restrictions on the parameters a and b the membership of L_0^{-1} in the Schatten class $\mathscr{S}_p$ for $p > 2/3$ was proved in [3]. Although the weaker result that the operator L_0^{-1} is of Schatten class $\mathscr{S}_p$ for $p > 1$ could be obtained directly from the factorization of the operator L_0 found in [5], the fact that the operator L_0^{-1} actually belongs to the class of nuclear operators $\mathscr{S}_1$ was crucial for the proof of completeness of eigenfunctions of the operator L_0 given in [3]. It was also shown numerically [4, 8] that the angle between the subspace spanned by the N-first eigenfunctions and the $(N+1)$-th eigenfunction of the operator L_0 tends to 0 as N tends to infinity. An analytical proof of this interesting geometrical property of the eigenfunctions is still an open question.

Under the more restrictive assumptions that $a, b > 0$ and $2a + b < 2$, the compactness of the operator L_0^{-1} is shown in [6].

2. The reduced resolvent of L

Assuming (1.2) we set

$$\alpha := \frac{a-1}{b} < 0, \quad \beta := \frac{a+1}{b} > 0, \quad \gamma := \frac{a}{b}. \tag{2.1}$$

We start by introducing an integral operator $T : C[-\pi,\pi] \to C[-\pi,\pi]$. Let $h \in C[-\pi,\pi]$. If $t \in (-\pi,0) \cup (0,\pi)$ we define

$$(Th)(t) := \frac{\operatorname{sgn} t}{2b} \sin^{\alpha}\!\left(\tfrac{|t|}{2}\right) \cos^{\beta}\!\left(\tfrac{t}{2}\right) \int_0^t \sin^{-\alpha-1}\!\left(\tfrac{|s|}{2}\right) \cos^{-\beta-1}\!\left(\tfrac{s}{2}\right) h(s)\, ds. \tag{2.2}$$

Moreover, we set

$$(Th)(0) := \frac{h(0)}{1-a}, \quad (Th)(\pi) := \frac{h(\pi)}{1+a}, \quad (Th)(-\pi) := \frac{h(-\pi)}{1+a}. \tag{2.3}$$

LEMMA 2.1. *If $h \in C[-\pi,\pi]$ then $Th \in C[-\pi,\pi]$, and Th is continuously differentiable on $(-\pi,0) \cup (0,\pi)$. Moreover, $y := Th$ satisfies*

$$(1 - a\cos t)y(t) + b \sin t\, y'(t) = h(t), \quad t \in (-\pi,0) \cup (0,\pi), \tag{2.4}$$

and this y is the only solution of (2.4) *in $C[-\pi,\pi]$. The linear operator $T : C[-\pi,\pi] \to C[-\pi,\pi]$ is bounded, where $C[-\pi,\pi]$ is equipped with the maximum-norm.*

Proof. Since $\alpha < 0$, $y = Th$ is well-defined and continuously differentiable on $(-\pi,0) \cup (0,\pi)$. A direct calculation shows that y solves the differential equation (2.4) on $(-\pi,0) \cup (0,\pi)$. We omit the easy proof that y is continuous on $[-\pi,\pi]$.

The general solution of equation (2.4) on $(0,\pi)$ is

$$Y(t) = c\sin^{\alpha}(\tfrac{t}{2})\cos^{\beta}(\tfrac{t}{2}) + y(t),$$

where c is a constant. Since $\alpha < 0$, the only solution which admits the limit $\lim_{t\to 0+} Y(t)$ is $y(t)$. A similar argument applies to the interval $(-\pi,0)$.

Finally, we have

$$|Th|_{\infty} \leqslant |T1|_{\infty}|h|_{\infty} = \frac{1}{1-|a|}|h|_{\infty}. \qquad \square$$

LEMMA 2.2. *Let $g \in H = L^2(-\pi,\pi)$. Set*

$$h(t) := \int_0^t g(s)\,ds, \tag{2.5}$$

and $y := Th$. Then $y \in C[-\pi,\pi]$, $y(0) = 0$, y and y' are locally absolutely continuous on $(-\pi,0)\cup(0,\pi)$ and

$$\frac{d}{dt}\big((1-a\cos t)y(t) + b\sin t\, y'(t)\big) = g(t), \quad t \in (-\pi,0)\cup(0,\pi)\, a.e. \tag{2.6}$$

This y is uniquely determined by these properties. y satisfies the boundary condition $y(-\pi) = y(\pi)$ if and only if $g \in H_0$.

Proof. Since $h(0) = 0$ we have $y(0) = 0$. The other properties of y and the uniqueness of y follow from Lemma 2.1. $y(\pi) = y(-\pi)$ is equivalent to $h(-\pi) = h(\pi)$ and this in turn is equivalent to $g \in H_0$. $\square$

We now return to the operators L and L_0 introduced in Section 1.

THEOREM 2.3. *L is a closed operator with range H_0, L_0^{-1} is compact and L has compact resolvent.*

Proof. Lemma 2.2 shows that the range of L is H_0. The kernel of L is spanned by the positive function $T1$; see [6]. Therefore, the operator $L_0 : D(L_0) = D(L)\cap H_0 \to H_0$ is bijective. Its inverse is

$$L_0^{-1}g = Th - \frac{\langle Th, 1\rangle}{\langle T1, 1\rangle}T1,$$

where h is defined by (2.5). The map $g \mapsto h$ is bounded linear from H_0 to $H^1(-\pi,\pi)$. The embedding of $H^1(-\pi,\pi)$ in $C[-\pi,\pi]$ is compact, T is bounded linear and the embedding of $C[-\pi,\pi]$ in $L^2(-\pi,\pi)$ is bounded. Therefore, L_0^{-1} is a compact operator in H_0. The compactness of the reduced resolvent L_0^{-1} implies that L has compact resolvent (see [6]) so L is closed. $\square$

Our next goal is to show that $D(L) \subset AC[-\pi,\pi]$.

LEMMA 2.4. *Let $c > 0$ and $f \in L^2(0,c)$. Set*

$$F(x) := \frac{1}{x}\int_0^x f(t)\,dt.$$

Then

$$\int_0^c |F(x)|^2\,dx \leqslant 4\int_0^c |f(x)|^2\,dx.$$

Proof. See [15, page 20]. $\square$

THEOREM 2.5. *If $y \in D(L)$ then $y \in H^1(-\frac{\pi}{2},\frac{\pi}{2})$.*

Proof. Let $y \in D(L)$. It will be sufficient to show that $y' \in L^2(-\frac{\pi}{2},\frac{\pi}{2})$. We show only that $y' \in L^2(0,\frac{\pi}{2})$ (the proof of $y' \in L^2(-\frac{\pi}{2},0)$ is similar.) There is $h \in H^1(-\pi,\pi)$ such that $y = Th$. A calculation involving integration by parts shows that

$$2by'(t) = -(1+a)\tan(\tfrac{t}{2})y(t) + \sin^{\alpha-1}(\tfrac{t}{2})\cos^{\beta+1}(\tfrac{t}{2})\int_0^t \sin^{-\alpha}(\tfrac{s}{2})H'(s)\,ds,$$

where

$$H(s) := \cos^{-\beta-2}(\tfrac{s}{2})h(s) \in H^1(0,\tfrac{\pi}{2}).$$

Therefore, there is a constant C such that, for $t \in (0,\frac{\pi}{2}]$,

$$|y'(t)| \leqslant C\left(|y(t)| + \frac{1}{t}\int_0^t |H'(s)|\,ds\right).$$

Lemma 2.4 implies that $y' \in L^2(0,\frac{\pi}{2})$. $\square$

It is not always true that $D(L) \subset H^1(-\pi,\pi)$. For example, consider

$$y(t) := \cos^{\beta}(\tfrac{t}{2}).$$

Then

$$(1-a\cos t)y(t) + b\sin t\, y'(t) = (1-a)\cos^{\beta+2}(\tfrac{t}{2})$$

so $y \in D(L)$. But if $\beta \leqslant \frac{1}{2}$ then $y \notin H^1(-\pi,\pi)$.

THEOREM 2.6. *$D(L) \subset AC[-\pi,\pi]$.*

Proof. Let $y \in D(L)$. We plan to show that $y' \in L^1(-\pi,\pi)$ which implies that $y \in AC[-\pi,\pi]$. In view of Theorem 2.5, it is enough to show that y' is integrable on $(-\pi,-\frac{\pi}{2})$ and on $(\frac{\pi}{2},\pi)$. We prove the latter. We choose $h \in H^1(-\pi,\pi)$ such that $y = Th$. For $t \in [\frac{\pi}{2},\pi)$, we write the integral in (2.2) as $\int_0^t = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^t$ leading to a decomposition $y = y_1 + y_2$. Clearly, y_1' is integrable on $(\frac{\pi}{2},\pi)$ so we will show that y_2'

is integrable on $(\frac{\pi}{2},\pi)$. Integrating by parts, we find that in order to show that y_2' is integrable on $(\frac{\pi}{2},\pi)$ it is enough to prove that the function

$$f(t) := \sin^{\alpha+1}(\tfrac{t}{2})\cos^{\beta-1}(\tfrac{t}{2})\int_{\frac{\pi}{2}}^{t}\cos^{-\beta}(\tfrac{s}{2})H'(s)\,ds,$$

where

$$H(s) := \sin^{-\alpha-2}(\tfrac{s}{2})h(s)\in H^1(\tfrac{\pi}{2},\pi)$$

is integrable on $(\frac{\pi}{2},\pi)$.

There is a constant C such that, for $t\in[\frac{\pi}{2},\pi)$,

$$|f(t)| \leqslant C(\pi-t)^{\beta-1}\int_{\frac{\pi}{2}}^{t}(\pi-s)^{-\beta}|H'(s)|\,ds. \tag{2.7}$$

If $\beta\in(0,\frac{1}{2})$ then (2.7) shows that f is in $L^2(\frac{\pi}{2},\pi)$. If $\beta>\frac{1}{2}$ then the Cauchy-Schwarz inequality applied to the integral in (2.7) gives

$$|f(t)| \leqslant \tilde{C}(\pi-t)^{-1/2}.$$

A similar estimates holds for $\beta=\frac{1}{2}$. Therefore, f is integrable on $(\frac{\pi}{2},\pi)$. The proof is complete. $\square$

3. A core of L

Let T be a closed linear operator in a Hilbert space. Let E be a linear subspace of $D(T)$, and let S denote the restriction of T to E. Then E is called a core of T if the closure of S equals T; see [12, page 166].

Consider

$$E := \operatorname{span}\{e_n : n\in\mathbb{Z}\}\subset D(L),$$

where $e_n(t)=e^{int}$. We want to decide whether E is a core of L.

We introduce the linear space

$$W := \{w\in H : \langle Le_n, w\rangle = 0 \text{ for all } n\in\mathbb{Z}\}.$$

Note that

$$Le_n = ine_n + \frac{i}{2}(n+1)(nb-a)e_{n+1} + \frac{i}{2}(1-n)(nb+a)e_{n-1}. \tag{3.1}$$

If $w\in H$ has Fourier expansion

$$w=\sum_{n\in\mathbb{Z}} c_n e_n,$$

then $w\in W$ if and only if

$$nc_n + \frac{1}{2}(n+1)(nb-a)c_{n+1} + \frac{1}{2}(1-n)(nb+a)c_{n-1} = 0 \quad \text{for all } n\in\mathbb{Z}. \tag{3.2}$$

Obviously, (3.2) does not involve c_0. Note that if $\{c_n\}$ is a solution of (3.2) then $\{c_{-n}\}$ is a solution, too.

THEOREM 3.1. *E is a core of L if and only if* $\dim W = 1$.

Proof. Suppose $\dim W > 1$. Then there is $0 \neq w \in W \cap H_0$. By Theorem 2.3, the range of L is H_0. Therefore, there is $y \in D(L)$ such that $w = Ly$. Let $\{y_k\}$ be any sequence in E such that $y_k \to y$ in H. Then $\langle Ly_k, w\rangle = 0$ for each k so Ly_k cannot converge to $Ly = w$. This shows that E is not a core of L.

Suppose $\dim W = 1$. If follows that $L(E)$ is dense in the range of L. Let $y \in D(L)$. We write $y = ce_0 + u$ with $u \in H_0$. There is a sequence $u_k \in E \cap H_0$ such that $Lu_k \to Lu$. By Theorem 2.3, the reduced resolvent L_0^{-1} is a bounded linear operator. Applying L_0^{-1} we obtain $u_k \to u$. Therefore, $y_k := ce_0 + u_k \to y$ and $Ly_k \to Ly$. $\square$

The further analysis depends on whether $\gamma = a/b$ is a nonnegative integer or not. We consider first the case

$$a \in (-1,1), \quad b > 0, \quad \frac{a}{b} \notin \mathbb{N}_0. \tag{3.3}$$

Let $\{c_n\}_{n\in\mathbb{Z}}$ be a solution of the recursion (3.2). It follows from (3.3) that if $c_1 = 0$ then $c_n = 0$ for all $0 \neq n \in \mathbb{Z}$. Therefore, it is sufficient to consider a solution $\{c_n\}$ with $c_1 = 1$. All c_n with $n \neq 0$ are then uniquely determined while c_0 remains arbitrary. Since $c_{-1} = c_1$ we obtain that $c_n = c_{-n}$ for all n. We consider the formal power series

$$f(x) = \sum_{n=1}^{\infty} c_n x^n, \quad c_1 = 1, \tag{3.4}$$

and its derivative

$$h(x) := f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}.$$

It follows from (3.2) that h satisfies the differential equation

$$\frac{b}{2}x(1-x^2)h' + \left(-\frac{a}{2} + x - bx^2 - \frac{a}{2}x^2\right)h = -\frac{a}{2}. \tag{3.5}$$

Since $\frac{a}{b} \notin \mathbb{N}_0$, this equation has a unique solution which is holomorphic at $x = 0$ and this solution agrees with h. Therefore, h is holomorphic in the unit disk $|x| < 1$ and can be extended to a holomorphic function on $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$. A fundamental solution of the homogeneous equation

$$\frac{b}{2}x(1-x^2)h' + \left(-\frac{a}{2} + x - bx^2 - \frac{a}{2}x^2\right)h = 0 \tag{3.6}$$

is

$$h_0(x) := (1-x)^{-\alpha-1}(1+x)^{-\beta-1}x^{\gamma},$$

where α, β, γ are from (2.1). This gives

$$h(x) = -\gamma(1-x)^{-\alpha-1}(1+x)^{-\beta-1}x^{\gamma}\int(1-x)^{\alpha}(1+x)^{\beta}x^{-\gamma-1}\,dx. \tag{3.7}$$

The anti-derivative is taken in such a way that h becomes holomorphic at $x=0$, that is, if

$$g(t) := (1-t)^{\alpha}(1+t)^{\beta} = \sum_{n=0}^{\infty} u_n t^n,$$

then

$$x^{\gamma}\int g(x)x^{-\gamma-1}\,dx = \sum_{n=0}^{\infty}\frac{u_n}{n-\gamma}x^n.$$

Equation (3.5) has a unique solution h_1 which is holomorphic at $x=-1$. Since $\beta>0$ we can write this solution in the form

$$h_1(x) = \gamma(1-x)^{-\alpha-1}(1+x)^{-\beta-1}(-x)^{\gamma}\int_{-1}^{x} g(t)(-t)^{-\gamma-1}\,dt \tag{3.8}$$

for $-1<x<0$.

LEMMA 3.2. *Suppose* (3.3). *The functions h, h_1 are different on $(-1,0)$, that is, $h-h_1$ is a nonzero multiple of h_0.*

Proof. Suppose first that $a<0$. If $-1<x<0$ then h can be written as

$$h(x) = \gamma(1-x)^{-\alpha-1}(1+x)^{-\beta-1}(-x)^{\gamma}\int_{0}^{x} g(t)(-t)^{-\gamma-1}\,dt.$$

If $h=h_1$ then (3.8) implies

$$\int_{-1}^{0} g(t)(-t)^{-\gamma-1}\,dt = 0$$

which is impossible because the integrand is positive.

Suppose now that $0<a<b$. Integrating by parts in (3.7) and (3.8) gives, for $-1<x<0$,

$$h(x) = (1-x)^{-\alpha-1}(1+x)^{-\beta-1}\left(g(x) - (-x)^{\gamma}\int_{0}^{x} g'(t)(-t)^{-\gamma}\,dt\right)$$

and a similar formula for h_1. If $h=h_1$ then

$$\int_{-1}^{0} g'(t)(-t)^{-\gamma}\,dt = 0$$

which is impossible because $g'(t)>0$.

We argue in a similar way when $(k-1)b<a<kb$ with $k=2,3,\ldots$ We integrate by parts k times and note that the kth derivative of g is positive on $(-1,0)$. $\square$

THEOREM 3.3. *Suppose* (3.3). *The function f from* (3.4) *belongs to the Hardy space H^2, that is, $\{c_n\}\in\ell^2(\mathbb{N})$, if and only if $a+1<\frac{b}{2}$.*

Proof. The function f is holomorphic on $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$. By Lemma 3.2, in a neighborhood of $x=-1$, f has the form

$$f(x) = f_1(x) + (1+x)^{-\beta} f_2(x),$$

where f_1, f_2 are holomorphic at $x=-1$ with $f_2(-1)\neq 0$. A similar analysis reveals that the singularity of $f(x)$ at $x=1$ is milder. Using Fatou's lemma and Lebesgue's dominated convergence theorem it can be shown that

$$\sum_{n=1}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2\, dt \tag{3.9}$$

regardless whether the sum or integral are finite or not. The nature of the singularities of f makes it clear that the right-hand side of (3.9) is finite if and only if $a+1<\frac{b}{2}$. This completes the proof. $\square$

THEOREM 3.4. *Suppose* (1.2)*. The linear space E forms a core of L if and only if $a+1\geqslant\frac{b}{2}$.*

Proof. First, suppose (3.3). It follows from Theorem 3.3 and the discussion at the beginning of this section that $\dim W = 1$ is equivalent to $a+1\geqslant\frac{b}{2}$. The desired statement follows from Theorem 3.1.

Let us now consider the remaining case

$$a\in[0,1),\quad b>0,\quad \ell := \frac{a}{b}\in\mathbb{N}_0. \tag{3.10}$$

Suppose $\{c_n\}_{n\in\mathbb{Z}}$ is a solution of the recursion (3.2). If $\ell\geqslant 1$ then the vector $(c_1,c_2,\dots,c_\ell)$ is in the kernel of the $\ell\times\ell$ matrix $(\langle Le_j,e_i\rangle)_{i,j=1}^{\ell}$. We saw in [7, Lemma 8] that this matrix is invertible so $c_1=c_2=\ldots=c_\ell=0$. The solution $\{c_n\}$ remains a solution if set $c_n=0$ for $n\leqslant 0$. Therefore, it will be sufficient to consider solutions of (3.2) with $c_n=0$ for $n\leqslant\ell$.

For a solution $\{c_n\}$ of (3.2) with $c_n=0$ for $n\leqslant\ell$ we consider the generating function

$$f(x)=\sum_{n=\ell+1}^{\infty} c_n x^n.$$

Then $h=f'$ solves the differential equation (3.6). We obtain that

$$h(x) = (1-x)^{-\ell-1+\frac{1}{b}}(1+x)^{-\ell-1-\frac{1}{b}}x^{\ell} \tag{3.11}$$

when we normalize our solution $\{c_n\}$ so that $c_{\ell+1}=\frac{1}{\ell+1}$. From (3.11) we read off the nature of singularities of h at $x=\pm1$. Note that under assumption (3.10) we do not need an analogue of Lemma 3.2. We argue now as before to show that f is in the Hardy space H^2 if and only if $a+1<\frac{b}{2}$, and we complete the proof. $\square$

In some special cases we can find solutions $\{c_n\}$ of (3.2) explicitly. For example, suppose $a+1=b$. In this case a solution of (3.2) is given by $c_n=(-1)^n$. The linear combinations of this solution and the trivial solution $c_n=0$ for $n\neq 0$ give us all solutions. Therefore, we have $\dim W=1$ in agreement with Theorem 3.4.

4. Matrix representation of $(-iL_0)^{-1}$

Our goal in this section is to find and estimate the matrix elements of the compact operator $(-iL_0)^{-1}$ with respect to the Fourier basis $\{e_n\}$. We will assume that

$$a \in [0,1), \quad b > 0. \tag{4.1}$$

This assumption is more restrictive than (1.2). The case $a \in (-1,0)$ may also be discussed, however, modifications will be necessary. For example, the definition of the constants c_n in (4.5) has to be changed because the integral would be infinite if $n+\gamma \leqslant -1$.

We consider the differential equation

$$\left(1-\frac{a}{2}\left(x+\frac{1}{x}\right)\right)w+\frac{b}{2}(x^2-1)w' = p(x), \tag{4.2}$$

where $p(x)$ is a polynomial. By assumption (4.1), equation (4.2) has a unique solution $w_0(x)$ which is holomorphic in $|x|<1$ and satisfies $w_0(0)=0$. We see this by substituting $w(x)=\sum_{k=1}^{\infty} u_k x^k$ into (4.2) and determining u_k recursively. Then we show that the power series $\sum_{k=1}^{\infty} u_k x^k$ has radius at least 1. There is another unique solution $w_1(x)$ which is holomorphic in $|x-1|<1$. For $0<x<1$ we have

$$w_0(x) = -\frac{2}{b}x^{-\gamma}(1-x)^{\alpha}(1+x)^{\beta}\int_0^x t^{\gamma}(1-t)^{-\alpha-1}(1+t)^{-\beta-1}p(t)\,dt,$$
$$w_1(x) = -\frac{2}{b}x^{-\gamma}(1-x)^{\alpha}(1+x)^{\beta}\int_1^x t^{\gamma}(1-t)^{-\alpha-1}(1+t)^{-\beta-1}p(t)\,dt.$$

Therefore, $w_0 = w_1$ if and only if

$$\int_0^1 t^{\gamma}(1-t)^{-\alpha-1}(1+t)^{-\beta-1}p(t)\,dt = 0. \tag{4.3}$$

Let

$$h(x) = \sum_{k=0}^{\infty}\phi_k x^k, \quad \phi_0 = 0 \tag{4.4}$$

be the solution $w_0(x)=h(x)$ of (4.2) with $p(x)=1$. For $n\in\mathbb{N}$ let $w_0(x)=f_n(x)$ be the solution of (4.2) with $p(x)=x^n-\frac{c_n}{c_0}$, where

$$c_n := \int_0^1 t^{n+\gamma}(1-t)^{-\alpha-1}(1+t)^{-\beta-1}\,dt. \tag{4.5}$$

Since $p(x)$ satisfies condition (4.3), $f_n(x)$ is holomorphic on $\mathbb{C}\setminus(-\infty,-1]$.

For $n\in\mathbb{N}$ we set

$$\delta_n := \phi_n\psi_n + \phi_{n-1}\psi_n\frac{1}{2}(-a+b(n-1)) - \phi_n\psi_{n+1}\frac{1}{2}(a+b(n+1)), \tag{4.6}$$

where

$$f_1(x) = \sum_{k=0}^{\infty}\psi_k x^k, \quad \psi_0 = 0.$$

LEMMA 4.1. *For $n \in \mathbb{N}$,*

$$\delta_n = -\frac{2}{a+b}\prod_{k=1}^{n-1}\frac{a-kb}{a+(k+1)b}. \tag{4.7}$$

Proof. For $n \in \mathbb{N}$ we have

$$\phi_n - \frac{1}{2}(a-(n-1)b)\phi_{n-1} - \frac{1}{2}(a+(n+1)b)\phi_{n+1} = 0. \tag{4.8}$$

Therefore,

$$\delta_n = \frac{1}{2}\tau_n(a+(n+1)b), \tag{4.9}$$

where

$$\tau_n := \phi_{n+1}\psi_n - \phi_n\psi_{n+1}.$$

Also, for $n \geqslant 2$,

$$\psi_n - \frac{1}{2}(a-(n-1)b)\psi_{n-1} - \frac{1}{2}(a+(n+1)b)\psi_{n+1} = 0. \tag{4.10}$$

Therefore, for $n \geqslant 2$,

$$\delta_n = \frac{1}{2}\tau_{n-1}(a-(n-1)b). \tag{4.11}$$

Now (4.9) and (4.11) imply

$$\tau_n = \tau_1\prod_{k=2}^{n}\frac{a-(k-1)b}{a+(k+1)b}. \tag{4.12}$$

Since $\phi_1 - (\frac{a}{2}+b)\phi_2 = 0$ and $\psi_1 - (\frac{a}{2}+b)\psi_2 = 1$ we obtain

$$\tau_1 = \phi_2\psi_1 - \phi_1\psi_2 = \phi_2 = -\frac{4}{(a+b)(a+2b)}. \tag{4.13}$$

Now (4.9), (4.12), (4.13) imply (4.7). $\square$

We note that $\delta_n \neq 0$ if $\frac{a}{b} \notin \mathbb{N}$.

LEMMA 4.2. *For $n \in \mathbb{N}$ we have*

$$\delta_n f_n(x) = \psi_n\sum_{k=1}^{n}\phi_k x^k + \phi_n\sum_{k=n+1}^{\infty}\psi_k x^k. \tag{4.14}$$

Proof. Let $g_n(x)$ denote the right-hand side of (4.14). It follows from (4.8) and (4.10) that

$$\left(1-\frac{a}{2}\left(x+\frac{1}{x}\right)\right)g_n(x) + \frac{b}{2}(x^2-1)g_n'(x) = -\frac{1}{2}(a+b)\phi_1\psi_n + \delta_n x^n.$$

The function $g_n(x)$ is a linear combination of $f_1(x)$ and a polynomial, so it is holomorphic at $x=0$ and $x=1$. Therefore, by comparison with the definition of f_n, we obtain $\delta_n f_n = g_n$. Since $\phi_1 = -\frac{2}{a+b}$ we also get that $\delta_n c_n = -\psi_n c_0$. $\square$

REMARK 4.3. For $n\in\mathbb{N}$, $\delta_n c_n = -\psi_n c_0$.

Proof. This was mentioned in the proof of Lemma 4.2. $\square$

Let

$$\rho_{m,n} := \langle(-iL_0)^{-1}e_n, e_m\rangle \quad \text{for } m,n\in\mathbb{Z}\backslash\{0\} \tag{4.15}$$

denote the entries of the matrix representing the compact operator $(-iL_0)^{-1}$ in the Fourier basis $\{e_n\}$. These entries are given by the following theorem.

THEOREM 4.4. *Suppose* (4.1) *and* $\frac{a}{b}\notin\mathbb{N}$. *Then* $\rho_{m,n} = -\rho_{-m,-n}$ *for all* m,n, *and* $\rho_{m,n}=0$ *if* $mn<0$. *If* $m,n\in\mathbb{N}$ *then*

$$\rho_{m,n} = \frac{1}{n\delta_n}\begin{cases}\phi_m\psi_n & \text{if } m\leqslant n\\ \phi_n\psi_m & \text{if } n<m.\end{cases} \tag{4.16}$$

Proof. Let $n\in\mathbb{N}$. For $t\in(-\pi,\pi)$, define

$$y_n(t) := \frac{1}{n}f_n(e^{it}).$$

This function is real-analytic on $(-\pi,\pi)$. We define numbers $\sigma_{m,n}$, $m,n\in\mathbb{N}$ by

$$f_n(x) = \sum_{m=1}^{\infty}\sigma_{m,n}x^m.$$

Equation (4.2) with $p(x)=0$ has a regular singularity at $x=-1$ with exponent $\beta>0$. By Darboux's method [14, Section 8.9], for fixed n,

$$\sigma_{m,n} = O(m^{-\beta-1}).$$

Therefore,

$$y_n(t) = \frac{1}{n}\sum_{m=1}^{\infty}\sigma_{m,n}e^{imt},$$

where the series converges absolutely and uniformly. It follows that $y_n(t)$ can be extended continuously onto the interval $[-\pi,\pi]$ and $y_n(-\pi)=y_n(\pi)$.

By definition, f_n satisfies (4.2) with right-hand side $p(x) = x^n - \frac{c_n}{c_0}$. Substituting $x=e^{it}$ and differentiating with respect to t we obtain $y_n\in D(L_0)$ and $-iL_0y_n = e^{int}$. Therefore, $\rho_{m,n} = \frac{1}{n}\sigma_{m,n}$ and the statement of the theorem follows from Lemma 4.2 when $n\in\mathbb{N}$. Setting $z_n(t) := y_n(-t)$ we see immediately that $z_n\in D(L_0)$ and $-iL_0z_n = e^{-int}$. Therefore, for $n\in\mathbb{N}$, $\rho_{-m,-n} = -\rho_{m,n}$. $\square$

THEOREM 4.5. *Suppose* (4.1) *and* $\frac{a}{b} \notin \mathbb{N}$. *There is a constant* K *depending only on* a *and* b *such that, for* $m,n \in \mathbb{N}$,

$$|\rho_{m,n}| \leqslant K \begin{cases} m^{-\alpha-1}n^{\alpha-1} & \text{if } m \leqslant n \\ m^{-\beta-1}n^{\beta-1} & \text{if } n < m. \end{cases}$$

Proof. From Darboux's method we obtain

$$\phi_n = O(n^{-\alpha-1}), \quad \psi_n = O(n^{-\beta-1}). \tag{4.17}$$

By Lemma 4.1,

$$\delta_n = (-1)^n \frac{2}{a+b} \frac{\Gamma(n-\gamma)\Gamma(2+\gamma)}{\Gamma(n+1+\gamma)\Gamma(1-\gamma)}. \tag{4.18}$$

The asymptotics of the Gamma function yields

$$\delta_n = O(n^{-2\gamma-1}). \tag{4.19}$$

Now Theorem 4.4, (4.17), (4.19) imply the statement of the theorem. □

A compact linear operator T in a separable Hilbert space is said to be of Schatten class $\mathscr{S}_p$ for some $p > 0$ if its sequence of singular numbers is p-summable; see [10, page 87]. A criterion of Gohberg and Markus [11] states that, for $0 < p \leqslant 2$, T is of Schatten class $\mathscr{S}_p$ if and only if there is an orthonormal basis $\{e_j\}$ for which the sequence of norms $\{\|Te_j\|\}$ is p-summable.

THEOREM 4.6. *Suppose* (4.1) *and* $\frac{a}{b} \notin \mathbb{N}$. *If* $2a+2 \geqslant b$ *then* $(-iL_0)^{-1}$ *is of Schatten class* S_p *for* $p > \frac{2}{3}$. *If* $2a+2 < b$ *then* $(-iL_0)^{-1}$ *is of Schatten class* $\mathscr{S}_p$ *for* $p > \frac{b}{a+b+1}$. *Therefore, in both cases,* $(-iL_0)^{-1}$ *is of trace class* $\mathscr{S}_1$, *that is,* $(-iL_0)^{-1}$ *is nuclear.*

Proof. By Theorem 4.5,

$$\begin{aligned} \sum_{n=1}^{\infty} \rho_{mn}^2 &\leqslant K^2 m^{-2\beta-2} \sum_{n=1}^{m-1} n^{2\beta-2} + K^2 m^{-2\alpha-2} \sum_{n=m}^{\infty} n^{2\alpha-2} \\ &\leqslant K^2 m^{-2\beta-2} \sum_{n=1}^{m-1} n^{2\beta-2} + K_2 m^{-3} \\ &\leqslant K_3 \begin{cases} m^{-3} & \text{if } 2a+2 > b \\ m^{-3}\log m & \text{if } 2a+2 = b \\ m^{-2\beta-2} & \text{if } 2a+2 < b. \end{cases} \end{aligned}$$

The stated result follows from the mentioned Gohberg-Markus criterion. □

It follows directly from the theorem above and from Lidskii's theorem [Theorem V.2.3 [9]] that the set of eigenfunctions of L_0 is complete in $L^2(-\pi,\pi)$ if $a = 0$ and $b > 0$.

Let us now consider the remaining case

$$a \in [0,1), \quad b > 0, \quad \ell := \frac{a}{b} \in \mathbb{N}. \tag{4.20}$$

Under these assumptions, we have

$$\delta_n = \begin{cases} -\frac{2}{b}\frac{(\ell-1)!\ell!}{(\ell-n)!(\ell+n)!} & \text{if } n = 1,2,\ldots,\ell, \\ 0 & \text{if } n > \ell. \end{cases} \tag{4.21}$$

Since $\delta_n \neq 0$ for $n = 1,2,\ldots,\ell$, formula (4.16) remains true for $n = 1,2,\ldots,\ell$. Note that Lemma 4.3 implies $\psi_n = 0$ for $n > \ell$. Therefore, $\rho_{m,n} = 0$ for $n = 1,2,\ldots,\ell$, $m > \ell$.

In order to obtain formulas for $\rho_{m,n}$ when $n > \ell$, we define a sequence $\{\widetilde{\psi}_k\}$ by

$$f_{\ell+1}(x) = \sum_{k=1}^{\infty} \widetilde{\psi}_k x^k.$$

Then we define $\widetilde{\delta}_n$ as in (4.6) but with ψ replaced by $\widetilde{\psi}$.

LEMMA 4.7. *Suppose* (4.20)*. If $n > \ell$ then*

$$\widetilde{\delta}_n = \phi_{\ell+2}\frac{b}{2}(-1)^{n+\ell+1}\frac{(n-\ell+1)!(2\ell+2)!}{(n+\ell)!}.$$

Proof. The proof is similar to the proof of Lemma 4.1 and will be omitted. □

THEOREM 4.8. *Suppose* (4.20)*. Then $\rho_{m,n} = -\rho_{-m,-n}$ for all m,n, and $\rho_{m,n} = 0$ if $mn < 0$ or if $|m| > |n| = 1,2\ldots,\ell$. If $n = 1,2,\ldots,\ell$ and $m \in \mathbb{N}$ then*

$$\rho_{m,n} = \frac{1}{n\delta_n}\begin{cases} \phi_m\psi_n & \text{if } m \leqslant n \\ \phi_n\psi_m & \text{if } n < m. \end{cases} \tag{4.22}$$

If $n > \ell$ and $m \in \mathbb{N}$ then

$$\rho_{m,n} = \frac{1}{n\widetilde{\delta}_n}\begin{cases} \phi_m\widetilde{\psi}_n & \text{if } m \leqslant n \\ \phi_n\widetilde{\psi}_m & \text{if } n < m. \end{cases} \tag{4.23}$$

Proof. We already mentioned (4.22). The proof of (4.23) is similar to the proof of Theorem 4.4 and is omitted. □

We now see that, under assumptions (4.1), Theorems 4.5 and 4.6 remain true.

Finally, let us determine the asymptotic behavior of the sequences ϕ_n, ψ_n, δ_n, c_n as $n \to \infty$ more precisely. We assume (4.1) and $\frac{a}{b} \notin \mathbb{N}$. At $x = 1$ the function $h(x)$ has the form

$$h(x) = -c_0\frac{2}{b}x^{-\gamma}(1-x)^{\alpha}(1+x)^{\beta} + \tilde{h}(x),$$

where $\tilde{h}$ is holomorphic at $x=1$. Therefore, the Darboux method yields

$$\phi_n = -\frac{2^{\beta+1}c_0}{b\Gamma(-\alpha)}n^{-\alpha-1} + O\left(n^{\min\{-\alpha-2,-\beta-1\}}\right), \tag{4.24}$$

Formula (4.18) gives

$$\delta_n = (-1)^n \frac{2}{b}\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)}n^{-2\gamma-1} + O(n^{-2\gamma-2}). \tag{4.25}$$

From (4.5) we obtain

$$c_n = 2^{-\beta-1}\Gamma(-\alpha)n^{\alpha} + O(n^{\alpha-1}). \tag{4.26}$$

Now Lemma 4.3 and (4.5), (4.6) give

$$\psi_n = (-1)^{n+1}\frac{2^{-\beta}}{bc_0}\frac{\Gamma(-\alpha)\Gamma(1+\gamma)}{\Gamma(1-\gamma)}n^{-\beta-1} + O(n^{-\beta-2}). \tag{4.27}$$

As an application we determine the asymptotic behavior of the diagonal entries of the matrix representing $(-iL_0)^{-1}$.

THEOREM 4.9. *Suppose* (4.1) *and* $\frac{a}{b} \notin \mathbb{N}$. *Then, as* $n \to \infty$,

$$\rho_{n,n} = \frac{1}{bn^2} + O\left(n^{\min\{-3,\alpha-\beta-2\}}\right). \tag{4.28}$$

Proof. This follows from Theorem 4.4, (4.24), (4.25), (4.27). □

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Marina Chugunova
Department of Mathematics
University of Toronto
Canada
e-mail: `chugunom@math.utoronto.ca`

Hans Volkmer
Department of Mathematical Sciences
University of Wisconsin at Milwaukee
USA
e-mail: `volkmer@uwm.edu`