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ON AN EXTREMAL PROBLEM OF GARCIA AND ROSS

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Abstract. We show the equivalence of two extremal problems on Hardy spaces, thus answering a question posed by Garcia and Ross. The proof uses a slight generalization of complex symmetric operators.

1. Introduction

In [4] Garcia and Ross discuss a nonlinear extremal problem for functions in the Hardy space and its relation to a well studied linear extremal problem. Specifically, let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. For $p > 0$, let H^p denote the classical Hardy space on $\mathbb{D}$ (identified, as usual, with a closed subspace of $L^p = L^p(\mathbb{T})$). For fixed $\psi \in L^\infty$, the following nonlinear extremal problem is considered in [4]:

$$\Gamma(\psi) := \sup_{\substack{f \in H^2 \\ \|f\|_2 = 1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) f(z)^2 \, dz \right|. \tag{1}$$

This is closely related to the well known classical linear extremal problem

$$\Lambda(\psi) := \sup_{\substack{F \in H^1 \\ \|F\|_1 = 1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) F(z) \, dz \right|; \tag{2}$$

it is noted in [4] that we always have $\Gamma(\psi) \leqslant \Lambda(\psi)$, and it is proved that in some particular cases, including the case of rational ψ, we have equality. We show in this short note that equality actually holds for all $\psi \in L^\infty$, thus answering an open question stated in [4].

The two problems can be reformulated in terms of operators on a Hilbert space. Denote by P_+ the projection in L^2 onto H^2 and by P_- the projection onto $H^2_- := L^2 \ominus H^2$. The Hankel operator of symbol ψ is $\mathfrak{H}_\psi : H^2 \to H^2_-$, defined by $\mathfrak{H}_\psi f = P_- \psi f$.

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By changing the variable $z = e^{it}$ and denoting $\zeta(t) = e^{it}$, we have

$$\Gamma(\psi) = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t)\psi(e^{it}) f(e^{it})^2 \, dt \right| = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} |\langle \psi f, \bar{\zeta}\bar{f} \rangle| = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} |\langle \mathfrak{H}_\psi f, \bar{\zeta}\bar{f} \rangle|. \tag{3}$$

On the other hand, any function $F \in H^1$ may be written as $F = fg$ with $f, g \in H^2$ and $\|f\|_2 = \|g\|_2 = \|F\|_1$. Therefore we get

$$\Lambda(\psi) = \sup_{\substack{f,g \in H^2 \\ \|f\|_2=\|g\|_2=1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t)\psi(e^{it}) f(e^{it}) g(e^{it}) \, dt \right| = \sup_{\substack{f,g \in H^2 \\ \|f\|_2=\|g\|_2=1}} |\langle \psi f, \bar{\zeta}\bar{g} \rangle| = \|\mathfrak{H}_\psi\|. \tag{4}$$

Both problems (1) and (2) are thus rephrased in terms of Hankel operators. A convenient reference for these, including all results that we shall use below, is [9].

2. Complex symmetric operators and their relatives

In [2, 3] the authors introduce the notion of *complex symmetric* operator on a Hilbert space, which has since found several applications; in particular, complex symmetric operators are used in [4] to prove the equivalence, in a particular case, of the two extremal problems. We need an extension of some of these facts to operators acting between two different spaces.

Suppose then that $\mathscr{X}, \mathscr{Y}$ are two Hilbert spaces. Define $\mathfrak{c} : \mathscr{X} \to \mathscr{Y}$ to be an *antiunitary* operator if it is a conjugate linear surjective map which satisfies $\langle \mathfrak{c}x, \mathfrak{c}x' \rangle = \langle x', x \rangle$ for all $x, x' \in \mathscr{X}$. It is then immediate that $\mathfrak{c}^{-1} : \mathscr{Y} \to \mathscr{X}$ is also an antiunitary operator. A *conjugation* is an antiunitary operator which acts on the same space and is equal to its inverse. If $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, we say that T is $\mathfrak{c}$-*symmetric* if $T = \mathfrak{c}T^*\mathfrak{c}$. If $T \in \mathscr{L}(\mathscr{X})$ and there exists a conjugation C such that T is C-symmetric, then one says that T is *complex symmetric*; this is the class considered in [2, 3].

In order to go from complex symmetric to $\mathfrak{c}$-symmetric operators, the main tool is the following lemma.

LEMMA 2.1. *If $\mathfrak{c} : \mathscr{X} \to \mathscr{Y}$ is an antiunitary operator, then there exists a unitary operator $V : \mathscr{X} \to \mathscr{Y}$ (not uniquely defined) such that $C = V^*\mathfrak{c}$ is a conjugation on $\mathscr{X}$. If such a V is fixed, then the map $T \mapsto V^*T$ is a bijection between $\mathfrak{c}$-symmetric operators and C-symmetric operators.*

Proof. Take an orthonormal basis (e_n) in $\mathscr{X}$, and define V to be the unitary operator which maps e_n into $\mathfrak{c}e_n$. Then it is easily seen that $C = V^*\mathfrak{c}$ is precisely the conjugation on $\mathscr{X}$ associated with the basis (e_n).

Now, if $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, then

$$\begin{aligned} T = \mathfrak{c}T^*\mathfrak{c} \Leftrightarrow \quad & V^*T = V^*\mathfrak{c}T^*\mathfrak{c} \quad \Leftrightarrow \quad V^*T = V^*\mathfrak{c}T^*VV^*\mathfrak{c} \\ \Leftrightarrow \quad & V^*T = C(V^*T)^*C, \end{aligned}$$

which proves the second part of the lemma. $\square$

As a consequence, we obtain the result that interests us, namely the analogue of Theorem 1 in [4] (which deals with the complex symmetric case).

LEMMA 2.2. *Suppose* $\mathfrak{c} : \mathscr{X} \to \mathscr{Y}$ *is an antiunitary operator and* $T : \mathscr{X} \to \mathscr{Y}$ *is* $\mathfrak{c}$*-symmetric. Then:*

(i) $\|T\| = \sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x\rangle|$.

(ii) The supremum in (i) is attained if and only if T *attains its norm (or, equivalently, if* $\|T\|$ *is an eigenvalue for* $|T|$*.) In this case* $Tx = \omega\|T\|\mathfrak{c}x$ *for some unimodular constant* ω.

Proof. Suppose that V is the unitary operator and C is the conjugation given by Lemma 2.1; thus $T' := V^*T$ is C-symmetric. Theorem 1 in [4] says then that $\|T'\| = \sup_{\|x\|=1} |\langle T'x, Cx\rangle|$. Since $\|T\| = \|T'\|$ and

$$\sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x\rangle| = \sup_{\|x\|=1} |\langle V^*Tx, V^*\mathfrak{c}x\rangle| = \sup_{\|x\|=1} |\langle T'x, Cx\rangle|,$$

the first assertion is proved.

For the second, it is immediate by Schwarz's inequality that, if $\|x\| = 1$, then $\|T\| = |\langle Tx, \mathfrak{c}x\rangle|$ if and only if $Tx = \omega\|T\|\mathfrak{c}x$ for some unimodular constant ω. But it is a general fact (for any operator T) that T attains its norm if and only if $\|T\|$ is an eigenvalue of $|T|$, given that $\|T\| = \||T|\|$ and $\||T|\| = \sup_{\|x\|=1}\langle |T|x, x\rangle$. $\square$

It might be of independent interest to state, as a corollary, the corresponding version of Theorem 2 in [1], characterizing the spectrum of the modulus of a $\mathfrak{c}$-symmetric operator in terms of what Garcia [1] calls an approximate antilinear eigenvalue problem.

PROPOSITION 2.3. *Let* T *be a bounded* $\mathfrak{c}$*-symmetric operator and* $\lambda \geqslant 0$. *Then*

(i) λ *belongs to the spectrum of* $|T|$ *if and only if there exists a sequence of unit vectors* $(f_n)_n$ *such that* $\lim_{n\to\infty} \|(T - \lambda\mathfrak{c})f_n\| = 0$.

(ii) λ *is a singular value of* T *if and only if* $Tf = \lambda\mathfrak{c}f$ *has a nonzero solution* f.

3. Main result

We can now prove the equivalence of the two problems (1) and (2) in the general case.

THEOREM 3.1. *For any* $\psi \in L^\infty$ *we have* $\Gamma(\psi) = \Lambda(\psi)$.

Proof. We intend to apply Lemma 2.2 to the following situation: $\mathscr{X} = H^2$, $\mathscr{Y} = H^2_-$, $T = \mathfrak{H}_\psi$ and $\mathfrak{c} : H^2 \to H^2_-$ defined by $\mathfrak{c}f = \bar{\zeta}\bar{f}$. It is easy to see that $\mathfrak{c}$ is antiunitary. Note that $\mathfrak{c}^{-1} : H^2_- \to H^2$ is given formally by the same formula as $\mathfrak{c}$. To be more accurate, we will define $\mathfrak{C} : L^2 \to L^2$ by $\mathfrak{C}f = \bar{\zeta}\bar{f}$. Then $\mathfrak{c} = \mathfrak{C}|H^2 = P_-\mathfrak{C}|H^2$ and $\mathfrak{c}^{-1} = \mathfrak{C}|H^2_- = P_+\mathfrak{C}|H^2_-$. Moreover, we have $\mathfrak{C}P_+ = P_-\mathfrak{C}$.

Then $\mathfrak{H}_\psi$ is $\mathfrak{c}$-symmetric: $\mathfrak{H}_\psi^* : H^2_- \to H^2$ acts by the formula $\mathfrak{H}_\psi^* g = P_+ \bar{\psi} g$, so

$$
\begin{aligned}
(\mathfrak{c}\mathfrak{H}_\psi^*\mathfrak{c})(f) &= (\mathfrak{c}\mathfrak{H}_\psi^*)(\bar{\zeta}\bar{f}) = \mathfrak{c}(P_+\bar{\psi}\bar{\zeta}\bar{f}) = \mathfrak{C}P_+(\bar{\psi}\bar{\zeta}\bar{f}) \\
&= P_-\mathfrak{C}(\bar{\psi}\bar{\zeta}\bar{f}) = P_-(\bar{\zeta}\psi\zeta f) = P_-(\psi f) = \mathfrak{H}_\psi f.
\end{aligned}
$$

We may apply Lemma 2.2 (i), which gives:

$$
\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \mathfrak{H}_\psi f, \mathfrak{c}f\rangle| = \sup_{\|f\|=1} |\langle P_-(\psi f), \bar{\zeta}\bar{f}\rangle|.
$$

Since $\mathfrak{c}f = \bar{\zeta}\bar{f} \in H^2_-$, there is no need of P_- in the last scalar product, and therefore, by (3),

$$
\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \psi f, \bar{\zeta}\bar{f}\rangle| = \Gamma(\psi).
$$

Since $\|\mathfrak{H}_\psi\| = \Lambda(\psi)$ by (4), the theorem is proved. $\square$

Also, from the second part of Lemma 2.2 it follows that the existence of an extremal function (a function that realizes $\Gamma(\psi)$) is equivalent to the fact that the Hankel operator attains its norm. This happens, for instance, if $\mathfrak{H}_\psi$ is compact, which is equivalent, via Hartman's theorem [6], to $\psi \in H^\infty + C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the algebra of continuous functions on $\mathbb{T}$.

Note that in [4] the solution to the extremal problem is related to truncated Toeplitz operators. These are operators on $K_\Theta = H^2 \ominus \Theta H^2$ defined, for $\phi \in H^\infty$, by the formula

$$
A_\phi^\Theta(f) = P_\Theta \phi f, \quad f \in K_\Theta,
$$

where P_Θ is the orthogonal projection onto K_Θ. More precisely, it is shown in [4] that, if there is an inner function Θ such that $\psi\Theta \in H^\infty$, then

$$
\Lambda(\psi) = \Gamma(\psi) = \|A_{\psi\Theta}^\Theta\|.
$$

The relation with Theorem 3.1 above is made by the following observation. Consider the orthogonal decompositions $H^2 = K_\Theta \oplus \Theta H^2$ and $H^2_- = \bar{\Theta}K_\Theta \oplus \bar{\Theta}H^2_-$. With respect to them, the only nonzero entry of the matrix of $\mathfrak{H}_\psi$ is in the upper left corner, and it is equal to $A_{\psi\Theta} : K_\Theta \to K_\Theta$ followed by multiplication with $\bar{\Theta}$. Consequently, in this case most of the results for the Hankel operators can be translated in terms of the truncated Toeplitz operator. Moreover, this is an *analytic* truncated Toeplitz operator, that is, one whose symbol is in H^∞. Their theory is significantly simpler that in the case of general truncated Toeplitz operators, since we may apply Sarason's interpolation arguments.

4. Final remarks

This section has no claim of novelty; its purpose is to put some other results in [4] in a more general context.

4.1 First, note that it is immediate that $\Gamma(\psi) \leqslant \|\psi\|_\infty$. Obviously $\Gamma(\psi)$ depends only on the antianalytic part of ψ. Using the equivalence of (1) and (2), and Nehari's theorem [8], it follows that for each $\psi \in L^\infty$ there exists $\hat\psi$ such that $\psi - \hat\psi \in H^\infty$ and $\|\hat\psi\|_\infty = \Gamma(\psi)$. In the context of truncated Toeplitz operators used in [4], $\hat\psi$ corresponds to what is called therein a *norm attaining symbol*.

4.2 In case an extremal function exists (equivalently, when the Hankel operator attains its norm) one can say more. With the previous notations, suppose $g \in H^2$ is an extremal function with $\|g\|_2 = 1$; thus $\|\mathfrak{H}_{\hat\psi} g\|_2 = \|\mathfrak{H}_{\hat\psi}\|$. The sequence of inequalities

$$\|\hat\psi\|_\infty = \|\mathfrak{H}_{\hat\psi}\| = \|\mathfrak{H}_{\hat\psi} g\|_2 = \|P_-(\hat\psi g)\|_2 \leqslant \|\hat\psi g\|_2 \leqslant \|\hat\psi\|_\infty \|g\|_2 = \|\hat\psi\|_\infty$$

imply that $\|P_-(\hat\psi g)\|_2 = \|\hat\psi g\|_2 = \|\hat\psi\|_\infty$. It follows then, first that $\hat\psi$ has constant modulus, and secondly that $\hat\psi g \in H^2_-$ and thus $\hat\psi g_o \in H^2_-$, where g_o is the outer part of g. Then $\|\mathfrak{H}_{\hat\psi} g_o\|_2 = \|\hat\psi g_o\|_2 = \|\hat\psi\|_\infty = \|\mathfrak{H}_{\hat\psi}\|$. Therefore, in case an extremal function exists, one can chose it outer, and thus not having zeros in $\mathbb{D}$. This recaptures the result of [7] quoted in [4], which says that if the symbol is continuous then there exists an extremal function which is nonzero on $\mathbb{D}$ (as noted above, a Hankel operator with continuous symbol is compact and thus attains its norm).

4.3 In case there exists an inner function Θ such that $\phi = \hat\psi\Theta \in H^\infty$, then the result can be strengthened. With the above notations, we have then $\bar\Theta\phi g_o \in H^2_-$, which implies $\phi g_o \in K_\Theta$; thus ϕ is a scalar multiple of the inner part of a function in K_Θ. This is essentially noticed in the remarks after [4, Theorem 2].

4.4 Finally, let us note that, in case there exists no extremal function, norm attaining symbols might not have constant modulus. An example appears in [5, Ch. IV, Example 4.2]. Namely, suppose Θ is an inner function that does not extend analytically across the unit circle in the neighborhood of 1, while f is a nonconstant invertible outer function with $\|f\|_\infty = 1$ that has modulus 1 on an arc of $\mathbb{T}$ around 1. Then the only norm attaining symbol for the Hankel operator $\mathfrak{H}_{\bar\Theta f}$ is $\bar\Theta f$, which has not constant modulus.

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