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A FINITE INVERSE PROBLEM BY THE DETERMINANT METHOD

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Abstract. We are concerned with the problem of identifying an operator that depends on n parameters. To this end we use the Poincare determinant to form a characteristic function which relates the n free parameters to n given eigenvalues. Using the implicit function theorem we find a condition that guarantees the local solvability of the inverse eigenvalue problem.

1. Introduction

Consider the $\mu-$parameter family of operators defined by

$$L_\mu = A + \sum_{i=1}^{n} \mu_i B_i \tag{1}$$

where A and B_i are given operators and the parameters $\{\mu_i\}_{i=1}^n \in \mathbb{C}$ are complex numbers. We assume that for any given fixed set of parameters $\{\mu_i\}_{i=1}^n \subset \mathbb{C}$, L_μ is an operator acting in a certain separable Hilbert space, H say, and has a compact resolvent. Thus its spectrum is discrete and depends on the parameters μ_i, i.e. $\sigma_\mu = \{\lambda_n(\mu) : n = 1,2,...\}$. We are interested in the inverse problem of identifying the parameters $\{\mu_i\}_{i=1}^n$ from n given eigenvalues $\{\lambda_i(\mu)\}_{i=1}^n$. A simple example would be of a family of operators acting in $L^2(-1,1)$

$$\begin{cases} L_\mu(y) := \left(\left(x^2-1\right)y'(x)\right)' + \sum_{i=1}^n \mu_i \rho_i(x) y(x) = \lambda y(x) \quad \text{where} \quad -1 \leqslant x \leqslant 1 \\ \lim_{x\to\pm 1}\left(1-x^2\right)y'(x) = 0. \end{cases} \tag{2}$$

where $\rho_i \in L(-1,1)$, are given complex valued functions. Another example is the family of second order differential operators with periodic boundary conditions

$$\begin{cases} L_\mu := -y''(x) + \sum_{i=1}^n \mu_i w_i(x) y(x) = \lambda y(x) \qquad \text{where} \quad 0 \leqslant x \leqslant 2\pi \\ y(0) = y(2\pi) \quad \text{and} \quad y'(0) = y'(2\pi) \end{cases}$$

where $w_i \in L(0,\, 2\pi)$.

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Denote the nonlinear mapping $\Gamma : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Gamma : \{\mu_i\}_{i=1}^n \to \{\lambda_i(\mu)\}_{i=1}^n .$$

The inverse problem of recovering the operator L_μ amounts to identifying the coefficients μ_i, from the λ_n. In other words finding a formula or an approximation for the inverse mapping,

$$\Gamma^{-1} : \{\lambda_i(\mu)\}_{i=1}^n \to \{\mu_i\}_{i=1}^n . \tag{3}$$

Although seen as an inverse problem, (3) also falls as a direct multiparameter eigenvalue problem. Clearly since we are given the values λ_i, the set $(\mu_1, \mu_2, ..., \mu_n, \lambda_n)$ is a "multi-eigenvalue" of L_μ, [15, 14]. Denote by $\mathbb{S}_1$ the space of trace class operators and recall that their determinants can be approximated by finite rank operators. The key to our approach is to use Lidskii's theorem which makes use of the Poincare determinant, [6], to connect the parameters μ_i with the eigenvalue λ_i by an algebraic formula, i.e. the characteristic function $\Delta(\mu, \lambda) = 0$. This new approach allows the handling of singular and non-selfadjoint problems at no extra cost, and is computationally feasible, [3], since it calls on standard methods of numerical linear algebra. The approximation property by finite rank operators reduces (3) to a classical inverse eigenvalue problem, [4]. For example, in (2) the eigenvalues of the principal part, which is the Legendre operator, are $n(n+1)$ as $n \to \infty$. Its inverse is in $\mathbb{S}_1$ since $\sum \frac{1}{n(n+1)} < \infty$. The setting of Poincare determinants, together with the finite section of infinite matrices is more general and leads to practical algorithms with minimal or no numerical integration [1]. Recall that standard methods in inverse spectral theory, such as the Gelfand-Levitan theory, rely heavily on integral equations, [5, 9, 10, 12], and usually require infinite data such as spectral functions and so are not appropriate for finite data inverse problems.

2. Preliminaries

Let A be a self-adjoint operator acting in a separable Hilbert space H with discrete spectrum, [13], and $\{\varphi_n\}_{n\geqslant 1}$ be its eigenfunctions

$$A\varphi_n = \lambda_n \varphi_n \quad \text{for} \quad n \geqslant 1. \tag{4}$$

Recall that $\{\varphi_n\}_{n\geqslant 1}$ form an orthogonal basis for H. Let L_μ be the family of perturbation operators defined by (1) where B_i may not be symmetric operators in H and thus L_μ maybe non self-adjoint. As for its domain, Dom(L_μ), [7, see III. section 5.1], it is enough to assume that

$$\text{Dom}(A) \subset \text{Dom}(B_i) \quad \text{for } i = 1, ..., n.$$

As for the closedness of L_μ, it is also enough that each B_i is A-bounded with zero A-bound. In this case, the A-bound [7, Theorem 1.1, section IV. 1], of $\sum_{i=1}^n \mu_i B_i$ is also zero for all $\{\mu_i\}_{i=1}^n$ and so L_μ is closed if A is closed. For example if B_i are bounded operators in H, then their A-bound is zero. We also assume that for any given set $\{\mu_i\}_{i=1}^n$ in a certain domain of $\mathbb{C}$, L_μ is an operator acting in H, densely defined but with a compact resolvent.

For simplicity we assume that A is positive in order to define its square root operator $A^{1/2}$. This also can be achieved when A is bounded below, by simply translating its spectrum so it is positive. For a fixed set of parameters $\{\mu_i\}_{i=1}^{n}$, denote the eigenvalues of L_μ by λ_k, i.e.

$$Ay_k + \sum_{i=1}^{n} \mu_i B_i y_k = \lambda_k y_k \tag{5}$$

where $y_k \in H$. If $Ay_k \in H$, then $\left|\left|A^{1/2} y_k\right|\right|^2 = (Ay_k, y_k)$ implies $A^{1/2} y_k \in H$ and thus we can write

$$A^{1/2} y_k + \sum_{i=1}^{n} \mu_i A^{-1/2} B_i A^{-1/2} A^{1/2} y_k = \lambda_k A^{-1/2} y_k.$$

Set

$$\psi_k = A^{1/2} y_k \in H$$

to obtain

$$\psi_k + \sum_{i=1}^{n} \mu_i A^{-1/2} B_i A^{-1/2} \psi_k = \lambda_k A^{-1} \psi_k \tag{6}$$

$$\left(1 + \sum_{i=1}^{n} \mu_i A^{-1/2} B_i A^{-1/2} - \lambda_k A^{-1} \right) \psi_k = 0.$$

In case $A^{-1/2} B_i A^{-1/2},\ A^{-1} \in \mathbb{S}_1$, then Lidskii's theorem is applicable and we have

$$\det \left(1 + \sum_{i=1}^{n} \mu_i A^{-1/2} B_i A^{-1/2} - \lambda_k A^{-1} \right) = 0. \tag{7}$$

Observe that (7) is the characteristic function for L_μ.

PROPOSITION 1. *Assume that $A^{-1/2} B_i A^{-1/2},\ A^{-1} \in \mathbb{S}_1$ then the parameters μ_i and eigenvalues λ_k are related by (7).*

Since we are looking for n unknown parameters $\{\mu_i\}_{i=1}^{n}$, we need n equations, which are generated by n different eigenvalues, $\lambda_1, \ldots, \lambda_n$, i.e. we have a system

$$\Delta_j(\mu_1, \ldots, \mu_n) = \det \left(1 + \sum_{i=1}^{n} \mu_i A^{-1/2} B_i A^{-1/2} - \lambda_j A^{-1} \right) = 0 \quad \text{for} \quad j = 1, \ldots, n. \tag{8}$$

Thus the inverse eigenvalue problem reduces to solving a system of n algebraic nonlinear equations with n unknowns, which we denote by $\mathbb{C}^n \xrightarrow{F} \mathbb{C}^n$, i.e.

$$F(\mu) = 0 \tag{9}$$

where $F = (\Delta_1, \Delta_2, \ldots, \Delta_n)$. The first observation is that there can be no global uniqueness, for the following simple reason: If we fix λ_n in (5) then we can look at a new multi-parameter eigenvalue problem in μ. To see this, let $n=1$ and then we have new eigenvalue problem in μ

$$[A-\lambda_1]\,y_k = -\mu_{1_k} B_1 y_k.$$

In other words for the same eigenvalue λ_1, there correspond a sequence of parameters μ_{1_1}, μ_{1_2}, ..., μ_{1_k}, ... and so there are infinitely many solutions to this inverse problem. Nevertheless we can show that a local inversion is possible for the system defined by (8).

3. Approximation

Assume that $\mu^* \in \mathbb{C}^n$ is a solution to $F(\mu^*)=0$. In order to show that it is a unique local solution, i.e. F is a local diffeomorphism, we need to show that its Jacobian, the $n\times n$ matrix

$$F'(\mu) = \left(\frac{\partial \Delta_j}{\partial \mu_i}(\mu)\right), \tag{10}$$

is invertible for μ close to μ^*. In that case, μ^* can be obtained by Newton's iterations provided that $F'(\mu^*)$ has a bounded inverse, [8, Theorem 11.1, page 140], and

$$\lim_{\mu\to\mu^*} \left|\left|F'(\mu)-F'(\mu^*)\right|\right| = 0. \tag{11}$$

We now compute its Jacobian, by taking one component of F at a time,

$$\begin{aligned}\Delta_j(\mu_1,\ldots,\,\mu_n) &= \det\left(1+\sum_{\substack{k=1\\k\neq i}}^{n}\mu_k A^{-1/2}B_kA^{-1/2}-\lambda_jA^{-1}+\mu_iA^{-1/2}B_iA^{-1/2}\right)\\ &= \det(1+\mathbb{D}_{ij}+\mu_i\mathbb{K}_i)\end{aligned}$$

where

$$\mathbb{D}_{ij}=\sum_{\substack{k=1\\k\neq i}}^{n}\mu_kA^{-1/2}B_kA^{-1/2}-\lambda_jA^{-1}\in\mathbb{S}_1 \quad\text{and}\quad \mathbb{K}_i=A^{-1/2}B_iA^{-1/2}\in\mathbb{S}_1. \tag{12}$$

In order to proceed further, we assume that $(1+\mathbb{D}_{ij})^{-1}$ exists, i.e. $\det(1+\mathbb{D}_{ij})\neq 0$, to obtain

$$\begin{aligned}\Delta_j(\mu_1,\ldots,\,\mu_n) &= \det\left[(1+\mathbb{D}_{ij})\left(1+\mu_i(1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right)\right]\\ &= \det(1+\mathbb{D}_{ij})\det\left(1+\mu_i(1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right).\end{aligned}$$

Observe that the variable μ_i appears only once in the second determinant and thus by the Plemelj-Smithies formula [6, theorem 5.2, p61], we have

$$\det\left(1+\mu_i(1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right) = 1+\sum_{n\geqslant 1}\mu_i^nC_n\left((1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right)$$

$$\frac{\partial}{\partial\mu_i}\det\left(1+\mu_i(1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right) = \operatorname{tr}\left((1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right)+\sum_{n\geqslant 1}(n+1)\,\mu_i^nC_{n+1}\left((1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\right)$$

where

$$C_n(A) = \frac{1}{n!}\det\begin{bmatrix} \operatorname{tr}(A) & n-1 & 0 & \dots & 0 \\ \operatorname{tr}\left(A^2\right) & \operatorname{tr}(A) & n-2 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ \operatorname{tr}\left(A^{n-1}\right) & \operatorname{tr}\left(A^{n-2}\right) & \dots & \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^n) & \operatorname{tr}\left(A^{n-1}\right) & \dots & \operatorname{tr}\left(A^2\right) & \operatorname{tr}(A) \end{bmatrix}.$$

Thus the entries of the Jacobian in (10) are explicitly given by

$$\frac{\partial \Delta_j}{\partial \mu_i}(\mu_1,\dots,\mu_n) = \det(1+\mathbb{D}_{ij})\Big(\operatorname{tr}\big((1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\big) + \sum_{n\geqslant 1}(n+1)\mu_i^n C_{n+1}\big((1+\mathbb{D}_{ij})^{-1}\mathbb{K}_i\big)\Big). \tag{13}$$

Obviously if $F'(\mu^*)$ has an inverse, or equivalently $\det[F'(\mu^*)] \neq 0$ then Newton's iterations defined by

$$p_{n+1} = p_n - \left[F'(p_n)\right]^{-1} F(p_n) \tag{14}$$

converge to μ^*, which solves $F(\mu^*) = 0$ provided p_0 is close enough to μ^*. We now can state our first result.

PROPOSITION 2. *Assume that conditions of Proposition 1 hold and that both* $\det(1+\mathbb{D}_{ij}) \neq 0$, *for* $\mu = \mu^*$ *and* $\det[F'(\mu^*)] \neq 0$, *(*13*) then the iterations defined by (*14*) converge to the solution of* $F(\mu^*) = 0$.

Proof. It is easy to see from (5) and (7) that the condition $\det(1+\mathbb{D}_{ij}) \neq 0$ means that λ_j is not an eigenvalue of $A + \sum_{\substack{k=1 \\ k\neq i}}^n \mu_k^* B_k$. It is also readily seen that (11) holds since by (13) the entries $\frac{\partial \Delta_j}{\partial \mu_i}(\mu_1,\dots,\ \mu_n)$ depend continuously on $\{\mu_i\}_{i=1}^n$. Thus by [8] the sequence defined by (14) converges to μ^*. $\square$

Next we explain how to reduce those infinite matrices in order to approximate the determinants and solve the system in (8). Recall that the determinant is a continuous function over $\mathbb{S}_1$, i.e. if $\mathbb{F}_n \to \mathbb{F}$ in $\mathbb{S}_1$ then $\det(1+\mathbb{F}_n) \to \det(1+\mathbb{F})$, and in particular this is true when $\mathbb{F}_n$ are finite sections of $\mathbb{F}$. From the computational point of view, this would reduce (8) to determinant of finite matrices. Recall that from (8) we have

$$\Delta_j(\mu_1,\dots,\mu_n) = \det\left(1 + \sum_{i=1}^n \mu_i A^{-1/2} B_i A^{-1/2} - \lambda_j A^{-1}\right) = 0$$

and if we use the eigenfunctions of A, see (4) to represent those matrices, we then have

$$A_{nk} = (A\varphi_k, \varphi_n) = (\rho_k \varphi_k,\ \varphi_n) = \rho_n \delta_{nk}$$

and so

$$A = \operatorname{diag}(\rho_1,\ \rho_2,\dots,\rho_n,\dots.) \text{ and } A^m = \operatorname{diag}(\rho_1^m,\ \rho_2^m,\dots,\rho_n^m,\dots.) \text{ for } m \in \mathbb{R}.$$

The condition $A^{-1} \in \mathbb{S}_1$, then translates into

$$\sum_{i\geqslant 1}\frac{1}{\rho_i} < \infty \quad \text{and} \quad A^{-1/2} B_i A^{-1/2} = \left(\frac{1}{\sqrt{\rho_k \rho_n}}(B_i\varphi_k, \varphi_n)\right)_{n,k\geqslant 1}.$$

Using the projection $\mathbf{P}_m(c_n) = (c_1, c_2, c_3, ..., c_m, 0, 0, ...)$, we have

$$\mathbf{P}_m A^{-1/2} B_i A^{-1/2} \mathbf{P}_m = \left(\frac{1}{\sqrt{\rho_k \rho_n}} (B_i \varphi_k, \varphi_n) \right)_{m \geqslant n, k \geqslant 1}$$
$$= \operatorname{diag}\left(\frac{1}{\sqrt{\rho_n}} \right)_{m \geqslant n \geqslant 1} [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1} \operatorname{diag}\left(\frac{1}{\sqrt{\rho_k}} \right)_{m \geqslant k \geqslant 1}.$$

The finite section then leads to

$$\begin{aligned}
&\Delta_j(\mu_1, ..., \mu_n) \\
&= \det\left(1 + \sum_{i=1}^{n} \mu_i \operatorname{diag}\left(\frac{1}{\sqrt{\rho_n}}\right)_{m \geqslant n \geqslant 1} [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1} \operatorname{diag}\left(\frac{1}{\sqrt{\rho_k}} \frac{1}{\rho_n}\right)_{m \geqslant n \geqslant 1}\right) + o(1) \\
&= \det\left(\operatorname{diag}_m\left(\frac{1}{\sqrt{\rho_n}}\right) \times \left[\operatorname{diag}_m(\rho_n) + \sum_{i=1}^{n} \mu_i \, [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1} - \lambda_j \mathbf{I}_m\right] \times \operatorname{diag}_m\left(\frac{1}{\sqrt{\rho_n}}\right)\right) \\
&\qquad + o(1) \\
&= \left(\Pi_{i=1}^{m} \frac{1}{\rho_i}\right) \det\left[\operatorname{diag}_{m \geqslant n \geqslant 1}(\rho_n) + \sum_{i=1}^{n} \mu_i [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1} - \lambda_j \mathbf{I}_m\right] + o(1).
\end{aligned}$$

Thus one approximation for equation $\Delta_j(\mu_1, ..., \mu_n) = 0$ can be given

$$\begin{aligned}
\Delta_j^{(m)}(\mu_1, ..., \mu_n) &= \det\left[\operatorname{diag}_m(\rho_n) + \sum_{i=1}^{n} \mu_i [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1} - \lambda_j \mathbf{I}_m\right] \\
&= \det\left(\operatorname{diag}_m(\rho_n - \lambda_j) + \sum_{i=1}^{n} \mu_i [(B_i \varphi_k, \varphi_n)]_{m \geqslant n, k \geqslant 1}\right) = 0 \qquad (15)
\end{aligned}$$

which is computationally easier to handle by Newton's method, since B_i, φ_i are known and no inversion of the operator A is needed. Also the solvability of the new system in (15) is much easier to check since we are dealing with a polynomial of finite degree in $(\mu_1, \mu_2, ..., \mu_n)$.

REMARK. In [11], McCarthy and Rundell used a quasi-Newton scheme to approximate a potential q by its partial Fourier series $\sum_{k=1}^{2n} q_k \varphi_k$. The characteristic function was obtained numerically through the shooting method generated by two sets of eigenvalues $\{\lambda_i^1\}_{i=1}^n$ and $\{\lambda_i^2\}_{i=1}^n$. Our approach can also be used to approximate potentials by their partial Fourier series, just by looking at the Fourier coefficients as the sought parameters. This is implemented in the last example of the next section. Note that one advantage of the determinant method is that we can increase the computational precision just by increasing the size of the section and there is no inverse operator to be computed. Also when φ_n are special functions and B_i are simple operations, such as multiplication by a polynomial, then the entries $(B_i \varphi_k, \varphi_n)$ in (15) can be computed explicitly, for example by using the residue theorem, Laplace transform or classical integral tables. This avoids numerical integration which is the main source of roundoff error in computational inverse problem.

4. Applications

We now work out few examples using regular problems. For the sake of simplicity we shall restrict ourselves to real valued parameters as Newton's method solvers for complex valued functions are more involved computationally.

EXAMPLE 1. Consider

$$\begin{cases} -y'' + (\mu_1 \sin(x) + \mu_2 \cos(x) + \mu_3 x)\, y(x) = \lambda y(x) & 0 \leqslant x \leqslant \pi \\ y(0) = y(\pi) = 0 \end{cases}$$

In case $(\mu_1,\ \mu_2,\ \mu_3) = (-2,\ 5,\ 3)$ then the first eigenvalues [1]

$$\lambda_k \in \{3.824205647, 7.423205658, 12.43988095, 19.43802168, \\ 28.43814459, 39.43866830, \ldots.\}.$$

Using the first three eigenvalues lead to a system of 3 equations with 3 unknowns and using a 10×10 section of the matrix as defined by (15) we obtain

$$\mu_1 = -1.999940437, \mu_2 = 5.000001747,\ \mu_3 = 2.999972181,$$

which agrees well with the solution. Here we show the structure of a 3×3 matrix section of

$$\underset{4}{\mathrm{diag}}\,(\rho_n) + \sum_{i=1}^{3} \mu_i \left[(B_i \varphi_k, \varphi_n)\right]_{4 \geqslant n, k \geqslant 1}$$

is given by

$$\frac{1}{\pi} \begin{bmatrix} \pi + 2\left(\frac{4}{3}\mu_1 + \frac{1}{4}\mu_3\pi^2\right) & 2\left(-\frac{8}{9}\mu_3 + \frac{1}{4}\mu_2\pi\right) & \frac{8}{15}\mu_1 \\ 2\left(-\frac{8}{9}\mu_3 + \frac{1}{4}\mu_2\pi\right) & 4\pi + 2\left(\frac{16}{15}a + \frac{1}{4}\mu_3\pi^2\right) & 2\left(-\frac{24}{25}\mu_3 + \frac{1}{4}\mu_2\pi\right) \\ -\frac{8}{15}\mu_1 & 2\left(-\frac{24}{25}\mu_3 + \frac{1}{4}\mu_2\pi\right) & 9\pi + 2\left(\frac{36}{35}\mu_1 + \frac{1}{4}\mu_3\pi^2\right) \end{bmatrix}$$

The above setting is very similar to methods found in [4].

EXAMPLE 2. Consider the Legendre type operator

$$\begin{cases} L_\mu(y) := \left(\left(x^2 - 1\right) y'(x)\right)' + \left(\mu_1 x^2 + \mu_2 x^4 + \mu_3 x^7\right) y(x) = \lambda y(x) & -1 \leqslant x \leqslant 1 \\ \lim_{x \to \pm 1} \left(1 - x^2\right) y'(x) = 0. \end{cases}$$

In case $(\mu_1,\ \mu_2,\ \mu_3) = (6, -5,\ 8)$ then the first few eigenvalues

$$\lambda_n \in \{-0.0833635055999295688, 2.68399371307382050, 7.42034640498025588, \\ 13.5060634153541878, 21.3751308658871793, 31.2694477790219985, \ldots.\}$$

Using the first 3 eigenvalues, the solution obtained from a 10×10 section of (15) is

$$\mu_1 = 6.00028320652478995, \quad \mu_2 = -5.00049042118147839, \\ \mu_3 = 7.99989337567292291.$$

EXAMPLE 3. We now examine a periodic problem, [2]

$$\begin{cases} L_\mu(y) := -y''(x) + (\mu_1\cos(x) + \mu_2\cos(2x) + \mu_3\cos(4x))\,y(x) = \lambda y(x) & 0 \leqslant x \leqslant 2\pi \\ y(0) = y(2\pi) \text{ and } y'(0) = y'(2\pi). \end{cases}$$

Computing the eigenvalues when, $(\mu_1,\ \mu_2,\ \mu_3) = (7,\ -2,\ 3,)$ we get

$$\lambda_n \in \{-6.132170402, -2.969297944,\ 2.545342063,\ 5.242147858,\ 5.316225886,\ 9.145800069,\ 10.74692302, 16.72280808,\ 16.78007897, 26.28446391, 26.31767849, ...\}$$

Using the following eigenvalues

$$-2.9692979436940092845,\ 2.5453420625352851238,\ 5.2421478583611986637$$

the solutions

$$\mu_1 = 6.926442585, \quad \mu_2 = -1.875537355, \quad \mu_3 = 3.063530176.$$

A 9×9 finite section of the band Toeplitz type matrix representing of the operator L_μ, with the basis $\{e^{inx}\}_{n\in\mathbb{Z}}$, is given by

$$\begin{bmatrix} 16 & a & b & 0 & c & 0 & 0 & 0 & 0 \\ a & 9 & a & b & 0 & c & 0 & 0 & 0 \\ b & a & 4 & a & b & 0 & c & 0 & 0 \\ 0 & b & a & 1 & a & b & 0 & c & 0 \\ c & 0 & b & a & 0 & a & b & 0 & c \\ 0 & c & 0 & b & a & 1 & a & b & 0 \\ 0 & 0 & c & 0 & b & a & 4 & a & b \\ 0 & 0 & 0 & c & 0 & b & a & 9 & a \\ 0 & 0 & 0 & 0 & c & 0 & b & a & 16 \end{bmatrix}$$

where $2a = \mu_1,\ 2b = \mu_2,\ 2c = \mu_3$.

EXAMPLE 4. Here we treat a similar potential as in [11, figure 4], however with simpler boundary conditions. Consider

$$\begin{cases} L(y) := -y''(x) + x\left(\pi^2 - x^2\right) y(x) = \lambda y(x) & 0 \leqslant x \leqslant \pi \\ y(0) = y(\pi) = 0 \end{cases}$$

whose first eigenvalues are

$$\{9.919756273,\ 12.53511213,\ 17.20969538,\ 24.04164338,\ 32.94746813, ...\}. \tag{16}$$

The beginning of the Fourier sine expansion of $q(x) = x\left(\pi^2 - x^2\right)$ is

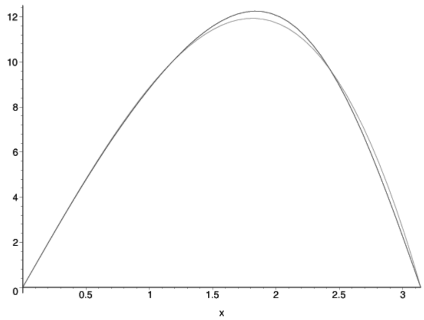
$$x\left(\pi^2 - x^2\right) = 12\sin(x) - 3/2\sin(2x) + 4/9\sin(3x) - 3/16\sin(4x) + 12/125\sin(5x) + \dots .$$

If we look for the first three Fourier coefficients, that is

$$\begin{cases} L(y) := -y''(x) + (\mu_1 \sin(x) + \mu_2 \sin(2x) + \mu_3 \sin(3x))\, y(x) = \lambda y(x) \quad 0 \leqslant x \leqslant \pi \\ y(0) = y(\pi) = 0. \end{cases}$$

and use the first three eigenvalues in (16), then we obtain the following values

$$\mu_1 = 12.091314365234747488; \quad \mu_2 = -1.5233958603626339028;$$



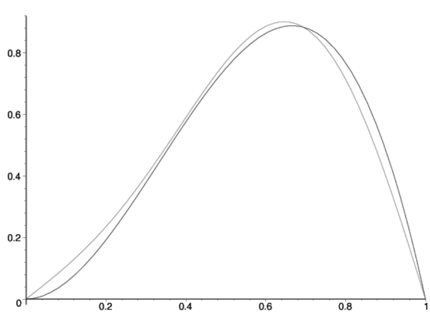
Example 4: $q(x) = x(\pi^2 - x^2)$ *and its approximation on* $[0, \pi]$

EXAMPLE 5. Consider the example, see figure 4 in [11]

$$\begin{cases} L(y) := -y''(x) + 6x^2(1-x)\, y(x) = \lambda y(x) \quad 0 \leqslant x \leqslant 1 \\ y(0) = y(1) = 0. \end{cases}$$

Using Fourier series in $L^2(0,1)$ we shall look for the first three coefficients

$$\begin{cases} L(y) := -y''(x) + (\mu_1 \sin(\pi x) + \mu_2 \sin(2\pi x) + \mu_3 \sin(3\pi x))\, y(x) = \lambda y(x) \quad 0 \leqslant x \leqslant 1 \\ y(0) = y(1) = 0. \end{cases}$$



Example 5: $q(x) = 6x^2(1-x)$ *and its approximation on* $[0, 1]$

Using the first three eigenvalues we get

$$\mu_1 = 0.7852952;\ \mu_2 = -0.2521436;\ \mu_3 = 0.01402773$$

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