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AGLER INTERPOLATION FAMILIES OF KERNELS

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Abstract. An abstract Pick interpolation theorem for a family of positive semi-definite kernels on a set X is formulated. The result complements those in [Ag] and [AM02] and will subsequently be applied to Pick interpolation on distinguished varieties [JKM].

1. Introduction

Let $s(z,w)$ denote Szegő's kernel; i.e.,

$$s(z,w) = \frac{1}{1-z\overline{w}},$$

for complex numbers z and w. The kernel s is the reproducing kernel for the Hardy space $H^2(\mathbb{D})$ of functions analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with square summable power series. Thus, an analytic function $f : \mathbb{D} \to \mathbb{C}$ with power series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

is, by definition, in $H^2(\mathbb{D})$ if and only if $\sum |f_n|^2$ converges. The Hardy space is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \overline{g_n}.$$

Evidently, for a fixed w, the function $s_w(z) = s(z,w)$ is in $H^2(\mathbb{D})$ and earns the title of reproducing kernel because, for $f \in H^2(\mathbb{D})$,

$$f(w) = \langle f, s_w \rangle.$$

Szegő's kernel is indispensable to the statement of

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THEOREM 1.1. (Pick Interpolation) *Let n be a positive integer. Given points $w_1,\dots,w_n;v_1,\dots,v_n \in \mathbb{D}$, there exists an analytic function $f:\mathbb{D}\to\mathbb{D}$ such that $f(w_j)=v_j$ if and only if Pick's matrix,*

$$\big((1-v_j\overline{v_\ell})s(w_j,w_\ell)\big)$$

is positive semi-definite.

Extensions of the Pick interpolation theorem to domains and settings more general than the disc $\mathbb{D}$ often involve replacing the Szegő kernel with a family of kernels. The references [AM02, AM99, AM03, DPRS, Ab, B, BB, BCV, BTV, P, CLW, FF, M, JK, MP, MS, P, R1, R2, S65] represent only a fraction of the results in this direction. For instance, in Abrahamse's [Ab] interpolation theorem on the annulus the Szegő kernel is replaced by a family of kernels $k^t(z,w)$ - parametrized by t in the unit circle $\mathbb{T}$ - identified by Sarason [S65]. See also [AD]. In a similar vein, the recent constrained Pick interpolation in [DPRS] [R1][R2] are stated in terms of a family of kernels over the disc canonically determined by the constraints.

The main result of this paper, Theorem 1.3 below, is a Pick theorem formulated, like the related results in [Ag] and [AM02], purely in terms of a collection of kernels. The result here has a natural operator algebraic interpretation which is exploited in the proof by using the fact that the quotient of an operator algebra by a two sided ideal is again an operator algebra. (This is a corollary of the Blecher-Ruan-Sinclair (BRS) theorem. See [Pa] or [BL] for an exposition of the BRS theorem and the related topics of completely positive maps, Arveson's extension theorem, and Stinespring's representation theorem.) In forthcoming work [JKM], Theorem 1.3 is applied to produce a Pick interpolation theorem on distinguished varieties [AM05] [AM03].

The statement of the main result requires the notion of a (positive semi-definite) matrix-valued kernel. Let M_n denote the $n\times n$ matrices with complex entries. An M_n-valued kernel on a set X is a function $k:X\times X\to M_n$ which is positive semi-definite in the sense that, for every finite subset $F\subset X$, the (block) matrix

$$\big(k(x,y)\big)_{x,y\in F}$$

is positive semi-definite.

DEFINITION 1.2. Fix a set X and a sequence $\mathscr{K}=(\mathscr{K}_n)$ where each $\mathscr{K}_n$ is a set of M_n-valued kernels on X.

The collection $\mathscr{K}$ is an *Agler interpolation family of kernels* provided:

(i) if $k_1\in\mathscr{K}_{n_1}$ and $k_2\in\mathscr{K}_{n_2}$, then $k_1\oplus k_2\in\mathscr{K}_{n_1+n_2}$;

(ii) if $k\in\mathscr{K}_n$, $z\in X$, $\gamma\in\mathbb{C}^n$, and $\gamma^*k(z,z)\gamma\neq 0$, then there exists an N, a kernel $\kappa\in\mathscr{K}_N$, and a function $G:X\to M_{n,N}$ such that

$$k'(x,y):=k(x,y)-\frac{k(x,z)\gamma\gamma^*k(z,y)}{\gamma^*k(z,z)\gamma}=G(x)\kappa(x,y)G(y)^*;$$

(iii) for each finite $F \subset X$ and for each $f : F \to \mathbb{C}$, there is a $\rho > 0$ such that, for each $k \in \mathscr{K}$,

$$F \times F \ni \mapsto (\rho^2 - f(x)f(y)^*)k(x,y)$$

is a positive semi-definite kernel on F; and

(iv) for each $x \in X$ there is a $k \in \mathscr{K}$ such that $k(x,x)$ is nonzero.

THEOREM 1.3. *Suppose $\mathscr{K}$ is an Agler interpolation family of kernels on X. Further suppose $Y \subset X$ is finite, $g : Y \to \mathbb{C}$ and $\rho \geqslant 0$. If for each $k \in \mathscr{K}$ the kernel*

$$Y \times Y \ni (x,y) \to (\rho^2 - g(x)g(y)^*)k(x,y) \tag{1.1}$$

is positive semi-definite, then there exists $f : X \to \mathbb{C}$ such that $f|_Y = g$ and for each $k \in \mathscr{K}$ the kernel

$$X \times X \ni \to (\rho^2 - f(x)f(y)^*)k(x,y) \tag{1.2}$$

is positive semi-definite.

The remainder of this introduction contains five subsections. The first gives alternate formulations and interpretations of the axioms of an Agler interpolation family. The second through fourth subsections compare Theorem 1.3 with the results results of [Ag] and [AM02] on *kernel structures*; with Quiggin's Theorem; and with the test function approach to interpolation like that found in [DMM]. We thank the referees for both suggesting and providing substantial contributions toward these topics - as well as for pointing out and fixing a rough patch in the proof of Theorem 1.3. The last subsection gives a road map to the rest of the paper. In particular, an initial discussion of the connections with the interpolation Theorem of Abrahamse is in Section 5.

1.1. The axioms

Condition (ii) in the definition of interpolation family asks that Schur complements of a kernel in the family is again in the family, at least up to conjugation.

Let $\mathscr{K}$ be a collection of kernels on X which doesn't necessarily satisfy condition (iii) of interpolation family. With Y a finite subset of X fixed, let $M(Y)$ denote those functions $f : Y \to \mathbb{C}$ for which there is a ρ such that

$$Y \times Y \ni (x,y) \to (\rho^2 - f(x)f(y)^*)k(x,y)$$

is positive semi-definite for all $k \in \mathscr{K}$. It is straightforward to check that $M(Y)$ is an algebra. Moreover, $M(Y)$ consists of all complex-valued functions on Y (condition (iii)) if and only if $M(Y)$ separates points if and only if for each $y \in Y$ the characteristic function of $\{y\}$ is in $M(Y)$.

Condition (iii) can be thought of as a type of (uniformly) full rank condition on the collection $\mathscr{K}$. With Y finite, k a kernel on Y, and $y \in Y$, view $k(\cdot,y) = (k(x,y))_{x \in Y}$ as a vector. For this single scalar kernel condition (iii) asks that the non-zero vectors in the set $\{k(\cdot,y) : y \in Y\}$ form a linearly independent set.

Condition (iv) says that, again for a finite subset Y of X,

$$\|f\|_Y = \inf\{\rho^2 : (\rho^2 - f(x)f(y)^*)k(x,y) \succeq 0,\ \ k \in \mathscr{K}\}$$

determines a norm, and not just a semi-norm, on the set of complex-valued functions on Y.

1.2. Kernel structures

In [AM02] (see also [Ag]), the notion of a kernel structure is introduced. A kernel structure consists of a collection $\mathscr{S}$ of scalar kernels on a set X satisfying a list of axioms which includes the requirement that a *normalized average* of kernels from the collection should again be in the collection. The resulting (Pick) interpolation theorem includes as a hypothesis a type of Schur complement condition in the spirit of (ii) in the definition of interpolation family. What is problematic in certain examples, including interpolation on distinguished varieties, is verifying that this Schur complement condition is compatible with averaging of kernels (of course it is always possible to take the closure of a given collection with respect to these operations, but at the expense of perhaps an undesired enlargement).

The direct sum condition (i) in the notion of an interpolation family necessarily introduces matrix-valued kernels and does in a sense play the role of averaging. However, it turns out that on a (nice) multiply connected domain R, the log modularity of the algebra $H^\infty(R)$ together with Theorem 1.3 recovers the Abrahamse interpolation Theorem. On the other hand, it seems likely that in other applications that either matrix-valued kernels must be included or the collection of scalar kernels must be enlarged, and perhaps significantly so. Indeed, it appears this is the situation for distinguished varieties.

1.3. NP kernels

It is interesting for several reasons to see that Quiggin's theorem [Q] (see also [AM00]) is a consequence of Theorem 1.3.

Let k be a positive definite (scalar-valued) kernel on a set X and assume that k is normalized by $k(x,b) = 1$ for some fixed base point $b \in X$. (The implicit assumption that there is a b such that $k(x,b)$ is never zero can be avoided, but the extra generality is not germane to this discussion). The kernel k is an NP kernel (NP for Nevanlinna-Pick) if there is a positive kernel $p(x,y)$ such that

$$k(x,y) - 1 = p(x,y)k(x,y).$$

It turns out that the choice of base point did not matter.

LEMMA 1.4. *For each $a \in X$ there is a positive kernel q_a such that*

$$k(a,a)k(x,y) - k(x,a)k(a,y) = q_a(x,y)k(x,y).$$

Using this lemma, it is straightforward to prove that the collection of kernels $\{I_{n+1} \otimes k : n \in \mathbb{N}\}$ (here I_{n+1} is the $(n+1) \times (n+1)$ identity matrix) is an Agler-interpolation family of kernels and consequently deduce Quiggin's theorem.

Sketch of proof of Lemma 1.4. Writing everything in terms of $\ell(x,y) = p(x,y) - 1$, it suffices to prove that

$$X \times X \ni (x,y) \mapsto \ell(x,y) + \frac{\ell(x,a)\ell(a,y)}{-\ell(a,a)}$$

is a positive semi-definite kernel. This conclusion follows from two facts. First, for a finite set F the matrix

$$(p(x,y) - 1)_{x,y \in F}$$

has exactly one negative eigenvalue, since it is a positive semi-definite matrix minus a rank one positive and moreover, $0 \leqslant p(x,x) < 1$ for all x. Second, if

$$\begin{pmatrix} -1 & b^* \\ b & D \end{pmatrix}$$

has exactly one negative eigenvalue, then $D + bb^*$ is positive semi-definite which follows from the computation,

$$\begin{pmatrix} 1 & 0 \\ b & I \end{pmatrix} \begin{pmatrix} -1 & b^* \\ b & D \end{pmatrix} \begin{pmatrix} 1 & b^* \\ 0 & I \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & D + bb^* \end{pmatrix}. \qquad \square$$

The proof of Quiggin's theorem in [Q] uses Parrot's Lemma to extend the domain of definition of the function g to a point y_0 not in X while preserving the positivity in (1.1) (for the single NP kernel k). A Zorn's Lemma argument then completes the proof. Unfortunately, this approach fails in the context of a collection of kernels (as opposed to a single NP kernel). To see the difficulty, suppose g and ρ satisfy (1.1) and let $k \in \mathscr{K}_n$ a vector $\gamma \in \mathbb{C}^n$ and a point $y_0 \notin Y$ be given. Schur complement and an application of Parrot's Lemma produces a $\lambda_{k,\gamma}$ such that the matrix

$$\begin{pmatrix} (1 - \lambda_{k,\gamma}\lambda_{k,\gamma}^*)\gamma^* k(y_0,y_0)\gamma & (1 - g(x)\lambda_{k,\gamma}^*)\gamma^* k(y_0,x)_{x \in Y} \\ (1 - \lambda_{k,\gamma} g(y)^*) k(x,y_0)\gamma & (1 - g(x)g(y)^*(k(y,x)_{x,y \in Y} \end{pmatrix}$$

is positive semi-definite. Unfortunately, it is not obvious from these considerations that there is a choice of λ independent of k and γ. Thus, to prove Theorem 1.3 it is not simply a matter of using a Schur complement argument to show that a function $g : Y \to \mathbb{C}$ satisfying condition (1.1) can be extended to a $g_0 : Y' = Y_0 \cup \{y_0\} \to \mathbb{C}$ (y_0 any point of X not already in Y) so that g_0 satisfies (1.1) on $Y' \times Y'$.

1.4. Test Functions

In [DMM] (see also [AM02]) a family of test functions Ψ on a set X is a collection of functions $f : X \to \mathbb{C}$ such that

(i) $\sup\{|\psi(x)| : \psi \in \Psi\} < 1$ for each $x \in X$;

(ii) for each finite subset Y of X, the algebra generated by the restrictions of $\psi \in \Psi$ to Y is the algebra of all complex functions on Y; and

(iii) Ψ is closed in the topology of pointwise convergence.

(Condition (iii) is not so important for the considerations here.)

Dualizing, let $\mathscr{K}(\Psi)$ denote the collection of kernels k such that

$$X \times X \to (1 - \psi(x)\psi(y)^*)k(x,y)$$

is positive semi-definite for every $\psi \in \Psi$.

PROPOSITION 1.5. *The collection $\mathscr{K}(\Psi)$ is an interpolation family.*

Sketch of proof. Condition (i) of Definition 1.2 is evidently satisfied.

To prove item (ii) observe, suppose Y is a finite set, k is an $n \times n$ matrix-valued kernel on Y, $f : Y \to \mathbb{C}$, and

$$Y \times Y \ni (x,y) \mapsto \ell(x,y) = (\rho^2 - f(x)f(y)^*)k(x,y)$$

is positive semi-definite. Given $z \in Y$ and a vector $\gamma \in \mathbb{C}^n$, the Schur complement

$$\ell'(x,y) = \ell(x,y) - \frac{\ell(x,z)\gamma\gamma^*\ell(z,y)}{\gamma^*\ell(z,z)\gamma}$$

is positive semi-definite. A bit of algebra gives,

$$\ell' = (\rho^2 - f(x)f(y)^*)k'(x,y) - (f(x) - f(z))(f(y) - f(z))^*L(x,y),$$

where k' is the kernel from item (ii) of the definition of interpolation family and L is a positive semidefinite kernel. It follows that if $k \in \mathscr{K}(\Psi)$, then so is k'.

That $\mathscr{K}(\Psi)$ satisfies condition (iii) of interpolation family follows immediately from item (ii) in the definition of test functions and remarks in Subsection 1.1.

Given $x \in X$, the kernel k_x defined by $k_x(x,x) = 1$ and $k_x(y,z) = 0$ if $(y,z) \neq (x,x)$ is in $\mathscr{K}(\Psi)$ and hence $\mathscr{K}(\Psi)$ satisfies condition (iv) in the definition of interpolation family. $\square$

1.5. The rest of the paper

The remainder of the paper is organized as follows. Section 2 contains some background on the connections between kernels and operators. The proof of Theorem 1.3 in the case that X is finite appears in Section 3. A main ingredient is the fact that the quotient of an operator algebra by a closed ideal is again an operator algebra. The induction step - passing from finite to infinite X and thus completing the proof of Theorem 1.3 - is the subject of Section 4.

In the forthcoming paper [JKM], Theorem 1.3 is applied to yield a Pick interpolation theorem for distinguished varieties. There are similarities to interpolation on multiply connected domains and the case of the annulus is discussed in Section 5, where the role of item (ii) of Definition 1.2 becomes apparent. The fact that every finite rank bundle shift over an annulus $\mathbb{A}$ is a direct sum of rank one bundle shifts [AD] implies directly that the family of kernels obtained by taking finite direct sums of Abrahamse/Sarason kernels over $\mathbb{A}$ is an interpolation family yielding Abrahamse's interpolation theorem on $\mathbb{A}$. For more general multiply connected domains (and for scalar interpolation) this is still true, but the proof is more involved and is not discussed here. Also, though Theorem 1.3 addresses only scalar-valued interpolation, the conditions of Definition 1.2 are formulated with matrix-valued interpolation in mind.

2. Operator Theoretic Preliminaries

The operator theoretic approach to interpolation associates to a positive semi-definite matrix-valued kernel k a Hilbert space $H^2(k)$. Functions satisfying, for this given k, the positivity condition of item (iii) of Definition 1.2 determine bounded operators on $H^2(k)$.

2.1. The Hilbert Space $H^2(k)$

In this section we review the basic machinery of reproducing kernel Hilbert spaces; see e.g. [AM02]. To a positive semi-definite kernel $k : X \times X \to M_n$, there is associated a Hilbert space $H^2(k)$ so that in the case that k is positive definite and X is finite, $H^2(k)$ is, as a set, all functions $F : X \to \mathbb{C}^n$. To construct $H^2(k)$, define a semi-inner product on functions $F, G : X \to \mathbb{C}^n$ of the form

$$F = \sum_{x \in X} k(\cdot, x) F_x,$$
$$G = \sum_{x \in X} k(\cdot, x) G_x,$$

by

$$\langle F, G \rangle = \sum_{x,y \in X} \langle k(x,y) F_y, F_x \rangle.$$

Let $H^2(k)$ denote the Hilbert space obtained by quotienting out null vectors and then forming the completion of the resulting pre-Hilbert space. When X is finite the quotient is finite dimensional and hence already complete. If moreover, k is positive definite, then the set of null vectors is trivial.

Condition (ii) in Definition 1.2 has a natural interpretation in terms of $H^2(k)$: if $\mathscr{N}$ is the subspace of $H^2(k)$ spanned by the nonzero vector $k(\cdot, z)\gamma$, then k' is the reproducing kernel for $\mathscr{N}^\perp$. Indeed, we have

$$P_{\mathscr{N}} = \frac{k(\cdot, z)\gamma (k(\cdot, z)\gamma)^*}{\langle k(\cdot, z)\gamma, k(\cdot, z)\gamma \rangle}.$$

Hence,

$$\begin{aligned}\langle P_{\mathscr{N}} k(\cdot,y)v, k(\cdot,x)u\rangle &= \frac{\langle k(\cdot,y)v, k(\cdot,z)\gamma\rangle\langle k(\cdot,z)\gamma, k(\cdot,x)u\rangle}{\langle k(z,z)\gamma,\gamma\rangle}\\ &= \frac{\langle k(z,y)v,\gamma\rangle\langle k(x,z)\gamma,u\rangle}{\langle k(z,z)\gamma,\gamma\rangle}\\ &= u^*\frac{k(x,z)\gamma\gamma^* k(z,y)}{\langle k(z,z)\gamma,\gamma\rangle}v.\end{aligned}$$

Thus, letting $\mathscr{M} = H^2(k)\ominus\mathscr{N}$ and using the notation of item (ii) in Definition 1.2,

$$\begin{aligned}\langle P_{\mathscr{M}} k(\cdot,y)v, k(\cdot,x)u\rangle &= \langle k(\cdot,y)v, k(\cdot,x)u\rangle - \frac{\langle k(x,z)\gamma\gamma^* k(z,y)v,u\rangle}{\langle k(z,z)\gamma,\gamma\rangle}\\ &= \langle k(x,y)v,u\rangle - \frac{\langle k(x,z)\gamma\gamma^* k(z,y)v,u\rangle}{\langle k(z,z)\gamma,\gamma\rangle}\\ &= \langle k'(x,y)v,u\rangle.\end{aligned}$$

Assuming k is a member of an Agler interpolation family $\mathscr{K}$, then, by item (ii) of Definition 1.2 there is an N, a $\kappa\in\mathscr{K}_N$, and a function $G: X\to M_{n,N}$ such that

$$\langle P_{\mathscr{M}} k(\cdot,y)v, k(\cdot,x)u\rangle = \langle G(x)\kappa(x,y)G(y)^* v,u\rangle.$$

LEMMA 2.1. *Let $\mathscr{K}$ be an Agler interpolation family of kernels on a finite set X. Suppose $k\in\mathscr{K}$, $Z\subset X$ and for each $z\in Z$ there is an associated subspace $\mathscr{J}_z\subset\mathbb{C}^n$. Let $\mathscr{G}_z = k(\cdot,z)\mathscr{J}_z$, let $\mathscr{N} = \sum\mathscr{G}_z\subset H^2(k)$, and let $\mathscr{M} = H^2(k)\ominus\mathscr{N}$. There is an N, a kernel $\kappa\in\mathscr{K}_N$, and a function $G: X\to M_{n,N}$ such that*

$$\langle P_{\mathscr{M}} k(\cdot,y)v, k(\cdot,x)u\rangle = \langle G(x)\kappa(x,y)G(y)^* u,v\rangle. \tag{2.1}$$

Moreover, there is a positive M_n-valued kernel k' such that, for $u,v\in\mathbb{C}^n$,

$$\langle k'(x,y)u,v\rangle = \langle P_{\mathscr{M}} k(\cdot,y)u, k(\cdot,x)v\rangle. \tag{2.2}$$

Finally, the mapping $W:\mathscr{M}\to H^2(\kappa)$ defined by

$$WP_{\mathscr{M}} k(\cdot,y)u = \kappa(\cdot,y)G(y)^* u$$

is (well defined and) an isometry.

Proof. Equation (2.1) follows by an induction argument based on the computation preceding the proof. The right hand side of equation (2.2) determines a (positive semi-definite) kernel. Finally, that W is an isometry follows immediately from equation (2.1). $\square$

2.2. The algebra $H^\infty(k)$

Let k be a positive semi-definite M_n-valued kernel on X and suppose for each $f : X \to \mathbb{C}$ there is a $\rho > 0$ such that

$$X \times X \ni (x,y) \mapsto (\rho^2 - f(x)f(y)^*)k(x,y)$$

is a positive semi-definite kernel on X. Let $H^\infty(k)$ denote the set of functions $f : X \to \mathbb{C}$ endowed with the norm,

$$\|f\|_k = \inf\{\rho > 0 : (\rho^2 - f(x)f(y)^*)k(x,y) \succeq 0 \text{ for all } k \in \mathscr{K}\}.$$

Here $\succeq 0$ means the relevant kernel is positive semi-definite.

An element f of $H^\infty(k)$ is identified with the operator $M_f : H^2(k) \to H^2(k)$ whose adjoint is determined by $M_k(f)^* k(\cdot,z)h = f(z)^* k(\cdot,z)h$. Indeed,

$$\|M_k(f)^*\|_k = \|f\|_k.$$

Hence $M_k : H^\infty(k) \to \mathscr{B}(H^2(k))$ defined by $f \mapsto M_k(f)$ is an isometric unital representation. Moreover, viewing $H^\infty(k)$ as a subalgebra of $\mathscr{B}(H^2(k))$ determines an operator algebra structure on $H^\infty(k)$.

LEMMA 2.2. *Suppose X is a finite set and $k : X \times X \to M_n$ is a (positive semi-definite) kernel. If for each $f : X \to \mathbb{C}$ there exists a $\rho > 0$ such that*

$$X \times X \ni (x,y) \to (\rho^2 - f(x)f(y)^*)k(x,y)$$

is positive semi-definite, then, for each $x \in X$ the mapping

$$Q_x k(\cdot,y)v = \begin{cases} k(\cdot,x)v & y = x; \\ 0 & y \neq x. \end{cases}$$

determines a well defined bounded linear mapping $Q_x : H^2(k) \to H^2(k)$.

Proof. Let χ_x denote the characteristic function of $\{x\}$ and note that $Q_x = M_{\chi_x}^*$ and thus the Lemma follows from the remarks above. $\square$

LEMMA 2.3. *Suppose X is finite. If $\mathscr{H}$ is a Hilbert space and $\tau : H^\infty(k) \to \mathscr{B}(\mathscr{H})$ is a completely contractive unital representation, then there is a Hilbert space $\mathscr{E}$ and an isometry $V : \mathscr{H} \to \mathscr{E} \otimes H^2(k)$ such that*

$$\tau(f) = V^*(I \otimes M_k(f))V.$$

Proof. Identify $H^\infty(k)$ with the subspace $\{M_k(f) : f \in H^\infty(k)\}$ of $\mathscr{B}(H^2(k))$. Since τ is completely contractive and unital, it extends to a completely contractive unital map $\Phi : \mathscr{B}(H^2(k)) \to \mathscr{B}(\mathscr{H})$. By Stinespring's representation theorem, there exists a Hilbert space $\mathscr{L}$, an isometry $V : \mathscr{H} \to \mathscr{L}$, and a representation $\pi : \mathscr{B}(H^2(k)) \to \mathscr{B}(\mathscr{L})$ such that

$$\Phi(T) = V^* \pi(T) V.$$

In particular, for $f \in H^\infty(k)$, we have $\tau(f) = V^* \pi(M_k(f)) V$.

Since $H^2(k)$ is finite dimensional (as X is finite), π is a multiple of the identity representation; i.e., up to unitary equivalence, $\pi(T) = I \otimes T$, and under this identification there is a Hilbert space $\mathscr{E}$ such that $\mathscr{L} = \mathscr{E} \otimes H^2(k)$. $\square$

3. The Proof for finite X

In this section we prove Theorem 1.3 first under the added hypothesis that X is a finite set. Accordingly, until Section 4, assume that X is finite.

3.1. Representations of quotients

Given $f : X \to \mathbb{C}$, let $\mathscr{Z}(f)$ denote the zero set of f.

Also, given $Y \subset X$ and a kernel $k : X \times X \to M_n$, let $k|_Y = k|_{Y \times Y}$. Thus $k|_Y$ is a kernel on Y, and, given a collection of kernels $\mathscr{K}$ on X we use the notation $\mathscr{K}|_Y$, for the collection of kernels of the form $k|_Y$ for $k \in \mathscr{K}$. In particular, if $\mathscr{K}$ is an interpolation family of kernels, then so is $\mathscr{K}|_Y$.

LEMMA 3.1. *Suppose*

(i) $\mathscr{K}$ *is an Agler interpolation family on the finite set* X;

(ii) $k \in \mathscr{K}_n$;

(iii) $\mathscr{H}$ *and* $\mathscr{E}$ *are Hilbert spaces, and* $V : \mathscr{H} \to \mathscr{E} \otimes H^2(k)$ *is an isometry;*

(iv) $\sigma : H^\infty(k) \to \mathscr{B}(\mathscr{H})$ *given by*

$$H^\infty(k) \ni f \mapsto V^*(I \otimes M_k(f))V$$

is a (unital) representation; and

(v) $Y \subset X$.

If $\sigma(g) = 0$ *whenever* $Y \subset \mathscr{Z}(g)$, *then, for each* $\psi \in H^\infty(k)$,

$$\|\sigma(\psi)\| \leqslant \sup\{\|M_\kappa(\psi|_Y)\|\} : \kappa \in \mathscr{K}|_Y\}.$$

REMARK 3.2. Note $\sigma(\psi)^*$ depends only upon $\psi|_Y$. In fact, σ induces a representation $\tilde{\sigma} : H^\infty(k)/I \to \mathscr{B}(\mathscr{H})$, where I is the ideal of functions in $H^\infty(k)$ which vanish on the complement, $\tilde{Y}$, of Y in X.

Proof. Fix $0 \neq \psi \in H^\infty(k)$ and $\varepsilon > 0$. Choose unit vectors h, γ in $\mathscr{H}$ such that

$$\|\sigma(\psi)^*\| \leqslant \langle \sigma(\psi)^* h, \gamma \rangle + \varepsilon. \tag{3.1}$$

Because X is a finite set, there exists a finite dimensional subspace $\mathscr{E}_0$ of $\mathscr{E}$ such that $0 \neq V\gamma \in \mathscr{E}_0 \otimes H^2(k)$. Let K denote the kernel $K : X \times X \to \mathscr{B}(\mathscr{E}_0) \otimes H^2(k)$ defined by

$$K(x,y)e \otimes v = e \otimes k(x,y)v.$$

Since $\mathscr{K}$ is closed with respect to direct sums, $K \in \mathscr{K}$. Indeed, K is the direct sum of k with itself m times, where m is the finite dimension of $\mathscr{E}_0$. Let $N = mn$ and view $K : X \times X \to \mathbb{C}^N$. Summarizing, $H^\infty(k) = H^\infty(K)$ (as operator algebras), $\mathscr{E} \otimes H^2(k)$ is canonically identified with $H^2(K) \oplus (\mathscr{E}_0^\perp \otimes H^2(k))$, and $V\gamma \in H^2(K)$.

Let $\mathbf{P}$ denote the projection onto $H^2(K)$. Thus $\mathbf{P} = P_{\mathscr{E}_0} \otimes I$, from which it follows that the subspace $H^2(K)$ reduces $(I_{\mathscr{E}} \otimes M_k(\varphi)^*)$ for each $\varphi \in H^\infty(k)$. Thus, for $h_* \in \mathscr{H}$,

$$\begin{aligned} \langle \sigma(\varphi)^* h_*, \gamma \rangle &= \langle V^*(I_{\mathscr{E}} \otimes M_k(\varphi)^*) V h_*, \gamma \rangle \\ &= \langle \mathbf{P}(I_{\mathscr{E}} \otimes M_k(\varphi)^*) V h_*, V\gamma \rangle \\ &= \langle (I_{\mathscr{E}_0} \otimes M_k(\varphi)^*) \mathbf{P} V h_*, V\gamma \rangle \\ &= \langle V^* M_K(\varphi)^* \mathbf{P} V h_*, \gamma \rangle, \end{aligned} \tag{3.2}$$

where $V\gamma = \mathbf{P}V\gamma$ was used in the second equality.

Because of item (iii) in the definition of interpolation family and Lemma 2.2, for $x \in X$,

$$Q_x K(\cdot, y)v = \begin{cases} K(\cdot, x)v & y = x; \\ 0 & y \neq x \end{cases}$$

determines a bounded operator $Q_x : H^2(K) \to H^2(K)$.

Next observe $Q_x^2 = Q_x$, the range of Q_x is $[K(\cdot,x)v : v \in \mathbb{C}^N]$, there is the (non-orthogonal) resolution $I = \sum_x Q_x$, and

$$M_K(\varphi)^* Q_x = \varphi(x)^* Q_x \tag{3.3}$$

for $\varphi \in H^\infty(k)$.

For $x \in X$, let

$$\mathscr{G}_x = Q_x \mathbf{P} V \mathscr{H}.$$

Observe $\mathscr{G}_x$ is invariant for $\{M_K(\phi)^* : \phi \in H^\infty(K)\}$ because of equation (3.3). Thus $\mathscr{G}_{\tilde{Y}} = \sum_{z \notin Y} \mathscr{G}_z$ is invariant for $\{M_K(\phi)^* : \phi \in H^\infty(k)\}$. Let $\mathscr{M} = H^2(K) \ominus \mathscr{G}_{\tilde{Y}}$.

If $g \in H^\infty(k)$ and $Y \subset \mathscr{Z}(g)$, and if $h_* \in \mathscr{H}$, then

$$\begin{aligned} 0 &= \langle \sigma(g)^* h_*, \gamma \rangle \\ &= \langle M_K(g)^* \mathbf{P} V h_*, V\gamma \rangle \\ &= \langle \sum_x g(x)^* Q_x \mathbf{P} V h_*, V\gamma \rangle \\ &= \langle \sum_{z \notin Y} g(z)^* Q_z \mathbf{P} V h_*, V\gamma \rangle. \end{aligned}$$

The first equality follows from the hypothesis on σ which gives $\sigma(g) = 0$; the second uses equation (3.2); the third uses equation (3.3) and $I = \sum Q_x$; and the fourth equality from the fact that $g(y) = 0$ for $y \in Y$. Fix a $z_0 \notin Y$ and use item (iii) in the definition of interpolation family to choose $g \in H^\infty(k)$ such that $g(z_0) = 1$ and $g(x) = 0$ otherwise to obtain

$$0 = \langle Q_{z_0}\mathbf{P}Vh_*, V\gamma\rangle.$$

Thus, $V\gamma$ is orthogonal to each $\mathscr{G}_{z_0}$ and therefore to $\mathscr{G}_{\tilde{Y}}$. Hence $V\gamma \in \mathscr{M}$.

Since $P_{\mathscr{M}}Q_z\mathbf{P}V\mathscr{H} = 0$ for $z \notin Y$, if $h_* \in \mathscr{H}$ and $\mathbf{P}Vh_*$ is written as

$$\mathbf{P}Vh_* = \sum_{y\in X} K(\cdot,y)v_y,$$

then, for $z \notin Y$,

$$P_{\mathscr{M}}K(\cdot,z)v_z = 0. \tag{3.4}$$

In particular,

$$P_{\mathscr{M}}\mathbf{P}Vh_* = \sum_{y\in Y} P_{\mathscr{M}}K(\cdot,y)v_y. \tag{3.5}$$

Thus, with $\mathscr{L}$ equal to the span of $\{K(\cdot,y)v : y \in Y,\ v \in \mathbb{C}^N\}$, it follows that $P_{\mathscr{M}}\mathbf{P}V\mathscr{H} \subset \mathscr{L}$.

From Lemma 2.1 there is an M, a kernel $\kappa \in \mathscr{K}_M$, and a function $G : X \to M_{N,M}$ such that

$$\langle P_{\mathscr{M}}K(\cdot,y)v, K(\cdot,x)u\rangle = \langle \kappa(x,y)G(y)^*v, G(x)^*u\rangle. \tag{3.6}$$

In particular, the map $W : \mathscr{L} \to H^2(\kappa|_Y)$ defined by $WP_{\mathscr{M}}K(\cdot,y)v = \kappa(\cdot,y)G(y)^*v$ is (well defined and) an isometry.

Returning to the vector $h \in \mathscr{H}$ in equation (3.1), there exists $h_x, \gamma_x \in \mathbb{C}^N$ such that

$$\begin{aligned} \mathbf{P}Vh &= \sum_{x\in X} Q_x\mathbf{P}Vh = \sum K(\cdot,x)h_x \\ V\gamma &= \sum_{x\in X} Q_xV\gamma = \sum K(\cdot,x)\gamma_x. \end{aligned} \tag{3.7}$$

Note that, since h and γ are unit vectors, $\|\mathbf{P}Vh\| \leqslant 1$ and $\|V\gamma\| = 1$.

With these notations and for $\varphi \in H^\infty(k)$,

$$\begin{aligned} \langle \sigma(\varphi)^*h, \gamma\rangle &= \langle M_K(\varphi)^*\mathbf{P}Vh, V\gamma\rangle \\ &= \sum_{y\in X} \langle \varphi(y)^*K(\cdot,y)h_y, P_{\mathscr{M}}V\gamma\rangle \\ &= \sum_{x,y\in X} \langle \varphi(y)^*K(\cdot,y)h_y, P_{\mathscr{M}}K(\cdot,x)\gamma_x\rangle \\ &= \sum_{x,y\in X} \langle \varphi(y)^*P_{\mathscr{M}}K(\cdot,y)h_y, P_{\mathscr{M}}K(\cdot,x)\gamma_x\rangle \\ &= \sum_{x,y\in Y} \langle \varphi(y)^*P_{\mathscr{M}}K(\cdot,y)h_y, P_{\mathscr{M}}K(\cdot,x)\gamma_x\rangle \\ &= \sum_{x,y\in Y} \langle \varphi(y)^*WP_{\mathscr{M}}K(\cdot,y)h_y, WP_{\mathscr{M}}K(\cdot,x)\gamma_x\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x,y\in Y} \langle \varphi(y)^* \kappa(\cdot,y) G(y)^* h_y, \kappa(\cdot,x) G(x)^* \gamma_x\rangle \\
&= \langle M_{\kappa|_Y}(\varphi|_Y)^* \sum_{y\in Y} k|_Y(\cdot,y) G(y)^* h_y, \sum_{x\in Y} k|_Y(\cdot,x) G(x)^* \gamma_x\rangle \\
&= \langle M_{\kappa|_Y}(\varphi|_Y)^* W P_{\mathscr{M}} \mathbf{P} V h, W V\gamma\rangle.
\end{aligned}
$$

Here the first equality follows from the definition of σ; the second uses equation (3.3) and $V\gamma \in \mathscr{M}$ as well as equation (3.7); the fifth uses equation (3.4); the sixth that $W : \mathscr{L} \to H^2(\kappa|_Y)$ is an isometry; the seventh the definition of W; and finally the last equality uses both the definition of W and equation (3.5).

Hence,

$$
\begin{aligned}
\|\sigma(\varphi)^*\| - \varepsilon \leqslant |\langle \sigma(\varphi)^* h, g\rangle| \\
&= |\langle M_{\kappa|_Y}(\varphi|_Y)^* W P_{\mathscr{M}} \mathbf{P} V h, V\gamma\rangle| \\
&\leqslant \|M_{\kappa|_Y}(\varphi|_Y)^*\| \, \|W P_{\mathscr{M}} \mathbf{P} V h\| \, \|W V\gamma\| \\
&\leqslant \|M_{\kappa|_Y}(\varphi|_Y)^*\| \, \|h\| \, \|\gamma\|.
\end{aligned}
$$

and the proof is complete. □

3.2. The end of the proof for finite X

In this subsection we complete the proof of Theorem 1.3 in the case that X is finite, in which case there exists m and $x_1, \dots, x_m$ such that $Y = X \setminus \{x_1, \dots, x_m\}$. Fix $g : Y \to \mathbb{C}$. Define

$$
\rho = \sup\{\|M_{k|_Y}(\psi|_Y)\| : k \in \mathscr{K}\}.
$$

By item (iii) in the definition of interpolation family, ρ is finite.

Let $\mathscr{K}_+$ denote those $\tilde{k} \in \mathscr{K}$ such that $\tilde{k}(x,x) \neq 0$ for each $x \in X$. Because of condition (iv) in the definition of interpolation family, for each $x \in X$ there is a k_x such that $k_x(x,x) \neq 0$. Hence, if $k \in \mathscr{K}$, then $\tilde{k} = k \oplus_{x\in X} k_x \in \mathscr{K}_+$. In particular, $\mathscr{K}_+$ is non-empty and if

$$
X \times X \ni (x,y) \mapsto (\rho^2 - f(x) f(y)^*) \tilde{k}(x,y)
$$

is positive semi-definite for all $\tilde{k} \in \mathscr{K}_+$, then

$$
X \times X \ni (x,y) \mapsto (\rho^2 - f(x) f(y)^*) k(x,y)
$$

for all $k \in \mathscr{K}$.

Let $\tilde{k}$ be a given element of $\mathscr{K}_+$. Let $\mathscr{I}_{\tilde{k}}$ denote the ideal of functions in $H^\infty(\tilde{k})$ which vanish on Y. The quotient $H^\infty(\tilde{k})/\mathscr{I}_{\tilde{k}}$ is a unital operator algebra and hence (by the BRS theorem) it has a completely isometric unital representation τ on a Hilbert space $\mathscr{H}$ (see [BL] Proposition 2.3.4).

The quotient mapping

$$
\pi : H^\infty(\tilde{k}) \to H^\infty(\tilde{k})/I_{\tilde{k}}
$$

is completely contractive and unital. Thus, $\sigma = \tau \circ \pi : H^\infty(\tilde{k}) \to \mathscr{B}(\mathscr{H})$ is a completely contractive representation. Further, because τ is a (complete) isometry,

$$\|\pi(\phi)\| = \|\sigma(\phi)\|$$

for $\phi \in H^\infty(\tilde{k})$. Since π is a unital completely contractive representation of $H^\infty(\tilde{k})$, π has the form given in Lemma 2.3. Hence, Lemma 3.1 applies to give

$$\|\pi(\psi)\| \leqslant \rho.$$

whenever $\psi|_Y = g$; also note that $\varphi|_Y = \psi|_Y$ if and only if $\pi(\varphi) = \pi(\psi)$ and since is completely isometric, we conclude that $\varphi|_Y = \psi|_Y$ if and only if $\sigma(\varphi) = \sigma(\psi)$.

Suppose now that $\rho' > \rho$. By the definition of the quotient norm, there exists a φ such that $\pi(\varphi) = \pi(\psi)$ and so that

$$X \times X \ni (x,y) \to [(\rho')^2 - \varphi(x)\varphi(y)^*]\tilde{k}(x,y) \tag{3.8}$$

is positive semi-definite.

Consider the set

$$C_{\tilde{k},\rho'} = \{(\varphi(x_1),\dots,\varphi(x_m)) : \pi(\varphi) = \pi(\psi) \text{ and equation (3.8) holds}\} \subset \mathbb{C}^m.$$

From above $C_{\tilde{k},\rho'}$ is nonempty. It is evidently closed, and because $\tilde{k} \in \mathscr{K}_+$, it is also bounded. Because $\mathscr{K}$ is closed with respect to direct sums, the collection $\{C_{\tilde{k},\rho'} : \tilde{k} \in \mathscr{K}_+\}$ has the finite intersection property. Hence, there exists a φ such that $\varphi|_Y = g$ and, for each $\kappa \in \mathscr{K}_+$, the kernel

$$X \times X \ni (x,y) \to [(\rho')^2 - \varphi(x)\varphi(y)^*]\kappa(x,y) \tag{3.9}$$

is positive semi-definite. Hence the same is true for all $k \in \mathscr{K}$.

To finish the proof, choose a sequence $\rho_\ell > \rho$ converging to ρ. There exists φ_ℓ such that the kernel in equation (3.9), with φ_ℓ in place of φ and ρ_ℓ in place of ρ', is positive semi-definite. Because φ_ℓ is uniformly bounded (again using item (iv) of the definition of interpolation family) it has a subsequence converging pointwise to some f which then satisfies the conclusion of the Theorem 1.3

4. The case of arbitrary X

The passage from finite X to infinite X involves a Zorn's Lemma argument.

Let $\mathscr{K}$ denote a given interpolation family on a set X. Let Y, a finite subset of X, $g : Y \to \mathbb{C}$ and $\rho > 0$ such that for each $k \in \mathscr{K}$ the kernel

$$Y \times Y \ni (x,y) \mapsto (\rho^2 - g(x)g(y)^*)k(x,y)$$

is positive semi-definite, be given.

Consider the collection $\mathscr{S}$ of pairs (U, f) where $Y \subset U \subset X$, $f : U \to \mathbb{C}$, $f|_Y = g$, and for each $k \in \mathscr{K}$ the kernel

$$U \times U \ni (x,y) \mapsto (\rho^2 - f(x)f(y)^*)k(x,y)$$

is positive semi-definite. (We do not assume U is finite.)

Partially order $\mathscr{S}$ as follows. Say $(U,f) \leqslant (W,h)$ if $U \subset W$ and $h|_U = f$. Suppose $\mathscr{C} = \{(U,f_U)\}$ is a chain from $\mathscr{S}$. To see that $\mathscr{C}$ has an upper bound, let $T = \cup U$ and define $h : T \to \mathbb{C}$ by $h(x) = f_U(x)$, where (U,f_U) is any element of $\mathscr{C}$ for which $x \in U$. The fact that $\mathscr{C}$ is linearly ordered implies that h is well defined. Further, if F is any finite subset of T, then there exists a $(U,f_U) \in \mathscr{C}$ such that $F \subset U$ and hence, for each $k \in \mathscr{K}$, the matrix

$$\begin{aligned} A_{k,F} &= \big((\rho^2 - f_U(x)f_U(y)^*)k(x,y)\big)_{x,y\in F} \\ &= \big((\rho^2 - h(x)h(x)^*)k(x,y)\big) \end{aligned}$$

is positive semi-definite. It follows that $(T,h) \in \mathscr{S}$ and is an upper bound for $\mathscr{C}$.

By Zorn's Lemma, $\mathscr{C}$ has a maximal element (W,h). Suppose $W \neq X$. In this case, there is a point $z \in X \setminus W$. Given a finite subset $F \subset Y$, let $G = F \cup \{z\}$. For each $u \in \mathbb{C}$, define a function $q : G \to \mathbb{C}$ by declaring $q|_F = h|_F$ and $q(z) = u$. Now define C_F to be the set of $u \in \mathbb{C}$ for which the kernel

$$G \times G \mapsto (\rho^2 - q(x)q(y)^*)k(x,y)$$

is positive semidefinite for all $k \in K$. The set C_F is nonempty by the finite case of Theorem 1.3 and is also closed. It is bounded by condition (iv) of Definition 1.2. Thus C_F is compact.

The collection $\{C_F : F \subset X,\ \ |F| < \infty\}$ has the finite intersection property and hence there is a u_* such that

$$u_* \in \cap\{C_F : F \subset X,\ \ |F| < \infty\}.$$

Define $h_* : Y \cup \{z\} \to \mathbb{C}$ by $h_*|_Y = h$ and $h_*(z) = u_*$. Then $(W \cup \{z\}, h_*) \in \mathscr{S}$ and is greater than (W,h), a contradiction which completes the proof.

5. Examples: the disc and the annulus

For the case of the disc, let $\mathscr{K}_n = \{s_n = I_n \otimes s\}$, where I_n is the identity $n \times n$ matrix and s is Szegő's kernel. Given a unit vector $\gamma \in \mathbb{C}^n$ and $\lambda \in \mathbb{D}$ let $Q = I - \gamma\gamma^*$, and let φ_λ denote a Möbius map of the disc sending λ to 0, and $G = \varphi_\lambda \gamma\gamma^* + Q$. It is readily verified that

$$\begin{aligned} k'(z,w) &= s_n(z,w) - \frac{s_n(z,\lambda)\gamma\gamma^* s_n(\lambda,w)}{\gamma^* s_n(\lambda,\lambda)\gamma} \\ &= G(w)^* s_n(z,w) G(z). \end{aligned}$$

Hence $\mathscr{K}$ is an Agler interpolation family.

Let $\mathbb{A}$ denote an annulus, $\{r < |z| < \frac{1}{r}\}$. There is a family $k_t(z,w)$ of scalar kernels parametrized by T in the unit circle $\mathbb{T}$ which collectively play a role on the annulus similar to that played by Szegő's kernel on the disc [S65]. These are the kernels appearing in Abrahamse's interpolation theorem on $\mathbb{A}$ [Ab]. It turns out that given

$t \in \mathbb{T}$ and $\lambda \in \mathbb{A}$ there is an $s \in \mathbb{T}$ (which can be explicitly described in terms of the Abel-Jacobi map) and an analytic function φ_λ such that

$$k_t(z,w) - \frac{k_t(z,\lambda)k_t(\lambda,w)}{k_t(\lambda,\lambda)} = \varphi_\lambda(w)^* k_s(z,w)\varphi_\lambda(z).$$

Moreover, to each t and s there is a λ such that the above identity holds, explaining, at least heuristically, the need to consider the whole Sarason collection of kernels when interpolating on $\mathbb{A}$.

Let $\mathscr{K}_n$ denote the collection of kernels of the form $k_{t_1} \oplus \ldots \oplus k_{t_n}$. The results in [AD] show that $\mathscr{K} = (\mathscr{K}_n)$ is an Agler interpolation family on $\mathbb{A}$. Moreover, interpolation with respect to this family is interpolation in $H^\infty(\mathbb{A})$ as in [Ab].

As a final remark, note that in the proof of Lemma 3.1 and using the notations there if k is a direct sum of kernels and if $\mathscr{G}_{\tilde{Y}} = \mathscr{L}$, then κ is also the direct sum of scalar kernels. If this were always the case, then there would be no need to consider direct sums in the definition of interpolation family. Thus, the fact that, for scalar interpolation on a multiply connected domain it suffices to consider scalar kernels only represents additional structure not modeled by Theorem 1.3.

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