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THE UNBOUNDED COMMUTANT OF AN OPERATOR OF CLASS C_0

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Abstract. We describe the closed, densely defined linear transformations commuting with a given operator T of class C_0 in terms of bounded operators in $\{T\}'$. Our results extend those of Sarason for operators with defect index 1, and Martin in the case of an arbitrary finite defect index.

1. Introduction

There has been some interest recently in the study of closed unbounded linear transformations in the commutant of a bounded operator. For instance, let T denote the restriction of the backward unilateral shift to a proper invariant subspace. Then Sarason [6] showed that any closed, densely defined linear transformation commuting with T is of the form $v(T)^{-1}u(T)$, where $u, v \in H^\infty$ and $v(T)$ is injective. This extends his earlier result [5] pertaining to bounded operators, for which one can take $v = 1$.

It is fairly easy to see for the above example that closed linear transformations commuting with T must in fact commute with every operator in $\{T\}'$. Therefore Sarason's theorem can be viewed as a particular case of a result of Martin [4], which we describe next. Assume that T is an operator of class $C_0(N)$ as defined in [7, Chapter III], and X is a closed, densely defined linear transformation commuting with every operator in $\{T\}'$. Then Martin [4] proved that $X = v(T)^{-1}u(T)$ with $u, v \in H^\infty$ such that $v(T)$ is injective. Thus these linear transformations are exactly the ones that can be obtained by applying the Sz.-Nagy–Foias functional calculus [7, Chapter IV] with unbounded functions.

Martin conjectured that his result would be true for operators T of class C_0 with finite multiplicity. We will show that it is in fact possible to extend this result to arbitrary contractions of class C_0. This follows from a more general description of closed, densely defined linear transformations X commuting with T. In case T has finite multiplicity, our result states that every such linear transformation X can be written as $X = v(T)^{-1}Y$, where Y is a bounded operator in $\{T\}'$, and $v \in H^\infty$ is such that $v(T)$ is injective.

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2. Preliminaries

We will denote by $\mathscr{B}(\mathscr{H},\mathscr{H}')$ the space of bounded linear operators $W:\mathscr{H}\to\mathscr{H}'$, where $\mathscr{H}$ and $\mathscr{H}'$ are complex Hilbert spaces. We will also write $\mathscr{B}(\mathscr{H})=\mathscr{B}(\mathscr{H},\mathscr{H})$. Recall that an operator $T\in\mathscr{B}(\mathscr{H})$ is a *quasiaffine transform* of $T'\in\mathscr{B}(\mathscr{H}')$ if there exists a *quasiaffinity*, i.e. an injective operator with dense range, $W\in\mathscr{B}(\mathscr{H},\mathscr{H}')$ satisfying $WT=T'W$. We write $T\prec T'$ if T is a quasiaffine transform of T'. The operators T and T' are *quasisimilar* if $T\prec T'$ and $T'\prec T$, in which case we write $T\sim T'$.

Assume that $T\in\mathscr{B}(\mathscr{H})$ is a contraction, i.e. $\|T\|\leqslant 1$, and it is completely nonunitary in the sense that it does not have any nontrivial unitary direct summand. The Sz.-Nagy–Foias functional calculus [7, Chapter III] is an algebra homomorphism $u\mapsto u(T)\in\mathscr{B}(\mathscr{H})$ of the algebra H^∞ of bounded analytic functions in the unit disk, and which extends the usual polynomial calculus. The operator T is said to be of class C_0 if $u(T)=0$ for some $u\in H^\infty\setminus\{0\}$. When T is of class C_0, the ideal $\{u\in H^\infty: u(T)=0\}$ is of the form mH^∞, where m is an inner function, uniquely determined up to a constant factor of absolute value 1, and called the *minimal function* of T. For any inner function m, there exist operators of class C_0 with minimal function m. The most basic example is constructed as follows. Denote by S the unilateral shift on the Hardy space H^2, i.e. $(Sf)(\lambda)=\lambda f(\lambda)$ for $f\in H^2$. The space $\mathscr{H}(m)=H^2\ominus mH^2$ is invariant for S^*, and the operator $S(m)\in\mathscr{B}(\mathscr{H}(m))$ is defined by the requirement that $S(m)^*=S^*|\mathscr{H}(m)$. The operator $S(m)$ has minimal function equal to m.

Quasisimilarity allows a complete classification of operators of class C_0. We will only need the facts collected in the following statement. We refer to [1, Theorem III.5.1] for (1-3), [1, Theorem VII.1.9] for (4), [1, Proposition III.5.33] for (5), [7, Proposition III.4.7] or [1, Proposition II.4.9] for (6), [1, Proposition VII.1.21] for (7), and [1, Theorem IV.1.2] for (8).

THEOREM 1. *Let $T\in\mathscr{B}(\mathscr{H})$ and $T'\in\mathscr{B}(\mathscr{H}')$ be operators of class C_0. Denote by m the minimal function of T.*

1. *We have $T\prec T'$ if and only if $T'\prec T$.*
2. *There exists a collection $\{m_i\}_{i\in I}$ of inner divisors of m such that $m=m_i$ for some i, and $T\sim\bigoplus_{i\in I}S(m_i)$.*
3. *If T has finite cyclic multiplicity n, we have $T\sim\bigoplus_{j=1}^{n}S(m_j)$, with $m_1=m$ and m_{j+1} divides m_j for $j=1,2,\ldots,n-1$.*
4. *If T has finite multiplicity, and $\mathscr{M}$ is an invariant subspace for T such that $T\sim T|\mathscr{M}$, then $\mathscr{M}=\mathscr{H}$.*
5. *Every invariant subspace $\mathscr{M}$ for T is of the form $\mathscr{M}=\overline{A\mathscr{H}}$, with A in the commutant $\{T\}'$ of T.*
6. *An operator of the form $v(T)$ with $v\in H^\infty$ is injective if and only if v and m have no nonconstant common inner factors. In this case, $v(T)$ is a quasiaffinity.*

7. *If T has finite multiplicity and $A \in \{T\}'$ is injective, then the map $\mathscr{M} \mapsto \overline{A\mathscr{M}}$ is an order preserving automorphism of the lattice of invariant subspaces of T.*

8. *For every Y in the double commutant $\{T\}''$ there exist $u, v \in H^\infty$ such that $v(T)$ is a quasiaffinity and $Y = v(T)^{-1}u(T)$.*

The following result appears in [3, Lemma 2.7] (see also [1, Proposition IV.1.13]), but unfortunately only for multiplicity 2. The argument here follows a different path.

PROPOSITION 2. *Assume that $T \in \mathscr{B}(\mathscr{H})$ is of class C_0 and has finite multiplicity. For every injective $A \in \{T\}'$ there exits another injective $B \in \{T\}'$, and a function $v \in H^\infty$ such that $AB = BA = v(T)$. The operators A, B and $v(T)$ are then quasiaffinities.*

Proof. As seen in [3], it suffices to consider operators of the form $T = \bigoplus_{j=1}^n S(m_j)$, where m_{j+1} divides m_j for $j = 1, 2, \ldots, n-1$. Let $A \in \{T\}'$ be an injective operator. By Theorem 1(7), the map $\mathscr{M} \mapsto \overline{A\mathscr{M}}$ is an order preserving automorphism of the lattice of invariant subspaces for T. Regard $\mathscr{H}(m_j)$ as subspaces of $\mathscr{H} = \bigoplus_{j=1}^n \mathscr{H}(m_j)$, and set $\mathscr{H}_j = \overline{A\mathscr{H}(m_j)}$, $\mathscr{K}_j = \bigvee_{i \neq j} \mathscr{H}_i$, and $\mathscr{H}_j' = \mathscr{H} \ominus \mathscr{K}_j$ for $j = 1, 2, \ldots, n$. We must then have $\bigcap_{j=1}^n \mathscr{K}_j = \{0\}$, $\mathscr{H}_j \cap \mathscr{K}_j = \{0\}$ and $\mathscr{H}_j \vee \mathscr{K}_j = \mathscr{H}$. The last two equalities imply that the operator $X_j \in \mathscr{B}(\mathscr{H}_j, \mathscr{H}_j')$ defined by $X_j = P_{\mathscr{H}_j'} | \mathscr{H}_j$ is a quasiaffinity. Moreover, this operator satisfies the equation $X(T|\mathscr{H}_j) = T_j X$, where $T_j \in \mathscr{L}(\mathscr{H}_j')$ is defined by the equality $T_j^* = T^*|\mathscr{H}_j'$. Thus $T|\mathscr{H}_j \prec T_j$, and since $S(m_j) \prec T|\mathscr{H}_j$ (via the operator $A|\mathscr{H}(m_j)$), there must exist a quasiaffinity $Y_j \in \mathscr{B}(\mathscr{H}_j', \mathscr{H}(m_j))$ satisfying $Y_j T_j = S(m_j) Y_j$. We define now an operator $C \in \{T\}'$ by setting

$$Ch = \bigoplus_{j=1}^n Y_j P_{\mathscr{H}_j'} h.$$

It is easy to verify that C is a quasiaffinity. Indeed, $Ch = 0$ implies that $P_{\mathscr{H}_j'} h = 0$, and hence $h \in \bigcap_{j=1}^n \mathscr{K}_j = \{0\}$. Also, $C\mathscr{H} = \bigvee_{j=1}^n Y_j \mathscr{H}_j' = \mathscr{H}$. The product AC leaves all the summands $\mathscr{H}(m_j)$ invariant, and therefore Sarason's generalized interpolation theorem [5] implies the existence of functions $u_j \in H^\infty$ such that $AC = \bigoplus_{j=1}^n u_j(S(m_j))$. Moreover, u_j and m_j have no nonconstant common inner factor because AC is injective. We deduce from [1, Theorem III.1.14] that there exist scalars t_j such that $v_j = u_j + t_j m_j$ has no nonconstant common inner factor with the minimal function m_1 of T. Note that we also have $AC = \bigoplus_{j=1}^n v_j(S(m_j))$. Define now $v = v_1 v_2 \cdots v_n \in H^\infty$ and operators $D, B \in \{T\}'$ by $D = \bigoplus_{j=1}^n (v/v_j)(S(m_j))$ and $B = CD$. We have $AB = v(T)$ and $A(BA - v(T)) = ABA - v(T)A = 0$ so that $BA = v(T)$ because A is injective. The operator $v(T)$ is a quasiaffinity because v and m_1 do not have nonconstant common inner divisors. $\square$

3. Unbounded linear transformations in the commutant

Consider a Hilbert space $\mathscr{H}$ and a linear transformation $X : \mathscr{D}(X) \to \mathscr{H}$, where $\mathscr{D}(X) \subset \mathscr{H}$ is a dense linear manifold. Recall that X is said to be closed if its graph

$$\mathscr{G}(X) = \{h \oplus Xh : h \in \mathscr{D}(X)\}$$

is a closed subspace in $\mathscr{H} \oplus \mathscr{H}$. The linear transformation X is closable if the closure $\overline{\mathscr{G}(X)}$ is the graph of a linear transformation, usually denoted $\overline{X}$ and called the closure of X.

Let now $T \in \mathscr{B}(\mathscr{H})$ be a completely nonunitary contraction, let $v \in H^\infty$ be such that $v(T)$ is a quasiaffinity, and let $A \in \{T\}'$. The linear transformation $X = v(T)^{-1}A$ with domain

$$\mathscr{D}(X) = \{h \in \mathscr{H} : Ah \in v(T)\mathscr{H}\}$$

has graph

$$\mathscr{G}(X) = \{h \oplus k : Ah = v(T)k\},$$

so that X is obviously closed. Moreover, since $v(T)A = Av(T)$, we have

$$\mathscr{G}(X) \supset \mathscr{G}(Av(T)^{-1}) = \{v(T)h \oplus Ah : h \in \mathscr{H}\}$$

and thus $\mathscr{D}(X) \supset v(T)\mathscr{H}$ is dense. If $v_1 \in H^\infty$ is another function such that $v_1(T)$ is a quasiaffinity, the equality $v(T)^{-1}Ah = v_1(T)^{-1}A_1h$ for h in a dense linear manifold $\mathscr{D} \subset \mathscr{D}(v(T)^{-1}A) \cap \mathscr{D}(v_1(T)^{-1}A_1)$ implies $v(T)^{-1}A = v_1(T)^{-1}A_1$. Indeed, we have $v_1(T)Ah = v(T)A_1h$ for $h \in \mathscr{D}$, hence $v_1(T)A = v(T)A_1$. Then we have

$$v_1(T)((v(T)k - Ah) = v(T)(v_1(T)k - A_1h),$$

so that $h \oplus k \in \mathscr{G}(v(T)^{-1}A)$ if and only if $h \oplus k \in \mathscr{G}(v_1(T)^{-1}A_1)$. These remarks apply more generally to linear transformations of the form $B^{-1}A$, where $A, B \in \{T\}'$, B is a quasiaffinity, and $AB = BA$. When A and B do not commute, the linear transformation $B^{-1}A$ is still closed, but might not be densely defined, while AB^{-1} is densely defined but perhaps not closable.

Linear transformations of the form $v(T)^{-1}A$, $A \in \{T\}'$, commute with T in the sense that $TX \subset XT$ or, equivalently, $\mathscr{G}(X)$ is invariant for $T \oplus T$.

PROPOSITION 3. *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator of class C_0, and let X be a closed, densely defined linear transformation commuting with T. There exist bounded operators $A, B \in \{T\}'$ such that B is a quasiaffinity and $X = \overline{AB^{-1}}$.*

Proof. The operator $T' = (T \oplus T)|\mathscr{G}(X)$ is of class C_0, and $T' \prec T$. Indeed, the operator $W \in \mathscr{B}(\mathscr{G}(X), \mathscr{H})$ defined by $W(h \oplus k) = h$ satisfies $WT' = TW$, and W is injective (because $\mathscr{G}(X)$ is a graph) and has dense range $\mathscr{D}(X)$. Theorem 1(1) implies the existence of an injective operator $V \in \mathscr{B}(\mathscr{H}, \mathscr{H} \oplus \mathscr{H})$ such that $\overline{V\mathscr{H}} = \mathscr{G}(X)$ and $(T \oplus T)V = VT$. Writing $Vh = Bh \oplus Ah$ for $h \in \mathscr{H}$, the operators A, B must belong to $\{T\}'$. Moreover, B is a quasiaffinity. Indeed, $Bh = 0$ implies $Ah = XBh = 0$, so that

$Vh = 0$ and hence $h = 0$ because V is injective. The fact that $V\mathscr{H}$ is dense in $\mathscr{G}(X)$ implies that $\overline{B\mathscr{H}} \supset \mathscr{D}(X)$, and hence B has dense range. Obviously, $\mathscr{G}(AB^{-1}) = V\mathscr{H}$, and hence $X = \overline{AB^{-1}}$. $\square$

For operators with finite multiplicity, a stronger result can be proved.

THEOREM 4. *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator of class C_0 with finite multiplicity, and let X be a closed, densely defined linear transformation commuting with T. There exist $A \in \{T\}'$ and $v \in H^\infty$ such that $v(T)$ is a quasiaffinity and $X = v(T)^{-1}A$.*

Proof. By Proposition 3, we can find $A_0, B \in \{T\}'$ such that B is a quasiaffinity and $X \supset A_0B^{-1}$. Proposition 2 implies the existence of $v \in H^\infty$ and of a quasiaffinity $C \in \{T\}'$ such that $BC = CB = v(T)$. Setting now $A = A_0C$, we have

$$Av(T)^{-1} = A_0C(BC)^{-1} \subset A_0B^{-1} \subset X.$$

We conclude the proof by showing that both $v(T)^{-1}A$ and X coincide with the closure of $Av(T)^{-1}$. For this purpose, define operators $T_1 = (T \oplus T)|\mathscr{G}(X)$, $T_2 = (T \oplus T)|\mathscr{G}(v(T)^{-1}A)$, and $T_3 = (T \oplus T)|\mathscr{G}(\overline{Av(T)^{-1}})$. As observed earlier, $T_1 \sim T_2 \sim T_3 \sim T$. Since $\mathscr{G}(\overline{Av(T)^{-1}})$ is an invariant subspace for T_1 and T_2, theorem 1(4) implies the desired conclusion that $X = v(T)^{-1}A$. $\square$

Our final result pertains to double commutants.

THEOREM 5. *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator of class C_0, and let X be a closed, densely defined linear transformation commuting with every $A \in \{T\}'$. Then there exist $u, v \in H^\infty$ such that $v(T)$ is a quasiaffinity and $X = v(T)^{-1}u(T)$.*

Proof. We first prove the result under the additional assumption that T has finite multiplicity. In this case, Theorem 4 yields $A_0 \in \{T\}'$ and $v_0 \in H^\infty$ such that $v_0(T)$ is a quasiaffinity and $X = v_0(T)^{-1}A_0$. We observe next that A_0 belongs to the double commutant $\{T\}''$. Indeed, for any $B \in \{T\}'$ and $h \in \mathscr{D}(X)$ we have $Bh \in \mathscr{D}(X)$ and $XBh = BXh$ so that

$$v_0(T)XBH = v_0(T)BXH = Bv_0(T)Xh$$

and therefore $A_0Bh = BA_0h$. We conclude that $A_0B = BA_0$ because $\mathscr{D}(X)$ is dense. By Theorem 1(8), there exist $u, v_1 \in H^\infty$ such that $v_1(T)$ is a quasiaffinity and $A_0 = v_1(T)^{-1}u(T)$. We reach the desired conclusion $X = v(T)^{-1}u(T)$ with $v = v_0v_1$.

Consider now an arbitrary operator of class C_0, and let m denote its minimal function. Let $\mathscr{M} \subset \mathscr{H}$ be an invariant subspace for T such that $T|\mathscr{M}$ has finite multiplicity and minimal function equal to m. By Theorem 1(5), $\mathscr{M} = \overline{C\mathscr{H}}$ for some $C \in \{T\}'$. We have $C\mathscr{D}(X) \subset \mathscr{D}(X) \cap \mathscr{M}$ and

$$X(C\mathscr{D}(X)) \subset CX\mathscr{D}(X) \subset C\mathscr{H} \subset \mathscr{M}.$$

Therefore there exists a closed densely defined linear transformation $X_{\mathscr{M}}$ on $\mathscr{M}$ such that

$$\mathscr{G}(X_{\mathscr{M}}) = \mathscr{G}(X) \cap (\mathscr{M} \oplus \mathscr{M}).$$

We claim that $\mathscr{D}(X_{\mathscr{M}}) = \mathscr{D}(X) \cap \mathscr{M}$. Indeed, let us set $T_1 = (T \oplus T)|\mathscr{G}(X_{\mathscr{M}})$ and $T_2 = (T \oplus T)|\mathscr{G}(X) \cap (\mathscr{M} \oplus \mathscr{H})$. The projection on the first component demonstrates the relations $T_1 \prec T|\mathscr{M}$ and $T_2 \prec T|\mathscr{M}$. The equality

$$\mathscr{G}(X_{\mathscr{M}}) = \mathscr{G}(X) \cap (\mathscr{M} \oplus \mathscr{H}),$$

and hence $\mathscr{D}(X_{\mathscr{M}}) = \mathscr{D}(X) \cap \mathscr{M}$, follows from Theorem 1(4). A similar argument shows that $\mathscr{G}(X_{\mathscr{M}})$ is the closure of $\{Ch \oplus CXh : h \in \mathscr{D}(X)\}$.

We show next that $X_{\mathscr{M}}$ commutes with every operator in the commutant of $T|\mathscr{M}$. Indeed, let $D \in \mathscr{B}(\mathscr{M})$ be such an operator. Then $DC \in \{T\}'$ so that $DCh \in \mathscr{D}(X)$ for every $h \in \mathscr{D}(X)$, and

$$XDCh = DCXh = DXCh.$$

Thus $D \oplus D$ leaves $\{Ch \oplus CXh : h \in \mathscr{D}(X)\}$ invariant, and hence it leaves its closure invariant as well, i.e. D commutes with $X_{\mathscr{M}}$.

The first part of the proof implies the existence of $u, v \in H^\infty$ such that $v(T_{\mathscr{M}})$ is a quasiaffinity, and $X_{\mathscr{M}} = v(T|\mathscr{M})^{-1}u(T|\mathscr{M})$. Note that $v(T)$ is a quasiaffinity as well since T and $T|\mathscr{M}$ have the same minimal function (cf. Theorem 1(6)). We claim that $X = v(T)^{-1}u(T)$. Indeed, consider arbitrary vectors $h_1 \in \mathscr{D}(X)$, $h_2 \in \mathscr{D}(v(T)^{-1}u(T))$, and let $\mathscr{M}_1 \supset \mathscr{M}$ be an invariant subspace for T such that $T|\mathscr{M}_1$ has finite multiplicity, and $h_1, h_2 \in \mathscr{M}_1$; for instance, once can take $\mathscr{M}_1$ to be the smallest invariant subspace containing $\mathscr{M}, h_1$ and h_2. The preceding argument, with $\mathscr{M}_1$ in place of $\mathscr{M}$, shows that $X_{\mathscr{M}_1} = v_1(T|\mathscr{M}_1)^{-1}u_1(T|\mathscr{M}_1)$ for some $u_1, v_1 \in H^\infty$ such that $v_1(T)$ is a quasiaffinity. Note now that, for $h \in \mathscr{D}(X) \cap \mathscr{M}$, we have both $v(T)Xh = u(T)h$ and $v_1(T)Xh = u_1(T)h,$ and therefore

$$(v_1(T)u(T) - v(T)u_1(T))h = v_1(T)v(T)Xh - v(T)v_1(T)Xh = 0$$

for such vectors. Since $\mathscr{D}(X) \cap \mathscr{M}$ is dense in $\mathscr{M}$, we have $(v_1u - u_1v)(T|\mathscr{M}) = 0$. We deduce that m, which is the minimal function of $T|\mathscr{M}$, divides $v_1u - vu_1$, and thus $v_1(T)u(T) = v(T)u_1(T)$. This implies that $v(T)^{-1}u(T) = v_1(T)^{-1}u_1(T)$, and therefore

$$h_1 \in \mathscr{D}(X) \cap \mathscr{M}_1 = \mathscr{D}(X_{\mathscr{M}_1}) = \mathscr{D}(v_1(T|\mathscr{M}_1)^{-1}u(T|\mathscr{M}_1)) \subset \mathscr{D}(v(T)^{-1}u(T)),$$

$$\begin{aligned} h_2 &\in \mathscr{D}(v(T)^{-1}u(T)) \cap \mathscr{M}_1 = \mathscr{D}(v(T|\mathscr{M}_1)^{-1}u(T|\mathscr{M}_1)) \\ &= \mathscr{D}(v_1(T|\mathscr{M}_1)^{-1}u(T|\mathscr{M}_1)) = \mathscr{D}(X_{\mathscr{M}_1}) \subset \mathscr{D}(X), \end{aligned}$$

and

$$Xh_j = v_1(T)^{-1}u_1(T)h_j = v(T)^{-1}u(T)h_j$$

for $j = 1, 2$. The desired equality $X = v(T)^{-1}u(T)$ follows. $\square$

When T has multiplicity 1, i.e. T has a cyclic vector, the algebra $\{T\}'$ is precisely the algebra generated by T and closed in the weak operator topology; see [1, Theorem IV.1.2]. Therefore Theorem 5 implies the following extension of Sarason's result [6].

COROLLARY 6. *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator of class C_0 with multiplicity 1, and let X be a closed, densely defined linear transformation commuting with T. Then there exist $u, v \in H^\infty$ such that $v(T)$ is a quasiaffinity and $X = v(T)^{-1}u(T)$.*

In Theorem 5, if we only assume that X is a densely defined linear transformation commuting with $\{T\}'$, the conclusion is that $X \subset v(T)^{-1}u(T)$ for some $u, v \in H^\infty$ such that $v(T)$ is a quasiaffinity. Indeed,the operator X must be closable by [2, Proposition 5.8]. As noted by Martin, in case $T = S(m)$ the closability of such linear transformations was also proved by Sarason [4, Lemma 3].

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