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THE KATO DECOMPOSITION OF QUASI–FREDHOLM RELATIONS

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Abstract. Quasi-Fredholm relations of degree $d \in \mathbb{N}$ in Hilbert spaces are defined in terms of conditions on their ranges and kernels. They are completely characterized in terms of an algebraic decomposition with a quasi-Fredholm relation of degree 0 and a nilpotent operator of degree d. The adjoint of a quasi-Fredholm relation of degree $d \in \mathbb{N}$ is shown to be quasi-Fredholm relation of degree $d \in \mathbb{N}$. The class of quasi-Fredholm relations contains the semi-Fredholm relations. Earlier results for quasi-Fredholm operators and semi-Fredholm operators are included.

1. Introduction

Semi-Fredholm operators were introduced by I.C. Gohberg and M.G. Kreĭn [4] and by T. Kato [7]. A closed linear operator A in a Hilbert space $\mathfrak{H}$ is said to be *semi-Fredholm* if

(S1) $\operatorname{ran} A$ is closed;

(S2) $\ker A$ or $\mathfrak{H}/\operatorname{ran} A$ is finite-dimensional.

Kato has shown that these operators allow an algebraic decomposition, the so-called Kato decomposition. The more general class of quasi-Fredholm operators was introduced and studied by J.-Ph. Labrousse [9]. A range space operator A in a Hilbert space $\mathfrak{H}$ (i.e., the graph of A is the range of a bounded linear operator from a Hilbert space $\mathfrak{K}$ to $\mathfrak{H} \times \mathfrak{H}$) is said to be *quasi-Fredholm of degree d* if there exists a nonnegative integer $d \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that

(Q1) d is the smallest number with $\operatorname{ran} A^n \cap \ker A = \operatorname{ran} A^d \cap \ker A$ for all $n \geqslant d$;

(Q2) $\ker A \cap \operatorname{ran} A^d$ is closed in $\mathfrak{H}$;

(Q3) $\operatorname{ran} A + \ker A^d$ is closed in $\mathfrak{H}$.

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A range space operator which is quasi-Fredholm of degree d is automatically closed. Large classes of operators have the quasi-Fredholm property. Labrousse [9] has shown that a quasi-Fredholm operator also allows an algebraic Kato decomposition, and that, in fact, if an operator has an algebraic Kato decomposition then the operator is quasi-Fredholm.

In the present work the concepts of semi-Fredholm and quasi-Fredholm operators are considered in the context of linear relations. A linear relation A in a linear space $\mathfrak{H}$ is a linear subspace of the Cartesian product $\mathfrak{H} \times \mathfrak{H}$; it can be interpreted as a linear multi-valued operator. All linear spaces in this paper are assumed to be complex. The domain, range, kernel, and multi-valued part are denoted by $\mathrm{dom} A$, $\mathrm{ran} A$, $\ker A$, and $\mathrm{mul} A$. The inverse A^{-1} of a linear relation A is defined by $A^{-1} = \{ \{y,x\} : \{x,y\} \in A \}$, so that $\mathrm{dom} A^{-1} = \mathrm{ran} A$, $\mathrm{ran} A^{-1} = \mathrm{dom} A$, $\ker A^{-1} = \mathrm{mul} A$, and $\mathrm{mul} A^{-1} = \ker A$. A relation A is (the graph of) an operator if and only if $\mathrm{mul} A = \{0\}$. Several decompositions of linear relations in linear spaces have been considered in the context of linear spaces in [17], [18], and in the context of Hilbert spaces in [5], [6]. The motivation for the study of linear relations is manifold: there is of course interest from a purely mathematical point of view, cf. [1], but there are also many fields where these objects appear naturally.

The notions of semi-Fredholm relation and quasi-Fredholm relation are introduced completely parallel to the operator case. The context of relations is quite natural since then A is quasi-Fredholm if and only if its adjoint A^* is quasi-Fredholm (without any additional conditions concerning the denseness of the domain). The main results of the paper are the Kato decompostion for quasi-Fredholm relations, and hence for semi-Fredholm relations, and, in fact, the characterization of quasi-Fredholm relations by their Kato decomposition. The theory of quasi-Fredholm operators in [9] has also been extended in different directions, cf. [12], [14], [15].

Quasi-Fredholm relations are defined in terms of range space relations, rather than in terms of closed relations. The notion of range space relation is a replacement for the notion of closed relation. Range space operators were considered in [9]. A range subspace in a Hilbert space is a subspace which is a Hilbert space in a stronger topology. The advantage of this notion is that the sum of range subspaces is a range subspace. Such spaces go back to J. Dixmier under the name of sous-espaces paracomplet. For an introduction to range space operators, see [3], [9]. In the present paper this notion is put in the context of linear relations.

The contents of the paper are as follows. Linear relations are discussed in Section 2; in particular, the degree of a relation is introduced. Section 3 contains a short introduction to range subspaces of Hilbert spaces. For the convenience of the reader the presentation is more or less self-contained. Linear relations in Hilbert spaces are discussed in Section 4; in particular, range space relations are studied. Quasi-Fredholm relations are introduced in Section 5, where also the corresponding decomposition result can be found. Equivalent definitions of quasi-Fredholm relations are given in Section 6. Results concerning the adjoints of quasi-Fredholm relations are studied in Section 7. Finally Section 8 contains a brief discussion of semi-Fredholm relations. A short announcement of the present results appeared in [11].

2. Linear relations in linear spaces

This section contains a brief introduction to linear relations in linear spaces. Linear relations in linear spaces are defined and direct sum decompositions of linear relations are considered, cf. [18]. Furthermore the degree of a linear relation is defined.

2.1. Linear relations in linear spaces

A *linear relation* A in a linear space $\mathfrak{H}$ is a linear subspace of the space $\mathfrak{H}\times\mathfrak{H}$, the Cartesian product of $\mathfrak{H}$ and itself. In the rest of this paper the term relation will refer to a linear relation. The notations $\operatorname{dom} A$ and $\operatorname{ran} A$ denote the domain and the range of A, defined by

$$\operatorname{dom} A = \{\, x : \{x,y\} \in A \,\}, \quad \operatorname{ran} A = \{\, y : \{x,y\} \in A \,\}.$$

Furthermore, $\ker A$ and $\operatorname{mul} A$ denote the kernel and the multi-valued part of A, defined by

$$\ker A = \{\, x : \{x,0\} \in A \,\}, \quad \operatorname{mul} A = \{\, y : \{0,y\} \in A \,\}.$$

The inverse relation A^{-1} is given by $\{\, \{y,x\} : \{x,y\} \in A \,\}$, so that

$$\operatorname{dom} A^{-1} = \operatorname{ran} A, \quad \operatorname{ran} A^{-1} = \operatorname{dom} A, \quad \ker A^{-1} = \operatorname{mul} A, \quad \operatorname{mul} A^{-1} = \ker A.$$

A relation A is (the graph of) an operator if and only if $\operatorname{mul} A = \{0\}$.

For relations A and B in a linear space $\mathfrak{H}$ the *component-wise sum* $A \mathbin{\widehat{+}} B$ is the relation in $\mathfrak{H}$ defined by

$$A \mathbin{\widehat{+}} B = \{\, \{x+u, y+v\} : \{x,y\} \in A,\ \{u,v\} \in B \,\};$$

this last sum is direct when $A \cap B = \{\{0,0\}\}$. The following identities are clear

$$A \cap (\mathfrak{H} \times \{0\}) = \ker A \times \{0\}, \quad A \mathbin{\widehat{+}} (\mathfrak{H} \times \{0\}) = \mathfrak{H} \times \operatorname{ran} A, \tag{2.1}$$

$$A \cap (\{0\} \times \mathfrak{H}) = \{0\} \times \operatorname{mul} A, \quad A \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H}) = \operatorname{dom} A \times \mathfrak{H}. \tag{2.2}$$

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two subspaces of $\mathfrak{H}$. The notation $A_{\mathfrak{M}\times\mathfrak{N}}$ will be used for the following subrelation of A: $A_{\mathfrak{M}\times\mathfrak{N}} = A \cap (\mathfrak{M}\times\mathfrak{N})$, so that

$$(A_{\mathfrak{M}\times\mathfrak{N}})^{-1} = (A^{-1})_{\mathfrak{N}\times\mathfrak{M}}.$$

In particular,

$$\ker A = \operatorname{dom} A_{\mathfrak{H}\times\{0\}}, \quad \operatorname{mul} A = \operatorname{ran} A_{\{0\}\times\mathfrak{H}}.$$

For relations A and B in a linear space $\mathfrak{H}$ the *operator-wise sum* $A+B$ is the relation in $\mathfrak{H}$ defined by

$$A + B = \{\, \{x, y+z\} : \{x,y\} \in A,\ \{x,z\} \in B \,\},$$

so that $\operatorname{dom}(A+B) = \operatorname{dom} A \cap \operatorname{dom} B$. For $\lambda \in \mathbb{C}$ the relation λA in $\mathfrak{H}$ is defined by

$$\lambda A = \{\, \{x, \lambda y\} : \{x,y\} \in A \,\},$$

while $A-\lambda$ stands for $A-\lambda I$, where I is the identity operator on $\mathfrak{H}$. Observe that

$$(A-\lambda)^{-1}=\{\{y-\lambda x,x\}:\{x,y\}\in A\},$$

and that

$$\ker(A-\lambda)=\{x:\{x,\lambda x\}\in A\}.$$

For relations A and B in a linear space $\mathfrak{H}$ the *product* AB is defined as the relation

$$AB=\{\{x,y\}:\{x,z\}\in B,\,\{z,y\}\in A \text{ for some } z\in\mathfrak{H}\}.$$

For $\lambda\in\mathbb{C}$ the notation λA agrees in this sense with $(\lambda I)A$. The product of relations is clearly associative. Hence A^n, $n\in\mathbb{Z}$, is defined as usual with $A^0=I$ and $A^1=A$. Observe that

$$A\subset B\Rightarrow A^n\subset B^n. \tag{2.3}$$

Clearly for each $n\in\mathbb{N}_0$ the following inclusions hold

$$\operatorname{dom}A^{n+1}\subset\operatorname{dom}A^n,\quad \operatorname{ran}A^{n+1}\subset\operatorname{ran}A^n, \tag{2.4}$$

and

$$\ker A^n\subset\ker A^{n+1},\quad \operatorname{mul}A^n\subset\operatorname{mul}A^{n+1}. \tag{2.5}$$

A combination of (2.4) and (2.5) leads to

$$\ker A^n\subset\operatorname{dom}A^m,\quad \operatorname{mul}A^n\subset\operatorname{ran}A^m,\quad n,m\in\mathbb{N}_0. \tag{2.6}$$

2.2. Some results about component-wise and operator-wise sums

Let the linear space $\mathfrak{H}$ have the direct sum decomposition $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{H}_2$, $\mathfrak{H}_1\cap\mathfrak{H}_2=\{0\}$. A linear relation A in $\mathfrak{H}$ is said to be *completely reduced* by the pair $\mathfrak{H}_1$ and $\mathfrak{H}_2$ if

$$A=A_1\,\widehat{+}\,A_2,\quad \text{direct sum}, \tag{2.7}$$

where $A_i=A\cap(\mathfrak{H}_i\times\mathfrak{H}_i)$, $i=1,2$, cf. [18, Section 8]. Then

$$\operatorname{dom}A=\operatorname{dom}A_1+\operatorname{dom}A_2,\quad \operatorname{ran}A=\operatorname{ran}A_1+\operatorname{ran}A_2, \tag{2.8}$$

$$\operatorname{mul}A=\operatorname{mul}A_1+\operatorname{mul}A_2,\quad \ker A=\ker A_1+\ker A_2. \tag{2.9}$$

The following lemma shows a connection between the component-wise sum and the operator-wise sum of relations.

LEMMA 2.1. *Let the relation A be completely reduced by the pair $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as in* (2.7)*. Let P_1 and P_2 be the corresponding parallel projections onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then A allows an operator-wise sum decomposition*

$$A=A_1P_1+A_2P_2, \tag{2.10}$$

in which case

$$\operatorname{ran}A=\operatorname{ran}A_1P_1+\operatorname{ran}A_2P_2. \tag{2.11}$$

Proof. Let $\{x,y\} \in A$, then by (2.7)

$$\{x,y\} = \{s,t\} + \{u,v\}, \quad \{s,t\} \in A_1, \quad \{u,v\} \in A_2$$

where $s = P_1 x$ and $u = P_2 x$. Hence, $\{x,y\} = \{x, t+v\}$ and $\{x,t\} \in A_1 P_1$, $\{x,v\} \in A_2 P_2$. This shows $A \subset A_1 P_1 + A_2 P_2$. Conversely, let $\{x,y\} \in A_1 P_1 + A_2 P_2$. Then

$$\{x,y\} = \{x, t+v\}, \quad \{x,t\} \in A_1 P_1, \quad \{x,v\} \in A_2 P_2.$$

Let $s = P_1 x$ and $u = P_2 x$, then $\{s,t\} \in A_1$ and $\{u,v\} \in A_2$ with $u = P_2 x$. Hence $\{x,y\} = \{s,t\} + \{u,v\} \in A_1 \widehat{+} A_2 = A$. This shows $A_1 P_1 + A_2 P_2 \subset A$. Thus (2.10) has been shown. Now note that $\operatorname{ran} A_i P_i = \operatorname{ran} A_i$, $i = 1,2$, so that (2.11) follows from (2.8). □

COROLLARY 2.2. *Let the relation A be completely reduced by the pair $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as in* (2.7)*. Let P_1 and P_2 be the corresponding parallel projections onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then for each $j \in \mathbb{N}$*

$$A^j = A_1^j \widehat{+} A_2^j, \quad \textit{direct sum.} \tag{2.12}$$

In particular, A^j allows an operator-wise sum decomposition

$$A^j = A_1^j P_1 + A_2^j P_2, \tag{2.13}$$

so that

$$\operatorname{ran} A^j = \operatorname{ran} A_1^j P_1 + \operatorname{ran} A_2^j P_2. \tag{2.14}$$

Furthermore, $A_1^j P_1 = (A_1 P_1)^j$ and $A_2^j P_2 = (A_2 P_2)^j$, $j \in \mathbb{N}$.

Proof. First it is shown that (2.7) implies (2.12). Clearly, since A_1 and A_2 are contained in A, the righthand side of (2.12) is contained in the lefthand side; cf. (2.3). To see the reverse inclusion, let $\{x,y\} \in A^j$. Then with $x_0 = x$ and $x_j = y$ there exist elements $x_i \in \mathfrak{H}$ such that

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{j-1}, x_j\} \in A.$$

Apply the direct sum decomposition (2.7) to each $\{x_{i-1}, x_i\}$:

$$\{x_{i-1}, x_i\} = \{u_{i-1}, u_i\} + \{v_{i-1}, v_i\}, \quad i = 1, \ldots, j,$$

with $\{u_{i-1}, u_i\} \in A_1$ and $\{v_{i-1}, v_i\} \in A_2$. Then $\{x,y\} = \{u_0, u_j\} + \{v_0, v_j\}$ and

$$\{u_0, u_j\} \in A_1^j, \quad \{v_0, v_j\} \in A_2^j,$$

which implies the reverse inclusion.

The identities (2.13) and (2.14) follow from (2.12) and Lemma 2.1. Finally, the identities $A_i^j P_i = (A_i P_i)^j$ follow from the fact that $P_i x = x$, $x \in \mathfrak{H}_i$, $i = 1,2$. This completes the proof. □

The component-wise decomposition (2.7) and the corresponding operator-wise decomposition (2.10) occur frequently in this paper. Sometimes it happens that one of the components, say A_2P_2, is an everywhere defined operator, which can then be interpreted as a perturbation of the component A_1P_1. In special cases the domain and multivalued part of powers of such perturbations can be described explicitly. The following lemma turns out to be very useful.

LEMMA 2.3. *Let $\mathfrak{H}$ be a linear space, let D be a linear relation in $\mathfrak{H}$, and let T be an everywhere defined linear operator in $\mathfrak{H}$.*

(i) *If* $\operatorname{ran} T \subset \ker D$, *then*

$$\operatorname{dom}(D+T)^n = \operatorname{dom} D^n, \quad n \in \mathbb{N}. \tag{2.15}$$

(ii) *If* $\operatorname{ran} D \subset \ker T$, *then*

$$\operatorname{mul}(D+T)^n = \operatorname{mul} D^n, \quad n \in \mathbb{N}. \tag{2.16}$$

Proof. (i) Clearly, the identity in (2.15) is true for $n=0$ and $n=1$. Now assume that $n \geqslant 2$ and let $x_0 \in \operatorname{dom}(D+T)^n$, so that

$$\{x_i, x_{i+1}\} \in D+T, \quad 0 \leqslant i \leqslant n-1,$$

for some $x_i \in \mathfrak{H}$, $1 \leqslant i \leqslant n$. Then

$$\{x_i, x_{i+1} - Tx_i\} \in D, \quad 0 \leqslant i \leqslant n-1. \tag{2.17}$$

Furthermore, the hypothesis implies that

$$\{Tx_i, 0\} \in D, \quad 0 \leqslant i \leqslant n-2. \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$\{x_i - Tx_{i-1}, x_{i+1} - Tx_i\} = \{x_i, x_{i+1} - Tx_i\} - \{Tx_{i-1}, 0\} \in D, \quad 1 \leqslant i \leqslant n-1,$$

which together with $\{x_0, x_1 - Tx_0\} \in D$ implies that $\{x_0, x_n - Tx_{n-1}\} \in D^n$. This shows that $\operatorname{dom}(D+T)^n \subset \operatorname{dom} D^n$. A similar argument leads to the reverse inclusion, namely $\operatorname{dom} D^n \subset \operatorname{dom}(D+T)^n$. Thus (2.15) has been shown.

(ii) It is easily seen that the identity in (2.16) is true for $n=0$ and $n=1$. Now assume that $n \geqslant 2$ and let $m_n \in \operatorname{mul}(D+T)^n$, so that

$$\{m_i, m_{i+1}\} \in D+T, \quad 0 \leqslant i \leqslant n-1,$$

for some $m_i \in \mathfrak{H}$, $0 \leqslant i \leqslant n-1$ with $m_0 = 0$. Then

$$\{m_i, m_{i+1} - Tm_i\} \in D, \quad 0 \leqslant i \leqslant n-1. \tag{2.19}$$

Using the hypothesis it follows that $m_1 \in \operatorname{mul} D \subset \operatorname{ran} D \subset \ker T$, which shows that

$$m_2 = m_2 - Tm_1 \in \operatorname{ran} D \subset \ker T.$$

It can be shown inductively that

$$m_i \in \ker T, \quad 1 \leqslant i \leqslant n-1. \tag{2.20}$$

A combination of (2.19) and (2.20) leads to

$$\{m_i, m_{i+1}\} \in D, \quad 0 \leqslant i \leqslant n-1,$$

so that $m_n \in \operatorname{mul} D^n$. This shows that $\operatorname{mul}(D+T)^n \subset \operatorname{mul} D^n$. A similar argument leads to the reverse inclusion, namely $\operatorname{mul} D^n \subset \operatorname{mul}(D+T)^n$. Thus (2.16) has been shown. □

2.3. The degree of a relation in a linear space

Let A be a relation in a linear space space $\mathfrak{H}$. Then $\operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n$ for all $n \in \mathbb{N}_0$, cf. (2.4). As soon as there is equality for some n, there is equality for all larger n. The smallest n with this property is said to be the *descent* of A; cf. [18]. In this paper a modification of this notion is considered. For this purpose, define

$$\Delta(A) = \{n \in \mathbb{N} : \operatorname{ran} A^n \cap \ker A = \operatorname{ran} A^m \cap \ker A \text{ for all } m \geqslant n\}$$

for a relation A in a linear space $\mathfrak{H}$. This leads to the definition of the degree of a relation.

DEFINITION 2.4. Let A be a relation in a linear space space $\mathfrak{H}$. The degree $\delta(A)$ of A, is defined as

$$\delta(A) = \min \Delta(A) \quad \text{if} \quad \Delta(A) \neq \emptyset, \quad \text{and} \quad \delta(A) = \infty \quad \text{if} \quad \Delta(A) = \emptyset.$$

LEMMA 2.5. *Let A be a relation in a linear space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $d \in \Delta(A)$*;*

(ii) $\ker A^m \subset \ker A^d + \operatorname{ran} A^n$ *for all* $m, n \in \mathbb{N}$*;*

(iii) $\ker A^m \subset \ker A^d + \operatorname{ran} A$ *for all* $m \in \mathbb{N}$*;*

(iv) $\operatorname{ran} A^d \cap \ker A^n \subset \operatorname{ran} A^m \cap \ker A^n$ *for all* $m, n \in \mathbb{N}$*;*

(v) $\operatorname{ran} A^d \cap \ker A \subset \operatorname{ran} A^m \cap \ker A$ *for all* $m \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) If $m \leqslant d$ and $n \in \mathbb{N}$, then clearly

$$\ker A^m \subset \ker A^d \subset \ker A^d + \operatorname{ran} A^n.$$

Now assume the statement is true for some $m \geqslant d$ and $n \in \mathbb{N}$:

$$\ker A^m \subset \ker A^d + \operatorname{ran} A^n,$$

and proceed by induction. Let $u \in \ker A^{m+1}$, so that $\{u, v\} \in A^m$ and $\{v, 0\} \in A$ for some $v \in \mathfrak{H}$. Therefore, since $m \geqslant d$,

$$v \in \operatorname{ran} A^m \cap \ker A = \operatorname{ran} A^{m+n} \cap \ker A,$$

so that $\{w, v\} \in A^{m+n}$ for some $w \in \operatorname{dom} A^{m+n}$. But then $\{w, z\} \in A^n$ and $\{z, v\} \in A^m$ for some $z \in \operatorname{ran} A^n$. Hence

$$\{u - z, 0\} = \{u, v\} - \{z, v\} \in A^m,$$

implies that $u - z \in \ker A^m$, so that $u \in \ker A^m + \operatorname{ran} A^n \subset \ker A^d + \operatorname{ran} A^n$ (by the induction hypothesis). Hence,

$$\ker A^{m+1} \subset \ker A^d + \operatorname{ran} A^n,$$

which proves (ii).

(ii) $\Rightarrow$ (iii) Put $n = 1$ in (ii).

(iii) $\Rightarrow$ (iv) If $m \leqslant d$ then (iv) is clearly satisfied. It is sufficient to show that for any $m \geqslant d$ and $n \in \mathbb{N}$:

$$\operatorname{ran} A^m \cap \ker A^n \subset \operatorname{ran} A^{m+1} \cap \ker A^n.$$

Let $u \in \operatorname{ran} A^m \cap \ker A^n$, so that $\{u, 0\} \in A^n$ and $\{v, u\} \in A^m$ for some $v \in \mathfrak{H}$. Therefore $\{v, 0\} \in A^{m+n}$, and thus by (iii)

$$v \in \ker A^{m+n} \subset \ker A^d + \operatorname{ran} A,$$

so that $v = v_1 + v_2$ for some $v_1 \in \ker A^d$ and $v_2 \in \operatorname{ran} A$. Hence $\{w, v_2\} \in A$ for some $w \in \mathfrak{H}$, and as $m \geqslant d$ it follows that $\{v_1, 0\} \in A^m$ so that

$$\{v_2, u\} = \{v, u\} - \{v_1, 0\} \in A^m.$$

The last relation and the fact that $\{w, v_2\} \in A$ imply that $\{w, u\} \in A^{m+1}$, so that $u \in \operatorname{ran} A^{m+1}$. Therefore $\operatorname{ran} A^m \cap \ker A^n \subset \operatorname{ran} A^{m+1}$ and (iv) follows.

(iv) $\Rightarrow$ (v) Put $n = 1$ in (iv).

(v) $\Rightarrow$ (i) The relation (v) implies that

$$\operatorname{ran} A^d \cap \ker A = \operatorname{ran} A^m \cap \ker A, \quad m \geqslant d,$$

and $d \in \Delta(A)$ follows by definition. $\square$

COROLLARY 2.6. *Let A be a relation in a linear space $\mathfrak{H}$. If $\delta(A) < \infty$, then for all $m, n \in \mathbb{N}_0$:*

(i) $\ker A^{\delta(A)} + \operatorname{ran} A^n = \ker A^{m+\delta(A)} + \operatorname{ran} A^n$;

(ii) $\operatorname{ran} A^{\delta(A)} \cap \ker A^n = \operatorname{ran} A^{m+\delta(A)} \cap \ker A^n$.

Proof. If $\delta(A) < \infty$ the the equivalent statements of Lemma 2.5 are valid with $d = \delta(A)$.

(i) Observe that (ii) of Lemma 2.5 and the second inclusion in (2.5) give

$$\ker A^{m+\delta(A)} + \operatorname{ran} A^n \subset \ker A^{\delta(A)} + \operatorname{ran} A^n \subset \ker A^{m+\delta(A)} + \operatorname{ran} A^n,$$

so that equality prevails.

(ii) Observe that (iii) of Lemma 2.5 and the first inclusion in (2.4) give

$$\operatorname{ran} A^{\delta(A)} \cap \operatorname{ran} A^n \subset \operatorname{ran} A^{m+\delta(A)} \cap \operatorname{ran} A^n \subset \operatorname{ran} A^{\delta(A)} + \operatorname{ran} A^n,$$

so that equality prevails. □

It follows from the definition that $\delta(A) = 0$ if and only if $\Delta(A) = \mathbb{N}_0$. Useful equivalent statements are given in the following lemma.

LEMMA 2.7. *Let A be a relation in a linear space $\mathfrak{H}$. The following statements are equivalent:*

(i) $\delta(A) = 0$;

(ii) $\ker A \subset \operatorname{ran} A^m$ *for all* $m \in \mathbb{N}_0$;

(iii) $\ker A^n \subset \operatorname{ran} A$ *for all* $n \in \mathbb{N}_0$;

(iv) $\ker A^n \subset \operatorname{ran} A^m$ *for all* $n, m \in \mathbb{N}_0$.

Proof. (i) ⇒ (ii) Assume that $\delta(A) = 0$, so that $\Delta(A) = \mathbb{N}_0$. Then

$$\operatorname{ran} A^m \cap \ker A = \operatorname{ran} A^0 \cap \ker A = \ker A,$$

for all $m \in \mathbb{N}_0$. In other words, $\ker A \subset \operatorname{ran} A^m$ for all $m \in \mathbb{N}_0$.

(ii) ⇒ (iii) This implication is proved by induction. The case $n = 0$ is trivial and the case $n = 1$ is a direct consequence of (ii). Assume now that (iii) is valid for $n = k$. Let $u \in \ker A^{k+1}$, so that $\{u, v\} \in A^k$ and $\{v, 0\} \in A$. Therefore $v \in \ker A \subset \operatorname{ran} A^{k+1}$ by (ii), so that $\{w, v\} \in A^{k+1}$ for some $w \in \mathfrak{H}$. Hence, there exists $z \in \mathfrak{H}$ such that $\{w, z\} \in A$ and $\{z, v\} \in A^k$. Clearly,

$$\{u - z, 0\} = \{u, v\} - \{z, v\} \in A^k,$$

and then $u - z \in \ker A^k \subset \operatorname{ran} A$. Since $z \in \operatorname{ran} A$, it follows that $u \in \operatorname{ran} A$. Therefore $\ker A^{k+1} \subset \operatorname{ran} A$, and the induction step has been shown.

(iii) $\Rightarrow$ (iv) The case $m=0$ is obvious and the case $m=1$ follows from (iii). Now assume that (iv)is valid for $m=k$. Let $u \in \ker A^n$. It follows from (iii) that $u \in \operatorname{ran} A$, so that $\{v,u\} \in A$, which implies $v \in \ker A^{n+1}$. By induction hypothesis $\ker A^{n+1} \subset \operatorname{ran} A^k$ so $v \in \operatorname{ran} A^k$. Since $\{v,u\} \in A$, it follows that $u \in \operatorname{ran} A^{k+1}$. Hence (iv) holds with $m=k+1$.

(iv) $\Rightarrow$ (i) Take $n=1$ in (iv), so that $\ker A \subset \operatorname{ran} A^m$ for all $m \in \mathbb{N}$. Therefore $\ker A \cap \operatorname{ran} A^m = \ker A$ for all $m \in \mathbb{N}_0$. In other words, $\Delta(A) = \mathbb{N}_0$ and, hence, $\delta(A) = 0$. $\square$

3. Range subspaces of Hilbert spaces

This section provides a concise introduction to the notion of range subspaces of a Hilbert space. In the following a *subspace* of a Hilbert space is a linear manifold which is not necessarily closed. The *closure* of a subspace $\mathfrak{M}$ will be denoted by $\operatorname{clos}\mathfrak{M}$. The notion of range subspace is a useful generalization of the notion of closed subspace. Whereas the sum of two closed subspaces need not be closed, the sum of two range subspaces is again a range subspace. All results in this section can be found in [3], [9]; the present exposition is mostly self-contained and is meant to give the reader a smooth introduction to the subject.

When $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a Hilbert space, then the subspaces $\mathfrak{M}^\perp$ and $\mathfrak{N}^\perp$ are automatically closed. Recall the following well-known result.

LEMMA 3.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be (not necessarily closed) subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{M} + \mathfrak{N}$ *is closed;*

(ii) $\mathfrak{M} + \mathfrak{N} = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp$.

Proof. (i) $\Rightarrow$ (ii) For not necessarily closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ it follows that $(\mathfrak{M} + \mathfrak{N})^\perp = \mathfrak{M}^\perp \cap \mathfrak{N}^\perp$, and, consequently, $\operatorname{clos}(\mathfrak{M} + \mathfrak{N}) = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp$. The assumption that $\mathfrak{M} + \mathfrak{N}$ is closed leads to the identity in (ii).

(ii) $\Rightarrow$ (i) This is clear. $\square$

If $\mathfrak{M}$ and $\mathfrak{N}$ in Lemma 3.1 are replaced by $\mathfrak{M}^\perp$ and $\mathfrak{N}^\perp$, then it follows that $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed if and only if $\mathfrak{M}^\perp + \mathfrak{N}^\perp = (\operatorname{clos}\mathfrak{M} \cap \operatorname{clos}\mathfrak{N})^\perp$. Furthermore, recall that when $\mathfrak{M}$ and $\mathfrak{N}$ are closed subspaces, then the subspace $\mathfrak{M} + \mathfrak{N}$ is closed if and only if the subspace $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed; cf. [8, Chapter IV, Theorem 4.8.], [9, Proposition 1.3.1, Corollaire 1.3.2]. These observations and Lemma 3.1 lead to the following useful proposition.

PROPOSITION 3.2. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{M} + \mathfrak{N}$ *is closed;*

(ii) $\mathfrak{M} + \mathfrak{N} = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp$*;*

(iii) $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ *is closed;*

(iv) $\mathfrak{M}^\perp + \mathfrak{N}^\perp = (\mathfrak{M} \cap \mathfrak{N})^\perp$.

A pair of closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ of a Hilbert space $\mathfrak{H}$ is said to be *complementary* when $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$ and $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. For each pair of complementary subspaces $\mathfrak{M}$ and $\mathfrak{N}$ there exists a pair of bounded linear projections $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ onto $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Note that when the closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ are complementary, then also the closed subspaces $\mathfrak{M}^\perp$ and $\mathfrak{N}^\perp$ are complementary.

DEFINITION 3.3. A subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$ is said to be a range subspace of $\mathfrak{H}$ if there exist a Hilbert space $\mathfrak{K}$ and an operator $A \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ such that $\mathfrak{M} = \operatorname{ran} A$.

If $A \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ is injective, then $\mathfrak{M} = \operatorname{ran} A$ is easily seen to be a Hilbert space, when the space $\mathfrak{M}$ is equipped with the inner product

$$(Ax, Ay)_{\mathfrak{M}} = (x, y)_{\mathfrak{K}}, \quad x, y \in \mathfrak{K}.$$

In fact, it is no loss of generality to assume in Definition 3.3 that $\ker A = \{0\}$; just replace $\mathfrak{K}$ by $\mathfrak{K} \ominus \ker A$.

Assume that also $\mathfrak{M} = \operatorname{ran} A_1$ with $A_1 \in \mathbf{L}(\mathfrak{K}_1, \mathfrak{H})$ with a Hilbert space $\mathfrak{K}_1$ and $\ker A_1 = \{0\}$. Then the injective operators A_1 and A have the same range. Hence, by Douglas' lemma, there is an invertible operator $B \in \mathbf{L}(\mathfrak{K}_1, \mathfrak{K})$ such that $A_1 = AB$; cf. [3]. This shows that the topologies induced on $\mathfrak{M}$ by A_1 and A are equivalent.

Furthermore, if $A \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ is injective, then there is an estimate for the corresponding norms:

$$\|Ax\|_{\mathfrak{M}} \geqslant \frac{1}{\|A\|} \|Ax\|_{\mathfrak{H}}, \quad x \in \mathfrak{K}.$$

This estimate is a motivation for the following useful characterization. For further equivalent statements, see [3].

PROPOSITION 3.4. *Let $\mathfrak{M}$ be a subspace of a Hilbert space $\mathfrak{H}$. Then $\mathfrak{M}$ is a range subspace of $\mathfrak{H}$ if and only if there is an inner product $(\cdot,\cdot)_+$ on $\mathfrak{M}$, such that*

(i) *$\mathfrak{M}$ is a Hilbert space when equipped with $(\cdot,\cdot)_+$;*

(ii) *$\|u\|_+ \geqslant c\|u\|_{\mathfrak{H}}$, $u \in \mathfrak{M}$, for some $c > 0$.*

REMARK 3.5. In particular, if $\mathfrak{M}$ is a Hilbert space with a continuous injection ι in $\mathfrak{H}$, then $\iota(\mathfrak{M})$ is a range subspace of $\mathfrak{H}$. Any closed subspace of a Hilbert space $\mathfrak{H}$ is a range subspace of $\mathfrak{H}$.

COROLLARY 3.6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of a Hilbert space $\mathfrak{H}$. If $\mathfrak{M}$ is a range subspace and $\mathfrak{N}$ is a closed subspace, then $\mathfrak{M} \cap \mathfrak{N}$ is a closed subspace of $\mathfrak{M}$ as a range subspace.*

Proof. Let (x_k) be a sequence in $\mathfrak{M}\cap\mathfrak{N}$ which converges to $x\in\mathfrak{M}$ with respect to the range space topology of $\mathfrak{M}$. As the range space topology is stronger than the topology on $\mathfrak{M}$ induced by $\mathfrak{H}$, and since $\mathfrak{N}$ is closed in $\mathfrak{H}$, it follows that $x\in\mathfrak{N}$. Hence, $x\in\mathfrak{M}\cap\mathfrak{N}$. Thus $\mathfrak{M}\cap\mathfrak{N}$ is a closed subspace of $\mathfrak{M}$ as a range subspace. □

COROLLARY 3.7. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of a Hilbert space $\mathfrak{H}$. If $\mathfrak{N}$ is a range subspace of $\mathfrak{H}$ (so that $\mathfrak{N}$ is a Hilbert space in its own topology), and $\mathfrak{M}$ is a range subspace of $\mathfrak{N}$, then $\mathfrak{M}$ is a range subspace of $\mathfrak{H}$.*

The Cartesian product $\mathfrak{H}\times\mathfrak{H}$ of $\mathfrak{H}$ with itself will be provided with the componentwise inner product, so that $\mathfrak{H}\times\mathfrak{H}$ is a Hilbert space itself.

COROLLARY 3.8. *If $\mathfrak{M}$ and $\mathfrak{N}$ are range subspaces of a Hilbert space $\mathfrak{H}$, then $\mathfrak{M}\times\mathfrak{N}$ is a range subspace of $\mathfrak{H}\times\mathfrak{H}$. Moreover, if $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a Hilbert space $\mathfrak{H}$, and $\mathfrak{M}\times\mathfrak{N}$ is a range subspace of $\mathfrak{H}\times\mathfrak{H}$, then $\mathfrak{M}$ and $\mathfrak{N}$ are range subspaces of $\mathfrak{M}$.*

The following proposition implies that the range subspaces of a Hilbert space $\mathfrak{H}$ form a lattice; cf. [3], [9, Proposition 2.1.2].

PROPOSITION 3.9. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$. Then:*

(i) $\mathfrak{M}+\mathfrak{N}$ *is a range subspace of $\mathfrak{H}$;*

(ii) $\mathfrak{M}\cap\mathfrak{N}$ *is a range subspace of $\mathfrak{H}$.*

Proof. (i) The mapping $\Phi:\mathfrak{M}\times\mathfrak{N}\to\mathfrak{M}+\mathfrak{N}$ defined by

$$\Phi(\{u,v\})=u+v,\quad u\in\mathfrak{M},\,v\in\mathfrak{N},$$

is a continuous linear mapping onto $\mathfrak{M}+\mathfrak{N}$.

(ii) Provide the space $\mathfrak{M}\cap\mathfrak{N}$ with the sum inner product

$$(u,v)_{\mathfrak{M}\cap\mathfrak{N}}=(u,v)_{\mathfrak{M}}+(u,v)_{\mathfrak{N}},\quad u\in\mathfrak{M},\,v\in\mathfrak{N}.$$

It is straightforward to check that $\mathfrak{M}\cap\mathfrak{N}$ is a Hilbert space and that, in fact, $\mathfrak{M}\cap\mathfrak{N}$ is a range space in $\mathfrak{M}$ and in $\mathfrak{N}$. □

PROPOSITION 3.10. ('Neubauer's lemma') *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$. If the range subspaces $\mathfrak{M}+\mathfrak{N}$ and $\mathfrak{M}\cap\mathfrak{N}$ are closed in $\mathfrak{H}$, then $\mathfrak{M}$ and $\mathfrak{N}$ are closed subspaces of $\mathfrak{H}$.*

Proof. First consider the case $\mathfrak{M}\cap\mathfrak{N}=\{0\}$; cf. [3]. Let the mapping

$$\Phi:\mathfrak{M}\times\mathfrak{M}\subset\mathfrak{H}\times\mathfrak{H}\to\mathfrak{M}+\mathfrak{N}\subset\mathfrak{H}$$

be defined by

$$\Phi(\{u,v\}) = u+v, \quad u \in \mathfrak{M}, v \in \mathfrak{N}.$$

Then Φ is continuous onto the closed subspace $\mathfrak{M}+\mathfrak{N}$. The assumption $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ implies that Φ is one-to-one. Hence, by Banach's open mapping theorem, Φ is an open mapping from $\mathfrak{M}\times\mathfrak{N}$ onto $\mathfrak{M}+\mathfrak{N}$. Hence

$$\mathfrak{M} = \Phi(\mathfrak{M}\times\{0\}), \quad \mathfrak{N} = \Phi(\{0\}\times\mathfrak{N}),$$

are closed in $\mathfrak{M}+\mathfrak{N}$, and, therefore, in $\mathfrak{H}$.

Now consider the general case, cf. [9, Proposition 2.1.1]. Define the subspaces $\mathfrak{M}'$ and $\mathfrak{N}'$ by

$$\mathfrak{M}' = \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}, \quad \mathfrak{N}' = \mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp},$$

so that $\mathfrak{M}'$ and $\mathfrak{N}'$ are range subspaces by Corollary 3.6. Moreover, clearly,

$$\mathfrak{M}'\cap\mathfrak{N}' = (\mathfrak{M}\cap\mathfrak{N})\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}, \quad \mathfrak{M}'+\mathfrak{N}' = (\mathfrak{M}+\mathfrak{N})\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}$$

so that $\mathfrak{M}'\cap\mathfrak{N}'=\{0\}$ and $\mathfrak{M}'+\mathfrak{N}'$ is closed. It follows that $\mathfrak{M}'$ and $\mathfrak{N}'$ are closed by the first part of the proof. As $\mathfrak{M}=\mathfrak{M}'\oplus(\mathfrak{M}\cap\mathfrak{N})$ and $\mathfrak{N}=\mathfrak{N}'\oplus(\mathfrak{M}\cap\mathfrak{N})$, orthogonal sums, it is shown that $\mathfrak{M}$ and $\mathfrak{N}$ are closed. □

Proposition 3.10 is very useful: it gives special circumstances under which one can conclude the closedness of range subspaces. The following corollary is a weakened form of Proposition 3.10; cf. [9, Proposition 2.3.1].

COROLLARY 3.11. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$. Assume that $\mathfrak{M}+\mathfrak{N}$ is closed in $\mathfrak{H}$ and that $\mathfrak{M}\cap\mathfrak{N}$ is closed in $\mathfrak{M}$ with its own topology. Then $\mathfrak{N}$ is closed in $\mathfrak{H}$.*

Proof. Let $\mathfrak{M}_0$ be the orthogonal complement of $\mathfrak{M}\cap\mathfrak{N}$ in $\mathfrak{M}$, so that

$$\mathfrak{M} = (\mathfrak{M}\cap\mathfrak{N})\,[+]\,\mathfrak{M}_0,$$

since $\mathfrak{M}\cap\mathfrak{N}$ is closed in $\mathfrak{M}$. Clearly, $\mathfrak{M}_0\cap\mathfrak{N}\subset\mathfrak{M}_0\cap\mathfrak{M}\cap\mathfrak{N}=\{0\}$. Furthermore

$$\mathfrak{M}_0+\mathfrak{N}\subset\mathfrak{M}+\mathfrak{N}\subset\mathfrak{M}_0+\mathfrak{N},$$

which shows that $\mathfrak{M}_0+\mathfrak{N}=\mathfrak{M}+\mathfrak{N}$ is closed. Hence by Proposition 3.10 it follows that $\mathfrak{N}$ is closed in $\mathfrak{H}$. □

The next corollary is taken from [9, Proposition 2.3.3]; it contains [9, Proposition 2.2.1].

COROLLARY 3.12. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$ and assume that $\mathfrak{M}+\mathfrak{N}$ is closed in $\mathfrak{H}$. Then*

(i) $\operatorname{clos}\mathfrak{M} = \mathfrak{M}+\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})\cap\mathfrak{N} = \mathfrak{M}+\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})$;

(ii) $\operatorname{clos}\mathfrak{N} = \mathfrak{N} + \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})\cap\mathfrak{M} = \mathfrak{N} + \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})$;

(iii) $(\operatorname{clos}\mathfrak{M})\cap\mathfrak{N} = \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})\cap\mathfrak{N}$; $\quad \mathfrak{M}\cap(\operatorname{clos}\mathfrak{N}) = \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})\cap\mathfrak{M}$;

(iv) $(\operatorname{clos}\mathfrak{M})\cap(\operatorname{clos}\mathfrak{N}) = (\operatorname{clos}\mathfrak{M})\cap\mathfrak{N} + \mathfrak{M}\cap(\operatorname{clos}\mathfrak{N}) = \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N})$.

Proof. Define $\mathfrak{M}_0 = \mathfrak{M} + (\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))\cap\mathfrak{N}$. Then it is clear that

$$\mathfrak{M}_0 + \mathfrak{N} = \mathfrak{M} + \mathfrak{N}, \quad \mathfrak{M}_0\cap\mathfrak{N} \subset (\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))\cap\mathfrak{N} \subset \mathfrak{M}_0\cap\mathfrak{N}.$$

Thus $\mathfrak{M}_0 + \mathfrak{N}$ is closed since $\mathfrak{M}+\mathfrak{N}$ is assumed to be closed; furthermore $\mathfrak{M}_0\cap\mathfrak{N}$ is closed in $\mathfrak{N}$ with its own topology, due to $\mathfrak{M}_0\cap\mathfrak{N} = (\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))\cap\mathfrak{N}$ and Corollary 3.6. Hence $\mathfrak{M}_0$ is closed by Proposition 3.10. The inclusion $\mathfrak{M}\subset\mathfrak{M}_0\subset\operatorname{clos}\mathfrak{M}$ now implies that $\mathfrak{M}_0 = \operatorname{clos}\mathfrak{M}$. This gives (i) and (ii) follows by symmetry. The first identity in (iii) follows from $\mathfrak{M}_0\cap\mathfrak{N} = (\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))\cap\mathfrak{N}$ and $\mathfrak{M}_0 = \operatorname{clos}\mathfrak{M}$. The second identity in (iii) follows by symmetry. Finally observe that

$$\begin{aligned}(\operatorname{clos}\mathfrak{M})\cap(\operatorname{clos}\mathfrak{N}) &\subset (\mathfrak{M}\cap(\operatorname{clos}\mathfrak{N})) + (\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))\cap\mathfrak{N}\\ &\subset \mathfrak{M}\cap(\operatorname{clos}\mathfrak{N}) + (\operatorname{clos}\mathfrak{M})\cap\mathfrak{N},\end{aligned}$$

where the first inclusion follows from $\mathfrak{M}_0 = \operatorname{clos}\mathfrak{M}$ and the second inclusion is trivial. It follows from (iii) that

$$\mathfrak{M}\cap(\operatorname{clos}\mathfrak{N}) + (\operatorname{clos}\mathfrak{M})\cap\mathfrak{N} \subset \operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}) \subset (\operatorname{clos}\mathfrak{M})\cap(\operatorname{clos}\mathfrak{N}).$$

Now (iv) is clear. □

The next corollary is taken from [9, Corollaire 2.3.1]; see also [9, Proposition 2.3.1].

COROLLARY 3.13. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$ and assume that $\mathfrak{M}+\mathfrak{N}$ is closed. Then*

$$(\mathfrak{M}\cap\mathfrak{N})^\perp = \mathfrak{M}^\perp + \mathfrak{N}^\perp.$$

In particular, $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed.

Proof. The assumption that $\mathfrak{M}+\mathfrak{N}$ is closed leads to

$$\mathfrak{M}+\mathfrak{N} \subset \operatorname{clos}\mathfrak{M} + \operatorname{clos}\mathfrak{N} \subset \operatorname{clos}(\mathfrak{M}+\mathfrak{N}) = \mathfrak{M}+\mathfrak{N}.$$

Thus $(\operatorname{clos}\mathfrak{M}) + (\operatorname{clos}\mathfrak{N})$ is closed. Hence Proposition 3.2 and Corollary 3.12 may be applied to obtain

$$\begin{aligned}\mathfrak{M}^\perp + \mathfrak{N}^\perp &= (\operatorname{clos}\mathfrak{M})^\perp + (\operatorname{clos}\mathfrak{N})^\perp = ((\operatorname{clos}\mathfrak{M})\cap(\operatorname{clos}\mathfrak{N}))^\perp\\ &= ((\operatorname{clos}(\mathfrak{M}\cap\mathfrak{N}))^\perp = (\mathfrak{M}\cap\mathfrak{N})^\perp,\end{aligned}$$

which gives the identity. □

COROLLARY 3.14. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}+\mathfrak{N}$ be closed in $\mathfrak{H}$. If $\mathfrak{M}$ and $\mathfrak{N}$ are dense in $\mathfrak{H}$, then also $\mathfrak{M}\cap\mathfrak{N}$ is dense in $\mathfrak{H}$.*

PROPOSITION 3.15. *Let $\mathfrak{M}_j$, $j \in \mathbb{N}$, be range subspaces of a Hilbert space $\mathfrak{H}$ with $\mathfrak{M}_j \subset \mathfrak{M}_{j+1}$ for all $j \in \mathbb{N}$, such that $\bigcup_{j=1}^{\infty} \mathfrak{M}_j = \mathfrak{H}$. Then there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{M}_{j_0} = \mathfrak{H}$.*

Proof. See [9, Proposition 2.2.4]. □

4. Linear relations in Hilbert spaces

In this section linear relations are considered in Hilbert spaces. The concept of range space relation is introduced and an important observation of C. Foiaş for range space operators is placed in the context of range space relations, see [3, p. 258]. The concept of adjoint relation is defined and studied for range space relations. Finally, sufficient conditions are given for a range space relation A such that $\operatorname{ran} A^n + \ker A^m$ is closed.

4.1. Range space operators and relations

An operator or a relation A in a Hilbert space $\mathfrak{H}$ is said to be *closed* if A as a subspace of $\mathfrak{H}\times\mathfrak{H}$ is closed. The following definition gives a useful extension of this concept.

DEFINITION 4.1. An operator or a relation A in $\mathfrak{H}$ is said to be a range space operator or range space relation if its graph A is a range subspace of $\mathfrak{H}\times\mathfrak{H}$.

In particular, a closed relation in a Hilbert space $\mathfrak{H}$ is a range space relation in $\mathfrak{H}$. Clearly, with A also A^{-1} is a range space relation. Range space operators were considered in [3] and [9]. The concept of range space relation seems to be new; cf. [10].

LEMMA 4.2. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then* $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$, *and* $\operatorname{mul} A$ *are range subspaces of $\mathfrak{H}$.*

Proof. The subspaces $\mathfrak{H}\times\{0\}$ and $\{0\}\times\mathfrak{H}$ are closed in $\mathfrak{H}\times\mathfrak{H}$. Hence, if A is a range space relation, then it follows from Proposition 3.9 and the identities (2.1) and (2.2) that $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$, and $\operatorname{mul} A$ are range subspaces of $\mathfrak{H}$. □

If A is a range space operator or range space relation then the subset A of the Cartesian product $\mathfrak{H}\times\mathfrak{H}$ equals $\operatorname{ran} F$, where F is a bounded injective linear mapping from a Hilbert space $\mathfrak{K}$ to $\mathfrak{H}\times\mathfrak{H}$. Then A has its own inner product, denoted by $(\cdot,\cdot)_A$ and clearly there exists a constant $c>0$, so that

$$\|\{u,v\}\|_A \geqslant c\,(\|u\|+\|v\|), \quad \{u,v\}\in A, \tag{4.1}$$

cf. Proposition 3.4.

Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Define the relation J from A to $\operatorname{dom} A$ by

$$J = \{\,\{\{u,v\},u\} : \{u,v\} \in A\,\}, \tag{4.2}$$

so that J is (the graph of) an operator from A onto $\operatorname{dom} A \subset \mathfrak{H}$.

PROPOSITION 4.3. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then the operator J in* (4.2) *from A onto* $\operatorname{dom} A \subset \mathfrak{H}$ *is bounded. Moreover, if* $\operatorname{dom} A$ *is closed, then J is an open mapping.*

Proof. Let A be a range space relation with inner product $(\cdot,\cdot)_A$, so that $A \subset \mathfrak{H} \times \mathfrak{H}$. It follows from (4.1) that

$$\|\{u,v\}\|_A \geqslant c\,(\|u\| + \|v\|) \geqslant c\|u\|, \quad \{u,v\} \in A.$$

This shows that J is a bounded operator from A onto $\operatorname{dom} A \subset \mathfrak{H}$. If $\operatorname{dom} A$ is closed, then the open mapping theorem shows that J is an open mapping. □

Proposition 4.3 is an extension of an observation by C. Foiaş, see [3, p. 258] and [9, Proposition 2.1.5]. The original observation is the following corollary.

COROLLARY 4.4. (Foiaş) *Let A be a range space operator in a Hilbert space $\mathfrak{H}$. If* $\operatorname{dom} A$ *is closed in $\mathfrak{H}$, then A is bounded and, hence, closed.*

Proof. When A is an operator, the mapping J is injective. Since it is surjective as well, the open mapping may be applied to conclude that J^{-1} is bounded, implying that

$$c\,(\|u\| + \|v\|) \leqslant \|\{u,v\}\|_A \leqslant d\|u\|, \quad u \in \operatorname{dom} A.$$

In other words, with $v = Au$, it follows that

$$\|Au\| \leqslant C\|u\|, \quad u \in \operatorname{dom} A.$$

This completes the proof. □

COROLLARY 4.5. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then:*

(i) *if* $\operatorname{dom} A$ *is closed, then* $\ker A$ *is closed;*

(ii) *if* $\operatorname{ran} A$ *is closed, then* $\operatorname{mul} A$ *is closed.*

Proof. (i) Observe that J maps $A \cap (\mathfrak{H} \times \{0\}) = \ker A \times \{0\}$ one-to-one onto $\ker A$. The space $A \cap (\mathfrak{H} \times \{0\})$ is a closed subspace of the range space A by Corollary 3.6. According to Proposition 4.3 the mapping J is open, so that the image space $\ker A$ is closed in $\operatorname{dom} A$.

(ii) If $\operatorname{ran} A$ is closed, then A^{-1} is a range space relation for which $\operatorname{ran} A = \operatorname{dom} A^{-1}$. The conclusion now follows from (i) and $\operatorname{mul} A = \ker A^{-1}$. □

LEMMA 4.6. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then:*

(i) *if* $\ker A$ *and* $\operatorname{ran} A$ *are closed, then* A *is closed;*

(ii) *if* $\operatorname{mul} A$ *and* $\operatorname{dom} A$ *are closed, then* A *is closed.*

Proof. Assume the conditions in (i). It follows from (2.1) that $A \cap (\mathfrak{H} \times \{0\})$ and $A \widehat{+} (\mathfrak{H} \times \{0\})$ are closed. The conclusion now follows from Proposition 3.10. The statement in (ii) follows likewise from (2.2). □

Let A and B be range space relations in a Hilbert space $\mathfrak{H}$. It follows from Proposition 3.9 that the component-wise sum $A \widehat{+} B$ is a range space relation.

PROPOSITION 4.7. *Let A and B be range space relations in a Hilbert space $\mathfrak{H}$. Then the operator-wise sum $A + B$ is a range space relation in $\mathfrak{H} \oplus \mathfrak{H}$.*

Proof. Denote by S the continuous mapping of $A \oplus B$ into $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$ given by

$$(\{x,y\}, \{s,t\}) \mapsto \{x, x-s, y+t\}.$$

It follows from Remark 3.5 that $\operatorname{ran} S$ is a range subspace in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$, and therefore so is $\operatorname{ran} S \cap (\mathfrak{H} \oplus \{0\} \oplus \mathfrak{H})$ which is isomorphic to $A + B$. □

PROPOSITION 4.8. *Let A and B be range space relations in a Hilbert space $\mathfrak{H}$. Then the product BA is a range space relation in $\mathfrak{H}$.*

Proof. Denote by T the continuous mapping of $A \times B$ into $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$ given by

$$(\{x,y\}, \{s,t\}) \mapsto \{x, y-s, t\}$$

Then from Remark 3.5 it follows that $\operatorname{ran} T$ is a range subspace in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$, and therefore so is $\operatorname{ran} T \cap (\mathfrak{H} \oplus \{0\} \oplus \mathfrak{H})$ which is isomorphic to BA. □

COROLLARY 4.9. *If A is a range space relation then A^n is a range space relation for all $n \in \mathbb{N}$.*

Suitable restrictions of range space relations are again range space relations.

LEMMA 4.10. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}$ and $\mathfrak{N}$ be range subspaces of $\mathfrak{H}$. Then*

(i) $A_{\mathfrak{M} \times \mathfrak{N}}$ *is a range space relation;*

(ii) $\operatorname{dom} A_{\mathfrak{M} \times \mathfrak{N}}$ *and* $\operatorname{ran} A_{\mathfrak{M} \times \mathfrak{N}}$ *are range subspaces;*

(iii) $\ker A_{\mathfrak{M} \times \mathfrak{N}}$ *and* $\operatorname{mul} A_{\mathfrak{M} \times \mathfrak{N}}$ *are range subspaces.*

Proof. If $\mathfrak{M}$ and $\mathfrak{N}$ are range subspaces of $\mathfrak{H}$, then $\mathfrak{M}\times\mathfrak{N}$ is a range subspace of $\mathfrak{H}\times\mathfrak{H}$, by (iii) of Proposition 3.9. Clearly, $A_{\mathfrak{M}\times\mathfrak{N}} = A\cap(\mathfrak{M}\times\mathfrak{N})$ is a range subspace of $\mathfrak{H}\times\mathfrak{H}$ by (i) of Proposition 3.9. This shows (i); the items (ii) and (iii) are now straightforward consequences of (i). □

PROPOSITION 4.11. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then*

(i) *if* $\operatorname{ran} A$ *is closed, then* $\operatorname{clos} A = A \,\widehat{+}\, (\overline{\ker} A\times\{0\})$*;*

(ii) *if* $\operatorname{dom} A$ *is closed, then* $\operatorname{clos} A = A \,\widehat{+}\, (\{0\}\times\overline{\operatorname{mul}} A)$.

Proof. Observe that

$$A \,\widehat{+}\, (\mathfrak{H}\times\{0\}) = \mathfrak{H}\times\operatorname{ran} A, \quad A\cap(\mathfrak{H}\times\{0\}) = \ker A\times\{0\}.$$

If $\operatorname{ran} A$ is closed, then apply Corollary 3.12 with $\mathfrak{M}=A$ and $\mathfrak{N}=\mathfrak{H}\times\{0\}$ to obtain the identity in (i). The identity in (ii) can be shown in a similar way. □

If A is a range space operator and $\operatorname{ran} A$ is closed, then $\operatorname{clos} A$ is (the graph of) an operator if and only if $\operatorname{dom} A\cap\overline{\ker} A=\{0\}$, cf. [9, Proposition 2.2.2].

4.2. Adjoint relations in Hilbert spaces

Let A be a relation in a Hilbert space $\mathfrak{H}$. The adjoint A^* of A is the closed (linear) relation defined by

$$A^* = \{\,\{f,f'\}\in\mathfrak{H}\times\mathfrak{H} : (f',h)=(f,h') \text{ for all } \{h,h'\}\in A\,\},$$

so that A^* is a range space relation. Note that A^{**} is the closure of A. The following identities are straightforward:

$$(\operatorname{dom} A)^\perp = \operatorname{mul} A^*, \quad (\operatorname{ran} A)^\perp = \ker A^*, \tag{4.3}$$

or, equivalently,

$$\overline{\operatorname{dom}} A = (\operatorname{mul} A^*)^\perp, \quad \overline{\operatorname{ran}} A = (\ker A^*)^\perp.$$

If the relation A is closed, then

$$(\ker A)^\perp = \overline{\operatorname{ran}} A^*, \quad (\operatorname{mul} A)^\perp = \overline{\operatorname{dom}} A^*, \tag{4.4}$$

or, equivalently,

$$\overline{\ker} A = (\operatorname{ran} A^*)^\perp, \quad \overline{\operatorname{mul}} A = (\operatorname{dom} A^*)^\perp.$$

The next result is a slight extension of a well-known result. The operator version of this result can be found in [9, Proposition 2.3.2].

LEMMA 4.12. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then*

(i) *if* $\operatorname{dom} A$ *is closed, then* $\operatorname{dom} A^* = (\operatorname{mul} A)^\perp$*, so that* $\operatorname{dom} A^*$ *is closed;*

(ii) *if* $\mathrm{dom}A^*$ *is closed, then* $\mathrm{dom}A^{**} = (\mathrm{mul}A^*)^\perp$*, so that* $\mathrm{dom}A^{**}$ *is closed;*

(iii) *if* $\mathrm{ran}A$ *is closed, then* $\mathrm{ran}A^* = (\ker A)^\perp$*, so that* $\mathrm{ran}A^*$ *is closed;*

(iv) *if* $\mathrm{ran}A^*$ *is closed, then* $\mathrm{ran}A^{**} = (\ker A^*)^\perp$*, so that* $\mathrm{ran}A^{**}$ *is closed.*

Proof. (i) Let the operator K be defined by $K\{u,v\} = \{v,-u\}$, so that $A^* = (KA)^\perp$. Clearly, the relation KA is a range space relation. Moreover, the subspace

$$(KA) \widehat{+} (\mathfrak{H} \times \{0\}) = \mathfrak{H} \times \mathrm{dom}A$$

is closed in $\mathfrak{H} \times \mathfrak{H}$, since $\mathrm{dom}A$ is closed. Furthermore, the intersection is given by

$$(KA) \cap (\mathfrak{H} \times \{0\}) = \mathrm{mul}A \times \{0\}$$

(which is closed in the range space KA, due to Corollary 3.6). Now apply Corollary 3.13 with $\mathfrak{M} = KA$ and $\mathfrak{N} = \mathfrak{H} \times \{0\}$ as subspaces of $\mathfrak{H} \times \mathfrak{H}$. Then

$$(\mathrm{mul}A)^\perp \times \mathfrak{H} = (\mathfrak{M} \cap \mathfrak{N})^\perp = \mathfrak{M}^\perp \widehat{+} \mathfrak{N}^\perp = A^* \widehat{+} (\{0\} \times \mathfrak{H}) = \mathrm{dom}A^* \times \mathfrak{H}.$$

This implies that $\mathrm{dom}A^* = (\mathrm{mul}A)^\perp$ and $\mathrm{dom}A^*$ is closed.

(ii) This follows from (i) by going from A to A^* and observing that A^* is closed, so that A^* is a range space relation.

(iii) This follows from (i) by going from A to A^{-1} and observing that A^{-1} is also a range space relation.

(iv) This follows from (iii) by going from A to A^* and observing that A^* is closed, so that A^* is a range space relation. □

COROLLARY 4.13. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Then*

(i) $\mathrm{dom}A$ *is closed if and only if* $\mathrm{dom}A^*$ *is closed;*

(ii) $\mathrm{ran}A$ *is closed if and only if* $\mathrm{ran}A^*$ *is closed.*

The behaviour of the adjoint in sums and products is given in the following lemma. Of course, equality can be shown under less strict conditions.

LEMMA 4.14. *Let A and B be relations in a Hilbert space $\mathfrak{H}$. Then*

$$A^* + B^* \subset (A+B)^*, \quad B^*A^* \subset (AB)^*. \tag{4.5}$$

If A is a bounded everywhere defined operator, then

$$(A+B)^* = A^* + B^*, \quad B^*A^* = (AB)^*. \tag{4.6}$$

In particular, note that

$$A^{*n} \subset A^{n*}. \tag{4.7}$$

4.3. Some auxiliary results

This subsection contains some results of a technical nature, which will be used in Section 7. The following construction is an extension of the construction of Labrousse [9, p. 210] to the context of relations. Recall that for a closed relation A in a Hilbert space $\mathfrak{H}$ the subspaces $\ker A$ and $\operatorname{mul} A$ are closed.

LEMMA 4.15. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Let $m \in \mathbb{N}$, $m \geqslant 2$, and assume that*

$$\operatorname{mul} A^i \quad \text{is closed for} \quad i = 2, \dots, m.$$

Then for each $x \in \ker A^m$ there exist unique vectors $x_1, \dots, x_{m-1} \in \mathfrak{H}$, such that

$$\{x, x_1\}, \{x_i, x_{i+1}\}, \{x_{m-1}, 0\} \in A, \quad 1 \leqslant i \leqslant m-2,$$

and such that

$$x_i \perp \operatorname{mul} A^i \cap \ker A, \quad 1 \leqslant i \leqslant m-1.$$

In particular, the subspace $\ker A^m$ provided with the norm

$$\|x\|_m = \left(\|x\|^2 + \sum_{i=1}^{m-1} \|x_i\|^2 \right)^{\frac{1}{2}}, \quad x \in \ker A^m,$$

is a range subspace of the Hilbert space $\mathfrak{H}$.

Proof. For $x \in \ker A^m$ there exists a sequence $y_1, \dots, y_{m-1} \in \mathfrak{H}$ such that $\{x, y_1\}$, $\{y_i, y_{i+1}\}$, $1 \leqslant i \leqslant m-2$, and $\{y_{m-1}, 0\}$ belong to A. Let z_{m-1} be the orthogonal projection of y_{m-1} into $\operatorname{mul} A^{m-1} \cap \ker A$. Then there exists a sequence of vectors $z_1, \dots, z_{m-2}$ such that $\{0, z_1\}$, $\{z_i, z_{i+1}\}$, $i = 1, \dots, m-2$, and $\{z_{m-1}, 0\}$ belong to A. It follows that $\{x, y_1 - z_1\}$, $\{y_i - z_i, y_{i+1} - z_{i+1}\}$ for $1 \leqslant i \leqslant m-2$, and $\{y_{m-1} - z_{m-1}, 0\}$ belong to A. Let $x_{m-1} = y_{m-1} - z_{m-1}$, then $x_{m-1} \perp \operatorname{mul} A^{m-1} \cap \ker A$. To continue in an inductive way, assume that for some $n < m-1$ the vectors $\{x, y_1\}$, $\{y_i, y_{i+1}\}$ for $1 \leqslant i \leqslant n-1$, and $\{y_n, x_{n+1}\}$ belong to A with $x_{n+1} \perp \operatorname{mul} A^{n+1} \cap \ker A$. If z_n is the orthogonal projection of y_n into $\operatorname{mul} A^n \cap \ker A$, then $x_n = y_n - z_n$ is orthogonal to $\operatorname{mul} A^n \cap \ker A$, and there exist as above vectors $\{x, y'_1\}$, $\{y'_i, y'_{i+1}\}$ for $1 \leqslant i \leqslant n-2$, $\{y'_{n-1}, x_n\}$ and $\{x_n, x_{n+1}\}$ belonging to A.

To show the uniqueness of the sequence $x_1, \dots, x_{m-1}$, assume that the elements $\{x, x_1\}$, $\{x_i, x_{i+1}\}$, $\{x_{m-1}, 0\}$ and $\{x, x'_1\}$, $\{x'_i, x'_{i+1}\}$, $\{x'_{m-1}, 0\}$ belong to A, and both x_i and x'_i are orthogonal to $\operatorname{mul} A^i \cap \ker A$. Then $\{0, x_1 - x'_1\}$, $\{x_i - x'_i, x_{i+1} - x'_{i+1}\}$, $\{x_{m-1} - x'_{m-1}, 0\}$ belong to A, and thus $x_{m-1} - x'_{m-1} \in \operatorname{mul} A^{m-1} \cap \ker A$. Hence, $x_{m-1} = x'_{m-1}$. The continuation of this argument shows that $x_i = x'_i$ for $1 \leqslant i \leqslant m-1$.

To see that $\ker A^m$ is complete with respect to the norm $\|\cdot\|_m$, note that A is closed, and if (x^n) is a Cauchy sequence in $\ker A^m$ with respect to $\|\cdot\|_m$ then $(\{x_i^n, x_{i+1}^n\}$ is a Cauchy sequence with respect to the graph norm. The inequality $\|\cdot\|_m \geqslant \|\cdot\|$ shows that $\ker A^m$ is a range subspace of $\mathfrak{H}$. □

The following proposition gives sufficient conditions so that subspaces of the form $\operatorname{ran} A^n + \ker A^m$ are closed.

PROPOSITION 4.16. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Assume that*

(i) $d \in \Delta(A)$;

(ii) $\operatorname{ran} A + \ker A^d$ *is closed in* $\mathfrak{H}$;

(iii) $\operatorname{mul} A^n$ *is closed for all* $n \in \mathbb{N}_0$.

Then $\operatorname{ran} A^n + \ker A^m$ *is closed for all* $m, n \in \mathbb{N}_0$ *such that* $m + n \geqslant d$.

Proof. The proof involves range subspace arguments. It will be given in three steps for different ranges of the values of the parameters m and n.

Step 1. In this step it will be shown that

$$\operatorname{ran} A^n + \ker A^d \quad \text{is closed.} \tag{4.8}$$

The proof is done by induction. If $n = 0$ then $\operatorname{ran} A^0 = \mathfrak{H}$ and the result is obvious, and for $n = 1$ it holds by assumption. Now assume that $\operatorname{ran} A^n + \ker A^d$ is closed in $\mathfrak{H}$ for some $n \in \mathbb{N}_0$. Define the subspace $\mathfrak{M}$ of $\operatorname{dom} A^n$ by

$$\mathfrak{M} = \operatorname{dom} A^n \cap [\operatorname{ran} A + \ker A^d].$$

Since A is closed, it is a range space relation and it follows that $\operatorname{dom} A^n$ is a range subspace of $\mathfrak{H}$. Furthermore, $\mathfrak{M}$ is a closed subspace of $\operatorname{dom} A^n$ endowed with its range space topology; cf. Corollary 3.6. Let $\mathfrak{M}'$ be the orthogonal complement of $\mathfrak{M}$ in $\operatorname{dom} A^n$ in the range space topology, so that

$$\operatorname{dom} A^n = \mathfrak{M} \, [+] \, \mathfrak{M}'.$$

Introduce the domain restriction B of A^n by

$$B = A^n \cap (\mathfrak{M}' \times \mathfrak{H}).$$

Clearly B is a range space relation, so that also $\operatorname{ran} B$ is a range subspace of $\mathfrak{H}$. Observe that $\operatorname{ran} A^{n+1}$ and $\ker A^d$ are also range subspaces. It will be proved that the range subspaces $\operatorname{ran} A^{n+1} + \ker A^d$ and $\operatorname{ran} B$ are closed by showing that they satisfy the conditions of Proposition 3.10.

First it will be shown that

$$\operatorname{ran} A^{n+1} + \ker A^d + \operatorname{ran} B = \operatorname{ran} A^n + \ker A^d. \tag{4.9}$$

Clearly, the lefthand side of (4.9) is included in the lefthand side. For the converse inclusion, let $u \in \operatorname{ran} A^n$, so that $\{v, u\} \in A^n$ for some $v \in \operatorname{dom} A^n$. Then $v = w + z$ for some $w \in \mathfrak{M}'$ and some $z \in \mathfrak{M}$, so that $z \in \operatorname{dom} A^n$ and $z = z_1 + z_2$ for some $z_1 \in \operatorname{ran} A$ and $z_2 \in \ker A^d$. Since $\ker A^d \subset \operatorname{dom} A^n$, cf. (2.6), it follows that $z_1 \in \operatorname{ran} A \cap \operatorname{dom} A^n$. Clearly, $z_2 \in \ker A^{d+n}$. Hence there exist $s, t \in \mathfrak{H}$ such that $\{z_1, s\} \in A^n$, and $\{z_2, t\} \in A^n$, so that $s \in \operatorname{ran} A^{n+1}$ and $t \in \ker A^d$. Therefore

$$\{z, s + t\} = \{z_1, s\} + \{z_2, t\} \in A^n,$$

and since $\{w+z,u\} = \{v,u\} \in A^n$, this implies that $\{w,u-s-t\} \in A^n$ or in other words that $u-s-t \in \operatorname{ran} B$. Consequently, $u \in \operatorname{ran} A^{n+1} + \ker A^d + \operatorname{ran} B$, so that (4.9) is proved.

Next it will be shown that

$$[\operatorname{ran} A^{n+1} + \ker A^d] \cap \operatorname{ran} B = \operatorname{mul} A^n. \tag{4.10}$$

Since $\operatorname{mul} A^n \subset \operatorname{ran} A^{n+1}$ and $\operatorname{mul} A^n \subset \operatorname{ran} B$ it follows that the righthand side of (4.10) is included in its lefthand side. Conversely, let $u \in [\operatorname{ran} A^{n+1} + \ker A^d] \cap \operatorname{ran} B$, so that $\{v,u\} \in A^n$ for some $v \in \mathfrak{M}' \subset \operatorname{dom} A^n$. Furthermore, $u = w+y$ with $w \in \operatorname{ran} A^{n+1}$ and $y \in \ker A^d$. Hence, there exists $w' \in \operatorname{dom} A^{n+1}$ such that $\{w',w\} \in A^{n+1}$. Since $u \in \operatorname{ran} A^n$ and $w \in \operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n$, it follows that $y = u - w \in \operatorname{ran} A^n \cap \ker A^d$. Therefore $\{z,y\} \in A^n$ with $z \in \ker A^{n+d}$. Also note that there is an element w'' such that $\{w',w''\} \in A$ and $\{w'',w\} \in A^n$; i.e., $w'' \in \operatorname{ran} A \cap \operatorname{dom} A^n$. Then

$$\{v-z-w'',0\} = \{v,w+y\} - \{z,y\} - \{w'',w\} \in A^n,$$

so that $x = v-z-w'' \in \ker A^n$. Thus,

$$v = z + w'' + x \in \ker A^{n+d} + \operatorname{ran} A + \ker A^n = \operatorname{ran} A + \ker A^{n+d},$$

so that $v \in \operatorname{ran} A \cap \ker A^d$ by (ii) of Lemma 2.5. Together with $v \in \operatorname{dom} A^n$ this shows that $v \in \mathfrak{M}$. Therefore, $v \in \mathfrak{M} \cap \mathfrak{M}' = \{0\}$ which implies that $v = 0$ and hence that $u \in \operatorname{mul} A^n$, so that (4.10) is proved.

According to the induction hypothesis $\operatorname{ran} A^n + \ker A^d$ is closed in $\mathfrak{H}$ and according to assumption (ii) $\operatorname{mul} A^n$ is closed in $\mathfrak{H}$. Hence (4.9) and (4.10) together with Proposition 3.10 show that $\operatorname{ran} A^{n+1} + \ker A^d$ and $\operatorname{ran} B$ are closed in $\mathfrak{H}$. In particular, this completes the induction argument.

Step 2. The present step is concerned with a first simple extension of the range of values for which $\operatorname{ran} A^n + \ker A^d$ is closed. It is based on the identity

$$\operatorname{ran} A^n + \ker A^d = \operatorname{ran} A^n + \ker A^{d+m}, \quad n,m \in \mathbb{N}_0, \quad d \in \Delta(A), \tag{4.11}$$

cf. Corollary 2.6. Hence (4.8) and (4.11) show that

$$\operatorname{ran} A^n + \ker A^{d+m} \quad \text{is closed,} \qquad n,m \in \mathbb{N},$$

in other words for $m \geqslant d$ one has that

$$\operatorname{ran} A^n + \ker A^m \quad \text{is closed.} \tag{4.12}$$

Step 3. Assume now that the statement in (4.12) is valid for n and m, $n+m \geqslant d$. Then the statement will be shown for $n+1$ and $m-1$ if $1 \leqslant m \leqslant d$. For this purpose the space $\ker A^m$ will be provided with the inner product in Lemma 4.15, so that $\ker A^m$ is a range subspace of $\mathfrak{H}$.

Define the following subspace of $\ker A^m$:

$$\mathfrak{M} = [\operatorname{ran} A^{n+1} + \ker A^{m-1}] \cap \ker A^m. \tag{4.13}$$

It is trivial to check that (4.13) leads to

$$\mathfrak{M} = [\operatorname{ran} A^{n+1} \cap \ker A^m] + \ker A^{m-1}. \tag{4.14}$$

Now let $u \in \ker A^m$, and let $u_1, \dots, u_{m-1}$ be the unique sequence in Lemma 4.15 belonging to u. It will be shown that

$$u \in \mathfrak{M} \quad \Longleftrightarrow \quad u_{m-1} \in \operatorname{ran} A^{m+n} \cap \ker A. \tag{4.15}$$

Observe that u belongs to $\mathfrak{M}$ if and only if $u = v + w$ with $v \in \operatorname{ran} A^{n+1} \cap \ker A^m$ and $w \in \ker A^{m-1}$; cf. (4.14). Let $v_1, \dots, v_{m-1}$ and $w_1, \dots, w_{m-2}$ be the unique sequences in Lemma 4.15 belonging to v and w; furthermore put $w_{m-1} = 0$. Then Lemma 4.15 implies that $u_i = v_i + w_i$ for $1 \leqslant i \leqslant m-2$ and $u_{m-1} = v_{m-1}$. It follows that $u_{m-1} \in \operatorname{ran} A^{m+n} \cap \ker A$. Conversely, let $u \in \ker A^m$ be such that $u_{m-1} \in \operatorname{ran} A^{m+n} \cap \ker A$. Then there exists some $v \in \operatorname{ran} A^{n+1}$ such that $\{v, u_{m-1}\} \in A^{m-1}$ and $v \in \ker A^m$. Note that by construction $\{u, u_{m-1}\} \in A^{m-1}$, so that $\{u - v, 0\} \in A^{m-1}$. Let $w = u - v$, then $u = v + w$ shows that $u \in \mathfrak{M}$, as $v \in \operatorname{ran} A^{n+1} \cap \ker A^m$ and $w \in \ker A^{m-1}$; cf. (4.14).

The space $\mathfrak{M}$ is closed in the topology of $\ker A^m$. In order to prove this, let (x^n) be a sequence from $\mathfrak{M}$ such that x^n converges to $x \in \ker A^m$ in the topology of $\ker A^m$. Decompose each of these elements as in Lemma 4.15. In particular, it follows that $x^n_{m-1} \to x_{m-1}$ in $\mathfrak{H}$, and as $x^n_{m-1} \in \operatorname{ran} A^{m+n} \cap \ker A$ (see (4.15)), and $\operatorname{ran} A^{m+n} \cap \ker A$ is closed in $\mathfrak{H}$ because of $m + n \geqslant d$, one finds that $x_{m-1} \in \operatorname{ran} A^{m+n} \cap \ker A$. This implies that $x \in \mathfrak{M}$ (see (4.15)).

Since the space $\mathfrak{M}$ is closed in the topology of $\ker A^m$, there is the following orthogonal decomposition

$$\ker A^m = \mathfrak{M} \, [+] \, \mathfrak{M}', \tag{4.16}$$

where $\mathfrak{M}'$ is the orthogonal complement of $\mathfrak{M}$ in $\ker A^m$, with respect to its own inner product.

The proof of the statement for $n+1$ and $m-1$ if $1 \leqslant m \leqslant d$ is now based on Proposition 3.10, once two identities have been shown. First observe that the following identity

$$\operatorname{ran} A^{n+1} + \ker A^{m-1} + \mathfrak{M}' = \operatorname{ran} A^{n+1} + \ker A^m \tag{4.17}$$

is satisfied. Clearly, the lefthand side of (4.17) is included in the righthand side of (4.17). For the converse inclusion, it suffices to show that $\ker A^m$ is included in the lefthand side. Let $u \in \ker A^m$, so that by the decomposition (4.16) $u = v + w$ for some $v \in \mathfrak{M}$ and for some $w \in \mathfrak{M}'$. It follows from (4.14) that $\mathfrak{M} \subset \operatorname{ran} A^{n+1} + \ker A^{m-1}$, from which the desired inclusion follows. Next observe that

$$[\operatorname{ran} A^{n+1} + \ker A^{m-1}] \cap \mathfrak{M}' \subset [\operatorname{ran} A^{n+1} + \ker A^{m-1}] \cap \ker A^m = \mathfrak{M},$$

so that the following identity is clear:

$$[\operatorname{ran} A^{n+1} + \ker A^{m-1}] \cap \mathfrak{M}' \subset \mathfrak{M} \cap \mathfrak{M}' = \{0\}. \tag{4.18}$$

The induction hypothesis implies that the righthand side of (4.17) is closed. It follows from the identities (4.17) and (4.18), and Proposition 3.10 that $\operatorname{ran} A^{n+1} + \ker A^{m-1}$ and $\mathfrak{M}'$ are closed in $\mathfrak{H}$. This completes the induction argument. □

REMARK 4.17. Lemma 4.15 and Proposition 4.16 were stated for the case of closed relations A. This required some assumptions on the closedness of $\operatorname{mul} A^j$. These requirements are of course superfluous in the case of operators, cf. [9].

5. Quasi-Fredholm relations

The following definition of quasi-Fredholm relations is formulated for range space relations. It will be shown in Theorem 5.2 below that any quasi-Fredholm relation is automatically closed.

DEFINITION 5.1. A relation A in a Hilbert space $\mathfrak{H}$ is said to be quasi-Fredholm if A is a range space relation and if there exists an integer $d \in \mathbb{N}_0$ for which:

(Q1) $\delta(A) = d$;

(Q2) $\operatorname{ran} A^d \cap \ker A$ is closed in $\mathfrak{H}$;

(Q3) $\operatorname{ran} A + \ker A^d$ is closed in $\mathfrak{H}$.

In this case the relation A is said to be a quasi-Fredholm relation of degree d.

A range space relation A in a Hilbert space $\mathfrak{H}$ is a quasi-Fredholm relation of degree 0 if and only if the following conditions are satisfied:

(N1) $\ker A \subset \operatorname{ran} A^m$, $m \in \mathbb{N}$;

(N2) $\ker A$ is closed in $\mathfrak{H}$;

(N3) $\operatorname{ran} A$ is closed in $\mathfrak{H}$;

cf. Lemma 2.7. In fact, the condition in (i) can be replaced by any of the equivalent conditions (iii) or (iv) in Lemma 2.7. It is an immediate consequence of Lemma 4.6 that a quasi-Fredholm relation of degree 0 is closed.

The following decomposition result goes back to Kato [7] for semi-Fredholm operators in the setting of pairs of operators; it was extended to the case of quasi-Fredholm operators by Labrousse [9].

THEOREM 5.2. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$, which is quasi-Fredholm of degree $d \in \mathbb{N}_0$. Then A is closed and there exist two closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ such that:*

(i) $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$;

(ii) $\operatorname{ran} A^d \subset \mathfrak{M}$;

(iii) $\mathfrak{N} \subset \ker A^d$ *and, if* $d \geqslant 1$, $\mathfrak{N} \not\subset \ker A^{d-1}$;

(iv) $A = A_{\mathfrak{M}\times\mathfrak{M}} \,\widehat{+}\, A_{\mathfrak{N}\times\mathfrak{N}}$, *direct sum;*

(v) $A_{\mathfrak{M}\times\mathfrak{M}}$ *is a quasi-Fredholm relation of degree* 0 *with* $A_{\mathfrak{M}\times\mathfrak{M}} = A_{\mathfrak{M}\times\mathfrak{H}}$ *and, in particular,* $\operatorname{mul} A_{\mathfrak{M}\times\mathfrak{M}} = \operatorname{mul} A$*;*

(vi) $A_{\mathfrak{N}\times\mathfrak{N}}$ *is a bounded operator defined on all of* $\mathfrak{N}$ *with* $(A_{\mathfrak{N}\times\mathfrak{N}})^d = \mathfrak{N}\times\{0\}$.

Proof. Observe that for $d=0$ one can choose $\mathfrak{M}=\mathfrak{H}$ and $\mathfrak{N}=\{0\}$. Hence in the rest of the proof it is assumed that $d\in\mathbb{N}$.

The proof consists of four steps. In the first step the range subspace $\mathfrak{M}$ of $\mathfrak{H}$ will be constructed inductively. In the second step the range subspace $\mathfrak{N}$ of $\mathfrak{H}$ will be constructed inductively. In the third step it will be shown that $\mathfrak{H}$ has a decomposition in terms of $\mathfrak{M}$ and $\mathfrak{N}$. The last step is to show that A is closed.

Step 1. Construction of the space $\mathfrak{M}$. Define the following sequence of subspaces $\mathfrak{M}_j$ of $\mathfrak{H}$ for $j\in\mathbb{N}_0$:

$$\mathfrak{M}_0=\mathfrak{H}, \quad \mathfrak{M}_{j+1} = (\operatorname{ran} A + \ker A^d)^\perp + \left\{ v : \{u,v\}\in A,\, u\in\mathfrak{M}_j \right\}. \tag{5.1}$$

These spaces $\mathfrak{M}_j$ are all range subspaces in $\mathfrak{H}$. For $j=0$ this is clear. Assume that $\mathfrak{M}_j$ is a range subspace for $j=m$. Then $A\cap(\mathfrak{M}_j\times\mathfrak{H})$ is a range relation and its range is a range subspace; cf. Lemma 4.10. Furthermore, $(\operatorname{ran} A+\ker A^d)^\perp$ is a closed subspace and, hence, a range subspace. Since the sum of two range subspaces is again a range subspace (see Proposition 3.9), it follows from (5.1) that $\mathfrak{M}_j$ is a range subspace for $j=m+1$.

These linear subspaces $\mathfrak{M}_j$ form a decreasing sequence:

$$\mathfrak{M}_j \subset \mathfrak{M}_{j-1}, \quad j\in\mathbb{N}. \tag{5.2}$$

The case $j=1$ is clear. Assume now that the inclusion is true for $j=m$; it will be proved for $j=m+1$. Let $x\in\mathfrak{M}_{m+1}$, then $x=w+v$, where $w\in(\operatorname{ran} A+\ker A^d)^\perp$ and $\{u,v\}\in A$ for some $u\in\mathfrak{M}_m$. By induction hypothesis $u\in\mathfrak{M}_{m-1}$, but then $x\in\mathfrak{M}_m$. Hence (5.2) is proved.

Furthermore, the linear subspaces $\mathfrak{M}_j$ satisfy the inclusions:

$$\operatorname{ran} A^j \subset \left\{ v : \{u,v\}\in A,\, u\in\mathfrak{M}_{j-1} \right\} \subset \mathfrak{M}_j, \quad j\in\mathbb{N}. \tag{5.3}$$

The second inclusion is clear from (5.1). It suffices to prove the first inclusion. The case $j=1$ is clear. Now assume that the first inclusion holds for $j=m$. Let $v\in\operatorname{ran} A^{m+1}$ so that $\{w,v\}\in A^{m+1}$ for some $w\in\mathfrak{H}$. Then there exists an element u such that $\{w,u\}\in A^m$ and $\{u,v\}\in A$. By the induction hypothesis $u\in\mathfrak{M}_m$. Hence the first inclusion in (5.3) has been proved for $j=m+1$.

The linear subspaces $\mathfrak{M}_j$ satisfy the following stability property:

$$\left\{ v : \{u,v\}\in A,\, u\in\mathfrak{M}_j \right\} + \ker A^d = \operatorname{ran} A + \ker A^d, \quad j\in\mathbb{N}. \tag{5.4}$$

Clearly, the lefthand side is contained in the righthand side, so it suffices to show that

$$\operatorname{ran} A \subset \operatorname{ran} A_{\mathfrak{M}_j\times\mathfrak{H}} + \ker A^d, \quad j\in\mathbb{N}. \tag{5.5}$$

The case $j=0$ is obvious. Assume that (5.5) holds true for $j=m$ and let $v \in \operatorname{ran} A$. Therefore $\{u,v\} \in A$ for some $u \in \operatorname{dom} A$. The assumption (Q3) leads to the following orthogonal decomposition

$$u = x+y, \quad x \in (\operatorname{ran} A + \ker A^d)^\perp, \quad y \in \operatorname{ran} A + \ker A^d.$$

Note that $y = t+s$ with $t \in \operatorname{ran} A$ and $s \in \ker A^d$. It follows from the induction hypothesis (5.5) with $j=m$ that $t \in \operatorname{ran} A_{\mathfrak{M}_m \times \mathfrak{H}}$. Define $w = x+t$. Then $w \in \mathfrak{M}_{m+1}$ and

$$w = u - s \in \operatorname{dom} A + \ker A^d \subset \operatorname{dom} A,$$

so that $\{w,w'\} \in A$ for some $w' \in \mathfrak{H}$. In fact, $w' \in \operatorname{ran} A_{\mathfrak{M}_{m+1} \times \mathfrak{H}}$. Since $\{w,w'\} \in A$ and $\{u,v\} \in A$ it follows that

$$\{s, v-w'\} = \{u,v\} - \{w,w'\} \in A.$$

Furthermore, $s \in \ker A^d$ implies that $\{s,t'\} \in A$ for some $t' \in \ker A^{d-1} \subset \ker A^d$. Now,

$$\{0, v-w'-t'\} = \{s, v-w'\} - \{s,t'\} \in A$$

shows that $v - w' - t' \in \operatorname{mul} A$, and hence,

$$v \in \operatorname{ran} A_{\mathfrak{M}_{m+1}} + \ker A^d + \operatorname{mul} A = \operatorname{ran} A_{\mathfrak{M}_{m+1}} + \ker A^d,$$

which leads to (5.5).

The space $\mathfrak{H}$ allows the following decomposition:

$$\mathfrak{M}_j + \ker A^d = \mathfrak{H}, \quad j \in \mathbb{N}_0. \tag{5.6}$$

The statement (5.6) is clear for $j=0$. It follows from the assumption (Q3) and the identity (5.4) that:

$$\begin{aligned}\mathfrak{H} &= (\operatorname{ran} A + \ker A^d)^\perp + (\operatorname{ran} A + \ker A^d) \\ &= (\operatorname{ran} A + \ker A^d)^\perp + \left\{ v : \{u,v\} \in A, u \in \mathfrak{M}_j \right\} + \ker A^d \\ &= \mathfrak{M}_{j+1} + \ker A^d, \quad j \in \mathbb{N},\end{aligned}$$

which leads to (5.6) with $j \in \mathbb{N}$.

The decomposition in (5.6) is not direct in general. In fact one has the following identity:

$$\mathfrak{M}_j \cap \ker A^k = \operatorname{ran} A^j \cap \ker A^k, \quad j,k \in \mathbb{N}_0. \tag{5.7}$$

Indeed, $\operatorname{ran} A^j \cap \ker A^k \subset \mathfrak{M}_j \cap \ker A^k$ is obvious for $j=0$ and follows from (5.3) for $j \in \mathbb{N}$ and any $k \in \mathbb{N}_0$. The converse inclusion,

$$\mathfrak{M}_j \cap \ker A^k \subset \operatorname{ran} A^j \cap \ker A^k, \quad j,k \in \mathbb{N}_0. \tag{5.8}$$

will be proved by induction. The case $j=0$ is clear for all $k \in \mathbb{N}_0$. Now assume that (5.8) holds for $j=m$ and for all k. Let $u \in \mathfrak{M}_{m+1} \cap \ker A^k$ so that $u = w + v'$ for some

$w \in (\operatorname{ran} A + \ker A^d)^\perp$ and $\{v, v'\} \in A$ with $v \in \mathfrak{M}_m$. Using Lemma 2.5 (ii), it follows that

$$w = u - v' \in \operatorname{ran} A + \ker A^k \subset \operatorname{ran} A + \ker A^d,$$

so that

$$w \in (\operatorname{ran} A + \ker A^d)^\perp \cap (\operatorname{ran} A + \ker A^d),$$

which shows that $w = 0$. Therefore $v' = u$, and since $u \in \ker A^k$, it follows that $v \in \ker A^{k+1}$. Now $v \in \mathfrak{M}_m$ and the assumption (5.8) shows that $v \in \operatorname{ran} A^m \cap \operatorname{ran} A^{k+1}$. In particular, it follows that $u \in \operatorname{ran} A^{m+1}$. Hence (5.8) is valid for $j = m+1$ and all k.

The sequence of linear subspaces $\mathfrak{M}_j$ is eventually stable:

$$\mathfrak{M}_j = \mathfrak{M}_d, \quad j \geqslant d. \tag{5.9}$$

Indeed, it follows from (5.2) that $\mathfrak{M}_j \subset \mathfrak{M}_d$, $j \geqslant d$. Hence, it suffices to show that

$$\mathfrak{M}_d \subset \mathfrak{M}_j, \quad j \geqslant d. \tag{5.10}$$

Let $u \in \mathfrak{M}_d$. The decomposition (5.6) implies that

$$u = v + w, \quad v \in \mathfrak{M}_j, \quad w \in \ker A^d.$$

Observe that $v \in \mathfrak{M}_j \subset \mathfrak{M}_d$. Hence, (5.7) and Lemma 2.5 (iii) imply that

$$w = u - v \in \mathfrak{M}_d \cap \ker A^d = \operatorname{ran} A^d \cap \ker A^d \subset \operatorname{ran} A^j \cap \ker A^d.$$

Therefore $w \in \operatorname{ran} A^j$ and, by (5.3), $w \in \mathfrak{M}_j$, so that $u \in \mathfrak{M}_j$. Hence (5.10) is now proved and therefore (5.9) has been shown.

The eventual stability of the sequence $\mathfrak{M}_j$ leads to the following definition:

$$\mathfrak{M} = \mathfrak{M}_d, \tag{5.11}$$

and the above construction shows that $\mathfrak{M}$ is a range subspace in $\mathfrak{H}$. The definitions (5.1) and (5.11) show that

$$\mathfrak{M} = (\operatorname{ran} A + \ker A^d)^\perp + \{\, v : \{u, v\} \in A,\, u \in \mathfrak{M} \,\}. \tag{5.12}$$

It follows from (5.1) and (5.3) that

$$\operatorname{ran} A^d \subset \mathfrak{M}, \quad \operatorname{ran} A_{\mathfrak{M} \times \mathfrak{H}} \subset \mathfrak{M}, \tag{5.13}$$

and from (5.4) that

$$\operatorname{ran} A_{\mathfrak{M} \times \mathfrak{H}} + \ker A^d = \operatorname{ran} A + \ker A^d. \tag{5.14}$$

Furthermore, it follows from (5.6) that

$$\mathfrak{M} + \ker A^d = \mathfrak{H}. \tag{5.15}$$

Finally, it follows from (5.7) that

$$\mathfrak{M} \cap \ker A^k = \operatorname{ran} A^d \cap \ker A^k. \tag{5.16}$$

Step 2. Construction of the space $\mathfrak{N}$. Define the following sequence of subspaces $\mathfrak{N}_j$ of $\mathfrak{H}$ for $j \in \mathbb{N}_0$:

$$\mathfrak{N}_0 = \{0\}, \quad \mathfrak{N}_{j+1} = \left\{ u \in (\operatorname{ran} A^d \cap \ker A)^\perp : \{u,v\} \in A, \, v \in \mathfrak{N}_j \right\}. \tag{5.17}$$

These spaces $\mathfrak{M}_j$ are all range subspaces in $\mathfrak{H}$. For $j = 0$ this is clear. Now assume that $\mathfrak{N}_j$ is a range subspace for $j = m$. The space $(\operatorname{ran} A^d \cap \ker A)^\perp$ is closed, and hence a range subspace. Therefore, also $(\operatorname{ran} A^d \cap \ker A)^\perp \times \mathfrak{N}_m$ is a range subspace by Proposition 3.9. Hence, $\mathfrak{N}_{m+1} = A \cap \left((\operatorname{ran} A^d \cap \ker A)^\perp \times \mathfrak{N}_m \right)$ is a range relation by Lemma 4.10.

The linear subspaces $\mathfrak{N}_j$ form an increasing sequence:

$$\mathfrak{N}_j \subset \mathfrak{N}_{j+1}, \quad j \in \mathbb{N}_0. \tag{5.18}$$

The inclusion is clearly true for $j = 0$. Assume now that the inclusion is true for $j = m$; it will be proved for $j = m+1$. Let $u \in \mathfrak{N}_{m+1}$, then $u \in (\operatorname{ran} A^d \cap \ker A)^\perp$ and there exists an element $v \in \mathfrak{N}_m$ such that $\{u,v\} \in A$. Since, by assumption $\mathfrak{N}_m \subset \mathfrak{N}_{m+1}$, it follows that $v \in \mathfrak{N}_{m+1}$, and, hence, $u \in \mathfrak{N}_{m+2}$. Thus (5.18) has been shown.

Furthermore, the linear subspaces $\mathfrak{N}_j$ satisfy the inclusions:

$$\mathfrak{N}_j \subset \ker A^j, \quad j \in \mathbb{N}_0. \tag{5.19}$$

This is clear for $j = 0$. Assume now that the inclusion is true for $j = m$; it will be proved for $j = m+1$. Let $u \in \mathfrak{N}_{m+1}$, so that there exists $v \in \mathfrak{N}_m$ such that $\{u,v\} \in A$, i.e., $\{u,v\} \in A$ and $\{v,0\} \in A^m$, since $\mathfrak{N}_m \subset \ker A^m$. Hence, $\{u,0\} \in A^{m+1}$. Thus (5.19) has been shown.

The linear subspaces $\mathfrak{N}_j$ satisfy the following stability property:

$$\ker A^j = \mathfrak{N}_j + (\operatorname{ran} A^d \cap \ker A^j), \quad j \in \mathbb{N}_0. \tag{5.20}$$

It follows from (5.19) that the righthand side is contained in the lefthand side. So it suffices to show that

$$\ker A^j \subset \mathfrak{N}_j + (\operatorname{ran} A^d \cap \ker A^j), \quad j \in \mathbb{N}_0. \tag{5.21}$$

For $j = 0$ this is clear. To see the case $j = 1$, let $u \in \ker A$. Then the assumption (Q2) leads to the following orthogonal decomposition:

$$u = u_1 + u_2, \quad u_1 \in (\operatorname{ran} A^d \cap \ker A)^\perp, \quad u_2 \in \operatorname{ran} A^d \cap \ker A.$$

Since $u, u_2 \in \ker A$, it follows that $u_1 \in \ker A$, which gives $u_1 \in \mathfrak{N}_1$ by definition, cf. (5.17). Hence, (5.21) has been shown for the case $j = 1$. Assume now that (5.21) is true for $j = m \geqslant 1$. Let $u \in \ker A^{m+1}$ so that $\{u,v\} \in A$ and $\{v,0\} \in A^m$ for some $v \in \mathfrak{H}$. Thus $v \in \ker A^m$ and by (5.21) with $j = m$ there is a decomposition

$$v = v_1 + v_2, \quad v_1 \in \mathfrak{N}_m, \quad v_2 \in \operatorname{ran} A^d \cap \ker A^m.$$

Clearly, $v_1 = v - v_2 \in \operatorname{ran} A$. Hence $\{u, v_1\} \in A$ for some $u \in \mathfrak{H}$. The assumption (Q2) leads to the following orthogonal decomposition:

$$u = u_1 + u_2, \quad u_1 \in (\operatorname{ran} A^d \cap \ker A)^\perp, \quad u_2 \in \operatorname{ran} A^d \cap \ker A.$$

Since $\{u_2, 0\} \in A$ it follows that $\{u_1, v_1\} \in A$. Now $v_1 \in \mathfrak{N}_m$ implies by (5.17) that $u_1 \in \mathfrak{N}_{m+1}$. Now $v_2 \in \operatorname{ran} A^d \cap \ker A^m$ and Lemma 2.5 (iii) lead to $v_2 \in \operatorname{ran} A^{d+1} \cap \ker A^m$. Because of $v_2 \in \operatorname{ran} A^{d+1}$ there is an element w such that $\{w, v_2\} \in A^{d+1}$ which implies the existence of an element u_2 such that $\{w, u_2\} \in A^d$ and $\{u_2, v_2\} \in A$. Moreover, $v_2 \in \ker A^m$ implies that $u_2 \in \ker A^{m+1}$, and hence $u_2 \in \operatorname{ran} A^d \cap \ker A^{m+1}$. Now the decomposition

$$\{u, v\} = \{u_1, v_1\} + \{u_2, v_2\} + \{u_3, 0\}, \quad u_3 = u - (u_1 + u_2),$$

and $\{u, v\}, \{u_1, v_1\}, \{u_2, v_2\} \in A$ show that $\{u_3, 0\} \in A$. Furthermore, (5.21) with $j = 1$, (5.18), and $\ker A \subset \ker A^{m+1}$, imply that

$$u_3 \in \ker A = \mathfrak{N}_1 + (\operatorname{ran} A^d \cap \ker A) \subset \mathfrak{N}_{m+1} + (\operatorname{ran} A^d \cap \ker A^{m+1}).$$

As $u_1 \in \mathfrak{N}_{m+1}$ and $u_2 \in \operatorname{ran} A^d \cap \ker A^{m+1}$, it follows that

$$u = u_1 + u_2 + u_3 \in \mathfrak{N}_{m+1} + (\operatorname{ran} A^d \cap \ker A^{m+1}).$$

Therefore (5.21) with $j = m+1$ has been proved.

The decomposition in (5.20) is direct:

$$\mathfrak{N}_j \cap (\operatorname{ran} A^d \cap \ker A^j) = \{0\}, \quad j \in \mathbb{N}_0. \tag{5.22}$$

The case $j = 0$ is obvious and so is the case $j = 1$. Proceed by induction on j and assume that (5.22) holds true for $j = m \geqslant 1$. Let $u \in \mathfrak{N}_{m+1} \cap (\operatorname{ran} A^d \cap \ker A^{m+1})$. Since $u \in \mathfrak{N}_{m+1}$ it follows that $\{u, v\} \in A$ for some $v \in \mathfrak{N}_m$. Furthermore, (5.19) implies that $v \in \ker A^m$. Since $u \in \operatorname{ran} A^d$, it is clear that $v \in \operatorname{ran} A^{d+1}$. Hence, $v \in \operatorname{ran} A^{d+1} \cap \ker A^m = \operatorname{ran} A^d \cap \ker A^m$ according to Lemma 2.5 (ii). Therefore,

$$v \in \mathfrak{N}_m \cap \operatorname{ran} A^d \cap \ker A^m = \{0\},$$

which implies that $u \in \ker A$. Since $u \in \mathfrak{N}_{m+1} \subset (\operatorname{ran} A^d \cap \ker A)^\perp$, it follows that

$$u \in (\operatorname{ran} A^d \cap \ker A) \cap (\operatorname{ran} A^d \cap \ker A)^\perp,$$

which implies that $u = 0$. Hence $\mathfrak{N}_{m+1} \cap (\operatorname{ran} A^d \cap \ker A^{m+1}) = \{0\}$, and (5.22) is proved. Note that (5.22) is equivalent to

$$\mathfrak{N}_j \cap \operatorname{ran} A^d = \{0\}, \quad j \in \mathbb{N}_0, \tag{5.23}$$

which follows from (5.19).

The sequence of linear subspaces $\mathfrak{N}_j$ is eventually stable:

$$\mathfrak{N}_j = \mathfrak{N}_d, \quad j \geqslant d. \tag{5.24}$$

The relation (5.18) implies that $\mathfrak{N}_d \subset \mathfrak{N}_j$. To show the converse inclusion, note that (5.19) and Lemma 2.5 (ii) lead to

$$\mathfrak{N}_j \subset \ker A^j \subset \ker A^d + \operatorname{ran} A^d,$$

and, since $\ker A^d \subset \ker A^j$, this in turn leads to

$$\mathfrak{N}_j \subset \ker A^d + (\operatorname{ran} A^d \cap \ker A^j).$$

Now substitute (5.20) with $j = d$ for the first term on the righthand side, so that

$$\mathfrak{N}_j \subset \mathfrak{N}_d + (\operatorname{ran} A^d \cap \ker A^d) + (\operatorname{ran} A^d \cap \ker A^j) \subset \mathfrak{N}_d + (\operatorname{ran} A^d \cap \ker A^j).$$

Since $\mathfrak{N}_d \subset \mathfrak{N}_j$ and $\mathfrak{N}_j \cap (\operatorname{ran} A^d \cap \ker A^j) = \{0\}$ (by (5.22)), this shows that $\mathfrak{N}_j \subset \mathfrak{N}_d$. Hence, (5.24) has been shown.

The eventual stability of the sequence $\mathfrak{N}_j$ leads to the following definition:

$$\mathfrak{N} = \mathfrak{N}_d, \tag{5.25}$$

and the above construction shows that $\mathfrak{N}$ is a range subspace in $\mathfrak{H}$. The definitions (5.17) and (5.25) show that

$$\mathfrak{N} = \{\, u \in (\operatorname{ran} A^d \cap \ker A)^\perp : \{u,v\} \in A,\ v \in \mathfrak{N} \,\}. \tag{5.26}$$

It follows from (5.19) that

$$\mathfrak{N} \subset \ker A^d, \tag{5.27}$$

and it follows from (5.23) that

$$\mathfrak{N} \cap \operatorname{ran} A^d = \{0\}. \tag{5.28}$$

Furthermore, (5.20) and (5.27) show that

$$\ker A^d + \operatorname{ran} A^d \subset \mathfrak{N} + \operatorname{ran} A^d \subset \ker A^d + \operatorname{ran} A^d,$$

and now (5.28) leads to the direct sum decomposition

$$\ker A^d + \operatorname{ran} A^d = \mathfrak{N} + \operatorname{ran} A^d, \quad \text{direct sum}. \tag{5.29}$$

Step 3. Decomposition by means of the spaces $\mathfrak{M}$ and $\mathfrak{N}$. The linear subspaces $\mathfrak{M}$ and $\mathfrak{N}$ have been defined in (5.11) and (5.25). Both $\mathfrak{M}$ and $\mathfrak{N}$ are range subspaces of $\mathfrak{H}$. The first inclusion in (5.13), (5.29), and (5.15) lead to

$$\begin{aligned}\mathfrak{M} + \mathfrak{N} &= \mathfrak{M} + \operatorname{ran} A^d + \mathfrak{N} \\ &= \mathfrak{M} + \operatorname{ran} A^d + \ker A^d \\ &= \mathfrak{M} + \ker A^d = \mathfrak{H}.\end{aligned}$$

Furthermore, (5.19), (5.16), and (5.28) imply that

$$\begin{aligned}\mathfrak{M}\cap\mathfrak{N} &= \mathfrak{M}\cap\ker A^d\cap\mathfrak{N}\\ &= \operatorname{ran}A^d\cap\ker A^d\cap\mathfrak{N}\\ &= \operatorname{ran}A^d\cap\mathfrak{N}=\{0\}.\end{aligned}$$

Hence, there is a direct sum decomposition $\mathfrak{H}=\mathfrak{M}+\mathfrak{N}$, $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, and the summands are range subspaces.

By Proposition 3.10 it follows that both $\mathfrak{M}$ and $\mathfrak{N}$ are closed subspaces of $\mathfrak{H}$. The inclusions $\operatorname{ran}A^d\subset\mathfrak{M}$ and $\mathfrak{N}\subset\ker A^d$ have been shown in (5.27) and the first identity in (5.13). Hence (i), (ii), and the first part of (iii) of the theorem have been demonstrated.

The second identity in (5.13) shows that $A_{\mathfrak{M}\times\mathfrak{H}}\subset A_{\mathfrak{M}\times\mathfrak{M}}$, and the reverse inclusion is obvious. Hence, the part of the statement (v) of the theorem concerning $A_{\mathfrak{M}\times\mathfrak{H}}=A_{\mathfrak{M}\times\mathfrak{M}}$ has been shown.

The identity (5.26) implies that $\operatorname{dom}A_{\mathfrak{N}\times\mathfrak{N}}=\mathfrak{N}$. It follows from (5.12) that $\operatorname{mul}A\subset\mathfrak{M}$. Hence $A_{\mathfrak{N}\times\mathfrak{N}}$ is an operator. The identity (5.28) implies that $\operatorname{ran}(A_{\mathfrak{N}\times\mathfrak{N}})^d$ $=\operatorname{ran}A^d\cap\mathfrak{M}=\{0\}$, that is

$$(A_{\mathfrak{N}\times\mathfrak{N}})^d=\mathfrak{N}\times\{0\}, \tag{5.30}$$

which completes the proof of (vi).

Next it will be shown that A satisfies the decomposition in (iv). To show the decomposition, it suffices to show that

$$A\subset A_{\mathfrak{M}\times\mathfrak{M}}\mathbin{\widehat{+}}A_{\mathfrak{N}\times\mathfrak{N}}, \tag{5.31}$$

as the reverse inclusion is obvious. Assume that $\{u,v\}\in A$. Then there is a unique decomposition $u=u_1+u_2$ with $u_1\in\mathfrak{M}$ and $u_2\in\mathfrak{N}$. Since $\mathfrak{N}=\operatorname{dom}A_{\mathfrak{N}\times\mathfrak{N}}\subset\operatorname{dom}A$, it follows that also $u_1=u-u_2\in\operatorname{dom}A$. Moreover, since $A_{\mathfrak{N}\times\mathfrak{N}}$ is an operator, there is a unique $v_2\in\mathfrak{N}$ such that $\{u_2,v_2\}\in A_{\mathfrak{N}\times\mathfrak{N}}$. Let $v_1=v-v_2$, then $\{u_1,v_1\}=\{u,v\}-\{u_2,v_2\}\in A_{\mathfrak{M}\times\mathfrak{H}}=A_{\mathfrak{M}\times\mathfrak{M}}$. Hence,

$$\{u,v\}=\{u_1,v_1\}+\{u_2,v_2\}\in A_{\mathfrak{M}\times\mathfrak{M}}\mathbin{\widehat{+}}A_{\mathfrak{N}\times\mathfrak{N}},$$

and (5.31) follows. To show that the sum in (iv) is direct, note that

$$A_{\mathfrak{M}\times\mathfrak{M}}\cap A_{\mathfrak{N}\times\mathfrak{N}}\subset(\mathfrak{M}\times\mathfrak{M})\cap(\mathfrak{N}\times\mathfrak{N})=\{0,0\}.$$

Hence, the decomposition in (iv) has been shown.

Now it will be shown that $A_{\mathfrak{M}\times\mathfrak{M}}$ is quasi-Fredholm of degree 0. Observe that it is an easy consequence of the decomposition (5.31) that

$$\mathfrak{M}\cap\operatorname{dom}A=\operatorname{dom}A_{\mathfrak{M}\times\mathfrak{H}},\quad \mathfrak{M}\cap\operatorname{ran}A=\operatorname{ran}A_{\mathfrak{M}\times\mathfrak{H}}. \tag{5.32}$$

The first identity is trivial. As to the second identity, note that the inclusion $\operatorname{ran}A_{\mathfrak{M}\times\mathfrak{H}}\subset\mathfrak{M}\cap\operatorname{ran}A$ is clear. To see the converse inclusion, let $v\in\mathfrak{M}\cap\operatorname{ran}A$. Hence there is an element u such that $\{u,v\}\in A$ and decompose $\{u,v\}$ according to (5.31):

$$\{u,v\}=\{u_1,v_1\}+\{u_2,v_2\},\quad \{u_1,v_1\}\in A_{\mathfrak{M}\times\mathfrak{M}},\quad \{u_2,v_2\}\in A_{\mathfrak{N}\times\mathfrak{N}}.$$

Since $v \in \mathfrak{M}$ and $v_1 \in \mathfrak{M}$ it follows that $v_2 = 0$. This implies that $v = v_1 \in \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}}$.

In particular, (5.32) gives the inclusion

$$\operatorname{ran} A^d \subset \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}}. \tag{5.33}$$

Note that $\operatorname{ran} A^d \subset \operatorname{ran} A$ and according to (5.13) $\operatorname{ran} A^d \subset \mathfrak{M}$. It follows that $\operatorname{ran} A^d \subset \mathfrak{M} \cap \operatorname{ran} A$, so that (5.32) leads to (5.33).

Hence with (5.33), (5.29), and (5.14) it follows that

$$\begin{aligned}\operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \mathfrak{N} &= \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \operatorname{ran} A^d + \mathfrak{N} \\ &= \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \operatorname{ran} A^d + \ker A^d \\ &= \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \ker A^d \\ &= \operatorname{ran} A + \ker A^d,\end{aligned} \tag{5.34}$$

which is closed by assumption (Q3). Furthermore, observe that

$$\operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} \cap \mathfrak{N} \subset \mathfrak{M} \cap \mathfrak{N} = \{0\}. \tag{5.35}$$

It follows from (5.34), (5.35), and Proposition 3.10 that

$$\operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} \quad \text{is closed.} \tag{5.36}$$

Taking $k = 1$ in (5.16) it follows that

$$\ker A_{\mathfrak{M}\times\mathfrak{H}} = \ker A \cap \mathfrak{M} = \ker A \cap \operatorname{ran} A^d. \tag{5.37}$$

Hence, it follows from assumption (Q2) and (5.37) that

$$\ker A_{\mathfrak{M}\times\mathfrak{H}} \quad \text{is closed.} \tag{5.38}$$

Furthermore, (5.37) leads to

$$\ker A_{\mathfrak{M}\times\mathfrak{H}} = \ker A \cap \operatorname{ran} A^j \subset \operatorname{ran} A^j, \quad j \geqslant d. \tag{5.39}$$

The decomposition result in (iv) remains valid for powers of the relation A:

$$A^j = (A_{\mathfrak{M}\times\mathfrak{M}})^j \,\widehat{+}\, (A_{\mathfrak{N}\times\mathfrak{N}})^j, \quad \text{direct sum}, \quad j \geqslant 1, \tag{5.40}$$

cf. Corollary 2.2.

From (5.30) and (5.40) it follows that

$$\operatorname{ran} (A_{\mathfrak{M}\times\mathfrak{M}})^j = \operatorname{ran} A^j, \quad j \geqslant d. \tag{5.41}$$

Now, (5.39) and (5.41) imply that

$$\ker A_{\mathfrak{M}\times\mathfrak{H}} \subset \operatorname{ran} A^j_{\mathfrak{M}\times\mathfrak{M}}, \quad j \geqslant d. \tag{5.42}$$

Due to the shrinking of the powers of the ranges (see (2.5)) it follows from (5.36), (5.38), and (5.42) that $A_{\mathfrak{M}\times\mathfrak{M}}$ is a quasi-Fredholm relation of degree 0. This completes the proof of (iv).

It remains to show that $\mathfrak{N} \not\subset \ker A^{d-1}$. To see this, assume that $\mathfrak{N} \subset \ker A^{d-1}$. It will be proved that this implies that

$$\operatorname{ran} A^{d-1} \cap \ker A \subset \operatorname{ran} A^{d} \cap \ker A, \tag{5.43}$$

so that there is actually equality. This leads to $d-1 \in \Delta(A)$, which is a contradiction, since A is a quasi-Fredholm relation of degree d. In order to demonstrate (5.43), let $y \in \operatorname{ran} A^{d-1} \cap \ker A$, so that $\{x,y\} \in A^{d-1}$ and $\{y,0\} \in A$. According to the decomposition (5.40) with $j = d-1$, there exist elements

$$\{x_{\mathfrak{M}}, y_{\mathfrak{M}}\} \in (A_{\mathfrak{M}\times\mathfrak{M}})^{d-1}, \quad \{x_{\mathfrak{N}}, y_{\mathfrak{N}}\} \in (A_{\mathfrak{N}\times\mathfrak{N}})^{d-1},$$

such that

$$\{x,y\} = \{x_{\mathfrak{M}}, y_{\mathfrak{M}}\} + \{x_{\mathfrak{N}}, y_{\mathfrak{N}}\}. \tag{5.44}$$

The fact that $\{y,0\} \in A$ implies that

$$\{y_{\mathfrak{M}}, -z\} \in A_{\mathfrak{M}\times\mathfrak{M}}, \quad \{y_{\mathfrak{N}}, z\} \in A_{\mathfrak{N}\times\mathfrak{N}},$$

for some $z \in \mathfrak{H}$. Since $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ it follows that $z = 0$. Therefore, the elements in the decomposition (5.44) satisfy

$$\{y_{\mathfrak{M}}, 0\} \in A_{\mathfrak{M}\times\mathfrak{M}}, \quad \{y_{\mathfrak{N}}, 0\} \in A_{\mathfrak{N}\times\mathfrak{N}}. \tag{5.45}$$

Since $x_{\mathfrak{N}} \in \mathfrak{N}$, the assumption $\mathfrak{N} \subset \ker A^{d-1}$ implies that $\{x_{\mathfrak{N}}, 0\} \in A^{d-1}$, so that, in fact, according to the decomposition (5.40) with $j = d-1$:

$$\{x_{\mathfrak{N}}, 0\} \in (A_{\mathfrak{N}\times\mathfrak{N}})^{d-1}. \tag{5.46}$$

Now (5.44) and (5.46) show that

$$\{0, y_{\mathfrak{N}}\} \in (A_{\mathfrak{N}\times\mathfrak{N}})^{d-1}. \tag{5.47}$$

Since $A_{\mathfrak{N}\times\mathfrak{N}}$ is an operator, (5.47) implies that $y_{\mathfrak{N}} = 0$. Thus, by (5.44) and (5.45), $y = y_{\mathfrak{M}} \in \ker A_{\mathfrak{M}\times\mathfrak{M}} \subset \operatorname{ran}(A_{\mathfrak{M}\times\mathfrak{M}})^{d}$; recall that $A_{\mathfrak{M}\times\mathfrak{M}}$ is a quasi-Fredholm relation of degree 0. In particular, $y \in \operatorname{ran} A^{d}$. Consequently, $y \in \operatorname{ran} A^{d} \cap \ker A$, which proves (5.43). Hence part (v) has been completely shown.

Step 4. The closedness of the relation A. The case $d = 0$ has been discussed already. Now consider the case $d \in \mathbb{N}$. Let $\{f_n, f_n'\} \in A$ converge to $\{f, f'\} \in \mathfrak{H} \times \mathfrak{H}$. Due to the decomposition $A = A_{\mathfrak{M}\times\mathfrak{M}} \widehat{+} A_{\mathfrak{N}\times\mathfrak{N}}$ it follows that

$$\{f_n, f_n'\} = \{g_n, g_n'\} + \{h_n, h_n'\}, \quad \{g_n, g_n'\} \in A_{\mathfrak{M}\times\mathfrak{M}}, \quad \{h_n, h_n'\} \in A_{\mathfrak{N}\times\mathfrak{N}}.$$

Since $\mathfrak{M} \times \mathfrak{M}$ and $\mathfrak{N} \times \mathfrak{N}$ are complementary spaces in $\mathfrak{H} \times \mathfrak{H}$, each of the sequences in this decomposition converges. Note $A_{\mathfrak{M}\times\mathfrak{M}}$ is closed because it is quasi-Fredholm of degree 0. The operator $A_{\mathfrak{N}\times\mathfrak{N}}$ is defined on all of $\mathfrak{N}$ and hence bounded in $\mathfrak{N}$ by Lemma 4.4, and therefore also closed. Hence,

$$\{g_n, g_n'\} \to \{g, g'\} \in A_{\mathfrak{M}\times\mathfrak{M}}, \quad \{h_n, h_n'\} \to \{h, h'\} \in A_{\mathfrak{N}\times\mathfrak{N}},$$

and as $\{f, f'\} = \{g, g'\} + \{h, h'\}$ it follows that $\{f, f'\} \in A$ and thus A is closed. This completes the proof $\square$

REMARK 5.3. Observe that all statements of Theorem 5.2, except the one concerning $\mathfrak{N} \not\subset \ker A^{d-1}$, have been proved under the assumptions that

1. $d \in \Delta(A)$;
2. $\operatorname{ran} A^d \cap \ker A$ is closed in $\mathfrak{H}$;
3. $\operatorname{ran} A + \ker A^d$ is closed in $\mathfrak{H}$.

The fact that $d = \delta(A)$ leads to the 'minimality condition' $\mathfrak{N} \not\subset \ker A^{d-1}$.

REMARK 5.4. The spaces $\mathfrak{M}$ and $\mathfrak{N}$ in the decomposition of a quasi-Fredholm relation A are not, in general, unique. In [12] it is shown that if $\mathfrak{M}_1$, $\mathfrak{N}_1$ and $\mathfrak{M}_2$, $\mathfrak{N}_2$ are two decompositions associated with a quasi-Fredholm operator A, then $\mathfrak{M}_1$, $\mathfrak{N}_2$ and $\mathfrak{M}_2$, $\mathfrak{N}_1$ are also decompositions associated with the operator A; in [12] there is also a condition for the uniqueness of the decomposition for the operator case.

REMARK 5.5. The definition of a quasi-Fredholm relation can be extended to the context of a Banach space, when in Definition 5.1 $\mathfrak{H}$ denotes a Banach space and the conditions (Q2) and (Q3) are replaced by

(Q2') $\operatorname{ran} A^d \cap \ker A$ is complemented in $\mathfrak{H}$;

(Q3') $\operatorname{ran} A + \ker A^d$ is complemented in $\mathfrak{H}$.

The case of a quasi-Fredholm operator in a Banach space was briefly discussed in a remark in [9, p. 206]. For a bounded quasi-Fredholm operator in a Banach space the analog of Theorem 5.2 can be found in [14].

6. Equivalent definitions of quasi-Fredholm relations

There are several ways to characterize quasi-Fredholm relations. It is possible to characterize such relations without an explicit reference to the degree. Another way to characterize these relations is via the decomposition as described in Theorem 5.2. Finally it will be shown that quasi-Fredholm relations can be seen as particular perturbations of quasi-Fredholm relations of degree 1.

6.1. A Baire type characterization of quasi-Fredholm relations

The first way to characterize quasi-Fredholm relations is a direct adaptation of a result of Labrousse [9, Proposition 3.3.6] for the operator case. Note that the degree does not explicitly show up in the characterization.

PROPOSITION 6.1. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then A is a quasi-Fredholm relation if and only if the spaces*

$$\operatorname{ran} A + \bigcup_{j=1}^{\infty} \ker A^j \quad \text{and} \quad \ker A \cap \left(\bigcap_{j=1}^{\infty} \operatorname{ran} A^j \right) \tag{6.1}$$

are closed in $\mathfrak{H}$.

Proof. Observe that for any relation A one has the identities

$$\operatorname{ran} A + \bigcup_{j=1}^{\infty} \ker A^j = \bigcup_{j=1}^{\infty} \left(\operatorname{ran} A + \ker A^j \right) \tag{6.2}$$

and

$$\ker A \cap \left(\bigcap_{j=1}^{\infty} \operatorname{ran} A^j \right) = \bigcap_{j=1}^{\infty} \left(\ker A \cap \operatorname{ran} A^j \right). \tag{6.3}$$

($\Rightarrow$) Let A be a quasi-Fredholm relation of degree d. Then Lemma 2.5 implies that $\ker A^j \subset \operatorname{ran} A + \ker A^d$ and it follows from (6.2) that

$$\operatorname{ran} A + \bigcup_{j=1}^{\infty} \ker A^j = \operatorname{ran} A + \ker A^d. \tag{6.4}$$

Furthermore, the sequence $\ker A \cap \operatorname{ran} A^j$ is decreasing as $j \in \mathbb{N}$ increases, and, in addition, $\ker A \cap \operatorname{ran} A^j = \ker A \cap \operatorname{ran} A^d$, $j \geqslant d$. Hence, it follows from (6.3) that

$$\ker A \cap \left(\bigcap_{j=1}^{\infty} \operatorname{ran} A^j \right) = \ker A \cap \operatorname{ran} A^d. \tag{6.5}$$

By definition the spaces $\operatorname{ran} A + \ker A^d$ and $\ker A \cap \operatorname{ran} A^d$ are closed. Hence (6.4) and (6.5) show that the spaces in (6.1) are closed.

($\Leftarrow$) Let A be a range space relation for which the spaces in (6.1) are closed. Define the subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ by the lefthand side of (6.2), so that $\mathfrak{H}_0$ is closed. Each space $\operatorname{ran} A + \ker A^j$ is a range subspace of $\mathfrak{H}_0$, and the sequence $\operatorname{ran} A + \ker A^n$ is increasing in $\mathfrak{H}_0$ as $n \in \mathbb{N}$ increases. Therefore it follows from Proposition 3.15 that there is an index j_0 such that $\operatorname{ran} A + \ker A^{j_0} = \mathfrak{H}_0$. Let d be the smallest index j_0 for which this identity holds:

$$\operatorname{ran} A + \ker A^d = \mathfrak{H}_0.$$

As the sequence $\operatorname{ran} A + \ker A^n$ is increasing in $\mathfrak{H}_0$ as $n \in \mathbb{N}$ increases, the construction implies that for all $n \geqslant d$

$$\operatorname{ran} A + \ker A^d = \operatorname{ran} A + \ker A^n \ (\subset \mathfrak{H}_0).$$

By Lemma 2.5, this leads to $\delta(A) = d < \infty$. Again by Lemma 2.5 it is seen that the identity

$$\ker A \cap \operatorname{ran} A^d = \ker A \cap \left(\bigcap_{j=1}^{\infty} \operatorname{ran} A^j \right)$$

holds. This shows that $\ker A \cap \operatorname{ran} A^d$ is closed. Therefore A is a quasi-Fredholm relation of degree d. □

6.2. Kato decomposable relations

A second way to characterize the decomposition of quasi-Fredholm relations in Theorem 5.2 is in fact an inverse result.

DEFINITION 6.2. A range space relation A in a Hilbert space $\mathfrak{H}$ is said to be Kato decomposable of degree d if there exist two closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ such that the properties (i)–(vi) in Theorem 5.2 are satisfied.

LEMMA 6.3. *Let A be a Kato decomposable relation in a Hilbert space $\mathfrak{H}$. Then its degree is well defined.*

Proof. Let $(\mathfrak{M}_1,\mathfrak{N}_1,d_1)$ and $(\mathfrak{M}_2,\mathfrak{N}_2,d_2)$ be two triplets which satisfy Definition 6.2 and assume that $d_1 < d_2$. It will be shown that $\mathfrak{N}_2 \subset \ker A^{d_1}$, leading to a contradiction. Let $u \in \mathfrak{N}_2$. Then the assumption concerning the triplet $(\mathfrak{M}_2,\mathfrak{N}_2,d_2)$ leads to $u \in \ker\,(A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_2}$. Observe that

$$\{u,0\} \in (A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_2} = (A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_2-d_1} \times (A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_1},$$

shows that

$$\{u,u'\} \in (A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_1} \subset A^{d_1}$$

for some

$$u' \in \ker\,(A_{\mathfrak{N}_2\times\mathfrak{N}_2})^{d_2-d_1} \subset \ker A^{d_2-d_1} \subset \ker A^{d_2}.$$

Note that, in particular, $u' \in \mathfrak{N}_2$. Therefore, the assumption concerning the triplet $(\mathfrak{M}_1,\mathfrak{N}_1,d_1)$ leads to $u = v+w$ for some $v \in \mathfrak{M}_1$ and $w \in \mathfrak{N}_1$. Observe that $w \in \mathfrak{N}_1 \subset \ker A^{d_1}$, so that

$$\{v,u'\} = \{u,u'\} - \{w,0\} \in A^{d_1},$$

and, therefore, $u' \in \operatorname{ran} A^{d_1} \subset \mathfrak{M}_1$. Hence,

$$u' \in \mathfrak{M}_1 \cap \ker A^{d_2} = \ker\,(A_{\mathfrak{M}_1\times\mathfrak{M}_1})^{d_2},$$

where the last equality follows from $A_{\mathfrak{M}_1\times\mathfrak{H}} = A_{\mathfrak{M}_1\times\mathfrak{M}_1}$. Since $A_{\mathfrak{M}_1\times\mathfrak{M}_1}$ is a quasi-Fredholm relation of degree 0 in $\mathfrak{M}_1$, one has that

$$\ker\,(A_{\mathfrak{M}_1\times\mathfrak{M}_1})^{d_2} \subset \operatorname{ran}\,(A_{\mathfrak{M}_1\times\mathfrak{M}_1})^{d_2} \subset \operatorname{ran} A^{d_2},$$

where the first inclusion follows from Lemma 2.7. Hence $u' \in \operatorname{ran} A^{d_2} \subset \mathfrak{M}_2$, which together with $u' \in \mathfrak{N}_2$ implies that $u' \in \mathfrak{M}_2 \cap \mathfrak{N}_2 = \{0\}$. Thus $u' = 0$ and then $u \in \ker A^{d_1}$. Therefore, it follows that $\mathfrak{N}_2 \subset \ker A^{d_1}$. The assumption $\mathfrak{N}_2 \not\subset \ker A^{d_2-1}$ then leads to $d_2 - 1 < d_1$, so that $d_2 \leqslant d_1$, a contradiction.

In the same way the inequality $d_1 > d_2$ leads to a contradiction. Therefore $d_1 = d_2$. $\square$

THEOREM 6.4. *Let A be a range space relation A in a Hilbert space $\mathfrak{H}$ and let $d \geqslant 1$. The relation A is Kato decomposable of degree d if and only if A is quasi-Fredholm of degree d.*

Proof. ($\Leftarrow$) Assume that A is quasi-Fredholm of degree d. Then Theorem 5.2 implies that A is Kato decomposable of degree d.

($\Rightarrow$) Assume that A is Kato decomposable of degree d. The decomposition result remains valid for powers of the relation A as in (5.40); cf. Corollary 2.2:

$$A^m = (A_{\mathfrak{M}\times\mathfrak{M}})^m \mathbin{\widehat{+}} (A_{\mathfrak{N}\times\mathfrak{N}})^m.$$

Hence it is clear that $\operatorname{ran} A^m = \operatorname{ran}(A_{\mathfrak{M}\times\mathfrak{M}})^m$ for all $m \geqslant d$. Thus, if $m \geqslant d$, then

$$\begin{aligned} \operatorname{ran} A^m \cap \ker A &= \operatorname{ran}(A_{\mathfrak{M}\times\mathfrak{M}})^m \cap \ker A \\ &= \operatorname{ran}(A_{\mathfrak{M}\times\mathfrak{M}})^m \cap \ker A_{\mathfrak{M}\times\mathfrak{M}} = \ker A_{\mathfrak{M}\times\mathfrak{M}}, \end{aligned} \tag{6.6}$$

as $A_{\mathfrak{M}\times\mathfrak{M}}$ is quasi-Fredholm of degree 0; cf. Lemma 2.7. It follows from (6.6) that for all $m \geqslant d$

$$\operatorname{ran} A^m \cap \ker A = \operatorname{ran} A^d \cap \ker A,$$

so that $\delta(A) \leqslant d$.

To show that A is quasi-Fredholm of degree $\delta(A)$, one has to prove that the subspaces $\operatorname{ran} A^{\delta(A)} \cap \ker A$ and $\operatorname{ran} A + \ker A^{\delta(A)}$ are closed, cf. Definition 5.1.

In order to show that $\operatorname{ran} A^{\delta(A)} \cap \ker A$ is closed, observe that $\operatorname{ran} A^{\delta(A)} \cap \ker A = \operatorname{ran} A^d \cap \ker A$. Hence, (6.6) implies that

$$\operatorname{ran} A^{\delta(A)} \cap \ker A = \ker A_{\mathfrak{M}\times\mathfrak{M}}.$$

Since $A_{\mathfrak{M}\times\mathfrak{M}}$ is quasi-Fredholm of degree 0 in $\mathfrak{M}$, the space $\ker A_{\mathfrak{M}\times\mathfrak{M}}$ is closed. Therefore, $\operatorname{ran} A^{\delta(A)} \cap \ker A$ is closed.

In order to show that $\operatorname{ran} A + \ker A^{\delta(A)}$ is closed, observe that Corollary 2.6 gives

$$\operatorname{ran} A + \ker A^{\delta(A)} = \operatorname{ran} A + \ker A^d.$$

Hence for the proof it suffices to show that

$$\operatorname{ran} A + \ker A^d = \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}} + \mathfrak{N}, \tag{6.7}$$

and that the righthand side is closed. Clearly the righthand side of (6.7) is contained in the lefthand side. For the reverse inclusion observe that

$$\operatorname{ran} A \subset \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}} + \operatorname{ran} A_{\mathfrak{N}\times\mathfrak{N}} \subset \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}} + \mathfrak{N}, \tag{6.8}$$

and that

$$\ker A^d \subset (\mathfrak{M} \cap \ker A^d) + \mathfrak{N} = \ker (A_{\mathfrak{M}\times\mathfrak{M}})^d + \mathfrak{N} \subset \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}} + \mathfrak{N}. \tag{6.9}$$

The first inclusion in (6.9) follows from $\mathfrak{N} \subset \ker A^d$ and $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$. The last inclusion follows from Lemma 2.7. The equality in (6.9) follows from

$$\mathfrak{M} \cap \ker A^d = \ker (A_{\mathfrak{M}\times\mathfrak{M}})^d,$$

which is clear since $A_{\mathfrak{M}\times\mathfrak{H}} = A_{\mathfrak{M}\times\mathfrak{M}}$. Observe that (6.8) and (6.9) complete the proof of (6.7). Now it will be shown that the righthand side of (6.7) is closed. As $A_{\mathfrak{M}\times\mathfrak{M}}$ is quasi-Fredholm of degree 0 in $\mathfrak{M}$, by definition $\operatorname{ran} A_{\mathfrak{M}\times\mathfrak{M}}$ is closed in $\mathfrak{M}$, and hence closed. Furthermore, $\mathfrak{M}$ and $\mathfrak{N}$ are complementary subspaces of $\mathfrak{H}$, which completes the argument.

It has been shown that A is a quasi-Fredholm relation of degree $\delta(A) \leqslant d$. Hence, Theorem 5.2 implies that A is Kato decomposable of degree $\delta(A)$. The original assumption is that A is Kato decomposable of degree d. It follows from Lemma 6.3 that $\delta(A) = d$. Hence, A is quasi-Fredholm of degree d. □

6.3. Normally decomposable linear relations

The third way to characterize the decomposition result in Theorem 5.2 is via a special operator-sum decomposition of the relation A; cf. [18].

DEFINITION 6.5. A range space relation A in a Hilbert space $\mathfrak{H}$ is said to be normally decomposable of degree d if there exist two range space relations D and T in $\mathfrak{H}$ such that:

(i) D is a quasi-Fredholm relation of degree at most 1;

(ii) T is an everywhere defined nilpotent operator of degree d;

(iii) $A = D + T$, $TD = \operatorname{dom} A \times \{0\}$, $DT = \mathfrak{H} \times \operatorname{mul} A$.

LEMMA 6.6. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$, which is normally decomposable as in Definition* 6.5. *Then*

$$\operatorname{ran} T \subset \ker D \quad \textit{and} \quad \operatorname{ran} D \subset \ker T, \tag{6.10}$$

so that

$$\operatorname{dom} A^k = \operatorname{dom}(D+T)^k = \operatorname{dom} D^k, \quad \operatorname{mul} A^k = \operatorname{mul}(D+T)^k = \operatorname{mul} D^k, \tag{6.11}$$

with $k \in \mathbb{N}$. Furthermore,

$$TD^j = \operatorname{dom} A^j \times \{0\}, \quad D^jT = \mathfrak{H} \times \operatorname{mul} A^j, \quad j \in \mathbb{N}. \tag{6.12}$$

In particular,

$$\operatorname{dom} TD^j = \operatorname{dom} D^j = \operatorname{dom} A^j, \quad \operatorname{mul} D^j T = \operatorname{mul} D^j = \operatorname{mul} A^j, \quad j \in \mathbb{N}.$$

Moreover,

$$A^n = D^n, \quad n \geqslant d. \tag{6.13}$$

Proof. Let $A = D + T$ with D and T as in Definition 6.5. Let $h \in \mathfrak{H}$ so that $\{h, 0\} \in \mathfrak{H} \times \operatorname{mul} A = DT$. Then $\{Th, 0\} \in D$ so that $Th \in \ker D$. Thus $\operatorname{ran} T \subset \ker D$. Moreover, let $d' \in \operatorname{ran} D$ so that $\{d, d'\} \in D$ for some $d \in \operatorname{dom} D$. Since $\{d', Td'\} \in T$ it follows that $\{d, Td'\} \in TD = \operatorname{dom} A \times \{0\}$ which shows that $Td' = 0$. Thus $d' \in \ker T$ which means that $\operatorname{ran} D \subset \ker T$. This proves (6.10).

Since (6.10) holds, Lemma 2.3 may be applied. In the operator-wise sum $A = D + T$ one has $\operatorname{dom} A = \operatorname{dom}(D+T) = \operatorname{dom} D$ and $\operatorname{mul} A = \operatorname{mul}(D+T) = \operatorname{mul} D$. Lemma 2.3 may be applied so that similar identities hold for powers, which leads to (6.11).

Consider the first identity in (6.12). The case $j = 1$ is clear by hypothesis. Now assume that $TD^k = \operatorname{dom} A^k \times \{0\}$ for some $k \in \mathbb{N}$. Let $\{x, y\} \in TD^{k+1}$, so that $\{x, z\} \in D$ and $\{z, y\} \in TD^k = \operatorname{dom} A^k \times \{0\} = \operatorname{dom} D^k \times \{0\}$; cf. (6.11). This implies that $z \in \operatorname{dom} D^k$ and $y = 0$. Therefore, it follows from $\{x, z\} \in D$ and $z \in \operatorname{dom} D^k$ that $x \in \operatorname{dom} D^{k+1} = \operatorname{dom} A^{k+1}$. This gives

$$TD^{k+1} \subset \operatorname{dom} A^{k+1} \times \{0\}. \tag{6.14}$$

Conversely, let $x \in \operatorname{dom} A^{k+1} = \operatorname{dom} D^{k+1}$; cf. (6.11). Then, $\{x, x''\} \in D^{k+1}$ for some $x'' \in \mathfrak{H}$. This implies that $\{x, x'\} \in D$ and $\{x', x''\} \in D^k$ for some $x' \in \mathfrak{H}$, so that $\{x', Tx''\} \in TD^k$. Therefore, $Tx'' = 0$ and then $\{x, 0\} \in TD^{k+1}$. Thus

$$\operatorname{dom} A^{k+1} \times \{0\} \subset TD^{k+1}. \tag{6.15}$$

Now (6.14) and (6.15) lead to the first identity in (6.12), and the induction process is finished.

Consider the second identity in (6.12). The case $j = 1$ is clear by hypothesis. Now assume that $D^k T = \mathfrak{H} \times \operatorname{mul} A^k$. Let $\{x, y\} \in D^{k+1} T$, so that $\{x, z\} \in D^k T$ and $\{z, y\} \in D$ for some $z \in \mathfrak{H}$. Hence, by assumption, $z \in \operatorname{mul} A^k = \operatorname{mul} D^k$, so that $\{0, z\} \in D^k$; cf. (6.11). Therefore, $\{0, y\} \in D^{k+1}$, i.e., $y \in \operatorname{mul} D^{k+1} = \operatorname{mul} A^{k+1}$. Hence $\{x, y\} \in \mathfrak{H} \times \operatorname{mul} A^{k+1}$. Thus,

$$D^{k+1} T \subset \mathfrak{H} \times \operatorname{mul} A^{k+1}. \tag{6.16}$$

Conversely, let $\{x, y\} \in \mathfrak{H} \times \operatorname{mul} A^{k+1}$. This implies that $\{x, 0\} \in \mathfrak{H} \times \{0\}$ and $\{0, y\} \in D^{k+1}$, so that $\{0, z\} \in D^k$ and $\{z, y\} \in D$ for some $z \in \mathfrak{H}$. Then, $\{0, z\} \in D^k T$, and since $\{x, 0\} \in D^k T$ by induction hypotheses, it follows that

$$\{x, z\} = \{x, 0\} + \{0, z\} \in D^k T,$$

so that $\{x, y\} \in D^{k+1} T$. Thus,

$$\mathfrak{H} \times \operatorname{mul} A^{k+1} \subset D^{k+1} T. \tag{6.17}$$

Finally, (6.16) and (6.17) lead to the second identity in (6.12), and the induction process is finished.

Now (6.13) will be shown. Assume that $n \geqslant d$. This implies that $T^n x = 0$ for all $x \in \mathfrak{H}$. To prove that $A^n \subset D^n$ consider $\{x_0, x_n\} \in A^n$, so that $\{x_i, x_{i+1}\} \in A$, $0 \leqslant i \leqslant n-1$, for some vectors $x_i \in \mathfrak{H}$, $1 \leqslant i \leqslant n-1$. This implies that

$$\{x_i, x_{i+1} - Tx_i\} \in D, \quad 0 \leqslant i \leqslant n-1, \tag{6.18}$$

which shows that $\{x_i, T(x_{i+1} - Tx_i)\} \in TD = \operatorname{dom} A \times \{0\}$. This leads to

$$Tx_{i+1} = T^2 x_i, \quad 0 \leqslant i \leqslant n-1. \tag{6.19}$$

Since $\operatorname{ran} T \subset \ker D$, it is also known that

$$\{Tx_i, 0\} \in D, \quad 0 \leqslant i \leqslant n. \tag{6.20}$$

A combination of (6.18) and (6.20) implies that

$$\{x_i - Tx_{i-1}, x_{i+1} - Tx_i\} = \{x_i, x_{i+1} - Tx_i\} - \{Tx_{i-1}, 0\} \in D,$$

for all $1 \leqslant i \leqslant n-1$, which together with $\{x_0, x_1 - Tx_0\} \in D$ leads to

$$\{x_0, x_n - Tx_{n-1}\} \in D^n. \tag{6.21}$$

Furthermore, it follows from (6.19) that

$$Tx_{n-1} = T^n x_0 = 0. \tag{6.22}$$

Then (6.21) and (6.22) imply that $\{x_0, x_n\} \in D^n$, which shows that $A^n \subset D^n$.

Conversely, let $\{y_0, y_n\} \in D^n$, so that $\{y_i, y_{i+1}\} \in D$, $0 \leqslant i \leqslant n-1$ for some vectors $y_i \in \mathfrak{H}$, $1 \leqslant i \leqslant n-1$, which implies that

$$\{y_i, y_{i+1} + Ty_i\} \in A, \quad 0 \leqslant i \leqslant n-1. \tag{6.23}$$

Furthermore, $\{y_i, Ty_{i+1}\} \in TD = \operatorname{dom} A \times \{0\}$, $0 \leqslant i \leqslant n-1$, so that

$$Ty_i = 0, \quad 1 \leqslant i \leqslant n. \tag{6.24}$$

It follows from (6.23) and (6.24) that

$$\{y_i, y_{i+1}\} \in A, \quad 1 \leqslant i \leqslant n-1. \tag{6.25}$$

Since $\operatorname{ran} T \subset \ker D$, one has $\{T^i y_0, 0\} \in D$, $1 \leqslant i \leqslant n-1$, which together with $\{T^i y_0, T^{i+1} y_0\} \in T$ leads to

$$\{T^i y_0, T^{i+1} y_0\} \in A, \quad 1 \leqslant i \leqslant n-1. \tag{6.26}$$

A combination of (6.25) and (6.26) implies that

$$\{y_i + T^i y_0, y_{i+1} + T^{i+1} y_0\} \in A, \quad 1 \leqslant i \leqslant n-1. \tag{6.27}$$

It follows from (6.23) that $\{y_0, y_1 + Ty_0\} \in A$ and thus (6.27) shows that

$$\{y_0, y_n + T^n y_0\} \in A^n,$$

so that $\{y_0, y_n\} \in A^n$. Hence, $D^n \subset A^n$. Therefore, $A^n = D^n$. □

THEOREM 6.7. *Let A be a range space relation in a Hilbert space $\mathfrak{H}$. Then A is normally decomposable of degree d if and only if A is quasi-Fredholm of degree d.*

Proof. ($\Leftarrow$) Assume that A is quasi-Fredholm of degree d. If $d = 0$ it suffices to consider $D = A$ and $T = 0$. Now suppose that $d \geqslant 1$ and apply Theorem 5.2. Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed complementary subspaces as in that theorem and let $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ be the projections corresponding to $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Define the linear relations D and T in $\mathfrak{H}$ by

$$D = A_{\mathfrak{M}\times\mathfrak{M}} P_{\mathfrak{M}}, \quad T = A_{\mathfrak{N}\times\mathfrak{N}} P_{\mathfrak{N}}, \tag{6.28}$$

where $A_{\mathfrak{M}\times\mathfrak{M}}$ and $A_{\mathfrak{N}\times\mathfrak{N}}$ are as in Theorem 5.2. The relations D and T are range space relations, due to Proposition 4.8. It follows from Theorem 5.2 and (6.28) that

$$D + T = A_{\mathfrak{M}\times\mathfrak{M}} \mathbin{\widehat{+}} A_{\mathfrak{N}\times\mathfrak{N}} = A, \tag{6.29}$$

see Corollary 2.2 for the first equality.

Now the identity $TD = \operatorname{dom} A \times \{0\}$ will be shown. Let $\{x, y\} \in TD$, so that $\{x, z\} \in D$ and $\{z, y\} \in T$ for some $z \in \mathfrak{H}$. Then, by definition, $\{P_{\mathfrak{M}}x, z\} \in A_{\mathfrak{M}\times\mathfrak{M}}$ and $\{P_{\mathfrak{N}}z, y\} \in A_{\mathfrak{N}\times\mathfrak{N}}$. Clearly, $P_{\mathfrak{M}}x \in \operatorname{dom} A$, so that $x = P_{\mathfrak{M}}x + P_{\mathfrak{N}}x \in \operatorname{dom} A$; recall that $\mathfrak{N} \subset \operatorname{dom} A$. Also observe that $z \in \mathfrak{M}$, so that $P_{\mathfrak{N}}z = 0$ and, consequently, $y = 0$. Hence, $TD \subset \operatorname{dom} A \times \{0\}$. To show the converse inclusion, let $\{x, 0\} \in \operatorname{dom} A \times \{0\}$. Then $P_{\mathfrak{M}}x \in \operatorname{dom} A_{\mathfrak{M}\times\mathfrak{M}}$, so that $\{P_{\mathfrak{M}}x, z\} \in A_{\mathfrak{M}\times\mathfrak{M}}$ for some $z \in \mathfrak{M}$, in other words $\{x, z\} \in D$. Furthermore, $\{z, 0\} \in T$ as $\{P_{\mathfrak{N}}z, 0\} = \{0, 0\} \in A_{\mathfrak{N}\times\mathfrak{N}}$. Thus it follows that, $\operatorname{dom} A \times \{0\} \subset TD$.

Next the identity $DT = \mathfrak{H} \times \operatorname{mul} A$ will be shown. Let $\{x, y\} \in DT$, so that $\{x, z\} \in T$ and $\{z, y\} \in D$ for some $z \in \mathfrak{H}$. By definition, $\{P_{\mathfrak{N}}x, z\} \in A_{\mathfrak{N}\times\mathfrak{N}}$ and $\{P_{\mathfrak{M}}z, y\} \in A_{\mathfrak{M}\times\mathfrak{M}}$. Now $z \in \mathfrak{N}$, so that $P_{\mathfrak{M}}z = 0$. Therefore, $y \in \operatorname{mul} A_{\mathfrak{M}\times\mathfrak{M}} = \operatorname{mul} A$. Hence, $DT \subset \mathfrak{H} \times \operatorname{mul} A$. To show the converse inclusion, let $\{x, y\} \in \mathfrak{H} \times \operatorname{mul} A$. Then $P_{\mathfrak{N}}x \in \mathfrak{N}$, so that $\{P_{\mathfrak{N}}x, z\} \in A_{\mathfrak{N}\times\mathfrak{N}}$ for some $z \in \mathfrak{N}$. This implies $\{P_{\mathfrak{M}}z, y\} = \{0, y\} \in A_{\mathfrak{M}\times\mathfrak{M}}$. Therefore, $\{x, z\} \in T$, $\{z, y\} \in D$, so that $\{x, y\} \in DT$. Hence, $\mathfrak{H} \times \operatorname{mul} A \subset DT$.

Now it will be shown that D is quasi-Fredholm of degree at most 1. First observe that

$$\ker D = (\mathfrak{M} \cap \ker A) + \mathfrak{N}. \tag{6.30}$$

To see this, let $x \in \ker D$, so that $\{P_{\mathfrak{M}}x, 0\} \in A$ and $x = P_{\mathfrak{M}}x + P_{\mathfrak{N}}x \in (\ker A \cap \mathfrak{M}) + \mathfrak{N}$. Conversely, if $x = x_1 + x_2$ with $x_1 \in \mathfrak{M} \cap \ker A$ and $x_2 \in \mathfrak{N}$, then $\{P_{\mathfrak{M}}x, 0\} = \{x_1, 0\} \in A_{\mathfrak{M}\times\mathfrak{M}}$ and $\{x, 0\} \in A_{\mathfrak{M}\times\mathfrak{M}} P_{\mathfrak{M}} = D$. Hence, $x \in \ker D$. It follows from (6.30) that

$$\mathfrak{M} \cap \ker D = \mathfrak{M} \cap \ker A = \ker A_{\mathfrak{M}\times\mathfrak{M}} \subset \operatorname{ran}(A_{\mathfrak{M}\times\mathfrak{M}})^j \subset \operatorname{ran} A^j, \quad j \in \mathbb{N}, \tag{6.31}$$

since $A_{\mathfrak{M}\times\mathfrak{M}}$ is quasi-Fredholm of degree 0. Recall that the decomposition (iv) in Theorem 5.2 implies the decomposition (5.40), so that by Corollary 2.2

$$A^j = (A_{\mathfrak{M}\times\mathfrak{M}})^j P_{\mathfrak{M}} + (A_{\mathfrak{N}\times\mathfrak{N}})^j P_{\mathfrak{N}}, \quad j \in \mathbb{N}. \tag{6.32}$$

Hence, $A^j = (A_{\mathfrak{M}\times\mathfrak{M}})^j P_{\mathfrak{M}}$ for $j \geqslant d$; note that $(A_{\mathfrak{M}\times\mathfrak{M}})^j P_{\mathfrak{M}} = (A_{\mathfrak{M}\times\mathfrak{M}} P_{\mathfrak{M}})^j$, by Corollary 2.2. Due to (6.28), it follows that

$$A^j = D^j \quad \text{for all} \quad j \geqslant d. \tag{6.33}$$

A combination of (6.31) and (6.33) leads to

$$\operatorname{ran} D \cap \ker D \subset \mathfrak{M} \cap \ker D \subset \operatorname{ran} A^j \cap \ker D = \operatorname{ran} D^j \cap \ker D,$$

for all $j \geqslant d$. Hence $\operatorname{ran} D \cap \ker D = \operatorname{ran} D^j \cap \ker D$ for all $j \geqslant 1$. This shows that $\delta(D) \leqslant 1$. Observe that $A^d = (A_{\mathfrak{M}\times\mathfrak{M}})^d$, so that $\operatorname{ran} A^d \cap \ker D = \operatorname{ran} A^d \cap \ker A$. Therefore,

$$\operatorname{ran} D \cap \ker D = \operatorname{ran} D^d \cap \ker D = \operatorname{ran} A^d \cap \ker D = \operatorname{ran} A^d \cap \ker A,$$

which shows that $\operatorname{ran} D \cap \ker D$ is closed. Furthermore, observe that

$$\operatorname{ran} D + \ker D = \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \ker A_{\mathfrak{M}\times\mathfrak{H}} + \mathfrak{N} = \operatorname{ran} A_{\mathfrak{M}\times\mathfrak{H}} + \mathfrak{N}.$$

The righthand side is closed due to (5.34), and hence $\operatorname{ran} D + \ker D$ is closed. Thus D is a quasi-Fredholm relation of degree at most 1.

Finally, it will be shown that T is nilpotent of degree d. Since $\mathfrak{N} \subset \ker A^d$ it follows that T is nilpotent of degree $d_0 \leqslant d$. Let $j \geqslant d_0$, so that $A^j = D^j$. This implies that $\operatorname{ran} A^j = \operatorname{ran} D^j \, (\subset \mathfrak{M})$, which further leads to

$$\operatorname{ran} A^j \cap \ker A = \operatorname{ran} D^j \cap \ker D = \operatorname{ran} D \cap \ker D,$$

since $\mathfrak{M} \cap \ker D = \ker A_{\mathfrak{M}\times\mathfrak{M}}$. Thus, $d = \delta(A) \leqslant d_0$, and therefore $d = d_0$. This allows to conclude that A is normally decomposable of degree d.

$(\Rightarrow)$ Assume that A is normally decomposable of degree d. First it will be shown that the identity

$$\operatorname{ran} D^j \cap \ker A = \operatorname{ran} D^j \cap \ker D \tag{6.34}$$

is valid. Let $y \in \operatorname{ran} D^j \cap \ker A$, so that $\{y,0\} \in A$ and $\{x,y\} \in D^j$ for some $x \in \mathfrak{H}$. It follows from $\{y,0\} \in A$ that $\{y,z\} \in D$ and $\{y,-z\} \in T$ for some $z \in \mathfrak{H}$. Then, $\{x,-z\} \in TD^j = \operatorname{dom} A^j \times \{0\}$, so that $z = 0$. Therefore $y \in \ker D$ which further leads to

$$\operatorname{ran} D^j \cap \ker A \subset \operatorname{ran} D^j \cap \ker D. \tag{6.35}$$

Conversely, let $y \in \operatorname{ran} D^j \cap \ker D$. Then $\{y,0\} \in D$ and $\{x,y\} \in D^j$ for some $x \in \mathfrak{H}$. Therefore $\{x, Ty\} \in TD^j = \operatorname{dom} A^j \times \{0\}$. This shows that $Ty = 0$ so that $y \in \ker A$. Thus

$$\operatorname{ran} D^j \cap \ker D \subset \operatorname{ran} D^j \cap \ker A. \tag{6.36}$$

A combination of (6.35) and (6.36) leads to (6.34).

Next it follows from Lemma 6.6 and (6.34) that

$$\operatorname{ran} A^j \cap \ker A = \operatorname{ran} D^j \cap \ker A = \operatorname{ran} D^j \cap \ker D = \operatorname{ran} D \cap \ker D$$

for all $j \geqslant d$, which implies that $\delta(A) \leqslant d$. Moreover, since $\operatorname{ran} D \cap \ker D$ is closed in $\mathfrak{H}$ it follows that $\operatorname{ran} A^{d_0} \cap \ker A$ is closed when $d_0 = \delta(A)$. Since D is quasi-Fredholm of degree 1, it follows from Corollary 2.6 that

$$\operatorname{ran} D + \ker D^d = \operatorname{ran} D + \ker D. \tag{6.37}$$

Now $A = D + T$ implies that $\operatorname{ran} A \subset \operatorname{ran} D + \operatorname{ran} T \subset \operatorname{ran} D + \ker D$, where the last inclusion follows from (6.10). Hence $\operatorname{ran} A \subset \operatorname{ran} D + \ker D^d$. Together with $A^d = D^d$ (cf. (6.13)) this leads to

$$\operatorname{ran} A + \ker A^d \subset \operatorname{ran} D + \ker D^d. \tag{6.38}$$

For the converse inclusion, observe that $D = A - T$ leads to

$$\operatorname{ran} D \subset \operatorname{ran} A + \operatorname{ran} T \subset \operatorname{ran} A + \ker D \subset \operatorname{ran} A + \ker D^d = \operatorname{ran} A + \ker A^d,$$

where, again, (6.10) and (6.13) have been used. Hence it follows

$$\operatorname{ran} D + \ker D^d \subset \operatorname{ran} A + \ker A^d. \tag{6.39}$$

Therefore, (6.38), (6.39), and (6.37) lead to

$$\operatorname{ran} A + \ker A^d = \operatorname{ran} D + \ker D^d = \operatorname{ran} D + \ker D.$$

As $d_0 = \delta(A)$, it follows from Corollary 2.6 that

$$\operatorname{ran} A + \ker A^{d_0} = \operatorname{ran} D + \ker D,$$

so that $\operatorname{ran} A + \ker A^{d_0}$ is closed in $\mathfrak{H}$. Therefore, A is a quasi-Fredholm relation in $\mathfrak{H}$ of degree $d_0 \leqslant d$. Now using the first part of the proof it follows that there exist two range space relations D' and T' satisfying the items (i), (ii), and (iii) of Definition 6.5, with T' nilpotent of degree d_0, and $d_0 \geqslant d$. This shows that $d = d_0$. This completes the proof. □

7. Adjoints of quasi-Fredholm relations

Let A be a quasi-Fredholm relation of degree $d \in \mathbb{N}$ in a Hilbert space $\mathfrak{H}$. The adjoint A^* of A is a closed relation and, in particular, a range space relation. It will be shown that A^* is also a quasi-Fredholm relation of degree d, i.e.,

1. $\delta(A^*) = d$;

2. $\operatorname{ran} A^{*d} \cap \ker A^*$ is closed in $\mathfrak{H}$;

3. $\operatorname{ran} A^* + \ker A^{*d}$ is closed in $\mathfrak{H}$.

For $d = 0$ the first condition can be replaced by the equivalent condition $\ker A^* \subset \operatorname{ran} A^{*n}$, $n \in \mathbb{N}$; cf. Lemma 2.7. The cases where $d = 0$ or $d = 1$ will be treated first; the general case follows then by means of the normal decomposability of A.

PROPOSITION 7.1. *Let A be a quasi-Fredholm relation of degree $d \leqslant 1$ in a Hilbert space $\mathfrak{H}$. Then A^* is a quasi-Fredholm relation of degree d. Moreover, for all $n \in \mathbb{N}$:*

$$(\ker A^n)^\perp = \operatorname{ran} A^{*n}, \quad n \in \mathbb{N}, \tag{7.1}$$

and

$$(\ker A^{*n})^\perp = \operatorname{ran} A^n, \quad n \in \mathbb{N}. \tag{7.2}$$

Proof. The proof consists of several steps; in particular, the equality (7.1) will be proved for the cases $d = 0$ and $d = 1$ separately. The equality (7.2) follows by symmetry.

Step 1. First observe that for any relation A in a Hilbert space $\mathfrak{H}$ the following inclusion holds

$$\operatorname{ran} A^{*n} \subset (\ker A^n)^\perp, \quad n \in \mathbb{N}. \tag{7.3}$$

Note that the inclusion (7.3) follows from (4.7) and (4.4):

$$\operatorname{ran} A^{*n} \subset \operatorname{ran} (A^n)^* \subset \overline{\operatorname{ran}}\, (A^n)^* = (\ker A^n)^\perp, \quad n \in \mathbb{N}.$$

Hence, in order to show (7.1), in view of (7.3) it suffices to prove that

$$(\ker A^n)^\perp \subset \operatorname{ran} A^{*n}, \quad n \in \mathbb{N}. \tag{7.4}$$

This will be done by induction for the cases $d = 0$ and $d = 1$ respectively.

Step 2. Assume that A is a quasi-Fredholm relation of degree $d = 0$. Then A is a range space relation for which $\ker A$ and $\operatorname{ran} A$ are closed. It follows from Lemma 4.6 that A is closed. The adjoint A^* is a closed relation and, in particular, a range space relation. The fact that A^* is closed implies that $\ker A^*$ is closed, whereas $\operatorname{ran} A^*$ is closed due to Lemma 4.12.

The inclusion (7.4) will be shown for the case $d = 0$. It is clear for $n = 0$; furthermore, it is clear for $n = 1$ due to (4.4) and the fact that $\operatorname{ran} A^*$ is closed. Assume that (7.4) is satisfied for some $n \in \mathbb{N}$, $n \geqslant 1$, and let $u \in (\ker A^{n+1})^\perp$. Since $\ker A^n \subset \ker A^{n+1}$ it follows that $u \in (\ker A^n)^\perp$, and by the induction hypothesis, that $\{v,u\} \in A^{*n}$ for some $v \in \operatorname{dom} A^{*n}$. Let $x \in \ker A$, then $\{x,0\} \in A$, and $\ker A \subset \operatorname{ran} A^n$, $n \in \mathbb{N}$, implies that $\{y,x\} \in A^n$ and $y \in \ker A^{n+1}$. Since $\{y,x\} \in A^n$ and $\{v,u\} \in A^{*n} \subset A^{n*}$ (cf. (4.7)), it follows that $0 = (y,u) = (x,v)$ for all $x \in \ker A$. In other words,

$v \in (\ker A)^{\perp} = \operatorname{ran} A^*$, which implies $u \in \operatorname{ran} A^{*(n+1)}$. Hence (7.4) has been proved, which completes the proof of (7.1).

Since A is a quasi-Fredholm relation of degree 0, it follows from Lemma 2.7 that $\ker A^n \subset \operatorname{ran} A$, $n \in \mathbb{N}$, which implies by (4.4) and (7.1) that

$$\ker A^* = (\operatorname{ran} A)^{\perp} \subset (\ker A^n)^{\perp} = \operatorname{ran} A^{*n}, \quad n \in \mathbb{N}.$$

Hence, A^* is a quasi-Fredholm relation of degree 0.

Step 3. Assume that A is a quasi-Fredholm relation of degree $d = 1$. Then A is a range space relation for which $\operatorname{ran} A + \ker A$ and $\operatorname{ran} A \cap \ker A$ are closed. Proposition 3.10 implies that $\operatorname{ran} A$ and $\ker A$ are closed. It follows from Lemma 4.6 that A is closed. The adjoint A^* is a closed relation and, in particular, a range space relation. The fact that A^* is closed implies that $\ker A^*$ is closed, whereas $\operatorname{ran} A^*$ is closed due to Lemma 4.12. In particular it is clear that $\operatorname{ran} A^* \cap \ker A^*$ is closed. Furthermore, since $\operatorname{ran} A + \ker A$ is closed, it follows from Proposition 3.2 that also the subspace

$$\operatorname{ran} A^* + \ker A^* = \overline{\operatorname{ran}} A^* + \overline{\ker} A^* = (\operatorname{ran} A)^{\perp} + (\ker A)^{\perp} \tag{7.5}$$

is closed; cf. (4.3) and (4.4).

Now the inclusion (7.4) will be shown for the case $d = 1$. It is clear for $n = 0$; furthermore, it is clear for $n = 1$ due to (4.4) and the fact that $\operatorname{ran} A^*$ is closed. Assume that (7.4) is satisfied for some $n \in \mathbb{N}$, $n \geqslant 1$, and let $u \in (\ker A^{n+1})^{\perp}$. Since $\ker A^n \subset \ker A^{n+1}$ it follows that $u \in (\ker A^n)^{\perp}$, and by the induction hypothesis, that $\{v,u\} \in A^{*n}$ for some $v \in \operatorname{dom} A^{*n}$. Let $x \in \operatorname{ran} A^n \cap \ker A$, so that $\{x,0\} \in A$ and $\{y,x\} \in A^n$ for some $y \in \mathfrak{H}$. Note that $y \in \ker A^{n+1}$. Since $\{y,x\} \in A^n$ and $\{v,u\} \in A^{*n} \subset A^{n*}$ (cf. (4.7)), it follows that $0 = (y,u) = (x,v)$ for all $x \in \operatorname{ran} A^n \cap \ker A$. Hence $v \in (\operatorname{ran} A^n \cap \ker A)^{\perp} = (\operatorname{ran} A \cap \ker A)^{\perp}$, where the last equality is by definition, since $d = 1$. Observe that

$$v \in \operatorname{clos}(\overline{\operatorname{ran}} A^* + \overline{\ker} A^*) = \operatorname{clos}(\operatorname{ran} A^* + \ker A^*) = \operatorname{ran} A^* + \ker A^*,$$

cf. (7.5). It follows that $v \in \operatorname{ran} A^* + \ker A^*$, so that

$$v = v_1 + v_2, \quad v_1 \in \operatorname{ran} A^*, \quad v_2 \in \ker A^*.$$

Since $\{v_2, 0\} \in A^{*n}$, it follows from $\{v,u\} \in A^{*n}$ that $\{v_1,u\} \in A^{*n}$. Since $v_1 \in \operatorname{ran} A^*$, it follows that $u \in \operatorname{ran} A^{*(n+1)}$. Hence (7.4) has been proved, which completes the proof of (7.1).

Since A is a quasi-Fredholm relation of degree 0, it follows from Corollary 2.6 that

$$\ker A^n + \operatorname{ran} A = \ker A + \operatorname{ran} A. \tag{7.6}$$

The identities (4.3), (7.1), and (7.6) show that

$$\begin{aligned} \ker A^* \cap \operatorname{ran} A^{*n} &= (\operatorname{ran} A)^{\perp} \cap (\ker A^n)^{\perp} = (\operatorname{ran} A + \ker A^n)^{\perp} \\ &= (\operatorname{ran} A + \ker A)^{\perp} = \ker A^* \cap \operatorname{ran} A^*. \end{aligned}$$

Therefore $\delta(A^*) \leqslant 1$. This leads to $\delta(A^*) = 1$; otherwise $\delta(A^*) = 0$, which implies $\delta(A) = \delta(A^{**}) = 0$, a contradiction. Hence, A^* is a quasi-Fredholm relation of degree 0.

Step 4. Let A be a quasi-Fredholm relation of degree $d \leqslant 1$. Then A is closed and A^* is a quasi-Fredholm relation of degree d. Hence the statement in (ii) follows from (i) by replacing A by A^*. Hence one may replace A by A^* in (7.1) to obtain (7.2). □

COROLLARY 7.2. *Let A be a quasi-Fredholm relation of degree d in a Hilbert space $\mathfrak{H}$. Then:*

(i) $\operatorname{ran} A^n$ *is closed for all* $n \geqslant d$*;*

(ii) $\operatorname{mul} A^n$ *is closed for all* $n \in \mathbb{N}_0$*;*

Proof. Since A is a quasi-Fredholm relation of degree d, it follows from Theorem 6.7 that it is normally decomposable. Therefore, $A = D + T$, with D a quasi-Fredholm relation of degree at most 1 and T a nilpotent operator of degree d. By Proposition 7.1 it follows that $\operatorname{ran} D^n = (\ker D^{*n})^{\perp}$ is closed for all $n \in \mathbb{N}$.

(i) For $n \geqslant d$ it follows from Lemma 6.6 that $A^n = D^n$. In particular,

$$\operatorname{ran} A^n = \operatorname{ran} D^n, \quad n \geqslant d,$$

is closed. This proves (i).

(ii) It also follows from Lemma 6.6 that

$$\operatorname{mul} A^n = \operatorname{mul} D^n, \quad n \in \mathbb{N}.$$

Recall that $\operatorname{ran} D^n$, $n \in \mathbb{N}$, is closed in $\mathfrak{H}$. Hence it follows from Corollary 4.5 that $\operatorname{mul} D^n$, $n \in \mathbb{N}$, is closed in $\mathfrak{H}$. This proves (ii). □

THEOREM 7.3. *Let A be a quasi-Fredholm relation of degree d. Then A^* is a quasi-Fredholm relation of degree d.*

Proof. Since A is quasi-Fredholm of degree d, it follows from Theorem 6.7 that it is normally decomposable of the same degree. Therefore $A = D + T$ with D a quasi-Fredholm relation of degree at most 1 and T a nilpotent operator of degree d, as in Definition 6.5. Since T is an everywhere defined bounded operator, Lemma 4.14 shows that $A^* = D^* + T^*$. According to Proposition 7.1, D^* is a quasi-Fredholm relation of degree at most 1. Furthermore, T^* is a bounded everywhere defined nilpotent operator of degree d. It follows from $TD = \operatorname{dom} A \times \{0\}$ and Lemma 4.14 that

$$D^*T^* = (TD)^* = \mathfrak{H} \times \operatorname{mul} A^*. \tag{7.7}$$

Moreover, $DT = \mathfrak{H} \times \operatorname{mul} A$ and Lemma 4.14 imply that

$$T^*D^* \subset (DT)^* = \overline{\operatorname{dom}} A^* \times \{0\}. \tag{7.8}$$

It is clear that $\operatorname{dom} T^*D^* = \operatorname{dom} D^*$; furthermore $A^* = D^* + T^*$ implies that $\operatorname{dom} A^* = \operatorname{dom} D^*$. Hence $\operatorname{dom} T^*D^* = \operatorname{dom} A^*$, and it follows from (7.8) that

$$T^*D^* = \operatorname{dom} A^* \times \{0\}. \tag{7.9}$$

The decomposition $A^* = D^* + T^*$ together with (7.7) and (7.9) shows that A^* is a normally decomposable relation of degree $d_0 \leqslant d$, with $d_0 = \delta(A^*)$. Therefore, according to Theorem 6.7, A^* is quasi-Fredholm of degree d_0, which implies that $A^{**} = A$ is quasi-Fredholm relation of degree $d = \delta(A) = \delta(A^{**}) \leqslant \delta(A^*) = d_0$. Thus $\delta(A) = \delta(A^*)$ and A^* is a quasi-Fredholm relation of degree d. □

PROPOSITION 7.4. *Let A be a quasi-Fredholm relation of degree d in a Hilbert space $\mathfrak{H}$. Then:*

$$(\ker A^n)^\perp = \overline{\operatorname{ran}} A^{*n}, \quad (\ker A^{*n})^\perp = \overline{\operatorname{ran}} A^n, \quad n \in \mathbb{N}_0. \tag{7.10}$$

Moreover, $\operatorname{ran} A^n + \ker A^m$ *and* $\operatorname{ran} A^{*n} + \ker A^{*m}$ *are closed for all* $n, m \in \mathbb{N}_0$ *with* $n + m \geqslant d$. *In fact,*

$$\operatorname{ran} A^n + \ker A^m = (\ker A^{*n} \cap \operatorname{ran} A^{*m})^\perp, \tag{7.11}$$

$$\operatorname{ran} A^{*n} + \ker A^{*m} = (\ker A^n \cap \operatorname{ran} A^m)^\perp, \tag{7.12}$$

for all n, $m \in \mathbb{N}_0$ *with* $n + m \geqslant d$.

Proof. Step 1. The second identity in (7.10) will be shown. The first identity in (7.10) will then follow from the fact that A^* is quasi-Fredholm and closed.

Assume that A is quasi-Fredholm of degree d. If $d = 0$ then Proposition 7.1 gives the result. Now suppose that $d \geqslant 1$ and apply Theorem 5.2. Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed complementary subspaces as in that theorem and let $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ be the projections corresponding to $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Define the linear relations D and T in $\mathfrak{H}$ by (6.28) where $A_{\mathfrak{M}\times\mathfrak{M}}$ and $A_{\mathfrak{N}\times\mathfrak{N}}$ are as in Theorem 5.2. The relations D and T are range space relations, due to Proposition 4.8. It follows from Theorem 5.2 and (6.28) that the identities (6.29) and (6.32) hold:

$$A^n = D^n + T^n, \quad n \in \mathbb{N}. \tag{7.13}$$

In particular, due to the special structure of D and T:

$$\operatorname{ran} A^n = \operatorname{ran} D^n + \operatorname{ran} T^n, \tag{7.14}$$

cf. Lemma 2.1. It follows from the construction that $\operatorname{ran} D \subset \mathfrak{M}$ and $\operatorname{ran} T \subset \mathfrak{N}$, so that

$$\mathfrak{M}^\perp \subset \ker D^*, \quad \mathfrak{N}^\perp \subset \ker T^*.$$

As $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, it follows from Proposition 3.2 that $\mathfrak{H} = \mathfrak{M}^\perp + \mathfrak{N}^\perp$, $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$. Hence, this leads to

$$\mathfrak{H} \subset \ker D^* + \ker T^* \subset \ker D^{*n} + \ker T^{*n},$$

so that

$$\ker D^{*n} + \ker T^{*n} = \mathfrak{H}. \tag{7.15}$$

In particular, the space in the lefthand side of (7.15) is closed. It follows from (7.14), Proposition 7.1, and the fact that T is an everywhere defined bounded operator, that

$$\begin{aligned} (\operatorname{ran} A^n)^\perp &= (\operatorname{ran} D^n)^\perp \cap (\operatorname{ran} T^n)^\perp \\ &= \overline{\ker} D^{*n} \cap \ker T^{*n} \\ &= \operatorname{clos}(\ker D^{*n} \cap \ker T^{*n}), \end{aligned} \tag{7.16}$$

where Corollary 3.12 and (7.15) have been used in the last equality. Next, observe that (7.13) leads to $A^* = D^* + T^*$, cf. (4.6). Hence, clearly,

$$\ker D^{*n} \cap \ker T^{*n} \subset \ker A^{*n}. \tag{7.17}$$

Recalll from (4.7) that $\ker A^{*n} \subset \ker (A^n)^*$, so that a combination of (7.16) and (7.17) leads to

$$(\operatorname{ran} A^n)^\perp \subset \operatorname{clos}\ker A^{*n} \subset \ker (A^n)^* = (\operatorname{ran} A^n)^\perp.$$

It follows that

$$(\operatorname{ran} A^n)^\perp = \operatorname{clos}\ker A^{*n},$$

which gives the second identity in (7.10).

Step 2. Since A is a quasi-Fredholm relation of degree d, it is clear that $\delta(A) = d$ and that $\operatorname{ran} A + \ker A^d$ is closed (see Definition 5.1). Theorem 5.2 implies that A is closed. Corollary 7.2 implies that $\operatorname{mul} A^n$ is closed for all $n \in \mathbb{N}$. Hence the conditions (i), (ii), and (iii) of Proposition 4.16 are satisfied. Therefore the subspace $\operatorname{ran} A^n + \ker A^m$ is closed for all $m, n \in \mathbb{N}_0$ such that $m + n \geqslant d$. By symmetry a similar statement also follows for the adjoint A^*.

Step 3. Observe that $\operatorname{ran} A^n$, $\ker A^m$, $\ker A^{*n}$, and $\operatorname{ran} A^{*m}$ are all range subspaces. The fact that $\ker A^{*n} + \operatorname{ran} A^{*m}$ is closed for $m + n \geqslant d$ and Corollary 3.13 show that

$$(\ker A^{*n} \cap \operatorname{ran} A^{*m})^\perp = (\ker A^{*n})^\perp + (\operatorname{ran} A^{*m})^\perp.$$

By (7.10) and Corollary 3.12 it follows that

$$(\ker A^{*n} \cap \operatorname{ran} A^{*m})^\perp = \overline{\operatorname{ran}} A^n + \overline{\ker} A^m = \operatorname{clos}(\operatorname{ran} A^n + \ker A^m).$$

Now observe that $\operatorname{ran} A^n + \ker A^m$ is closed for $m + n \geqslant d$, so that (7.11) follows. The proof of (7.12) is similar. □

8. Semi-Fredholm relations

Labrousse has shown that semi-Fredholm operators are in fact quasi-Fredholm operators. A direct proof of this can be found in [9, p. 197]. Now the situation of semi-Fredholm relations will be considered. A closed linear relation A in a Hilbert space $\mathfrak{H}$ is said to be *semi-Fredholm* if

(S1) $\operatorname{ran} A$ is closed;

(S2) $\ker A$ or $\mathfrak{H}/\operatorname{ran} A$ is finite-dimensional.

Semi-Fredholm relations were studied in [2], [13]. The following result parallels Theorem 7.3.

PROPOSITION 8.1. *Let A be a closed linear relation in a Hilbert space. Then A is a semi-Fredholm relation if and only if A^* is a semi-Fredholm relation.*

Proof. Let A be a closed linear relation which is semi-Fredholm. Then $\operatorname{ran} A$ is closed, so that by Corollary 4.13 also $\operatorname{ran} A^*$ is closed. The identities

$$\ker A^* = \mathfrak{H} \ominus \operatorname{ran} A, \quad \mathfrak{H} \ominus \operatorname{ran} A^* = \ker A,$$

show that if $\ker A$ is finite-dimensional, then $\mathfrak{H}/\operatorname{ran} A^*$ is finite-dimensional, whereas if $\mathfrak{H}/\operatorname{ran} A$ is finite-dimensional, then $\ker A^*$ is finite-dimensional. Hence A^* is semi-Fredholm. The converse statement follows by symmetry. □

The next observation is a simple application of finite-dimensional arguments, see for instance [17], [18].

LEMMA 8.2. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$, such that*

(i) $\operatorname{ran} A$ *is closed;*

(ii) $\ker A$ *is finite-dimensional.*

Then A is a quasi-Fredholm relation.

Proof. Since $\ker A$ is finite-dimensional, it follows that $\dim(\ker A \cap \operatorname{ran} A^n)$ is a nonincreasing sequence and has therefore a limit. Hence, there is some smallest $d \in \mathbb{N}_0$ such that $\ker A \cap \operatorname{ran} A^d = \ker A \cap \operatorname{ran} A^{d+n}$ for all $n \in \mathbb{N}$, and thus $(Q1)$ is satisfied. The condition $(Q2)$ holds because $\ker A \cap \operatorname{ran} A^d$ is finite-dimensional, and as $\operatorname{ran} A$ is closed and $\dim \ker A^d \leqslant d \dim \ker A$ is finite (see [17], [18]), the condition $(Q3)$ is valid too. □

The following result is a straightforward consequence of Lemma 8.2.

LEMMA 8.3. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$, such that*

(i) $\operatorname{ran} A$ *is closed;*

(ii) $\mathfrak{H}/\operatorname{ran} A$ *is finite-dimensional.*

Then A is a quasi-Fredholm relation.

Proof. The adjoint A^* of A is a closed relation by definition. As A is closed and $\operatorname{ran} A$ is closed, it follows from Corollary 4.13 that $\operatorname{ran} A^*$ is closed. The identity

$$\ker A^* = \mathfrak{H} \ominus \operatorname{ran} A,$$

shows that $\ker A^*$ is finite-dimensional, since $\mathfrak{H}/\operatorname{ran} A$ is finite-dimensional. Now apply Lemma 8.2 with A replaced by A^*. This shows that A^* is quasi-Fredholm, and by Theorem 7.3 it follows that A is quasi-Fredholm. $\square$

A combination of Lemma 8.2 and Lemma 8.3 leads to the main result of this section.

PROPOSITION 8.4. *Let A be a semi-Fredholm relation in a Hilbert space $\mathfrak{H}$. Then A is a quasi-Fredholm relation.*

Hence the decomposition result of Kato for semi-Fredholm operators remains valid in the context of relations as follows from Theorem 5.2; see also Theorem 6.4.

COROLLARY 8.5. *A semi-Fredholm relation in a Hilbert space is Kato decomposable.*

REMARK 8.6. Parallel to semi-Fredholm relations one can introduce Browder relations and B-Fredholm relations. Results for the operator case can be found in, for instance, [14], [16].

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