UNITARY EQUIVALENCE TO A COMPLEX
SYMMETRIC MATRIX: GEOMETRIC CRITERIA

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Abstract. We develop several methods, based on the geometric relationship between the eigenspaces of a matrix and its adjoint, for determining whether a square matrix having distinct eigenvalues is unitarily equivalent to a complex symmetric matrix. Equivalently, we characterize those matrices having distinct eigenvalues which lie in the unitary orbit of the complex symmetric matrices.

1. Introduction

Our aim in this note is to develop simple geometric criteria for determining whether a given square matrix $T \in M_n(\mathbb{C})$ is unitarily equivalent to a complex symmetric matrix (UECSM). To be more specific, a *complex symmetric matrix* is a square matrix T with complex entries such that $T = T^t$ (the superscript t denotes the transpose operation) and two matrices $A, B \in M_n(\mathbb{C})$ are *unitarily equivalent* if there exists a unitary $U \in M_n(\mathbb{C})$ such that $A = U^*BU$.

Our primary motivation stems from the emerging theory of complex symmetric operators on Hilbert space (see [1, 2, 3, 4, 5, 8, 10], for instance). To be more specific, we say that a bounded operator T on a separable complex Hilbert space $\mathcal{H}$ is a *complex symmetric operator* if $T = CT^*C$ for some conjugation C (a conjugate-linear, isometric involution) on $\mathcal{H}$. The terminology stems from the fact that the preceding condition is equivalent to insisting that the operator have a complex symmetric matrix representation with respect to some orthonormal basis [3, Sect. 2.4-2.5].

From the preceding remarks, we see that the problem of determining whether a given matrix is UECSM is equivalent to determining whether that matrix represents a complex symmetric operator with respect to some orthonormal basis. From another perspective, we may view our main problem as part of a quest to determine the structure of the unitary orbit of the set of all complex symmetric matrices.

Complicating this endeavor, it is well-known that every $n \times n$ complex matrix is *similar* to a complex symmetric matrix [9, Thm. 4.4.9] (see also [4, Ex. 4] and [3, Thm. 2.3]). It follows that similarity invariants, such as the Jordan canonical form, are

useless when attempting to determine whether a given matrix is UECSM. This greatly complicates our work. For instance, one can show that among the matrices

$$\begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 6 \end{pmatrix},$$

all of which belong to the same similarity class, only the fourth is UECSM. In fact, prior to the recent advent of Tener's procedure UECSMTest [11], only a handful of matrices were known to be *not* UECSM.

In fact, we are partly motivated by Tener's UECSMTest. His procedure is based upon the diagonalization of the selfadjoint components A and B in the Cartesian decomposition $T = A + iB$. Although highly effective, it is often difficult to understand with this method, in simple geometric terms, *why* a given matrix is UECSM or not. In particular, studying the matrices A and B often gives little insight into the eigenstructure of T itself.

In this note, we proceed along a different route. We develop a number of procedures, based upon a direct examination of the eigenstructure of T, for testing whether T is UECSM or not. To this end, we require that T has distinct eigenvalues – a condition that is satisfied by all matrices outside of a set of Lebesgue measure zero in $M_n(\mathbb{C})$. On the other hand, Tener's UECSMTest requires that neither A nor B have a repeated eigenvalue. In Section 7, we consider several numerical examples and establish that neither our test nor UECSMTest subsumes the other. They should therefore be viewed as complimentary procedures.

2. Preliminary Setup

Let T be a $n \times n$ complex matrix having n *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let $u_1, u_2, \ldots, u_n$ denote *normalized* eigenvectors of T corresponding to the eigenvalues λ_i. Since

$$\det(T^* - \overline{\lambda_i}I) = \det[(T - \lambda_i I)^*] = \overline{\det(T - \lambda_i I)} = 0,$$

it follows immediately that T^* has the n *distinct* eigenvalues $\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}$. Let $v_1, v_2, \ldots, v_n$ denote *normalized* eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_i}$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, it follows that both $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ are bases for $\mathbb{C}^n$.

Based upon the data

$$\boxed{u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n,} \tag{1}$$

we wish to determine if T is unitarily equivalent to a complex symmetric matrix (UECSM). Before proceeding, we require a few preliminary lemmas.

LEMMA 1. *Under the hypotheses above we have* $\langle u_i, v_j \rangle = 0$ *whenever* $i \neq j$ *and* $\langle u_i, v_i \rangle \neq 0$.

Proof. If $i \neq j$, then

$$\lambda_i \langle u_i, v_j \rangle = \langle \lambda_i u_i, v_j \rangle = \langle Tu_i, v_j \rangle = \langle u_i, T^* v_j \rangle = \langle u_i, \overline{\lambda_j} v_j \rangle = \lambda_j \langle u_i, v_j \rangle$$

whence $\langle u_i, v_j \rangle = 0$ since $\lambda_i \neq \lambda_j$. On the other hand, if $\langle u_i, v_i \rangle = 0$ for some i, then by the preceding $\langle u_i, v_j \rangle = 0$ for $j = 1, 2, \ldots, n$. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{C}^n$, it would follow that $\langle u_i, x \rangle = 0$ for all $x \in \mathbb{C}^n$ whence $u_i = 0$. This contradiction shows that we must have $\langle u_i, v_i \rangle \neq 0$ for $i = 1, 2, \ldots, n$.

The following lemma allows us to easily express any $x \in \mathbb{C}^n$ in terms of the bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$:

LEMMA 2. *The following formulas hold for all $x \in \mathbb{C}^n$:*

$$x = \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{\langle v_j, u_j \rangle} v_j, \tag{2}$$

$$x = \sum_{j=1}^{n} \frac{\langle x, v_j \rangle}{\langle u_j, v_j \rangle} u_j. \tag{3}$$

Proof. By symmetry, it suffices to prove (3). Since $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{C}^n$ and since the expression (3) is linear in x, it suffices to verify (3) for $x = u_1, u_2, \ldots, u_n$. Since $\langle u_i, v_j \rangle = 0$ if $i \neq j$ and $\langle u_i, v_i \rangle \neq 0$, (3) can be verified immediately by setting $x = u_i$.

Lastly, we require a few words about a useful and practical way to view the property of being UECSM.

DEFINITION. A *conjugation* on $\mathbb{C}^n$ is a conjugate-linear operator $C : \mathbb{C}^n \to \mathbb{C}^n$ which is both *involutive* (i.e., $C^2 = I$) and *isometric* (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathbb{C}^n$).[1]

In particular, T is a complex symmetric matrix if and only if T is J-symmetric (i.e., $T = JT^*J$), where J denotes the *canonical conjugation*

$$J(z_1, z_2, \ldots, z_n) = (\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n}) \tag{4}$$

on $\mathbb{C}^n$. Moreover, the most general conjugation on $\mathbb{C}^n$ is easily seen to be of the form $C = SJ$ where S is a complex symmetric unitary matrix. Lastly, it is not hard to show that T is UECSM if and only if T is C-symmetric with respect to some conjugation C.

3. The angle test and its relatives

In this section we briefly outline several convenient geometric conditions which are necessary for a given $n \times n$ matrix T to be UECSM (unfortunately, none of these procedures is sufficient – see Example 5). Building upon this material, we present a condition in Section 5 which is both necessary *and* sufficient.

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[1] In light of the polarization identity, this is equivalent to $\|Cx\| = \|x\|$ for all $x \in \mathbb{C}^n$.

Recall that T is UECSM if and only if there exists a conjugation C on $\mathbb{C}^n$ such that $T = CT^*C$. If this holds, then it follows easily that

$$(T - \lambda I)^j x = 0 \quad \Leftrightarrow \quad (T^* - \overline{\lambda}I)^j (Cx) = 0. \tag{5}$$

Maintaining the notation and conventions of Section 2, we see that if T is C-symmetric, then the conjugation C maps the one-dimensional eigenspace of T corresponding to λ_i onto the one-dimensional eigenspace of T^* corresponding to $\overline{\lambda}_i$. This is where we invoke the hypothesis that the eigenvalues of T are distinct. Since C is isometric and the vectors u_i and v_i are normalized, it follows that there are *unimodular* constants α_i such that

$$Cu_i = \alpha_i v_i$$

for $i = 1, 2, \ldots, n$. Since C is isometric, this implies that

$$\begin{aligned}
\langle u_i, u_j \rangle &= \langle Cu_j, Cu_i \rangle \\
&= \langle \alpha_j v_j, \alpha_i v_i \rangle \\
&= \alpha_j \overline{\alpha_i} \langle v_j, v_i \rangle
\end{aligned} \tag{6}$$

for $1 \leqslant i, j \leqslant n$. Taking absolute values in the preceding and utilizing symmetry yields the following test which can be implemented easily in `Mathematica`:

THEOREM 1. (Angle Test) *Suppose that*

(i) *T is a $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

(ii) *$u_1, u_2, \ldots, u_n$ denote normalized eigenvectors of T corresponding to the eigenvalues λ_i,*

(iii) *$v_1, v_2, \ldots, v_n$ denote normalized eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda}_i$.*

Under these hypotheses, the condition $|\langle u_i, u_j \rangle| = |\langle v_i, v_j \rangle|$ for all $1 \leqslant i < j \leqslant n$ is necessary for T to be UECSM.

In light of the fact that Theorem 1 takes into consideration the (complex) angles between the eigenspaces of T and compares them to the (complex) angles between the eigenspaces of T^*, we refer to the procedure introduced in Theorem 1 as the *Angle Test*. One can interpret the Angle Test as asserting that the geometric relationship between the eigenspaces of T must precisely mirror the geometric relationship between the eigenspaces of T^*. In some sense, T and T^* must be perfect mirror images of each other. In Section 5, we present a refined version of Theorem 1 which yields a necessary and sufficient condition for T to be UECSM.

It turns out that the same principles can also be used in certain cases where the eigenvalues of T are not distinct. For instance in [4, Ex. 7], a similar argument is used to show that the matrix

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is not UECSM whenever $|a| \neq |b|$.

The condition (6) can also be interpreted in terms of Gram matrices. Let $U = (u_1|u_2|\cdots|u_n)$ and $V = (v_1|v_2|\cdots|v_n)$ and observe that (6) is equivalent to asserting that

$$(U^*U)^t = A^*(V^*V)A \tag{7}$$

holds where $A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes the diagonal unitary matrix having the unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ along the main diagonal. This leads us to the following test:

COROLLARY 1. (Grammian Test) *A necessary condition for T to be UECSM is that U^*U and V^*V have the same eigenvalues, repeated according to multiplicity.*

Proof. If T is UECSM, then (7) holds. Since U^*U is a positive matrix, it follows that U^*U and $(U^*U)^t$ are both unitarily equivalent to the same diagonal matrix, whence (7) implies that U^*U and V^*V are unitarily equivalent.

We should remark that Example 5 in Section 6 reveals that passing the Grammian Test is insufficient for a matrix to be UECSM. On the other hand, we show in Section 4 that (7) is both necessary and *sufficient* for T to be UECSM.

Taking the determinant of both sides of (7) immediately yields the following:

COROLLARY 2. (Parallelepiped Test) *Maintaining the notation above, if $|\det U| \neq |\det V|$, then T is not UECSM.*

The name *Parallelepiped Test* stems from the fact that $|\det U|^{\frac{1}{2}}$ and $|\det V|^{\frac{1}{2}}$ can be interpreted as the volumes of the generalized parallelepipeds in $\mathbb{C}^n$ spanned by the vectors $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, respectively.

The following example illustrates the preceding ideas:

EXAMPLE 1. We claim that the matrix

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is not UECSM. Letting $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$ we obtain the corresponding normalized eigenvectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \tag{8}$$

and

$$v_1 = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{9}$$

of T and T^*, respectively. Setting

$$U = \left(\begin{array}{c|c|c} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} \end{array} \right), \qquad V = \left(\begin{array}{c|c|c} -\frac{2}{3} & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{array} \right),$$

we immediately find that

$$|\det U| = \sqrt{\frac{2}{5}} \neq \frac{2}{3} = |\det V|$$

whence it follows from the Parallelepiped Test that T is not UECSM.

Moreover, we also have

$$(U^*U)^t = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} & 1 \end{pmatrix}, \qquad V^*V = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix} \qquad (10)$$

whence, by considering the moduli of the off-diagonal entries in (10), it is clear that no diagonal unitary matrix A exists which satisfies (7). Thus the Grammian Test also establishes that T is not UECSM.

Finally, let us take this opportunity to illustrate the Angle Test, which is less computationally intensive than either the Parallelepiped Test or the Grammian Test. A short calculation based upon the data (8) and (9) reveals that

$$|\langle u_1, u_2 \rangle| = \frac{1}{\sqrt{2}} \neq \frac{2}{3} = |\langle v_1, v_2 \rangle|$$

whence T is not UECSM. This can be also seen directly by examining the $(1,2)$ entry of the matrices in (10).

It is important to remark that none of the conditions described in Theorem 1, Corollary 1, or Corollary 2, are sufficient for T to be UECSM. This is illustrated in a series of rather involved computations (see Example 5) that we postpone until later. In Section 5 we remedy this situation and provide a test which is both necessary and sufficient.

4. Constructing a Conjugation

Under our running hypotheses, it turns out that the condition (6) is sufficient for T to be UECSM (in particular, so is the Gram matrix condition (7)). The following lemma is the main workhorse upon which the rest of this note is based:

LEMMA 3. *Let*

(i) *T be a $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

(ii) *$u_1, u_2, \ldots, u_n$ denote normalized eigenvectors of T corresponding to the eigenvalues λ_i,*

(iii) *$v_1, v_2, \ldots, v_n$ denote normalized eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_i}$.*

If unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ exist such that

$$\langle u_i, u_j \rangle = \overline{\alpha_i} \alpha_j \langle v_j, v_i \rangle \tag{11}$$

holds for $1 \leqslant i < j \leqslant n$, then T is UECSM.

Proof. First observe that if (11) holds for $1 \leqslant i < j \leqslant n$, then (11) holds whenever $1 \leqslant i, j \leqslant n$ by symmetry and the fact that we are considering normalized eigenvectors. Let $C u_i = \alpha_i v_i$ for $i = 1, 2, \ldots, n$ and extend this by conjugate-linearity to all of $\mathbb{C}^n$. We intend to show that C is a conjugation with respect to which T is C-symmetric. Since C is conjugate-linear by definition, it suffices to show that C is involutive and isometric.

Step 1: Show that C is involutive (i.e., $C^2 = I$).

Since C^2 is linear, it suffices to verify that $C^2 u_i = u_i$ for $i = 1, 2, \ldots, n$. By (3) it follows that

$$C u_i = \alpha_i v_i = \alpha_i \sum_{j=1}^{n} \frac{\langle v_i, v_j \rangle}{\langle u_j, v_j \rangle} u_j \tag{12}$$

whence

$$
\begin{aligned}
C^2 u_i &= C \left(\alpha_i \sum_{j=1}^{n} \frac{\langle v_i, v_j \rangle}{\langle u_j, v_j \rangle} u_j \right) && \text{by (12)} \\
&= \overline{\alpha_i} \sum_{j=1}^{n} \frac{\langle v_j, v_i \rangle}{\langle v_j, u_j \rangle} C u_j && \text{conjugate-linearity} \\
&= \overline{\alpha_i} \sum_{j=1}^{n} \frac{\langle v_j, v_i \rangle}{\langle v_j, u_j \rangle} \alpha_j v_j && \text{definition of } C \\
&= \overline{\alpha_i} \sum_{j=1}^{n} \frac{\alpha_i \overline{\alpha_j} \langle u_i, u_j \rangle}{\langle v_j, u_j \rangle} \alpha_j v_j && \text{by (11)} \\
&= \sum_{j=1}^{n} \frac{\langle u_i, u_j \rangle}{\langle v_j, u_j \rangle} v_j && \\
&= u_i. && \text{by (2)}
\end{aligned}
$$

Thus C is involutive.

Step 2: Show that C is isometric (i.e., $\|Cx\| = \|x\|$ for all $x \in \mathbb{C}^n$).

If $x = \sum_{i=1}^n c_i u_i$, then observe that

$$
\begin{aligned}
\|x\|^2 &= \langle x, x \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \langle u_i, u_j \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \overline{\alpha_i} \alpha_j \langle v_j, v_i \rangle && \text{by (11)} \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \langle \alpha_j v_j, \alpha_i v_i \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \langle Cu_j, Cu_i \rangle && \text{definition of } C \\
&= \sum_{i,j=1}^n \langle \overline{c_j} Cu_j, \overline{c_i} Cu_i \rangle \\
&= \langle \sum_{j=1}^n \overline{c_j} Cu_j, \sum_{i=1}^n \overline{c_i} Cu_i \rangle \\
&= \langle Cx, Cx \rangle \\
&= \|Cx\|^2.
\end{aligned}
$$

Thus C is isometric whence C is a conjugation on $\mathbb{C}^n$.

Step 3: Show that T is C-symmetric (i.e., $T = CT^*C$).

Since both T and CT^*C are linear, it suffices to prove that they agree on the basis $u_1, u_2, \ldots, u_n$. Having shown that $C^2 = I$, it now follows from the equation $Cu_i = \alpha_i v_i$ and the conjugate-linearity of C that $Cv_i = \alpha_i u_i$. Thus

$$
\begin{aligned}
CT^*Cu_i &= CT^*(\alpha_i v_i) = \overline{\alpha_i} CT^* v_i = \overline{\alpha_i} C \overline{\lambda_i} v_i \\
&= \overline{\alpha_i} \lambda_i C v_i = \overline{\alpha_i} \lambda_i \alpha_i u_i = \lambda_i u_i = Tu_i
\end{aligned}
$$

whence T is C-symmetric and hence UECSM.

The conjugation C constructed by Lemma 3 can be concretely realized as $C = SJ$ where S is a complex symmetric unitary matrix and J denotes the canonical conjugation (4) on $\mathbb{C}^n$. Let us briefly describe the construction of the matrix S.

First observe that C satisfies $Cu_i = \alpha_i v_i$, which is easily seen to be equivalent to $Cv_i = \alpha_i u_i$ for $i = 1, 2, \ldots, n$. As before, let $U = (u_1 | u_2 | \cdots | u_n)$ and $V = (v_1 | v_2 | \cdots | v_n)$ denote the matrices having the vectors $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, respectively, as columns. Since the columns of U and V form bases of $\mathbb{C}^n$, it follows that both of these

matrices are invertible. Next we note that

$$V^*U = \begin{pmatrix} \overline{v_1} \\ \overline{v_2} \\ \vdots \\ \overline{v_n} \end{pmatrix} (u_1|u_2|\cdots|u_n) = \begin{pmatrix} \langle u_1, v_1 \rangle & & & \\ & \langle u_2, v_2 \rangle & & \\ & & \ddots & \\ & & & \langle u_n, v_n \rangle \end{pmatrix} = E$$

by Lemma 1. Let

$$A = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \tag{13}$$

and

$$D = AE^{-1} = \begin{pmatrix} \frac{\alpha_1}{\langle u_1, v_1 \rangle} & & & \\ & \frac{\alpha_2}{\langle u_2, v_2 \rangle} & & \\ & & \ddots & \\ & & & \frac{\alpha_n}{\langle u_n, v_n \rangle} \end{pmatrix}. \tag{14}$$

We claim that $C = SJ$ where

$$S = UDU^t. \tag{15}$$

To prove (15), it suffices to show that the conjugate-linear operators C and SJ agree on each of the vectors v_i. In other words, we must show that $SJv_i = \alpha_i u_i$ for $i = 1, 2, \ldots, n$. Letting $s_1, s_2, \ldots, s_n$ denote the standard basis for $\mathbb{C}^n$ we have

$$\begin{aligned}
SJv_i &= UDU^t Jv_i & &\text{by (15)} \\
&= UAE^{-1}U^t Jv_i & &\text{by (14)} \\
&= UAE^{-1}JU^* v_i & &\text{since } JU^* = U^t J \\
&= UAJ\overline{E}^{-1}U^* v_i & &\text{since } J\overline{E}^{-1} = E^{-1}J \\
&= UAJV^{-1} v_i & &\text{since } V^{-1} = \overline{E}^{-1}U^* \\
&= UAJs_i & &\text{def. of } V \\
&= UAs_i & &\text{since } Js_i = s_i \\
&= U\alpha_i s_i & &\text{by (13)} \\
&= \alpha_i u_i. & &\text{def. of } U
\end{aligned}$$

Thus $C = SJ$. Since the matrix $D = AE^{-1}$ is diagonal, it is clear from (15) that S is symmetric. Since $S = CJ$ is the product of two conjugations, it is an invertible isometry and hence unitary (see also [5, Lem. 1]).

It is worth remarking that the condition $T = CT^*C$ implies that $T = SJT^*SJ = ST^t JSJ = ST^t S^*$ since S is symmetric (i.e., S is J-symmetric). Therefore the matrix S yields a unitary equivalence between T and its transpose T^t.

5. The Strong Angle Test

The main theorem of this article is the following necessary and sufficient condition for a matrix with distinct eigenvalues to be UECSM. The procedure introduced in the following theorem was implemented in Mathematica by the first author. We refer to this procedure as StrongAngleTest.

THEOREM 2. (Strong Angle Test) *If*

(i) *T is a $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

(ii) *$u_1, u_2, \ldots, u_n$ denote normalized eigenvectors of T corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

(iii) *$v_1, v_2, \ldots, v_n$ denote normalized eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}$,*

then T is UECSM if and only if the condition

$$\langle u_i, u_j \rangle \langle u_j, u_k \rangle \langle u_k, u_i \rangle = \overline{\langle v_i, v_j \rangle \langle v_j, v_k \rangle \langle v_k, v_i \rangle} \tag{16}$$

holds whenever $1 \leqslant i \leqslant j \leqslant k \leqslant n$ and not all of i, j, k are equal.[2]

Proof. The necessity of the condition (16) follows immediately from (6). The proof that (16) is sufficient for T to be UECSM is more complicated. First observe that if (16) holds for $1 \leqslant i \leqslant j \leqslant k \leqslant n$, then (16) holds whenever $1 \leqslant i, j, k \leqslant n$ by symmetry. Let us assume for the moment that $\langle u_i, u_j \rangle \neq 0$ (whence $\langle v_j, v_i \rangle \neq 0$) for $1 \leqslant i, j \leqslant n$. Later we will relax this restriction, but for the sake of clarity it will be easier to consider this special case first. Under this additional hypothesis, there exist n^2 unimodular constants β_{ij} uniquely determined by

$$\beta_{ij} = \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle} \tag{17}$$

for $1 \leqslant i, j \leqslant n$. Since $\|u_i\| = \|v_i\| = 1$ by hypotheses (ii) and (iii), it follows immediately that that $\beta_{ii} = 1$ for $1 \leqslant i \leqslant n$. Moreover, we also have

$$\beta_{ij} \langle v_j, v_i \rangle = \langle u_i, u_j \rangle = \overline{\langle u_j, u_i \rangle} = \overline{\beta_{ji} \langle v_i, v_j \rangle} = \overline{\beta_{ji}} \langle v_j, v_i \rangle,$$

whence $\beta_{ij} = \overline{\beta_{ji}}$. In other words, the matrix $B = (\beta_{ij})_{i,j=1}^n$ is selfadjoint and has constant diagonal 1. Suppose for the moment that, based on the hypothesis (16), we

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[2]Observe that setting $k = j$ in condition (16) leads to $|\langle u_i, u_j \rangle| = |\langle v_i, v_j \rangle|$ for $1 \leqslant i \leqslant j \leqslant n$. Thus Theorem 2 can be viewed as an extension of the original Angle Test (Theorem 1). Also note that if $i = j = k$, then (16) merely asserts that $\|u_i\| = \|v_i\|$ which is already known from conditions (ii) and (iii).

are able to establish that B enjoys a factorization of the form

$$\begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} \\ \beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2n} \\ \beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \tag{18}$$

(i.e., suppose that we are able to show that B is positive and has rank one). By (17) and the preceding factorization (18) it would then follow that the unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy

$$\langle u_i, u_j \rangle = \overline{\alpha_i} \alpha_j \langle v_j, v_i \rangle \tag{19}$$

for $1 \leqslant i, j \leqslant n$. At this point, we could invoke Lemma 3 to conclude that T is UECSM.

The difficulty in the approach outlined above lies in the fact that some of the inner products $\langle u_i, u_j \rangle$ or $\langle v_j, v_i \rangle$ may vanish. If this occurs, then we cannot immediately consider the associated unimodular constants β_{ij} defined by (17) since applying (16) with $k = i$ implies that $\langle v_j, v_i \rangle = 0$ if and only if $\langle u_i, u_j \rangle = 0$. On the other hand, observe that the hypothesis (16) implies that

$$\beta_{ij} \beta_{jk} \beta_{ki} = \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle} \frac{\langle u_j, u_k \rangle}{\langle v_k, v_j \rangle} \frac{\langle u_k, u_i \rangle}{\langle v_i, v_k \rangle} = 1$$

holds whenever $\beta_{ij}, \beta_{jk}, \beta_{ki}$ are well-defined by (17). In light of the fact that each β_{ij} is unimodular, we obtain the following *multiplicative property*

$$\beta_{ij} = \beta_{ik} \beta_{kj} \tag{20}$$

whenever the expressions above are well-defined by (17).

Regarding the matrix $B = (\beta_{ij})_{i,j=1}^n$ as being only partially defined by (17), suppose for the moment that we are able to define unimodular constants β_{ij} for those i and j for which $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle = 0$ such that the multiplicative property (20) holds for all $1 \leqslant i, j, k \leqslant n$. Under this hypothesis, we claim that the matrix $B = (\beta_{ij})_{i,j=1}^n$ has a factorization of the form (18). Indeed, use (20) and the fact that $\beta_{ji} = \overline{\beta_{ij}}$ for $1 \leqslant i, j \leqslant n$ to conclude that

$$B = \begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} \\ \beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2n} \\ \beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \overline{\beta_{12}} \\ \overline{\beta_{13}} \\ \vdots \\ \overline{\beta_{1n}} \end{pmatrix} \begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} \end{pmatrix}. \tag{21}$$

As suggested by (18), we now define the unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ by setting $\alpha_i = \beta_{1i}$ for $1 \leqslant i \leqslant n$. Next observe that

$$\overline{\alpha_i} \alpha_j = \overline{\beta_{1i}} \beta_{1j} = \beta_{i1} \beta_{1j} = \beta_{ij} = \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle}$$

holds whenever β_{ij} is well-defined by (17). Thus the desired condition (19) holds for all $1 \leqslant i, j \leqslant n$ (since it holds trivially if $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle = 0$) and T is UECSM by Lemma 3.

To complete the proof of Theorem 2 it suffices to demonstrate a procedure by which we may define unimodular constants β_{ij} for those i and j for which $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle = 0$ such that the multiplicative property (20) holds for all $1 \leqslant i, j \leqslant n$. This will lead us to the desired matrix factorization (21).

To define the constants β_{ij} we employ an inductive procedure. Consider the partially defined $n \times n$ matrix

$$
\left(
\begin{array}{ccccc|ccc}
1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} & * & \cdots & * \\
\beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2r} & * & \cdots & * \\
\beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3r} & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\beta_{r1} & \beta_{r2} & \beta_{r3} & \cdots & 1 & * & \cdots & * \\
\hline
* & * & * & \cdots & * & 1 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & * & \cdots & 1
\end{array}
\right)
\tag{22}
$$

where $*$ indicates either an entry β_{ij} already defined by (17) or an entry that is not defined in terms of (17) because $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle = 0$. As our inductive hypothesis, we assume that the multiplicative property (20) is satisfied by all triples $\beta_{ij}, \beta_{ik}, \beta_{kj}$ for which $1 \leqslant i, j, k \leqslant r$.

To complete the proof Theorem 2, we must devise a way to fill out the undefined entries in (22) with unimodular constants β_{ij} in such a way that (20) holds for these new entries. There are two cases to consider:

Case 1: Suppose that there exists an entry $\beta_{i(r+1)}$ with $1 \leqslant i \leqslant r$ in (22) that is already defined by (17). Without loss of generality, we may assume that it is the $\beta_{1(r+1)}$ is the entry that is well-defined by (17) since this situation may be obtained by permuting the indices $1, 2, \ldots, r$ and relabeling the eigenvectors $u_1, u_2, \ldots, u_r; v_1, v_2, \ldots, v_r$. We are thus left with the partially completed matrix

$$
\left(
\begin{array}{ccccc|c|ccc}
1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} & \beta_{1(r+1)} & * & \cdots & * \\
\beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2r} & * & * & \cdots & * \\
\beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3r} & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{r1} & \beta_{r2} & \beta_{r3} & \cdots & 1 & * & * & \cdots & * \\
\hline
\beta_{(r+1)1} & * & * & \cdots & * & 1 & * & \cdots & * \\
* & * & * & \cdots & * & * & 1 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & * & * & \cdots & 1
\end{array}
\right) .
$$

For each entry $\beta_{i(r+1)}$ with $2 \leqslant i \leqslant r$ (i.e., the entries immediately below $\beta_{1(r+1)}$ and above the 1 on the main diagonal) there are two possibilities:

Subcase 1.1: If $\beta_{i(r+1)}$ is already well-defined by (17), then do nothing.

Subcase 1.2: If $\beta_{i(r+1)}$ cannot be defined by (17) because $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle = 0$, then let

$$\beta_{i(r+1)} := \beta_{i1}\beta_{1(r+1)} \tag{23}$$

to obtain the partially defined matrix[3]

$$
\begin{pmatrix}
1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} & \beta_{1(r+1)} & * & \cdots & * \\
\beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2r} & \beta_{2(r+1)} & * & \cdots & * \\
\beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3r} & \beta_{3(r+1)} & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{r1} & \beta_{r2} & \beta_{r3} & \cdots & 1 & \beta_{r(r+1)} & * & \cdots & * \\
\beta_{(r+1)1} & \beta_{(r+1)2} & \beta_{(r+1)3} & \cdots & \beta_{(r+1)r} & 1 & * & \cdots & * \\
* & * & * & \cdots & * & * & 1 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & * & * & \cdots & 1
\end{pmatrix} . \tag{24}
$$

Case 2: Suppose that there does not exist an entry $\beta_{i(r+1)}$ with $1 \leqslant i \leqslant r$ in (22) that is already defined by (17). In other words, suppose that $\langle u_i, u_{r+1} \rangle = \langle v_{r+1}, v_i \rangle = 0$ whenever $1 \leqslant i \leqslant r$. We are thus left with the partially completed matrix

$$
\begin{pmatrix}
1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} & * & * & \cdots & * \\
\beta_{21} & 1 & \beta_{23} & \cdots & \beta_{2r} & * & * & \cdots & * \\
\beta_{31} & \beta_{32} & 1 & \cdots & \beta_{3r} & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{r1} & \beta_{r2} & \beta_{r3} & \cdots & 1 & * & * & \cdots & * \\
* & * & * & \cdots & * & 1 & * & \cdots & * \\
* & * & * & \cdots & * & * & 1 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & * & * & \cdots & 1
\end{pmatrix} .
$$

Select a complex number of unit modulus and assign this value to $\beta_{1(r+1)}$. Having done this, we define $\beta_{i(r+1)}$ for $1 \leqslant i \leqslant r$ as in (23) to obtain a partially defined matrix of the form (24).

To wrap-up the proof, we must show that in either case (23) defines the new entries $\beta_{i(r+1)}$ in a manner which is consistent with the multiplicative property (20). For $1 \leqslant i, k \leqslant r$ we employ the definition (23) to find that

$$\beta_{ik}\beta_{k(r+1)} = (\beta_{i1}\beta_{1k})(\beta_{k1}\beta_{1(r+1)}) \qquad \text{by inductive hypothesis and (23)}$$

$$= \beta_{i1}(\beta_{1k}\overline{\beta_{1k}})\beta_{1(r+1)} \qquad \text{hermitian symmetry}$$

---

[3]The entries $\beta_{(r+1)1}, \beta_{(r+1)2}, \beta_{(r+1)3}, \ldots, \beta_{r(r+1)}$ in the $(r+1)$st row are defined by conjugate symmetry: $\beta_{ij} = \overline{\beta_{ji}}$.

$$= \beta_{i1}\beta_{1(r+1)} \qquad \text{since } |\beta_{1k}| = 1$$
$$= \beta_{i(r+1)}. \qquad \text{by (23)}$$

Thus (23) defines $\beta_{i(r+1)}$ for $1 \leqslant i,j \leqslant r+1$ in a manner consistent with (20).

Starting with the upper left 1×1 block, repeated applications of the preceding inductive procedure eventually yields an $n \times n$ matrix $B = (\beta_{ij})_{i,j=1}^{n}$ whose entries satisfy the required multiplicative condition (20). This concludes the proof of Theorem 2.

COROLLARY 3. *Every 2×2 matrix is UECSM.*

Proof. Let T be a 2×2 matrix. If T has a repeated eigenvalue λ, then by Schur's Theorem on Unitary Upper Triangularization, it follows that $T - \lambda I$ is unitarily equivalent to a scalar multiple of a 2×2 nilpotent Jordan matrix. This Jordan matrix is C-symmetric with respect to $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ whence T is UECSM. We therefore restrict our attention to the case where T has two distinct eigenvalues. Upon applying Schur's Theorem, subtracting a suitable multiple of the identity, and normalizing, we may assume that

$$T = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} \tag{25}$$

for some complex constant a. A short computation reveals that normalized eigenvectors for T corresponding to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$ are

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{a}{\sqrt{1+|a|^2}} \\ \frac{1}{\sqrt{1+|a|^2}} \end{pmatrix}. \tag{26}$$

Similarly, we find that corresponding normalized eigenvectors for T^* are given by

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{1+|a|^2}} \\ \frac{-\overline{a}}{\sqrt{1+|a|^2}} \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{27}$$

By Theorem 2, T is UECSM if and only if (16) holds for all $1 \leqslant i \leqslant j \leqslant k \leqslant 2$ such that not all of i, j, k are equal. This leaves us only two ordered triples (i, j, k) to consider: $(1, 1, 2)$ and $(1, 2, 2)$. These values of i, j, k both lead to the condition $|\langle u_1, u_2 \rangle| = |\langle v_1, v_2 \rangle|$ which needs to be verified. Since

$$|\langle u_1, u_2 \rangle| = \frac{|a|}{\sqrt{1+|a|^2}} = |\langle v_1, v_2 \rangle|$$

follows immediately from (26) and (27), we conclude that T is UECSM.

The preceding corollary has been proved in a number of different ways by several different authors. For instance, one can reduce to the special case (25) as above and then construct the corresponding conjugation by straightforward computation [4, Ex. 6]. The procedure developed by J. Tener can also be used to establish Corollary 3

[11, Cor. 3]. We should also mention that Corollary 3 is the byproduct of more sophisticated theorems. For instance, it follows immediately from N. Chevrot, E. Fricain, and D. Timotin's study of the characteristic functions of complex symmetric contractions [1, Cor. 3.3]. More recently, the second author and W. Wogen established that every *binormal* operator (i.e., an operator that is unitarily equivalent to a 2×2 block operator whose entries are commuting normal operators) is complex symmetric [6]. Corollary 3 is a special case of this result.

6. A few examples

To illustrate the preceding ideas, we devote this section to the detailed consideration of several examples. In particular, Example 5 demonstrates that none of the simple conditions given in Section 3 is sufficient for T to be UECSM.

EXAMPLE 2. Let T be a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$. As before, let u_1, u_2 denote normalized eigenvectors of T corresponding to the eigenvalues λ_1, λ_2, respectively and let v_1, v_2 denote normalized eigenvectors of T^* corresponding to the eigenvalules $\overline{\lambda_1}, \overline{\lambda_2}$, respectively.

By Corollary 3, we know that T is UECSM and hence $|\langle u_1, u_2 \rangle| = |\langle v_2, v_1 \rangle|$. We may therefore find unimodular constants α_1 and α_2 such that

$$\langle u_1, u_2 \rangle = \overline{\alpha_1} \alpha_2 \langle v_2, v_1 \rangle.$$

For instance, if T is normal, then we may simply set $\alpha_1 = \alpha_2 = 1$ since $\langle u_1, u_2 \rangle = \langle v_2, v_1 \rangle = 0$. Letting

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

denote the matrix whose columns are the eigenvectors u_1, u_2 of T we follow the procedure outlined at the end of Section 4 to construct the conjugation

$$C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha_1 u_{11}^2}{\langle u_1, v_1 \rangle} + \dfrac{\alpha_2 u_{12}^2}{\langle u_2, v_2 \rangle} & \dfrac{\alpha_1 u_{11} u_{21}}{\langle u_1, v_1 \rangle} + \dfrac{\alpha_2 u_{12} u_{22}}{\langle u_2, v_2 \rangle} \\ \dfrac{\alpha_1 u_{11} u_{21}}{\langle u_1, v_1 \rangle} + \dfrac{\alpha_2 u_{12} u_{22}}{\langle u_2, v_2 \rangle} & \dfrac{\alpha_1 u_{21}^2}{\langle u_1, v_1 \rangle} + \dfrac{\alpha_2 u_{22}^2}{\langle u_2, v_2 \rangle} \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \tag{28}$$

with respect to which T is C-symmetric.

EXAMPLE 3. Applying the preceding formula to the matrix (25), where $a \neq 0$, and using the data (26) and (27) we find that

$$\langle u_1, u_2 \rangle = \frac{\overline{a}}{\sqrt{1 + |a|^2}}, \qquad \langle v_2, v_1 \rangle = -\frac{a}{\sqrt{1 + |a|^2}}.$$

Following the notation of Example 2, note that

$$u_{11} = 1 \qquad\qquad u_{12} = \frac{a}{\sqrt{1 + |a|^2}},$$

$$u_{21} = 0 \qquad\qquad u_{22} = \frac{1}{\sqrt{1+|a|^2}}.$$

One possible solution[4] to the equation

$$\langle u_1, u_2 \rangle = \overline{\alpha_1}\alpha_2 \langle v_2, v_1 \rangle$$

is given by

$$\alpha_1 = \frac{a}{|a|}, \qquad \alpha_2 = -\frac{\overline{a}}{|a|}.$$

Plugging this data into (28) we obtain the conjugation

$$C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a/|a|}{\sqrt{1+|a|^2}} & -\frac{|a|}{\sqrt{1+|a|^2}} \\ -\frac{|a|}{\sqrt{1+|a|^2}} & \frac{-\overline{a}/|a|}{\sqrt{1+|a|^2}} \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$$

with respect to which the matrix (25) is C-symmetric.

EXAMPLE 4. In [11, Ex. 3], the matrix

$$T = \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{pmatrix}$$

is demonstrated to be UECSM via Tener's `UECSMTest`. For the sake of comparison, let us also consider this matrix using the techniques discussed above. Letting $\lambda_0 = 6$, $\lambda_1 = 1$, and $\lambda_2 = 0$, we obtain the corresponding normalized eigenvectors

$$u_1 = \begin{pmatrix} -\frac{7}{11} \\ -\frac{6}{11} \\ \frac{6}{11} \end{pmatrix}, \qquad u_2 = \begin{pmatrix} \frac{7}{5\sqrt{2}} \\ \frac{1}{5\sqrt{2}} \\ 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

of T and

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -\frac{6}{55} \\ \frac{42}{55} \\ \frac{7}{11} \end{pmatrix}$$

of T^*, respectively. A short computation reveals that

$$\langle u_1, u_2 \rangle = -\frac{1}{\sqrt{2}}, \qquad \langle u_2, u_3 \rangle = \frac{7}{5\sqrt{2}}, \qquad \langle u_3, u_1 \rangle = -\frac{7}{11},$$

$$\langle v_1, v_2 \rangle = \frac{1}{\sqrt{2}}, \qquad \langle v_2, v_3 \rangle = \frac{7}{5\sqrt{2}}, \qquad \langle v_3, v_1 \rangle = \frac{7}{11},$$

---

[4]The other solutions will simply yield a unimodular multiple of S.

whence it is clear that (16) holds for all triples

$$(i, j, k) = (1,1,2), (1,1,3), (1,2,2), (1,2,3), (1,3,3), (2,2,3), (2,3,3)$$

required by `StrongAngleTest`. In particular, this proves that T is UECSM.

The corresponding matrices $U = (u_1|u_2|u_3)$ and $V = (v_1|v_2|v_3)$ are

$$U = \begin{pmatrix} -\frac{7}{11} & \frac{7}{5\sqrt{2}} & 1 \\ -\frac{6}{11} & \frac{1}{5\sqrt{2}} & 0 \\ \frac{6}{11} & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 & -\frac{6}{55} \\ 0 & \frac{1}{\sqrt{2}} & \frac{42}{55} \\ 1 & \frac{1}{\sqrt{2}} & \frac{7}{11} \end{pmatrix}.$$

As expected, T passes the Parallelepiped Test (Corollary 2) since

$$|\det U| = |\det V| = \frac{3\sqrt{2}}{55}.$$

Next, observe that

$$U^*U = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{7}{11} \\ -\frac{1}{\sqrt{2}} & 1 & \frac{7}{5\sqrt{2}} \\ -\frac{7}{11} & \frac{7}{5\sqrt{2}} & 1 \end{pmatrix}, \qquad V^*V = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{7}{11} \\ \frac{1}{\sqrt{2}} & 1 & \frac{7}{5\sqrt{2}} \\ \frac{7}{11} & \frac{7}{5\sqrt{2}} & 1 \end{pmatrix}$$

whence T passes the Grammian Test (Corollary 1) with the A from (7) being

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

In particular, this once again confirms that T is UECSM.

Let us, for the moment, examine the mechanics of the proof of Theorem 2, which establishes the theoretical underpinnings of the procedure `StrongAngleTest`. Using the data above, we find that the matrix $B = (\beta_{ij})$ from the proof of Theorem 2 is given by

$$B = \frac{(U^*U)^t}{V^*V} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} (1 \ -1 \ -1)$$

whence we again read that $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -1$. We remind the reader that the quotient appearing in the preceding equation is simply the entry-by-entry quotient of the matrices $(U^*U)^t$ and V^*V.

Based upon the preceding calculations, we can construct the corresponding conjugation $C = SJ$ where S is a complex symmetric unitary matrix which is to be determined (this matrix also has the property that $T = ST^tS^*$). Following the recipe

described at the end of Section 4 we obtain

$$E = V^*U = \begin{pmatrix} \frac{6}{11} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & -\frac{6}{55} \end{pmatrix}$$

so that

$$D = AE^{-1} = \begin{pmatrix} \frac{11}{6} & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & \frac{55}{6} \end{pmatrix}$$

Putting this all together we find that

$$S = UDU^t = \begin{pmatrix} \frac{6}{55} & -\frac{42}{55} & -\frac{7}{11} \\ -\frac{42}{55} & \frac{19}{55} & -\frac{6}{11} \\ -\frac{7}{11} & -\frac{6}{11} & \frac{6}{11} \end{pmatrix}.$$

It turns out that our S differs from the corresponding matrix obtained in [11, Ex. 3] by a unimodular multiplicative factor of $\frac{-19+6i\sqrt{74}}{55}$.

The following important example demonstrates that the Angle Test (Theorem 1), the Grammian Test (Corollary 1), and the Parallelepiped Test (Corollary 2) are insufficient to determine whether a given matrix is UECSM. In particular, this demonstrates the utility of the Strong Angle Test (Theorem 2), which provides a necessary *and* sufficient condition.

EXAMPLE 5. Consider the matrix

$$T = \begin{pmatrix} 5 & 0 & -1 & 3 \\ 2 & 4 & 1 & 2 \\ 2 & -2 & 6 & -2 \\ 0 & -2 & 1 & 4 \end{pmatrix},$$

which has the distinct eigenvalues

$$\lambda_1 = 5 + i\sqrt{5}, \quad \lambda_2 = 5 - i\sqrt{5}, \quad \lambda_3 = \tfrac{1}{2}(9 + i\sqrt{15}), \quad \lambda_4 = \tfrac{1}{2}(9 - i\sqrt{15}),$$

and corresponding unit eigenvectors

$$\underbrace{\begin{pmatrix} -\frac{2i}{3} - \frac{1}{3\sqrt{5}} \\ \frac{1}{15}\left(-5i - 2\sqrt{5}\right) \\ \frac{1}{15}\left(5i - \sqrt{5}\right) \\ \frac{1}{\sqrt{5}} \end{pmatrix}}_{u_1}, \quad \underbrace{\begin{pmatrix} \frac{2i}{3} - \frac{1}{3\sqrt{5}} \\ \frac{1}{15}\left(5i - 2\sqrt{5}\right) \\ \frac{1}{15}\left(-5i - \sqrt{5}\right) \\ \frac{1}{\sqrt{5}} \end{pmatrix}}_{u_2}, \quad \underbrace{\begin{pmatrix} -\frac{i(-5i + \sqrt{15})}{2\sqrt{30}} \\ 0 \\ \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \end{pmatrix}}_{u_3}, \quad \underbrace{\begin{pmatrix} \frac{i(5i + \sqrt{15})}{2\sqrt{30}} \\ 0 \\ -\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{15}} \end{pmatrix}}_{u_4}.$$

The matrix T^* has the eigenvalues

$$\overline{\lambda_1} = 5 - i\sqrt{5}, \quad \overline{\lambda_2} = 5 + i\sqrt{5}, \quad \overline{\lambda_3} = \tfrac{1}{2}(9 - i\sqrt{15}), \quad \overline{\lambda_4} = \tfrac{1}{2}(9 + i\sqrt{15}),$$

and corresponding unit eigenvectors

$$\underbrace{\begin{pmatrix} \frac{2}{\sqrt{15}} \\ \frac{-1-i\sqrt{5}}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix} \frac{2}{\sqrt{15}} \\ \frac{i(i+\sqrt{5})}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix} \frac{i}{2} - \frac{1}{2\sqrt{15}} \\ \frac{1}{10}\left(-5i + \sqrt{15}\right) \\ \frac{1}{30}\left(15i + \sqrt{15}\right) \\ \frac{1}{\sqrt{15}} \end{pmatrix}}_{v_3}, \quad \underbrace{\begin{pmatrix} -\frac{i}{2} - \frac{1}{2\sqrt{15}} \\ \frac{1}{10}\left(5i + \sqrt{15}\right) \\ \frac{1}{30}\left(-15i + \sqrt{15}\right) \\ \frac{1}{\sqrt{15}} \end{pmatrix}}_{v_4}.$$

The matrices $U = (u_1 | u_2 | u_3 | u_4)$ and $V = (v_1 | v_2 | v_3 | v_4)$ satisfy

$$|\det U| = \frac{2}{5\sqrt{3}} = |\det V|$$

whence T passes the Parallelepiped Test (Corollary 2).

A further computation reveals that the matrices

$$U^*U = \begin{pmatrix} 1 & -\frac{1}{3} - \frac{2i}{3\sqrt{5}} & \frac{4+\sqrt{3}}{6\sqrt{2}} - \frac{i(1+4\sqrt{3})}{6\sqrt{10}} & \frac{-4+\sqrt{3}}{6\sqrt{2}} - \frac{i(-1+4\sqrt{3})}{6\sqrt{10}} \\ -\frac{1}{3} + \frac{2i}{3\sqrt{5}} & 1 & \frac{-4+\sqrt{3}}{6\sqrt{2}} + \frac{i(-1+4\sqrt{3})}{6\sqrt{10}} & \frac{4+\sqrt{3}}{6\sqrt{2}} + \frac{i(1+4\sqrt{3})}{6\sqrt{10}} \\ \frac{4+\sqrt{3}}{6\sqrt{2}} + \frac{i(1+4\sqrt{3})}{6\sqrt{10}} & \frac{-4+\sqrt{3}}{6\sqrt{2}} - \frac{i(-1+4\sqrt{3})}{6\sqrt{10}} & 1 & -\frac{1}{4} - \frac{3}{4}i\sqrt{\frac{3}{5}} \\ \frac{-4+\sqrt{3}}{6\sqrt{2}} + \frac{i(-1+4\sqrt{3})}{6\sqrt{10}} & \frac{4+\sqrt{3}}{6\sqrt{2}} - \frac{i(1+4\sqrt{3})}{6\sqrt{10}} & -\frac{1}{4} + \frac{3}{4}i\sqrt{\frac{3}{5}} & 1 \end{pmatrix},$$

$$V^*V = \begin{pmatrix} 1 & \frac{1}{3} - \frac{2i}{3\sqrt{5}} & \frac{1}{2\sqrt{3}} + \frac{i(3+4\sqrt{3})}{6\sqrt{5}} & -\frac{1}{2\sqrt{3}} - \frac{i(-3+4\sqrt{3})}{6\sqrt{5}} \\ \frac{1}{3} + \frac{2i}{3\sqrt{5}} & 1 & -\frac{1}{2\sqrt{3}} + \frac{i(-3+4\sqrt{3})}{6\sqrt{5}} & \frac{1}{2\sqrt{3}} - \frac{i(3+4\sqrt{3})}{6\sqrt{5}} \\ \frac{1}{2\sqrt{3}} - \frac{i(3+4\sqrt{3})}{6\sqrt{5}} & -\frac{1}{2\sqrt{3}} - \frac{i(-3+4\sqrt{3})}{6\sqrt{5}} & 1 & -\frac{1}{2} + \frac{1}{2}i\sqrt{\frac{3}{5}} \\ -\frac{1}{2\sqrt{3}} + \frac{i(-3+4\sqrt{3})}{6\sqrt{5}} & \frac{1}{2\sqrt{3}} + \frac{i(3+4\sqrt{3})}{6\sqrt{5}} & -\frac{1}{2} - \frac{1}{2}i\sqrt{\frac{3}{5}} & 1 \end{pmatrix},$$

share the eigenvalues (given approximately by)

$$2.73115, \quad 0.932497, \quad 0.253856, \quad 0.0824931.$$

Thus T passes the Grammian Test (Corollary 1).

Recall that the ijth entries of U^*U and V^*V are $\langle u_j, u_i \rangle$ and $\langle v_j, v_i \rangle$, respectively. Therefore to check whether T passes the Angle Test (Theorem 1), we need only compare the moduli of the entries of U^*U and V^*V. The moduli of the entries of U^*U and

V^*V are equal, entry-by-entry, and given by

$$\begin{pmatrix} 1 & \frac{1}{5} & \frac{2}{15}\left(3+\sqrt{3}\right) & \frac{2}{15}\left(3-\sqrt{3}\right) \\ \frac{1}{5} & 1 & \frac{2}{15}\left(3-\sqrt{3}\right) & \frac{2}{15}\left(3+\sqrt{3}\right) \\ \frac{2}{15}\left(3+\sqrt{3}\right) & \frac{2}{15}\left(3-\sqrt{3}\right) & 1 & \frac{2}{5} \\ \frac{2}{15}\left(3-\sqrt{3}\right) & \frac{2}{15}\left(3+\sqrt{3}\right) & \frac{2}{5} & 1 \end{pmatrix}$$

Thus T passes the Angle Test (Theorem 1).

On the other hand, since

$$\langle u_1, u_2\rangle\langle u_2, u_3\rangle\langle u_3, u_1\rangle = \tfrac{2}{75}(5-i\sqrt{5}) \neq \tfrac{2}{75}(5+i\sqrt{5}) = \overline{\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle},$$

the Strong Angle Test (Theorem 2) asserts that T is *not* UECSM. Similar computations reveal that the desired condition (16) is violated for the triples $(i, j, k) = (1, 2, 4), (1, 3, 4),$ $(2, 3, 4)$ as well.

Working through the mechanics of the proof of Theorem 2, we find that the matrix $B = (\beta_{ij})$ is given by

$$B = \begin{pmatrix} 1 & -1 & \frac{11+6\sqrt{3}-2i\sqrt{5}-3i\sqrt{15}}{4\sqrt{2}(3+\sqrt{3})} & \frac{11-6\sqrt{3}-2i\sqrt{5}+3i\sqrt{15}}{4\sqrt{2}(-3+\sqrt{3})} \\ -1 & 1 & \frac{11-6\sqrt{3}+2i\sqrt{5}-3i\sqrt{15}}{4\sqrt{2}(-3+\sqrt{3})} & \frac{11+6\sqrt{3}+2i\sqrt{5}+3i\sqrt{15}}{4\sqrt{2}(3+\sqrt{3})} \\ \frac{11+6\sqrt{3}+2i\sqrt{5}+3i\sqrt{15}}{4\sqrt{2}(3+\sqrt{3})} & \frac{11-6\sqrt{3}-2i\sqrt{5}+3i\sqrt{15}}{4\sqrt{2}(-3+\sqrt{3})} & 1 & \frac{1}{8}\left(7-i\sqrt{15}\right) \\ \frac{11-6\sqrt{3}+2i\sqrt{5}-3i\sqrt{15}}{4\sqrt{2}(-3+\sqrt{3})} & \frac{11+6\sqrt{3}-2i\sqrt{5}-3i\sqrt{15}}{4\sqrt{2}(3+\sqrt{3})} & \frac{1}{8}\left(7+i\sqrt{15}\right) & 1 \end{pmatrix}$$

In particular, each entry of B is unimodular whence we once again see that T passes the Angle Test. Also observe that the rank of B is 4 and its eigenvalues are approximately

$$3.88114, \quad 0.694237, \quad -0.66798, \quad 0.0926015.$$

In particular, B is neither rank-one nor positive.

We should also mention that J. Tener's procedure UECSMTest also confirms, via entirely different methods (see Section 7), that T is not UECSM.

The preceding example was discovered by the first author during a search of 10 million random integer matrices. Such examples appear to be exceedingly rare and those which can be worked through in closed form rarer still. Moreover, we were unable to find a 3×3 matrix with the same properties.

7. Comparison with Tener's UECSMTest

J. Tener's procedure UECSMTest, introduced in [11], is an effective tool in determining whether a given matrix is UECSM. However, there are certain limitations

inherent in the procedure. To be more specific, `UECSMTest` cannot be applied if the given matrix T is 4×4 or larger *and* either Cartesian component A or B in the decomposition $T = A + iB$ (where $A = A^*$ and $B = B^*$) has a repeated eigenvalue.

On the other hand, the criterion for applying `StrongAngleTest` is simply that the matrix T have distinct eigenvalues. In this section, we compare the two procedures and demonstrate the existence of matrices, both UECSM and not, for which either `UECSMTest` or `StrongAngleTest` (possibly both) fail to apply. In particular, this demonstrates that `StrongAngleTest` and `UECSMTest` are complimentary procedures in the sense that neither test subsumes the other.

Obviously many normal matrices (e.g., the 4×4 identity matrix) do not satisfy the hypotheses of either test. This does not pose a problem, however, since the Spectral Theorem asserts that every normal matrix is unitarily equivalent to a *diagonal matrix* whence every normal matrix is UECSM. In light of the preceding remarks, we therefore focus our attention on producing examples which are *non-normal*.

Finding non-normal matrices for which `StrongAngleTest` is applicable and for which `UECSMTest` is not is relatively straightforward. Several examples are listed in Table 1 below (where $\sigma(T), \sigma(A), \sigma(B)$ denote the spectra of the operators T, A, B, respectively, in the decomposition $T = A + iB$, $A = A^*$, $B = B^*$).

T	$\sigma(T)$	$\sigma(A)$	$\sigma(B)$	UECSM?
$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$-2, 2, 4, 8$	$2, 4, 3 \pm \sqrt{29}$	$0, 0, \pm 2$	Yes
$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 8 & 0 & 0 \end{pmatrix}$	$2, 4, -2 \pm 2i\sqrt{3}$	distinct	$0, 0, \pm\sqrt{21}$	No
$\begin{pmatrix} 4 & 1 & -1 & -2 \\ 3 & 2 & -4 & 1 \\ -1 & -2 & 4 & 1 \\ -4 & 1 & 3 & 2 \end{pmatrix}$	$-2, 2, 4, 8$	$2, 4, 3 \pm \sqrt{29}$	$0, 0, \pm 2$	Yes
$\begin{pmatrix} 4 & -1 & 1 & -2 \\ -2 & 1 & -1 & 4 \\ -1 & 4 & -2 & 1 \\ 1 & -2 & 4 & -1 \end{pmatrix}$	$2, 4, -2 \pm 2i\sqrt{3}$	distinct	$0, 0, \pm\sqrt{21}$	No

Table 1: Examples of simple matrices for which `StrongAngleTest` is applicable and `UECSMTest` is not. The third and fourth matrices listed are, respectively, unitarily equivalent to the first and second matrices. The eigenvalues of the second and fourth matrices are distinct, but too long to display explicitly in the confines of the table.

In cases where T has repeated eigenvalues, one frequently finds that both A and B both have distinct eigenvalues. Such matrices are testable by `UECSMTest` but not by `StrongAngleTest`. Several simple examples are listed in Table 2 below.

T	$\sigma(T)$	$\sigma(A)$	$\sigma(B)$	UECSM?
$\begin{pmatrix} 0 & 18 & 0 \\ 0 & 0 & 18i \\ 0 & 0 & 0 \end{pmatrix}$	0,0,0	$0, \pm 9\sqrt{2}$	$0, \pm 9\sqrt{2}$	Yes
$\begin{pmatrix} 0 & 18 & 0 \\ 0 & 0 & 9i \\ 0 & 0 & 0 \end{pmatrix}$	0,0,0	$0, \pm \frac{9\sqrt{5}}{2}$	$0, \pm \frac{9\sqrt{5}}{2}$	No
$\begin{pmatrix} 8+4i & 4+8i & -8+8i \\ -8+2i & -4+4i & 8+4i \\ 4-4i & 2-8i & -4-8i \end{pmatrix}$	0,0,0	$0, \pm 9\sqrt{2}$	$0, \pm 9\sqrt{2}$	Yes
$\begin{pmatrix} 8+2i & 4+4i & -8+4i \\ -8+i & -4+2i & 8+2i \\ 4-2i & 2-4i & -4-4i \end{pmatrix}$	0,0,0	$0, \pm \frac{9\sqrt{5}}{2}$	$0, \pm \frac{9\sqrt{5}}{2}$	No

Table 2: Matrices for which `UECSMTest` *is applicable and* `StrongAngleTest` *is not. The third and fourth matrices listed are, respectively, unitarily equivalent to the first and second matrices.*

It is possible to construct matrices for which neither `StrongAngleTest` nor Tener's `UECSMTest` is applicable. To be more specific, we exhibit several matrices T such that

(i) T has repeated eigenvalues (so that `StrongAngleTest` is not applicable),

(ii) $T = A + iB$ is at least 4×4 and either A or B has repeated eigenvalues (so that `UECSMTest` is not applicable).

Although it is relatively straightforward to produce matrices T satisfying (i) and (ii), it is naturally quite difficult to check whether T is UECSM or not since by design neither `StrongAngleTest` nor `UECSMTest` are applicable. Fortunately, the set of matrices having properties (i) and (ii) has Lebesgue measure zero in $M_n(\mathbb{C})$.

We require a couple preliminary lemmas. The following can be found in [11, Ex. 1] or [6, Ex. 1]:

LEMMA 4. *The matrix*

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

is UECSM if and only if $ab = 0$ *or* $|a| = |b|$.

In particular, the matrix (29) is *not* UECSM whenever a and b are nonzero and satisfy $|a| \neq |b|$. In our construction, we intend to use (29) as a building block in conjunction with the following lemma from [7]:

LEMMA 5. T *is UECSM if and only if* $T \oplus 0$ *is UECSM.*

In the lemma above, $T \oplus 0$ denotes the orthogonal direct sum of T with a square zero matrix of any given size. Since T is UECSM if and only if $T - \lambda I$ is UECSM for any $\lambda \in \mathbb{C}$, it follows from Lemma 4 and Lemma 5 that the matrix

$$T = \begin{pmatrix} c & 0 & 0 & 0 \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

can be made UECSM or not according to our choice of a and b (the value of c is irrelevant). A short computation then reveals that $\sigma(T) = \{0,0,0,c\}$ and

$$\sigma(A) = \{0, \mathrm{Re}\, c, \pm\sqrt{|a|^2 + |b|^2}\},$$
$$\sigma(B) = \{0, \mathrm{Im}\, c, \pm\sqrt{|a|^2 + |b|^2}\},$$

whence if c is either real or purely imaginary condition (ii) holds. This leads us to the examples listed in Table 3 below:

T	$\sigma(T)$	$\sigma(A)$	$\sigma(B)$	UECSM?
$\begin{pmatrix} 4&0&0&0 \\ 0&0&8&0 \\ 0&0&0&8 \\ 0&0&0&0 \end{pmatrix}$	$0,0,0,4$	$0,4,\pm\sqrt{2}$	$0,0,\pm4\sqrt{2}$	Yes
$\begin{pmatrix} 8&0&0&0 \\ 0&0&4&0 \\ 0&0&0&8 \\ 0&0&0&0 \end{pmatrix}$	$0,0,0,8$	$0,8,\pm2\sqrt{5}$	$0,0,\pm2\sqrt{5}$	No
$\begin{pmatrix} 5&1&-3&1 \\ 1&-3&1&5 \\ 1&5&1&-3 \\ -3&1&5&1 \end{pmatrix}$	$0,0,0,4$	$0,4,\pm4\sqrt{2}$	$0,0,\pm4\sqrt{2}$	Yes
$\begin{pmatrix} 5&1&-1&3 \\ 3&-1&1&5 \\ 1&5&3&-1 \\ -1&3&5&1 \end{pmatrix}$	$0,0,0,8$	$0,8,\pm2\sqrt{5}$	$0,0,\pm2\sqrt{5}$	No

Table 3: Matrices for which neither `UECSMTest` *nor* `StrongAngleTest` *are applicable. The third and fourth matrices listed are, respectively, unitarily equivalent to the first and second matrices.*

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