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ON THE MATRICES THAT PRESERVE THE VALUE OF THE IMMANANT OF THE UPPER TRIANGULAR MATRICES

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Abstract. Let χ be an irreducible character of the symmetric group of degree n, let $M_n(\mathbb{F})$ be the linear space of n-square matrices with elements in the field $\mathbb{F}$ of characteristic zero, let $T_n^U(\mathbb{F})$ be the subset of $M_n(\mathbb{F})$ of the upper triangular matrices and let d_χ be the immanant associated with χ. We denote by $\mathscr{T}(S_n,\chi)$ the set of all $A \in M_n(\mathbb{F})$, such

$$d_\chi(AX) = d_\chi(X),$$

for all $X \in T_n^U(\mathbb{F})$. The purpose of this paper is to present, in some cases, a complete description of the matrices in the set $\mathscr{T}(S_n,\chi)$.

1. Introduction

Let S_n be the symmetric group of degree n, and let H be a subgroup of S_n. Let $\mathbb{F}$ be an arbitrary field of characteristic zero and let χ be an $\mathbb{F}$ valued irreducible character of H. If $X = [x_{ij}]$ is an $n \times n$ matrix over $\mathbb{F}$, the generalized matrix function $d_\chi^H(X)$ is defined by, [9], [11],

$$d_\chi^H(X) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}.$$

Let $M_n(\mathbb{F})$ be the linear space of n-square matrices with elements in $\mathbb{F}$. Matrices that satisfy certain polynomial identities always have found several applications in several areas of mathematics. One of this problems gets a special attention in the last years of the past century. The goal is to obtain a description of the $n \times n$ matrices A over $\mathbb{F}$ that satisfy

$$d_\chi^H(AX) = d_\chi^H(X), \text{ for all } X \in M_n(\mathbb{F}).$$

The study of this matrices was motivated by a problem in multilinear algebra: finding condition for equality of two nonzero decomposable symmetrized tensors (see [8], [10],[12], [5] [6], [7], [11] and [15]). This problem, suggests other problems similar to this one. Some of such problems were also motivated by certain mathematical problems. For instance the description of the $n \times n$ matrices A that satisfy

$$d_\chi^H(AX) = 0, \text{ for all } X \in M_n(\mathbb{F})$$

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was motivated by the multilinear algebra problem of finding conditions for a decomposable symmetrized tensor to be zero (see [13]). During the years the previous two problems, and others, were solved by several authors, while others problems remains unsolved. In [15], G. de Oliveira and J. A. Dias da Silva solved the first problem (characterize the $n\times n$ matrices A over $\mathbb{F}$ that satisfy $d_\chi^H(AX)=d_\chi^H(X)$ for all $X\in M_n(\mathbb{F})$). In [2], A. Duffner presented a description for the $n\times n$ matrices A that satisfy $d_\chi^H(AX)=0$, for all $X\in M_n(\mathbb{F})$, when $\mathbb{F}=\mathbb{C}$, the complex field and $H=S_n$. For the reader less familiar with this problems, [14] is a survey on this kind of problems.

This paper deals with a question similar to those problems. We want to obtain a description of the $n\times n$ matrices over $\mathbb{F}$ that satisfy

$$d_\chi^H(AX)=d_\chi^H(X),$$

for all $n\times n$ upper triangular matrices X over $\mathbb{F}$.

We denote the set of the $n\times n$ upper triangular matrices over $\mathbb{F}$ by $T_n^U(\mathbb{F})$. The set of the matrices that satisfy $d_\chi^H(AX)=d_\chi^H(X)$, for all $X\in T_n^U(\mathbb{F})$ is denoted by $\mathscr{T}(H,\chi)$. Hence

$$\mathscr{T}(H,\chi)=\{A\in M_n(\mathbb{F}): d_\chi^H(AX)=d_\chi^H(X),\ \text{ for all } X\in T_n^U(\mathbb{F})\}.$$

The paper [3] was the first, and the only, paper to deal with this problem and so it is the main reference for this paper. In that paper a characterization of the set $\mathscr{T}(H,\chi)$ was presented. However, this characterization gives rise new questions that remains unsolved until today. We are going now to make a resume of the main results of [3]. Let H be a subgroup of S_n and let χ be an irreducible character of H. The first important conclusion on the set $\mathscr{T}(H,\chi)$ was the fact that if $A\in\mathscr{T}(H,\chi)$ then A is nonsingular (proposition 2.5 of [3]). However, in general $\mathscr{T}(H,\chi)$ is not a group (see example 2.6 of [3]). It is well known that if $A\in M_n(\mathbb{F})$ is nonsingular, there are an upper triangular matrix R, a lower triangular matrix L and $\sigma\in S_n$ such

$$A=P(\sigma)LR,$$

where $P(\sigma)$ is the $n\times n$ permutation matrix whose (i,j) entry is $P(\sigma)_{ij}=\delta_{i\sigma(j)}$, $i,j\in\{1,\ldots,n\}$. Based on this fact she proved that $A\in\mathscr{T}(H,\chi)$ if and only if

$$A=P(\sigma)LR,$$

where σ is an element of H satisfying $\chi(\sigma^{-1})\neq 0$, R is an upper triangular matrix satisfying $\det(R)=\frac{\chi(id)}{\chi(\sigma^{-1})}$ and L is a lower triangular matrix with ones in the main diagonal satisfying

$$d_\chi^H(P(\sigma)LX)=d_\chi^H(P(\sigma)X)\ \text{ for all } X\in T_n^U(F). \tag{1}$$

Let $\sigma\in H$ such that $\chi(\sigma^{-1})\neq 0$. In [3] the author denoted by $V_\sigma(H,\chi)$ the set of matrices $L\in T_n^L(F)$ (the set of n-square lower triangular matrices) with diagonal elements equal to one, satisfying (1).

Using this notation we can state the previous conclusion as follows:

THEOREM 1.1. [3] *Let H be a subgroup of S_n and let χ be an irreducible character of H. Then,*

$$\mathscr{T}(H,\chi) = \bigcup_{\sigma\in H,\chi(\sigma^{-1})\neq 0} \left\{P(\sigma)LR:\ L\in V_\sigma(H,\chi), R\in T_n^U(\mathbb{F}), \det(R)=\frac{\chi(id)}{\chi(\sigma^{-1})}\right\}.$$

□

By this theorem we conclude that if we want to obtain a complete description of the set $\mathscr{T}(H,\chi)$ we have to somehow obtain a description of the sets $V_\sigma(H,\chi)$, for all $\sigma\in H$ such $\chi(\sigma^{-1})\neq 0$. That is obvious is that for any $\sigma\in H$ such $\chi(\sigma^{-1})\neq 0$, the $n\times n$ identity matrix, I_n, is in $V_\sigma(H,\chi)$. A natural question is to know if there are other matrices than I_n in $V_\sigma(H,\chi)$, and, if so, how can we describe them. This problem seems to be quite difficult. The most impressive results on this problem was obtained in [3] when $H=S_n$. One of the reasons that made this case more easy is the existence of a combinatorial algorithm, the Murnaghan-Nakayama rule (see [1]), that allow us to compute the value of an irreducible character of S_n in any conjugacy class. The Murnaghan-Nakayama rule was extensively used in [3]. Hence, from now on we assume that $H=S_n$ and χ is an irreducible character of S_n. A generalized matrix function of the form $d_\chi^{S_n}$ is called immanant and is denoted simply by d_χ.

NOTATION 1.2. *Let $\sigma\in S_n$ such that $\chi(\sigma)\neq 0$. Denote by $(S_n)_T^\sigma$ the subgroup of S_n generated by those transpositions, τ, of S_n satisfying*

$$\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}),$$

If there is no transposition τ in S_n such $\chi(\sigma^{-1}\tau)=-\chi(\sigma^{-1})$ then $(S_n)_T^\sigma=\{id\}$.

Observe that $\chi(\sigma^{-1}\tau)=\chi(\tau\sigma^{-1})$.

The next theorem is crucial to obtain a characterization of the sets $V_\sigma(S_n,\chi)$.

THEOREM 1.3. [3] *Let $L=[l_{ij}]\in T_n^L(F)$ with diagonal elements equal to 1 and let $\sigma\in S_n$ such that $\chi(\sigma)\neq 0$. If $L\in V_\sigma(S_n,\chi)$ then $l_{ij}=0$ whenever i and j belong to different orbits of $(S_n)_T^\sigma$.*

The converse of this theorem is not true (see example 2.8 of [3]).

Let x be an indeterminate over the field F and $E^{(i)+x(j)}$ the matrix obtained from the identity matrix by adding x times column j to column i. The next is very useful in the proof of some results:

PROPOSITION 1.4. [3] *Let $L=E^{(k)+x(k+1)}\in T_n^L(F)$ with $x\neq 0$ and $\sigma\in S_n$ such that $\chi(\sigma)\neq 0$. Then $L\in V_\sigma(S_n,\chi)$ if and only if $\chi((k,\ k+1)\sigma^{-1})=-\chi(\sigma^{-1})$.*

We define a partition α of n as $\alpha=(\alpha_1,\dots,\alpha_r)$ where the α_i's are integers, $\alpha_1\geqslant\dots\geqslant\alpha_r\geqslant 0$, and $\alpha_1+\dots+\alpha_r=n$. We do not distinguish between two partitions that differ by a sequence of zeros. If $\alpha=(\alpha_1,\dots,\alpha_r)$ is a partition of n and $\alpha_r>0$, we say that r is the length of α. Each partition $\alpha=(\alpha_1,\dots,\alpha_r)$ of length r is related to a Young diagram, denoted by $[\alpha]$, which consists of r left justified rows of boxes. The number of boxes in the ith row is α_i.

If $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition of n, the α_1-tuple $\alpha' = (\alpha'_1, \ldots, \alpha'_{\alpha_1})$ defined by

$$\alpha'_i = |\{j : \alpha_j \geqslant i\}|$$

is also a partition of n called the conjugate partition of α.

We say that a Young diagram is symmetric if it is associated with a partition α such that $\alpha = \alpha'$.

It is well known (see [1], [4] or [9]) that the irreducible characters of S_n are in a bijective correspondence with the ordered partitions of n. We identify the irreducible character λ with the partition that corresponds to λ. If λ is an irreducible character of S_n, the character λ' such that

$$\lambda'(\sigma) = \varepsilon(\sigma)\lambda(\sigma)$$

for all $\sigma \in S_n$ is an irreducible character called the character associated with λ. If $\lambda = \lambda'$ we say that λ is self-associated.

In [3], the author proved, by applying the Murnaghan-Nakayama rule, that if χ is a self-associated irreducible character of S_n then, for any $\sigma \in S_n$ such $\chi(\sigma) \neq 0$, there is no transposition $\tau \in S_n$ satisfying $\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1})$. Hence, $(S_n)^T_\sigma = \{id\}$ and by theorem 1.3 we conclude that

$$V_\sigma(S_n, \chi) = \{I_n\},$$

for all $\sigma \in S_n$ such $\chi(\sigma) \neq 0$. The same conclusion can be easily achieve if $\chi = 1$, that is, if χ the principal character of S_n. The converse of this statements also holds and we can summarize this conclusions in the following theorem:

THEOREM 1.5. [3] *Let χ be an irreducible character of S_n. Then*

$$\bigcup_{\sigma \in S_n,\ \chi(\sigma) \neq 0} V_\sigma(S_n, \chi) = \{I_n\}$$

if and only if

$$\chi = 1 \textit{ or } \chi \textit{ is self-associated}.$$

This theorem allowed us to solve the characterization of the set $\mathscr{T}(S_n, \chi)$ when χ is an irreducible self-associated character of S_n. In fact, using the previous theorem and theorem 1.1, we conclude that if χ is an irreducible self-associated character of S_n then $A \in \mathscr{T}(S_n, \chi)$ if and only if

$$A = P(\sigma)R,$$

where σ is an element of S_n satisfying $\chi(\sigma^{-1}) \neq 0$ and R is an upper triangular matrix satisfying $\det(R) = \frac{\chi(id)}{\chi(\sigma^{-1})}$.

If χ is not self-associated this problem remains unsolved. The principal result on this case was also obtained in [3] and is a description of the set $V_\sigma(S_n, \chi)$ where $\chi = (n-1, 1)$ and σ is a cycle with length $n-2$:

THEOREM 1.6. [3] *Let* $\chi = (n-1,1)$ *be the irreducible character of* S_n *with* $n > 3$. *Let* $\sigma \in S_n$ *be a cycle with length* $n-2$ *and* $L = [l_{ij}] \in T_n^L(F)$ *with diagonal elements equal to* 1. *Then*

$$L \in V_\sigma(S_n, \chi)$$

if and only if L satisfies the condition:

"For $r > p$, *if there exists an integer* k *such that* $p \leqslant k \leqslant r$ *and* $\sigma(k) \neq k$ *then* $l_{rp} = 0$."

The purpose of this paper is to go further on this problem and present a complete description of the matrices in the set $\mathcal{T}(S_n, \chi)$, with $\chi = (n-1,1)$ or $\chi = (n-2,2)$. Hence, the main results of this paper are the theorems 2.7 and 3.16 in the end of the second and third sections of this paper respectively. The strategy adopted is similar in both cases. In the first step we identify the permutations $\sigma \in S_n$, $\chi(\sigma) \neq 0$, such $(S_n)_\sigma^T \neq \{id\}$. Applying theorem 1.3 we know, for a given $L \in V_\sigma(H, \chi)$, the entries below the main diagonal that could be different from zero. After, using (1) with some appropriated upper triangular matrices we can show that some of this entries, that could be different of zero, must be, in fact, equal to zero. This allow us to obtain a description of the set $V_\sigma(S_n, \chi)$ and consequently a description of the set $\mathcal{T}(S_n, \chi)$.

2. The character $\chi = (n-1,1)$

In this section we characterize the set $V_\sigma(S_n, \chi)$ where χ is the irreducible character of S_n, $\chi = (n-1,1)$, and $\sigma \in S_n$ is such that $\chi(\sigma) \neq 0$. We are going to see that the proof of Theorem 1.6 is very important in this section. We suppose that $n > 3$ because if $n = 2$ then $\chi = (1,1) = \varepsilon$ and if $n = 3$ then $\chi = (2,1) = \chi'$ (Theorem 1.5).

Let $\rho \in S_n$. If $\rho = \rho_1 \ldots \rho_k$, where $\rho_1, \ldots, \rho_k$ are pairwise disjoint cycles with length $\pi_1, \ldots, \pi_k$ respectively, we denote by $C_{[\pi_1, \ldots, \pi_k]}$ the class where ρ belongs.

EXAMPLE 2.1. Let $\rho = (1243)(57)(6) \in S_7$. Then $\rho \in C_{[4,2,1]}$.

LEMMA 2.2. *Let* $\sigma \in S_n$ *such that* $\chi(\sigma) \neq 0$ *and* $\sigma \in C_{[\pi_1, \ldots, \pi_k]}$ *where* $k \geqslant 3$ *and* $\pi_{k-2} > \pi_{k-1} = \pi_k = 1$. *Let* $u, v \in \{1, \ldots, n\}$ *such that* $u \neq v$, $\sigma(u) = u$ *and* $\sigma(v) = v$. *Then*

$$(S_n)_T^\sigma = \langle (u,v) \rangle.$$

Proof. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\sigma \in C_{[\pi_1, \ldots, \pi_k]}$ where $k \geqslant 3$ and $\pi_{k-2} > \pi_{k-1} = \pi_k = 1$. Then σ has two fix point. Let $u, v \in \{1, \ldots, n\}$ such that $u \neq v$, $\sigma(u) = u$ and $\sigma(v) = v$ and let $\tau = (u,v)$. Then $\sigma^{-1}\tau \in C_{[\pi_1, \ldots, \pi_{k-2}, 2]}$ and so

$$\chi(\sigma^{-1}\tau) = -1 = -\chi(\sigma^{-1}).$$

Therefore, $(u,v) \in (S_n)_T^\sigma$. Let now $a, b \in \{1, \ldots, n\}$ such that $a \neq b$ and $\sigma(a) \neq a$ or $\sigma(b) \neq b$. Let $\tau' = (a,b)$. Then, $\sigma^{-1}\tau'$ has at least one fix point and so $\chi(\sigma^{-1}\tau') \neq -\chi(\sigma^{-1})$. Hence $\tau' \notin (S_n)_T^\sigma$. $\square$

LEMMA 2.3. *Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\sigma \in C_{[\pi_1,\ldots,\pi_k]}$ where $\pi_k = 2$. If $\sigma = \sigma_1 \ldots \sigma_k$ where $\sigma_1, \ldots, \sigma_k$ are pairwise disjoint cycles, then*

$$(S_n)_T^\sigma = \langle \sigma_i : \sigma_i \text{ is a transposition of } \sigma \rangle.$$

Proof. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\sigma \in C_{[\pi_1,\ldots,\pi_k]}$ where $\pi_k = 2$. Then, σ do not have fix points and so $\chi(\sigma) = -1 = \chi(\sigma^{-1})$. Let $u, v \in \{1, \ldots, n\}$ such that $u \neq v$ and $\tau = (u, v)$ is a transposition of σ. Consequently τ is a transposition of σ^{-1}. Then $\sigma^{-1}\tau \in C_{[\pi_1,\ldots,\pi_{k-1},1^2]}$ and so

$$\chi(\sigma^{-1}\tau) = 1 = -\chi(\sigma^{-1}).$$

Therefore, $(u, v) \in (S_n)_T^\sigma$. Let now $a, b \in \{1, \ldots, n\}$ such that $a \neq b$ and $\tau' = (a, b)$ is not a transposition of σ^{-1}. Then, $\sigma^{-1}\tau'$ do not have fix points, and so $\chi(\sigma^{-1}\tau') = -1 = \chi(\sigma^{-1})$ or $\sigma^{-1}\tau'$ has one fix point and so $\chi(\sigma^{-1}\tau') = 0$. In both cases we have $\chi(\sigma^{-1}\tau') \neq -\chi(\sigma^{-1})$. □

LEMMA 2.4. *Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\sigma \in C_{[\pi_1,\ldots,\pi_k]}$. Then, $(S_n)_T^\sigma = \{id\}$ if and only if $\pi_{k-2} = \pi_{k-1} = \pi_k = 1$ or $\pi_k \geqslant 3$.*

Proof. Assume that $\sigma \in S_n$ is such that $\sigma \in C_{[\pi_1,\ldots,\pi_k]}$, where $\pi_k \geqslant 3$. Then

$$\chi(\sigma) = -1 = \chi(\sigma^{-1}).$$

Let τ be a transposition of S_n. Then $\sigma^{-1}\tau$ has at most one fix point and so

$$\chi(\sigma^{-1}\tau) = -1 \qquad \text{or} \qquad \chi(\sigma^{-1}\tau) = 0.$$

Hence $\chi(\sigma^{-1}\tau) \neq -\chi(\sigma^{-1})$, and so $(S_n)_\sigma^T = \{id\}$.

Assume now that $\pi_{k-2} = \pi_{k-1} = \pi_k = 1$. Then $\chi(\sigma) = \chi(\sigma^{-1}) \geqslant 2$ and if τ' is a transposition of S_n, $\sigma^{-1}\tau'$ has at least one fix point and so $\chi(\sigma^{-1}\tau') \neq -\chi(\sigma^{-1})$ and $(S_n)_\sigma^T = \{id\}$.

Using Propositions 2.2 and 2.3 we can conclude this result. □

If $\sigma \in S_n$ is in the conditions of Proposition 2.4, using Theorem 1.3, $V_\sigma(S_n, \chi) = \{I_n\}$. Suppose that σ is in the conditions of Proposition 2.2.

PROPOSITION 2.5. *Let $\sigma \in S_n$ be a permutation in the conditions of Proposition* 2.2 *with $u > v$ and $L = [l_{ij}] \in T_n^L(F)$ with diagonal elements equal to* 1*. Then*

$$L \in V_\sigma(S_n, \chi)$$

if and only if L satisfies the condition:

"For $r > p$, if there exists an integer k such that $p \leqslant k \leqslant r$ and $\sigma(k) \neq k$ then $l_{rp} = 0$."

Proof. Necessity. Suppose that $L=[l_{ij}]\in V_\sigma(S_n,\chi)$. By Theorem 1.3, if $a>b$, $a,\ b\in\{1,\dots,n\}$ and $\sigma(a)\neq a$ or $\sigma(b)\neq b$ then $l_{ab}=0$.

Suppose there exists an integer k such that $u>k>v$ and $\sigma(k)\neq k$. Let Z be the matrix whose $(v+1)$th column is the vth column of I_n and the uth column of Z is the $(v+1)$th column of I_n, the remaining columns of Z are the columns of I_n. Since $l_{u,v+1}=0$ then

$$d_\chi(P(\sigma)LZ)=(\chi(\sigma^{-1}(v+1,\ u))+\chi(\sigma^{-1}(v+1,\ u,\ v)))l_{uv}.$$

Since $\sigma^{-1}(v+1,u)(v+1)\neq v+1$ and $\sigma^{-1}(v+1,u)(u)\neq u$ then $\sigma^{-1}(v+1,\ u)$ has one fix point. So,

$$\chi(\sigma^{-1}(v+1,\ u))=0.$$

But $\sigma^{-1}(v+1,\ u,\ v)$ does not have fix points, then $\chi(\sigma^{-1}(v+1,\ u,\ v))=-1$. Therefore, $d_\chi(P(\sigma)LZ)=-l_{uv}$. Since $L\in V_\sigma(S_n,\chi)$,

$$-l_{uv}=d_\chi(P(\sigma)LZ)=d_\chi(P(\sigma)Z)=0.$$

Consequently, $l_{uv}=0$ and we have the condition.

Sufficiency. Let $L=[l_{ij}]$ be a matrix satisfying the condition of the theorem. Then

$$L=\begin{cases} I_n & \text{if } u\neq v+1\\ E^{(v)+l_{uv}(u)} & \text{if } u=v+1\end{cases}$$

Let $X\in T_n^U$. If $u\neq v+1$,

$$d_\chi(P(\sigma)LX)=d_\chi(P(\sigma)I_nX)=d_\chi(P(\sigma)X),$$

and then $L\in V_\sigma(S_n,\chi)$.

If $u=v+1$, by Proposition 1.3, $L\in V_\sigma(S_n,\chi)$. □

PROPOSITION 2.6. *Let $\sigma\in S_n$ be a permutation in the conditions of Proposition* 2.3 *and $L=[l_{ij}]\in T_n^L(F)$ with diagonal elements equal to* 1. *Then*

$$L\in V_\sigma(S_n,\chi)$$

if and only if L satisfies the conditions:

1. *"For $r>p$, if $\sigma(rp)$ does not have two fix points, or there is an integer k such that $p<k<r$, then $l_{rp}=0$."*

2. *"For $r>p$, if $\sigma(r+1,r)(p+1,p)$ has four fix points then $l_{r+1,r}=0$ or $l_{p+1,p}=$* 0.

Proof. Necessity. Let $L=[l_{ij}]\in V_\sigma(S_n,\chi)$. If $\sigma(rp)$ does not have two fix points, using Theorem 1.3, $l_{rp}=0$. If there is two integers a,b such that $\sigma(r,p)(a)=a$ and $\sigma(r,p)(b)=b$ and there is an integer k such that $p<k<r$, we are going to prove that $l_{rp}=0$. Observe that $a=p$, $b=r$ and $(r,p+1)\notin(S_n)_T^\sigma$.

Let Z be the matrix whose $(p+1)$th column is the pth column of I_n and the rth column of Z is the $(p+1)$th column of I_n, the remaining columns of Z are the columns of I_n. Since $l_{r,p+1}=0$ then

$$d_\chi(P(\sigma)LZ) = (\chi(\sigma^{-1}(p+1,\ r)) + \chi(\sigma^{-1}(p+1,\ r,\ p)))l_{rp}.$$

Since $\sigma^{-1}(p+1,r)$ does not have fix points and r is the only fix point of $\sigma^{-1}(p+1,r,p)$, using the Murnaghan-Nakayama rule,

$$\chi(\sigma^{-1}(p+1,\ r)) = -1, \qquad \chi(\sigma^{-1}(p+1,\ r,\ p)) = 0.$$

Therefore, $d_\chi(P(\sigma)LZ) = -l_{rp}$. Since $L \in V_\sigma(S_n,\chi)$,

$$-l_{rp} = d_\chi(P(\sigma)LZ) = d_\chi(P(\sigma)Z) = 0.$$

Consequently, $l_{rp}=0$ and we have this condition.

Suppose that $r>p$ and there is four fix points k,h,l,m of $\sigma(r+1,r)(p+1,p)$, we are going to see that $l_{r+1,r}=0$ or $l_{p+1,p}=0$. Observe that $\{k,h,l,m\}=\{r,r+1,p,p+1\}$ and $p+1<r$. Let W be the matrix whose $(r+1)$th column is the rth column of I_n and the $p+1$th column of W is the pth column of I_n, the remaining columns of W are the columns of I_n. Then

$$\begin{aligned} d_\chi(P(\sigma)LW) = {} & (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(p+1,\ p)) + \chi(\sigma^{-1}(r+1,\ r)) \\ & + \chi(\sigma^{-1}(r+1,\ r)(p+1,\ p)))l_{r+1,r}l_{p+1,p}. \end{aligned}$$

Since $\sigma^{-1}(p+1,\ p)$ and $\sigma^{-1}(r+1,\ r)$ have two fix points and $\sigma^{-1}(r+1,\ r)(p+1,\ p)$ has four fix points, then

$$d_\chi(P(\sigma)LW) = (-1+1+1+3)l_{r+1,r}l_{p+1,p} = 4l_{r+1,r}l_{p+1,p}.$$

Since $L \in V_\sigma(S_n,\chi)$,

$$4l_{r+1,r}l_{p+1,p} = d_\chi(P(\sigma)LW) = d_\chi(P(\sigma)W) = 0.$$

Consequently, $l_{r+1,r}=0$ or $l_{p+1,p}=0$ and we have this condition.

Sufficiency. Let $L=[l_{ij}]$ be a matrix satisfying the conditions of the theorem. Then

$$L = \begin{cases} E^{(r)+l_{r+1,r}(r+1)} & \text{if } \sigma(r+1,r) \text{ has two fix points} \\ I_n & \text{otherwise} \end{cases}$$

Let $X \in T_n^U$.

If $\sigma(r+1,r)$ has two fix points, by Proposition 1.3, $L \in V_\sigma(S_n,\chi)$.

If $L=I_n$, then

$$d_\chi(P(\sigma)LX) = d_\chi(P(\sigma)I_nX) = d_\chi(P(\sigma)X).$$

Consequently, $L \in V_\sigma(S_n,\chi)$. $\square$

The next theorem is now an easy consequence of theorem 1.1 and propositions 2.5 and 2.6 and it gives a complete description of the set $\mathscr{T}(S_n,\chi)$, with $\chi=(n-1,1)$:

THEOREM 2.7. *If $\chi = (n-1,1)$ then $A \in \mathscr{T}(S_n,\chi)$ if and only if $A = P(\sigma)L_\sigma R$ such that $\sigma \in S_n$ satisfy $\chi(\sigma) \neq 0$, $R \in T_n^U(F)$ satisfy* $\det(R) = \frac{\chi(id)}{\chi(\sigma)}$ *and $L_\sigma = [l_{i,j}]$ is a lower triangular matrix with all diagonal elements equal to 1 and satisfying the following conditions:*

1. *$l_{rp} = 0$ whenever $r \notin \{p+1,p\}$;*
2. *$L_\sigma = I_n$ if $\sigma \notin C_{[\pi_1,\dots\pi_k,1^2]}$, $\pi_k \geqslant 2$ or $\sigma \notin C_{[\pi_1,\dots\pi_k,2^u,1^0]}$, $\pi_k \geqslant 3$ and $u \geqslant 1$;*
3. *There is at most one i, $1 \leqslant i \leqslant n-1$, such that $l_{i+1,i} \neq 0$, if $\{i,i+1\}$ is the set of the fix points of σ or, if σ don't have fix points, $(i,i+1)$ is a transposition of σ.*

□

3. The character $\chi = (n-2,2)$

In this section we present a complete characterization of the set $\mathscr{T}(S_n,\chi)$, when $\chi = (n-2,2)$. We start with an easy result:

LEMMA 3.1. *Let $\chi = (n-2,2)$ and let $\sigma \in C_{[\pi_1,\dots,\pi_k,2^v,1^u]}$ with $\pi_k \geqslant 3$. Then,*

$$\chi(\sigma) = \frac{1}{2}u(u-3)+v.$$

Proof. Easy from Murnagham-Nakayma rule.

LEMMA 3.2. *Let $\chi = (n-2,2)$ and let $\sigma \in C_{[\pi_1,\dots,\pi_k,2^v,1^u]}$, $\pi_k \geqslant 3$, such that $\chi(\sigma) \neq 0$. Then, $(S_n)_\sigma^T = \{id\}$ except in the following situations:*

1. *$v = 0$ and $u = 2$;*
2. *$v = 1$ and $u = 0$ or $u = 3$.*

If $v = 0$ and $u = 2$ then

$$(S_n)_T^\sigma = \langle (a,b) : a \text{ and } b \text{ be the fix points of } \sigma \rangle.$$

If $v = 1$ and $u = 0$ then

$$(S_n)_T^\sigma = \langle (a,b) : (a,b) \text{ is the unique transposition of } \sigma \rangle.$$

If $v = 1$ and $u = 3$ then

$$(S_n)_T^\sigma = \langle (a,b) : a \text{ is in the unique transposition of } \sigma \text{ and } b \text{ is a fix point of } \sigma \rangle.$$

Proof. Let $\chi = (n-2,2)$ and let $\sigma \in C_{[\pi_1,\dots,\pi_k,2^v,1^u]}$, with $\pi_k \geqslant 3$, such that $\chi(\sigma) \neq 0$. Then $\sigma^{-1} \in C_{[\pi_1,\dots,\pi_k,2^v,1^u]}$. Let τ be a transposition of S_n. Then, we have the following cases:

1. $\sigma^{-1}\tau \in C_{[\pi_1',\dots,\pi_k',2^v,1^u]}$ with $\pi_k' \geqslant 3$;

2. $\sigma^{-1}\tau \in C_{[\pi_1,\dots,\pi_k,2^{v-1},1^{u+2}]}$, if $v \geqslant 1$;

3. $\sigma^{-1}\tau \in C_{[\pi_1,\dots,\pi_k,2^{v+1},1^{u-2}]}$, if $u \geqslant 2$;

4. $\sigma^{-1}\tau \in C_{[\pi_1',\dots,\pi_k',2^{v-1},1^{u-1}]}$ with $\pi_k' \geqslant 3$, if $v,u \geqslant 1$;

5. $\sigma^{-1}\tau \in C_{[\pi_1',\dots\pi_k',,2^{v-1},1^u]}$ with $\pi_k' \geqslant 3$, if $v \geqslant 1$;

6. $\sigma^{-1}\tau \in C_{[\pi_1',\dots,\pi_k',2^v,1^{u-1}]}$ with $\pi_k' \geqslant 3$, if $u \geqslant 1$;

In first case we have $\chi(\sigma^{-1}\tau) = \chi(\sigma^{-1})$. Assume that $v \geqslant 1$ and $\sigma^{-1}\tau \in C_{[\pi_1,\dots,\pi_k,2^{v-1},1^{u+2}]}$. Then,

$$\begin{aligned}\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}) &\iff \tfrac{1}{2}(u+2)(u-1)+v-1 = -\tfrac{1}{2}u(u-3)-v\\ &\iff (u+2)(u-1)+2v-2 = -u(u-3)-2v\\ &\iff u^2-u+2(v-1)=0\\ &\iff u = \tfrac{1\pm\sqrt{9-8v}}{2}.\end{aligned}$$

In this case we conclude that if $v>1$ then $\chi(\sigma^{-1}\tau) \neq -\chi(\sigma^{-1})$. If $v=1$ then $u=0$ or $u=1$. But if $v=1$ and $u=1$ we have $\chi(\sigma)=0$. So we only have to consider the case $v=1$, $u=0$ and $\tau=(a,b)$ where (a,b) is the unique transposition of σ.

If $u \geqslant 2$ and $\sigma^{-1}\tau \in C_{[\pi_1,\dots,\pi_k,2^{v+1},1^{u-2}]}$ we have

$$\begin{aligned}\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}) &\iff \tfrac{1}{2}(u-2)(u-5)+v+1 = -\tfrac{1}{2}u(u-3)-v\\ &\iff (u-2)(u-5)+2v+2 = -u(u-3)-2v\\ &\iff u^2-5u+6+2v=0\\ &\iff u = \tfrac{5\pm\sqrt{1-8v}}{2}.\end{aligned}$$

If $v>0$ we conclude that $\chi(\sigma^{-1}\tau) \neq -\chi(\sigma^{-1})$, for all transpositions $\tau \in S_n$ that satisfy this condition. If $v=0$ then $u=3$ or $u=2$. If $u=3$ we have $\chi(\sigma)=0$. So in this case, if $\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1})$ then $v=0$, $u=2$ and $\tau=(a,b)$ where a and b be the fix points of σ.

If $v,u \geqslant 1$ and if $\sigma^{-1}\tau \in C_{[\pi_1',\ldots,\pi_k',2^{v-1},1^{u-1}]}$ with $\pi_k' \geqslant 3$, we obtain

$$\begin{aligned}\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}) &\iff \tfrac{1}{2}(u-1)(u-4)+v-1 = -\tfrac{1}{2}u(u-3)-v\\ &\iff (u-1)(u-4)+2v-2 = -u(u-3)-2v\\ &\iff u^2-4u+2v+1=0\\ &\iff u = \tfrac{4\pm\sqrt{12-8v}}{2}.\end{aligned}$$

If $v>1$ we have $\chi(\sigma^{-1}\tau) \neq -\chi(\sigma^{-1})$. If $v=1$ then $u=3$ or $u=1$. But if $v=1$ and $u=1$ then $\chi(\sigma)=0$. So we only have to consider the case $v=1$ and $u=3$.

If $v \geqslant 1$, $\sigma^{-1}\tau \in C_{[\pi_1',\ldots,\pi_k',2^{v-1},1^{u}]}$ with $\pi_k' \geqslant 3$, then

$$\chi(\sigma^{-1}\tau) = \chi(\sigma^{-1})-1,$$

and so

$$\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}) \Longleftrightarrow \chi(\sigma^{-1}) = \frac{1}{2},$$

which in impossible.

Finally, if $u \geqslant 1$ and $\sigma^{-1}\tau \in C_{[\pi_1',\ldots,\pi_k',2^{v},1^{u-1}]}$ with $\pi_k' \geqslant 3$, then

$$\begin{aligned}\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1}) &\iff \tfrac{1}{2}(u-1)(u-4)+v = -\tfrac{1}{2}u(u-3)-v\\ &\iff (u-1)(u-4)+2v = -u(u-3)-2v\\ &\iff u^2-4u+2(v+1)=0\\ &\iff u = \tfrac{4\pm\sqrt{8-8v}}{2}.\end{aligned}$$

If $v>1$ we have $\chi(\sigma^{-1}\tau) \neq -\chi(\sigma^{-1})$. If $v=1$ then $u=2$. But if $v=1$ e $u=2$, $\chi(\sigma)=0$.

Therefore, we only have to consider the following cases:

1. $v=0$ and $u=2$;
2. $v=1$ and $u=0$ or $u=3$.

By the previous computations we have:

If $v=0$ and $u=2$ then

$$(S_n)_T^{\sigma} = \langle (a,b) : a \text{ and } b \text{ be the fix points of } \sigma\rangle.$$

If $v=1$ and $u=0$ then

$$(S_n)_T^{\sigma} = \langle (a,b) : (a,b) \text{ is the unique transposition of } \sigma\rangle.$$

If $v=1$ and $u=3$ then

$$\begin{aligned}(S_n)_T^{\sigma} = \langle (a,b) : a &\text{ is in the unique transposition of } \sigma \text{ and}\\ &b \text{ is a fix point of } \sigma\rangle.\end{aligned}$$

In all the other cases we have

$$(S_n)_T^\sigma = \{id\}.$$

$\square$

PROPOSITION 3.3. *Let* $\chi = (n-2,2)$, *let* $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^v,1^u]}$, $\pi_k \geqslant 3$, *such that* $v = 0$ *and* $u = 2$ *(* $\chi(\sigma) \neq 0$ *). Let* $L = [l_{i,j}] \in T_n^L(F)$, *with diagonal elements equal to one. Then,* $L \in V_\sigma(S_n,\chi)$ *if and only if there exists at most one* $a \in \{2,\ldots,n\}$ *such* $l_{a,a-1} \neq 0$, *being* a *and* $a-1$ *the fix points of* σ.

Proof. By the previous proposition we have

$$(S_n)_T^\sigma = \langle (a,b) : a \text{ and } b \text{ be the fix points of } \sigma \rangle.$$

Assume that $L \in V_\sigma(S_n,\chi)$. By Theorem 1.3 L as at most one non zero entries below the main diagonal. Suppose that there exists a $k \in \{1,\ldots,n\}$ such $a > k > b$. Let X be the upper triangular matrix whose $(b+1)$th column is the bth column of I_n, the (a)th column is the $(b+1)$th column of I_n, and the remaining columns of X are the correspondent columns of I_n. Since $l_{a,b+1} = 0$ then,

$$d_\chi(P(\sigma)LX) = (\chi(\sigma^{-1}(b,b+1,a)) + \chi(\sigma^{-1}(b+1,a)))l_{a,b}.$$

Bearing in mind that $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^0,1^2]}$, $(S_n)_T^\sigma = \langle (a,b) \rangle$, where a and b be the fix points of σ, and attending that there exists a $k \in \{1,\ldots,n\}$ such $a > k > b$, we conclude that

$$\sigma^{-1}(b,b+1,a) \in C_{[\pi_1',\ldots,\pi_k',2^0,1^0]} \quad \text{and} \quad \sigma^{-1}(b+1,a) \in C_{[\pi_1'',\ldots,\pi_k'',2^0,1^1]}.$$

Hence, $\chi(\sigma^{-1}(b,b+1,a)) = 0$ and $\chi(\sigma^{-1}(b+1,a)) = -1$ and then

$$d_\chi(P(\sigma)LX) = -l_{a,b}.$$

Since $L \in V_\sigma(S_n,\chi)$ we have

$$d_\chi(P(\sigma)LX) = d_\chi(P(\sigma)X) = 0,$$

because X has a zero row and so we must have

$$l_{a,b} = 0.$$

Using Proposition 1.4 we have that all matrices $L = [l_{i,j}] \in T_n^L(F)$ such that there exists most one $a \in \{2,\ldots,n\}$ that satisfies $l_{a,a-1} \neq 0$, being a and $a-1$ the fix points of σ, and all the other entries below the main diagonal are null, are in $V_\sigma(S_n,\chi)$. $\square$

PROPOSITION 3.4. *Let* $\chi = (n-2,2)$, *let* $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^v,1^u]}$, $\pi_k \geqslant 3$, *such that* $v = 1$ *and* $u = 0$ *(* $\chi(\sigma) \neq 0$ *). Let* $L = [l_{i,j}] \in T_n^L(F)$, *with diagonal elements equal to one. Then,* $L \in V_\sigma(S_n,\chi)$ *if and only if there exists at most one* $a \in \{1,\ldots,n-1\}$ *such* $l_{a+1,a} \neq 0$, *being* $(a,a+1)$ *the unique transposition of* σ.

Proof. By Lemma 3.2, if $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^1,1^0]}$,

$$(S_n)_T^\sigma = \langle (a,b) \rangle,$$

where (a,b) is the unique transposition of σ. If $L \in V_\sigma(S_n,\chi)$, by Theorem 1.3, L has at most one non zero entry below to the main diagonal; the entry (a,b). Assume that $b > a$ and assume that there exists a $k \in \{1,\ldots,n\}$ such $a > k > b$. Let X be the upper triangular matrix whose $(a+1)$th column is the ath of I_n, and the bth column is the $(a+1)$th column of I_n. Since

$$\sigma^{-1}(b,b+1,a) \in C_{[\pi_1',\ldots,\pi_k',2^0,1^1]} \quad \text{and} \quad \sigma^{-1}(b+1,a) \in C_{[\pi_1'',\ldots,\pi_k'',2^0,1^0]},$$

we have $\chi(\sigma^{-1}(b,b+1,a)) = -1$ and $\chi(\sigma^{-1}(b+1,a)) = 0$ and then

$$d_\chi(P(\sigma)LX) = -l_{a,b} = 0.$$

To complete the proof we only have to show that all matrices with this form are in $V_\sigma(S_n,\chi)$. Using Proposition 1.4 we have that all matrices $L = [l_{i,j}] \in T_n^L(F)$ such that there exists most one $a \in \{2,\ldots,n\}$ that satisfies $l_{a,a-1} \neq 0$, being $(a,a-1)$ the unique transposition of σ, and all the other entries below the main diagonal are null, are in $V_\sigma(S_n,\chi)$. $\square$

To complete the characterization of the set $\mathscr{T}(S_n,\chi)$, where $\chi = (n-2,2)$ we only have characterize the set $V_\sigma(S_n,\chi)$, with $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$. By Lemma 3.2,

$$(S_n)_T^\sigma = \langle (a,c),(a,d),(a,e),(b,c),(b,d),(b,e) \rangle,$$

where (a,b) is the unique transposition of σ and c,d and e be the fix points of σ. We will consider several cases:

1. $a > b > c > d > e$;
2. $a > c > b > d > e$;
3. $a > c > d > b > e$;
4. $a > c > d > e > b$;
5. $c > a > d > e > b$;
6. $c > d > a > e > b$;
7. $c > d > e > a > b$;
8. $c > a > b > d > e$;
9. $c > d > a > b > e$;
10. $c > a > d > b > e$.

We are going to proof the first three of these cases. The proof of the others is similar.

PROPOSITION 3.5. *Let* $\chi = (n-2,2)$, *let* $\sigma \in C_{[\pi_1,\dots,\pi_k,2^1,1^3]}$, $\pi_k \geqslant 3$, *and let* $L = [l_{i,j}] \in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and let* c,d *and* e *be the fix points of* σ. *Assume that* $a > b > c > d > e$. *Then,* $L = [l_{i,j}] \in V_\sigma(S_n,\chi)$ *if and only if* L *has at most one nonzero entry bellow the main diagonal; the entry* $l_{b,c}$ *if* $b = c+1$.

Proof. Let $L = [l_{i,j}] \in T_n^L(F)$, with diagonal elements equal to one. Assume that $L \in V_\sigma(S_n,\chi)$. Then, by Theorem 1.3, the entries of L , below the main diagonal, that could be different from zero are $l_{a,b}, l_{a,c}, l_{c,d}, l_{b,d}, l_{a,d}, l_{a,e}, l_{b,e}, l_{c,e}$ and $l_{d,e}$. We will show that all this entries are zero except eventually $l_{b,c}$.

Let Z_1 be the upper triangular matrix whose ath column is the bth column of I_n and the other columns of Z_1 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_1) = (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(a,b)))l_{a,b}.$$

Since $\sigma^{-1}(a,b) \in C_{[\pi_1,\dots,\pi_k,2^0,1^5]}$ we have, by Lemma 3.1, that $\chi(\sigma^{-1}(a,b)) = 5$. Hence

$$d_\chi(P(\sigma)LZ_1) = 6l_{a,b},$$

and since $L \in V_\sigma(S_n,\chi)$,

$$d_\chi(P(\sigma)LZ_1) = d_\chi(P(\sigma)Z_1) = 0,$$

because Z_1 has a zero row. Then, $l_{a,b} = 0$.

Let Z_2 be the upper triangular matrix whose dth column is the eth column of I_n and the other columns of Z_2 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_2) = (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(e,d)))l_{d,e}.$$

Since d and e are fix points of σ we have $\sigma^{-1}(e,d) \in C_{[\pi_1,\dots,\pi_k,2^2,1]}$ and by Lemma 3.1, that $\chi(\sigma^{-1}(e,d)) = 1$. Hence

$$d_\chi(P(\sigma)LZ_2) = 2l_{d,e},$$

and since $L \in V_\sigma(S_n,\chi)$,

$$d_\chi(P(\sigma)LZ_2) = d_\chi(P(\sigma)Z_2) = 0,$$

because Z_2 has a zero row. Then, $l_{d,e} = 0$.

Let Z_3 be the upper triangular matrix whose cth column is the eth column of I_n and the other columns of Z_3 are the correspondent columns of I_n, and let Z_4 be the upper triangular matrix whose cth column is the dth column of I_n and the other columns of Z_4 are the correspondent columns of I_n. By similar computations we have $l_{c,e} = l_{c,d} = 0$.

Let Z_5 be the upper triangular matrix whose dth column is the eth column of I_n, the bth column is the dth column of I_n and the other columns of Z_5 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_5) = (\chi(\sigma^{-1}(e,d,b)) + \chi(\sigma^{-1}(d,b)))l_{b,e}.$$

Since $\sigma^{-1}(e,d,b) \in C_{[\pi_1,\ldots,\pi_k,2^0,1]}$ and $\sigma^{-1}(d,b) \in C_{[\pi_1,\ldots,\pi_k,2^0,1^2]}$ we have, by Lemma 3.1 that $\chi(\sigma^{-1}(d,b)) = \chi(\sigma^{-1}(e,d,b)) = -1$. Therefore

$$d_\chi(P(\sigma)LZ_5) = -2l_{b,e}.$$

Since $L \in V_\sigma(S_n,\chi)$,

$$d_\chi(P(\sigma)LZ_5) = d_\chi(P(\sigma)Z_5) = 0,$$

because Z_5 has a zero row, and so $l_{b,e} = 0$.

By a similar process we obtain $l_{a,e} = l_{b,d} = l_{a,d} = 0$.

Let Z_9 be the upper triangular matrix whose bth column is the cth column of I_n and the ath column of Z_9 is the bth column of I_n. The remaining columns of Z_9 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma^{-1}LZ_9)) = (\chi(\sigma^{-1}(c,b,a)) + \chi(\sigma^{-1}(a,b)))l_{a,c} = 0.$$

Since $\chi(\sigma^{-1}(b,c,a)) + \chi(\sigma^{-1}(a,b)) = 6$, we conclude that $l_{a,c} = 0$.

Hence L has at most one nonzero element bellow the main diagonal; $l_{b,c}$. Assume that there exists an integer k such $c < k < b$. Let Z_{10} be the upper triangular matrix whose $(c+1)$th column is the cth column of I_n and the bth column is the $(c+1)$th column of I_n. The other columns of Z_{10} are the correspondent columns of I_n. Since $\chi(\sigma^{-1}(c+1,b)) = 0$ we obtain

$$d_\chi(P(\sigma)LZ_{10}) = \chi(\sigma^{-1}(c,c+1,b))l_{b,c} = -l_{b,c} = 0.$$

The sufficiency of the condition follows from Proposition 1.4. □

PROPOSITION 3.6. *Let $\chi = (n-2,2)$, let $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k \geqslant 3$, such $\chi(\sigma) \neq 0$ and let $L = [l_{i,j}] \in T_n^L(F)$, with diagonal elements equal to one. Let (a,b) be the unique transposition of σ and c,d and e be the fix points of σ. Assume that $a > c > b > d > e$. Then, $L = [l_{i,j}] \in V_\sigma(S_n,\chi)$ if and only if:*

1. *$L = I_n$;*

2. *L has at most one nonzero entry bellow the main diagonal: the entry $l_{b,d}$ if $b = d+1$, the entry $l_{c,b}$ if $c = b+1$ or the entry $l_{a,c}$ if $a = c+1$.*

3. *L has at most two nonzero entries below the main diagonal, the entry $l_{c,b}$ if $c = b+1$ and the entry $l_{a,c}$ if $a = c+1$.*

Proof. Let $L=[l_{i,j}]\in T_n^L(F)$, with diagonal elements equal to one. Assume that $L\in V_\sigma(S_n,\chi)$. Then, by Theorem 1.3, the entries of L , below the main diagonal, that could be different from zero are $l_{a,c},l_{c,b},l_{a,b},l_{c,d},l_{b,d},l_{a,d},l_{a,e},l_{c,e},l_{b,e}$ and $l_{d,e}$. We will show that all this entries are zero except eventually $l_{b,d}$, $l_{c,b}$ and $l_{a,c}$.

Let Z_1 be the upper triangular matrix whose dth column is the eth column of I_n and the other columns of Z_1 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_1)=(\chi(\sigma^{-1})+\chi(\sigma^{-1}(e,d)))l_{d,e}=2l_{d,e}=0.$$

Hence $l_{d,e}=0$.

Let Z_2 be the upper triangular matrix whose dth column is the eth column of I_n and bth column is the dth column of I_n. The other columns of Z_2 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_2)=(\chi(\sigma^{-1}(e,d,b))+\chi(\sigma^{-1}(b,d)))l_{b,e}=-2l_{b,e}=0.$$

Hence $l_{b,e}=0$.

Let Z_3 be the upper triangular matrix whose cth column is the eth column of I_n and the other columns of Z_3 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_3)=(\chi(\sigma^{-1})+\chi(\sigma^{-1}(c,e)))l_{c,e}=2l_{c,e}=0,$$

and so $l_{c,e}=0$.

Let Z_4 be the upper triangular matrix whose dth column is the eth column of I_n and the ath column of Z_4 is the dth column of I_n. The remaining columns of Z_4 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_4)=(\chi(\sigma^{-1}(e,d,a))+\chi(\sigma^{-1}(d,a)))l_{a,e}=-2l_{a,e}=0,$$

and so $l_{a,e}=0$.

Let Z_5 be the upper triangular matrix whose bth column is the dth column of I_n and the cth column is the bth column of I_n. The other columns of Z_5 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_5)=(\chi(\sigma^{-1})+\chi(\sigma^{-1}(d,b)))l_{b,d}l_{c,b}+(\chi(\sigma^{-1}(d,b,c))+\chi(\sigma^{-1}(b,c)))l_{c,d}.$$

Since $\chi(\sigma^{-1})+\chi(\sigma^{-1}(d,b))=0$ we obtain

$$d_\chi(P(\sigma)LZ_5)=-2l_{c,d}=0,$$

and so $l_{c,d}=0$.

Let Z_6 be the upper triangular matrix whose ath column is the bth column of I_n and the other columns of Z_6 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_6)=(\chi(\sigma^{-1})+\chi(\sigma^{-1}(a,b)))l_{a,b}+(\chi(\sigma^{-1}(b,a,c))+\chi(\sigma^{-1}(a,c)))l_{a,c}l_{c,b}.$$

Since $\chi^{-1}(\sigma^{-1}(b,a,c))+\chi(\sigma^{-1}(a,c))=0$ we obtain

$$d_\chi(P(\sigma)LZ_6)=6l_{a,b}=0,$$

and so $l_{a,b}=0$.

Let Z_7 be the upper triangular matrix whose bth column is the dth column of I_n and the ath column is the bth column of I_n. The remaining columns of Z_7 are the correspondent columns of I_n. Then

$$\begin{aligned} d_\chi(P(\sigma)LZ_7) &= (\chi(\sigma^{-1}(a,b))+\chi(\sigma^{-1}(d,b,a)))l_{a,d} \\ &\quad +(\chi(\sigma^{-1}(a,c))+\chi(\sigma^{-1}(a,c)(b,d)))l_{b,d}l_{c,b}l_{a,c} \\ &= 6l_{a,d}-2l_{b,d}l_{c,b}l_{a,c} \\ &= 0. \end{aligned}$$

Therefore

$$3l_{a,d}=l_{b,d}l_{c,b}l_{a,c}. \tag{2}$$

Let Z_8 be the upper triangular matrix whose cth column is the dth column of I_n and the other columns of Z_8 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_8)=(\chi(\sigma^{-1}(d,c,b))+\chi(\sigma^{-1}(c,b)))l_{b,d}l_{c,b}=-2l_{b,d}l_{c,b}=0.$$

Hence $l_{b,d}l_{c,b}=0$ and by (2) we conclude that $l_{a,d}=0$.

We have shown that in this case, if $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ then L has at most three nonzero entries bellow the main diagonal; the entry $l_{b,d}$, the entry $l_{c,b}$ and the entry $l_{a,c}$. We also have

$$l_{b,d}l_{c,b}=0. \tag{3}$$

Let Z_9 be the upper triangular matrix whose bth column is the dth column of I_n and the ath column is the cth column of I_n. The other columns of Z_9 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_9)=-2l_{b,d}l_{a,c}=0,$$

and so

$$l_{b,d}l_{a,c}=0. \tag{4}$$

If $l_{b,d}\neq 0$, by (3) and (4) we conclude $l_{a,c}=l_{c,b}=0$. Assume that $l_{b,d}\neq 0$ and there exists an integer k such $d<k<b$. Let Z_{10} be the upper triangular matrix whose $d+1$th column is the dth column of I_n and the bth column is the $d+1$th column of I_n. The other columns of Z_{10} are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_{10})=(\chi(\sigma^{-1}(d,d+1,b))+\chi(\sigma^{-1}(d+1,b)))l_{b,d}.$$

Since $\chi(\sigma^{-1}(d+1,b))=0$ we obtain

$$d_\chi(P(\sigma)LZ_{10})=-l_{b,d}=0,$$

and so $l_{b,d}=0$, which is a contradiction. We have proved that if $l_{b,d}\neq 0$ then $b=d+1$.

If $l_{c,b}\neq 0$ and $l_{a,c}\neq 0$, by (3) or (4), $l_{d,b}=0$. If there exists an integer k such $b<k<c$ or if there exists an integer k such $c<k<a$ we conclude, by a similar away, that $l_{c,b}=0$ or $l_{a,c}=0$, and the proof of the necessity of the condition is complete.

To prove the sufficiency of the conditions, bearing in mind the Proposition 1.4, we only have to prove that if L has at most two nonzero entries below the main diagonal, the entry $l_{c,b}$ if $c=b+1$ and the entry $l_{a,c}$ if $a=c+1$ then $L\in V_\sigma(S_n,\chi)$. Let $X=[x_{i,j}]\in T_n^U(F)$. Then

$$\begin{aligned}
d_\chi(P(\sigma)LX) &= \chi(\sigma^{-1})\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq c}} x_{i,i}\right)(l_{c,b}x_{b,c}+x_{c,c})(l_{a,c}x_{c,a}+x_{a,a})\\
&\quad +\chi(\sigma^{-1}(a,c))\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq c}} x_{i,i}\right)(l_{c,b}x_{b,a}+x_{c,a})l_{a,c}x_{c,c}\\
&\quad +\chi(\sigma^{-1}(b,c))\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq b, i\neq c}} x_{i,i}\right)x_{b,c}l_{c,b}x_{b,b}(l_{a,c}x_{c,a}+x_{a,a})\\
&\quad +\chi(\sigma^{-1}(b,a,c))\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq b, i\neq c}} x_{i,i}\right)x_{b,a}l_{a,c}x_{c,c}l_{c,b}x_{b,b}\\
&= \left(\prod^n_{\substack{i=1\\ i\neq a, i\neq c}} x_{i,i}\right)(l_{c,b}x_{b,c}l_{a,c}x_{c,a}+l_{c,b}x_{b,c}x_{a,a}+x_{c,c}l_{a,c}x_{c,a}+x_{a,a}x_{c,c})\\
&\quad -\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq c}} x_{i,i}\right)(l_{c,b}x_{b,a}l_{a,c}x_{c,c}+x_{c,a}l_{a,c}x_{c,c})\\
&\quad -\left(\prod^n_{\substack{i=1\\ i\neq a, i\neq b, i\neq c}} x_{i,i}\right)(x_{b,c}l_{c,b}x_{b,b}l_{a,c}x_{c,a}+x_{b,c}l_{c,b}x_{b,b}x_{a,a})\\
&\quad +\left(\prod^n_{\substack{i=1\\ i\neq a}} x_{i,i}\right)x_{b,a}l_{a,c}l_{c,b}\\
&= \prod_{i=1}^n x_{i,i}\\
&= d_\chi(P(\sigma)X),
\end{aligned}$$

and the proof is complete. □

PROPOSITION 3.7. *Let* $\chi=(n-2,2)$, *let* $\sigma\in C_{[\pi_1,\dots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, *such* $\chi(\sigma)\neq 0$ *and let* $L=[l_{i,j}]\in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and* c,d *and* e *be the fix points of* σ. *Assume that* $a>c>d>b>e$. *Then,* $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ *if and only if:*

1. $L=I_n$;
2. *L has at most one nonzero entry bellow the main diagonal, the entry $l_{b,e}$ if $b=e+1$, the entry $l_{d,b}$ if $d=b+1$, the entry $l_{c,b}$ if $c=b+2$ or the entry $l_{a,c}$ if $a=c+1$.*
3. *L has at most two nonzero entries below the main diagonal, the entry $l_{a,c}$ if $a=c+1$ and the entry $l_{c,b}$ if $c=b+2$.*

Proof. Let $L=[l_{i,j}]\in T_n^L(F)$, with diagonal elements equal to one. Assume that $L\in V_\sigma(S_n,\chi)$. Then, by Theorem 1.3, the entries of L , below the main diagonal, that could be different from zero are $l_{a,c},l_{c,b},l_{a,b},l_{c,d},l_{b,d},l_{a,d},l_{a,e},l_{c,e},l_{b,e}$ and $l_{d,e}$. We will show that all this entries are zero except eventually $l_{b,e}$, $l_{d,b}$, $l_{c,b}$ and $l_{a,c}$.

Let Z_1 be the upper triangular matrix whose cth column is the dth column of I_n. The other columns of Z_1 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_1) = (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(c,d)))l_{c,d} = 2l_{c,d} = 0,$$

because $L \in V_\sigma(S_n,\chi)$ and Z_1 has a zero row. Therefore, $l_{c,d} = 0$.

Let Z_2 be the upper triangular matrix whose cth column is the dth column of I_n and the ath column is the cth column of I_n. The other columns of Z_2 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_2) = (\chi(\sigma^{-1}(d,c,a)) + \chi(\sigma^{-1}(c,a)))l_{a,d} = -2l_{a,d} = 0,$$

and so $l_{a,d} = 0$.

Let Z_3 be the upper triangular matrix whose ath column is the bth column of I_n. The other columns of Z_3 are the correspondent columns of I_n. Then

$$d_\chi(P(\sigma)LZ_3) = (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(a,b)))l_{a,b} + (\chi(\sigma^{-1}(a,c)) + \chi(\sigma^{-1}(b,a,c)))l_{c,b}l_{a,c}.$$

Since $\chi(\sigma^{-1}(a,c)) + \chi(\sigma^{-1}(b,a,c)) = 0$ and $\chi(\sigma^{-1}) + \chi(\sigma^{-1}(a,b)) = 6$ we conclude that $l_{a,b} = 0$.

By a similar away we can prove that $l_{d,e} = l_{c,e} = 0$.

Let Z_6 be the upper triangular matrix whose bth column is the eth column of I_n and the ath columns the bth column of I_n. The other columns of Z_6 are the correspondent column of I_n. Then

$$\begin{aligned} d_\chi(P(\sigma)LZ_6) &= (\chi(\sigma^{-1}(a,c)) + \chi(\sigma^{-1}(e,b)(a,c)))l_{b,e}l_{c,b}l_{a,c} \\ &\quad + (\chi(\sigma^{-1}(e,b,a)) + \chi(\sigma^{-1}(a,b))l_{a,e} \\ &= -2l_{b,e}l_{c,b}l_{a,c} + 6l_{a,e} \\ &= 0. \end{aligned}$$

Therefore

$$3l_{a,e} = l_{b,e}l_{c,b}l_{a,c} \tag{5}$$

Let Z_7 be the upper triangular matrix whose bth column is the eth column of I_n and the ath columns the cth column of I_n. The other columns of Z_7 are the correspondent column of I_n. Then

$$\begin{aligned} d_\chi(P(\sigma)LZ_7) &= (\chi(\sigma^{-1}) + \chi(\sigma^{-1}(e,b)) + \chi(\sigma^{-1}(a,c)) + \chi(\sigma^{-1}(e,b)(a,c)))l_{b,e}l_{a,c} \\ &= -2l_{b,e}l_{a,c} \\ &= 0. \end{aligned}$$

Hence, $l_{b,e}l_{a,c} = 0$ and by (5) we conclude that $l_{a,e} = 0$.

By a similar process we can prove that

$$l_{d,b}l_{a,c} = 0,\ \ l_{d,b}l_{c,b} = 0,\ \ l_{b,e}l_{d,b} = 0 \ \text{ and } \ l_{b,e}l_{c,b} = 0. \tag{6}$$

Assume that $l_{b,e} \neq 0$ and assume that there is an integer such $e < k < b$. Let Z_{11} the upper triangular matrix whose $(e+1)$th column is the eth column of I_n and the bth column is the $(e+1)$ th column of I_n. Then

$$d_\chi(P(\sigma)LZ_{11}) = (\chi(\sigma^{-1}(e,e+1,b)) + \chi(\sigma^{-1}(e+1,b))l_{b,e}.$$

Since $\chi(\sigma^{-1}(e+1,b)) = 0$ we obtain

$$d_\chi(P(\sigma)LZ_{11}) = -l_{b,e} = 0,$$

which is a contradiction. We have proved that if $l_{b,e} = 0$ then $b = e+1$.

By a similar process we prove that if $l_{a,c} \neq 0$ then $a = c+1$, if $l_{d,b} \neq 0$ then $d = b+1$ and if $l_{c,b} \neq 0$ then $c = b+2$ and the proof of the necessity of the conditions is now complete.

To prove the sufficiency of the conditions, bearing in mind the Proposition 1.4, we only have to prove that if L has the (c,b) entry different from zero, with $c = b+2$, and the other entries below the main diagonal are null then $L \in V_\sigma(S_n,\chi)$. Let $X = [x_{i,j}] \in T_n^U(F)$. Then

$$\begin{aligned}
d_\chi(P(\sigma)LX) &= \chi(\sigma^{-1})\left(\prod_{\substack{i=1\\ i\neq b+2}}^n x_{i,i}\right)(l_{c,b}x_{b,b+2} + x_{b+2,b+2})\\
&\quad +\chi(\sigma^{-1}(b,c))\left(\prod_{\substack{i=1\\ i\neq b, i\neq b+2}}^n x_{i,i}\right) l_{c,b}x_{b,b}x_{b,b+2}\\
&\quad +\chi(\sigma^{-1}(d,c))\left(\prod_{\substack{i=1\\ i\neq b+1, i\neq b+2}}^n x_{i,i}\right) l_{c,b}x_{b,b+1}x_{b+1,b+2}\\
&\quad +\chi(\sigma^{-1}(b,d,c))\left(\prod_{\substack{i=1\\ i\neq b, i\neq b+1, i\neq b+2}}^n x_{i,i}\right) l_{c,b}x_{b,b}x_{b,b+1}x_{b+1,b+2}\\
&= \prod_{i=1}^n x_{i,i}\\
&= d_\chi(P(\sigma)X),
\end{aligned}$$

Assume now that $l_{a,c} \neq 0$ and $l_{c,b} \neq 0$. Let $X = [x_{i,j}] \in T_n^U(F)$. Bearing in mind that $c = a-1 = b+2$, we have

$$\begin{aligned}
d_\chi(P(\sigma)LX) &= \chi(\sigma^{-1})\left(\prod_{\substack{i=1\\ i\neq a, i\neq c}}^n x_{ii}\right)(l_{a,c}x_{c,a} + x_{a,a})(l_{c,b}x_{b,c} + x_{c,c})\\
&\quad + \chi(\sigma^{-1}(b,d,c))\left(\prod_{\substack{i=1\\ i\neq a, i\neq c, i\neq d}}^n x_{ii}\right)(l_{a,c}x_{c,a} + x_{a,a})l_{c,b}x_{b,d}x_{d,c}\\
&\quad + \chi(\sigma^{-1}(b,c))\left(\prod_{\substack{i=1\\ i\neq a, i\neq c}}^n x_{ii}\right)(l_{a,c}x_{c,a} + x_{a,a})l_{c,b}x_{b,c}\\
&\quad + \chi(\sigma^{-1}(d,c))\left(\prod_{\substack{i=1\\ i\neq a, i\neq c, i\neq d}}^n x_{ii}\right)(l_{a,c}x_{c,a} + x_{a,a})l_{c,b}x_{b,d}x_{d,c}\\
&\quad + \chi(\sigma^{-1}(a,c))\left(\prod_{\substack{i=1\\ i\neq a}}^n x_{ii}\right)(l_{c,b}x_{b,a} + x_{c,a})l_{a,c}
\end{aligned}$$

$$
\begin{aligned}
&+\chi(\sigma^{-1}(a,c,b,d))\left(\prod^{n}_{\substack{i=1\\ i\neq a, i\neq d}} x_{ii}\right) l_{c,b}x_{b,d}x_{d,a}l_{a,c}\\
&+\chi(\sigma^{-1}(a,c,b))\left(\prod^{n}_{\substack{i=1\\ i\neq a}} x_{ii}\right) l_{c,b}x_{b,a}l_{a,c}\\
&+\chi(\sigma^{-1}(a,c,d))\left(\prod^{n}_{\substack{i=1\\ i\neq a, i\neq d}} x_{ii}\right) l_{c,b}x_{d,a}x_{b,d}l_{a,c}\\
=&\prod_{i=1}^{n} x_{i,i}\\
=&\,d_{\chi}(P(\sigma)X),
\end{aligned}
$$

and the is proof is complete. □

PROPOSITION 3.8. *Let $\chi=(n-2,2)$, let $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, such $\chi(\sigma)\neq 0$ and let $L=[l_{i,j}]\in T_n^L(F)$, with diagonal elements equal to one. Let (a,b) be the unique transposition of σ and c,d and e be the fix points of σ. Assume that $a>c>d>e>b$. Then, $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ if and only if:*

1. *$L=I_n$;*
2. *L has at most one nonzero entry bellow the main diagonal, the entry $l_{e,b}$ if $e=b+1$, the entry $l_{d,b}$ if $d=b+2$ or the entry $l_{a,c}$ if $a=c+1$.*

PROPOSITION 3.9. *Let $\chi=(n-2,2)$, let $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, such $\chi(\sigma)\neq 0$ and let $L=[l_{i,j}]\in T_n^L(F)$, with diagonal elements equal to one. Let (a,b) be the unique transposition of σ and c,d and e be the fix points of σ. Assume that $c>a>d>e>b$. Then, $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ if and only if:*

1. *$L=I_n$;*
2. *L has at most one nonzero entry bellow the main diagonal, the entry $l_{e,b}$ if $e=b+1$, the entry $l_{d,b}$ if $d=b+2$, the entry $l_{a,d}$ if $a=d+1$ or the entry $l_{c,a}$ if $c=a+1$.*
3. *L has at most two nonzero entries below the main diagonal, the entry $l_{a,d}$ if $a=d+1$ and the entry $l_{d,b}$ if $d=b+2$.*

PROPOSITION 3.10. *Let $\chi=(n-2,2)$, let $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, such $\chi(\sigma)\neq 0$ and let $L=[l_{i,j}]\in T_n^L(F)$, with diagonal elements equal to one. Let (a,b) be the unique transposition of σ and c,d and e be the fix points of σ. Assume that $c>d>a>e>b$. Then, $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ if and only if:*

1. *$L=I_n$;*
2. *L has at most one nonzero entry bellow the main diagonal, the entry $l_{e,b}$ if $e=b+1$, the entry $l_{a,e}$ if $a=e+1$, the entry $l_{d,a}$ if $d=a+1$, the entry $l_{d,b}$ if $d=b+3$ or the entry $l_{c,a}$ if $c=a+2$.*

3. *L has at most two nonzero entries below the main diagonal, the entry* $l_{d,b}$ *if* $d=b+3$ *and the entry* $l_{d,a}$ *if* $d=a+1$ *or the entry* $l_{e,b}$ *if* $e=b+1$ *and the entry* $l_{a,e}$ *if* $a=e+1$.

PROPOSITION 3.11. *Let* $\chi=(n-2,2)$, *let* $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, *such* $\chi(\sigma)\neq 0$ *and let* $L=[l_{i,j}]\in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and* c,d *and* e *be the fix points of* σ. *Assume that* $c>d>e>a>b$. *Then,* $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ *if and only if:*

1. $L=I_n$;

2. *L has at most one nonzero entry bellow the main diagonal, the entry* $l_{d,a}$ *if* $d=a+2$, *the entry* $l_{e,a}$ *if* $e=a+1$ *or the entry* $l_{e,b}$ *if* $e=b+2$.

3. *L has at most two nonzero entries below the main diagonal, the entry* $l_{e,b}$ *if* $e=b+2$ *and the entry* $l_{e,a}$ *if* $e=a+1$.

PROPOSITION 3.12. *Let* $\chi=(n-2,2)$, *let* $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, *such* $\chi(\sigma)\neq 0$ *and let* $L=[l_{i,j}]\in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and* c,d *and* e *be the fix points of* σ. *Assume that* $c>a>b>d>e$. *Then,* $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ *if and only if:*

1. $L=I_n$;

2. *L has at most one nonzero entry bellow the main diagonal, the entry* $l_{b,d}$ *if* $b=d+1$, *the entry* $l_{c,b}$ *if* $c=b+2$ *and the entry* $l_{c,a}$ *if* $c=a+1$.

3. *L has at most two nonzero entries below the main diagonal, the entry* $l_{c,b}$ *if* $c=b+2$ *and the entry* $l_{c,a}$ *if* $c=a+1$.

PROPOSITION 3.13. *Let* $\chi=(n-2,2)$, *let* $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, *such* $\chi(\sigma)\neq 0$ *and let* $L=[l_{i,j}]\in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and* c,d *and* e *be the fix points of* σ. *Assume that* $c>d>a>b>e$. *Then,* $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ *if and only if:*

1. $L=I_n$;

2. *L has at most one nonzero entry bellow the main diagonal, the entry* $l_{b,e}$ *if* $b=e+1$ *or* $l_{c,a}$ *if* $c=a+2$, *the entry* $l_{d,a}$ *if* $d=a+1$ *or the entry* $l_{d,b}$ *if* $d=b+2$.

3. *L has at most two nonzero entries below the main diagonal, the entry* $l_{d,b}$ *if* $d=b+2$ *and the entry* $l_{d,a}$ *if* $d=a+1$.

PROPOSITION 3.14. *Let* $\chi=(n-2,2)$, *let* $\sigma\in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k\geqslant 3$, *such* $\chi(\sigma)\neq 0$ *and let* $L=[l_{i,j}]\in T_n^L(F)$, *with diagonal elements equal to one. Let* (a,b) *be the unique transposition of* σ *and* c,d *and* e *be the fix points of* σ. *Assume that* $c>a>d>b>e$. *Then,* $L=[l_{i,j}]\in V_\sigma(S_n,\chi)$ *if and only if:*

1. $L = I_n$;

2. L has at most one nonzero entry bellow the main diagonal, the entry $l_{b,e}$ if $b = e+1$ or $l_{d,b}$ if $d = b+1$, the entry $l_{a,d}$ if $a = d+1$, the entry $l_{c,a}$ if $c = a+1$ or the entry $l_{c,b}$ if $c = b+3$.

3. L has at most two nonzero entries below the main diagonal, the entry $l_{d,b}$ if $d = b+1$ and the entry $l_{a,d}$ if $a = d+1$ or the entry $l_{c,b}$ if $c = b+3$ and the entry $l_{c,a}$ if $c = a+1$.

It is possible to rewrite the 10 cases in a Theorem.

THEOREM 3.15. *Let $\chi = (n-2,2)$, let $\sigma \in C_{[\pi_1,\ldots,\pi_k,2^1,1^3]}$, $\pi_k \geqslant 3$, such $\chi(\sigma) \neq 0$ and let $L = [l_{i,j}] \in T_n^L(F)$, with diagonal elements equal to one. Let (a,b) be the unique transposition of σ with $b < a$. Then, $L = [l_{i,j}] \in V_\sigma(S_n,\chi)$ if and only if except at most one of the following conditions, all the other elements below the main diagonal of L are zero:*

1. *if $l_{b,b-1} \neq 0$ then $b-1$ is a fix point of σ;*

2. *if $l_{b+1,b} \neq 0$ then $b+1$ is a fix point of σ;*

3. *if $l_{b+2,b} \neq 0$ then $b+1$ and $b+2$ are fix points of σ;*

4. *if $l_{b+2,b} \neq 0$ and/or $l_{b+3,b+2} \neq 0$ then $b+1$ and $b+2$ are fix points of σ and $a = b+3$;*

5. *if $l_{b+1,b} \neq 0$ and/or $l_{b+2,b+1} \neq 0$ then $b+1$ is a fix point of σ and $a = b+2$;*

6. *if $l_{b+3,b} \neq 0$ and/or $l_{b+3,b+2} \neq 0$ then $b+1$ and $b+3$ are fix points of σ and $a = b+2$;*

7. *if $l_{b+2,b} \neq 0$ and/or $l_{b+2,b+1} \neq 0$ then $b+2$ is a fix point of σ and $a = b+1$;*

8. *if $l_{a,a-1} \neq 0$ then $a-1$ is a fix point of σ;*

9. *if $l_{a+1,a} \neq 0$ then $a+1$ is a fix point of σ;*

10. *if $l_{a+2,a} \neq 0$ then $a+1$ and $a+2$ are fix points of σ;*

The next theorem is now an easy consequence of theorem 1.1 and propositions 2.5 and 2.6 and it gives a complete description of the set $\mathscr{T}(S_n,\chi)$, with $\chi = (n-2,2)$:

THEOREM 3.16. *If $\chi = (n-2,2)$ then $A \in \mathscr{T}(S_n,\chi)$ if and only if $A = P(\sigma)L_\sigma R$ such $\sigma \in S_n$ satisfy $\chi(\sigma) \neq 0$, $R \in T_n^U(F)$ satisfy $\det(R) = \frac{\chi(id)}{\chi(\sigma)}$ and $L_\sigma = [l_{i,j}]$ is a lower triangular matrix with all diagonal elements equal to 1 and satisfying the following conditions:*

1. *$l_{rp} = 0$ whenever $r \notin \{p, p+1, p+2, p+3\}$;*

2. *$L_\sigma = I_n$ if $\sigma \notin C_{[\pi_1,\dots\pi_k,2^0,1^2]}$, $\sigma \notin C_{[\pi_1,\dots\pi_k,2^1,1^0]}$ or $\sigma \notin C_{[\pi_1,\dots\pi_k,2^1,1^3]}$;*

3. *There is at most one i, $1 \leqslant i \leqslant n-1$, such that $l_{i+1,i} \neq 0$, if $\{i, i+1\}$ is the set of the fix points of σ or, if σ don't have fix points and $(i, i+1)$ is the unique transposition of σ or, if i is in the unique transpositions of σ and $i+1$ is a fix point of σ or, if $i+1$ is in the unique transposition of σ and i is a fix point of σ.*

4. *There exists at most one i, $1 \leqslant i \leqslant n-2$, such that $l_{i+2,i} \neq 0$, being $i+1$ and $i+2$ fix points of σ and i is in the unique transposition of σ;*

5. *There is at most one i, $1 \leqslant i \leqslant n-3$ such $l_{i+2,i} \neq 0$ and/or $l_{i+3,i+2} \neq 0$ being $i+1$ and $i+2$ fix points of σ and $(i, i+3)$ is the unique transposition of σ;*

6. *There is at most one i, $1 \leqslant i \leqslant n-2$ such $l_{i+1,i} \neq 0$ and/or $l_{i+2,i+1} \neq 0$ being $i+1$ a fix point of σ and $(i, i+2)$ is the unique transposition of σ;*

7. *There is at most one i, $1 \leqslant i \leqslant n-3$ $l_{i+3,i} \neq 0$ and/or $l_{i+3,i+2} \neq 0$ being $i+1$ and $i+3$ fix points of σ and $(i, i+2)$ is the unique transposition of σ ;*

8. *There is at most one i, $1 \leqslant i \leqslant n-2$ such $l_{i+2,i} \neq 0$ and/or $l_{i+2,i+1} \neq 0$ being $i+2$ a fix point of σ and $(i, i+1)$ is the unique transposition of σ;*

$\square$

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