Operators
and
Matrices
Volume 4, Number 1 (2010), 119–125

ON OPERATORS WITH LARGE SELF–COMMUTATORS

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(*Communicated by C.-K. Li*)

Abstract. We study the "size" of self-commutator of Hilbert space operator. Different properties of operators, having large self-commutators are established. Possible values of the numerical radius of such operators are investigated. Two necessary and sufficient conditions of equality $\|A\| = 2w(A)$ are mentioned.

1. Let A be a linear bounded operator, acting in a Hilbert space $(\mathscr{H}, \langle \bullet, \bullet \rangle)$. The well-known and important class of normal operators is characterized by the equality $AA^* = A^*A$. The difference $AA^* - A^*A = C(A)$ is said to be the self-commutator of the operator A . This notion was investigated first probably by Halmos in [4]. If $C(A)$ is semi-definite, the operator A is said to be semi-normal, particularly, if $C(A) \geqslant \mathbf{0}$, then A is hyponormal. According to Putnam's inequality [5] for any semi-normal operator

$$\|AA^* - A^*A\| \leqslant \frac{1}{\pi} mes_2(SpA),$$

where SpA is the spectrum of A and mes_2 means the plane Lebesgue measure. As the spectrum of any operator is contained in the circle centered at the origin of coordinate system and of radius $\|A\|$, the last inequality implies

$$\|AA^* - A^*A\| \leqslant \|A\|^2. \tag{1}$$

PROPOSITION 1. *For any operator A inequality* (1) *holds.*

Proof. As $C(A)$ is self-adjoint

$$\|C(A)\| = \sup_{\|x\|=1} |\langle (AA^* - A^*A)x, x \rangle| = \sup_{\|x\|=1} \left| \|Ax\|^2 - \|A^*x\|^2 \right|$$
$$\leqslant \sup_{\|x\|=1} \left(\max \left\{ \|Ax\|^2, \|A^*x\|^2 \right\} \right) \leqslant \|A\|^2. \quad \square$$

EXAMPLE 1. Let S be the operator of the simple unilateral shift. Then $S^*S - SS^*$ is the operator of orthogonal projection on the first element of the basis, shifted by S, so

$$\|S^*S - SS^*\| = \|S\|^2.$$

Mathematics subject classification (2000): 47A12, 47B47.
Keywords and phrases: Self-commutator, numerical radius.

Paper OaM-04-05

Definition 1. Operator A is said to have a large self-commutator, if

$$\|AA^* - A^*A\| = \|A\|^2. \tag{2}$$

Denote

$$W(A) = \sup_{x \neq \theta} \frac{|\langle Ax, x\rangle|}{\|x\|^2}$$

and $w(A) = \sup\limits_{\lambda \in W(A)} |\lambda| -$ the numerical range and the numerical radius of A respectively. Let $A = H + iJ$ be the Cartesian decomposition of A ,

$$H = \mathrm{Re}A = \frac{A + A^*}{2},\ J = \mathrm{Im}A = \frac{A - A^*}{2i}.$$

PROPOSITION 2. *Let $\|H\| = \|J\| = \frac{\|A\|}{2}$. Then the operator A has a large self-commutator.*

Proof. First we establish a useful equality. We have

$$HJ = \frac{A + A^*}{2} \cdot \frac{A - A^*}{2i} = \frac{A^2 - AA^* + A^*A - A^{*2}}{4i}$$

and

$$(HJ)^* = JH = \frac{A - A^*}{2i} \cdot \frac{A + A^*}{2} = \frac{A^2 + AA^* - A^*A - A^{*2}}{4i},$$

so

$$\mathrm{Im}(HJ) = \frac{AA^* - A^*A}{4}.$$

From the condition of the Proposition 2 we get $\|A\| = \|H + iJ\| \leqslant \|H\| + \|J\| = \|A\|$, hence $\|A\| = \|H\| + \|iJ\|$. According to a theorem of Barraa and Boumazgour [2] this condition is equivalent to $\|H\| \cdot \|J\| \in \overline{W}(HiJ)$ (the upper bar denotes the closure of the set in the topology of $\mathbb{C}$.) It means that $-i\|H\| \cdot \|J\| \in \overline{W}(HJ)$ or

$$\|H\| \cdot \|J\| \in \mathrm{Im}\left(\overline{W}(HJ)\right) = \overline{W}(\mathrm{Im}(HJ)),$$

which implies $\|H\| \cdot \|J\| = \|\mathrm{Im}(HJ)\|$ and

$$\|AA^* - A^*A\| = \|A\|^2. \quad \square$$

The next example shows that the condition in the above proposition, in general, is not necessary.

EXAMPLE 2. Let

$$\mathscr{J}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

be the Jordan cell. It is easy to check that

$$\mathscr{J}_n \mathscr{J}_n^* - \mathscr{J}_n^* \mathscr{J}_n = \begin{pmatrix} 1\,0\,0 \cdots & 0 \\ 0\,0\,0 \cdots & 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\; \cdot & \cdot \\ 0\,0\,0 \cdots & 0 \\ 0\,0\,0 \cdots & -1 \end{pmatrix},$$

so $\mathscr{J}_n$ has a large self-commutator, although $\|\mathrm{Re}\,\mathscr{J}_n\| = \|\mathrm{Im}\,\mathscr{J}_n\| = 1$. As $\|\mathscr{J}_n\| = 1$, $w(\mathscr{J}_n) = \cos\frac{\pi}{n+1}$, the ratio $w(\mathscr{J}_n)/\|\mathscr{J}_n\|$, starting by value 1/2, may be arbitrary close to 1. Note that for the operator S of the unilateral shift this ratio is equal to 1.

Let

$$A_t = Ae^{it},\ t \in [0, 2\pi), H_t = \mathrm{Re}A_t,\ J_t = \mathrm{Im}A_t.$$

It is known that [7]

$$w(A) = \sup_{t\in[0,2\pi)} \|H_t\| = \sup_{t\in[0,2\pi)} \|J_t\|. \tag{3}$$

We have

$$\|A\| = \left\|e^{it}A\right\| = \|H_t + iJ_t\| \leqslant \|H_t\| + \|J_t\|,$$

thus

$$\|A\| - w(A) \leqslant \|H_t\|. \tag{4}$$

PROPOSITION 3. *Conditions*

$$\|A\| = 2w(A) \tag{5}$$

and $\|H_t\| \equiv \|J_t\| \equiv \frac{\|A\|}{2}$ *are equivalent.*

Proof. Let first $\|A\| = 2w(A)$. According to (4) $w(A) \leqslant \|H_t\|$ and by (3) we deduce $\|H_t\| \equiv w(A)$. The inverse implication is elementary. □

COROLLARY. *Equality* $\|A\| = 2w(A)$ *takes place if and only if* $\overline{W}(A)$ *coincides with the disk* $\overline{D}\left(0, \frac{\|A\|}{2}\right)$ *with the center at the origin of the coordinate system and of radius* $\frac{\|A\|}{2}$.

In fact, operators satisfying equality (5) have extra large self-commutators.

PROPOSITION 4. *Equality* (5) *is satisfied if and only if*

$$\|AA^* - A^*A\| = 4w^2(A). \tag{6}$$

Proof. Let (6) be satisfied. Then $\|A\|^2 \geqslant \|AA^* - A^*A\| = 4w^2(A) \geqslant \|A\|^2$, hence $\|A\| = 2w(A)$. If this condition is satisfied, then by Propositions 3 and 2 condition (6) holds. □

PROPOSITION 5. *Let the operator* A *be nilpotent with the index of nilpotency equal to 2, i.e.* $A^2 = \mathbf{0}$. *Then* A *satisfies* (6).

LEMMA. *The conditions* $A^2 = \mathbf{0}$ *and* $R(A) \perp R(A^*)$ *are equivalent.*

Proof of Lemma. Let $A^2 = \mathbf{0}$. Then for any $x, y \in H$ we have

$$\langle Ax, A^*y\rangle = \langle A^2x, y\rangle = 0.$$

On the other hand, if $R(A) \perp R(A^*)$, then $\left\|A^2x\right\|^2 = \langle A^2x, A^2x\rangle = \langle Ax, A^*A^2x\rangle = 0$. □

Proof of Proposition 5. Since $A^2 = \mathbf{0}$, we note that

$$\|AA^* - A^*A\| = \sup_{\|x\|=1} \|(AA^* - A^*A)x\| = \sup_{\|x\|=1} \|(AA^* + A^*A)x\| = \|AA^* + A^*A\|.$$

As it is well known ([3], Theorem 4.3)

$$\left\|\frac{A + A^*}{2}\right\|^2 \leqslant \frac{1}{2}\left(\left\|\frac{AA^* + A^*A}{2}\right\| + w\left(A^2\right)\right)$$

therefore

$$\|\mathrm{Re}\,A\|^2 \leqslant \frac{1}{4}\|AA^* + A^*A\| \leqslant \frac{\|A\|^2}{4}.$$

Putting A_t instead of A in the above inequality, we get

$$\|H_t\| \leqslant \frac{\|A\|}{2},$$

completing the proof. □

EXAMPLE 3. Let V be the operator in $L^2(-1; 1)$, defined by the formula

$$(Vf) = \int_{-x}^{x} f(t)\,dt.$$

Evidently $V^2 = \mathbf{0}$. We have

$$w(V) = \frac{\|V\|}{2} = \frac{2}{\pi}.$$

Another description of operators, satisfying (5) is known. In [7] it has been shown that the condition (5) is equivalent to

$$\|H_t\| + \|J_t\| = \|A\|, \ \forall t \in [0, 2\pi).$$

As it is noticed by the author, the fulfillment of this equality at only one point does not imply the above mentioned relation between the norm and the numerical radius of the operator, namely for self-adjoint operator the numerical radius and the norm are equal. Nevertheless for an operator with a large self-commutator some estimate may be proposed.

PROPOSITION 6. *Let A have a large self-commutator and satisfy the following condition*

$$\|A\| = \|H\| + \|J\|. \tag{7}$$

Then

$$w(A) \leqslant \frac{\sqrt{2}}{2}\|A\|.$$

Proof. From the proof of Proposition 2 we have

$$4\|H\| \cdot \|J\| = \|AA^* - A^*A\| = \|A\|^2$$

and equality (7) gives $\|H\| = \|J\| = \frac{\|A\|}{2}$.

It is easy to check that

$$H_t = H\cos t - J\sin t.$$

We get

$$w(A) = \sup_{t\in[0,2\pi)} \|H_t\| \leqslant \sup_{t\in[0,2\pi)} (\|H\| \cdot |\cos t| + \|J\| \cdot |\sin t|) = \frac{\sqrt{2}}{2}\|A\|. \quad \square$$

The next example shows that this upper bound is attainable.

EXAMPLE 4. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1+i}{2} \end{pmatrix}.$$

We have

$$\|A\| = 1, \|H\| = \|J\| = \frac{1}{2}.$$

Elementary calculations show that

$$AA^* - A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

implying $\|AA^* - A^*A\| = 1$.

For the numerical radius one gets

$$w(A) = \frac{\sqrt{2}}{2}.$$

2. Now we consider another problem, related to the self-commutator. As $C(A - \lambda I) = C(A)$ for any $\lambda \in \mathbb{C}$, inequality (1) may be sharpened

$$\|AA^* - A^*A\| \leqslant \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|^2. \tag{8}$$

Let

$$n(A) = \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|.$$

PROPOSITION 7. *For any operator A*

$$\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \sup_{x\neq\theta}\sqrt{\frac{\|Ax\|^2}{\|x\|^2}-\frac{|\langle Ax,x\rangle|^2}{\|x\|^4}}. \tag{9}$$

Proof. We have

$$\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \inf_{\lambda\in\mathbb{C}}\sup_{x\neq\theta}\frac{\|(A-\lambda I)x\|}{\|x\|}.$$

According to a theorem of Asplund and Ptak [1]

$$\inf_{\lambda\in\mathbb{C}}\sup_{x\neq\theta}\frac{\|(A-\lambda I)x\|}{\|x\|} = \sup_{x\neq\theta}\inf_{\lambda\in\mathbb{C}}\frac{\|(A-\lambda I)x\|}{\|x\|}.$$

The inner expression is equal to

$$\inf_{\lambda\in\mathbb{C}}\frac{\|(A-\lambda I)x\|}{\|x\|} = \sqrt{\frac{\|Ax\|^2}{\|x\|^2}-\frac{|\langle Ax,x\rangle|^2}{\|x\|^4}},$$

so

$$\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \sup_{x\neq\theta}\sqrt{\frac{\|Ax\|^2}{\|x\|^2}-\frac{|\langle Ax,x\rangle|^2}{\|x\|^4}}. \quad \square$$

Let t be the number such that $\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \|A-tI\|$ and $\{x_n\}$ – a sequence of unit vectors, realizing the supremum in the expression $\sqrt{\|Ax\|^2-|\langle Ax,x\rangle|^2}$. Then

$$\begin{aligned}|t-\langle Ax_n,x_n\rangle|^2 &= \|(A-tI)x_n\|^2-\|Ax_n\|^2+|\langle Ax_n,x_n\rangle|^2\\ &\leqslant n^2(A)-\|Ax_n\|^2+|\langle Ax_n,x_n\rangle|^2\to 0.\end{aligned}$$

Therefore

$$t=\lim_{n\to\infty}\langle Ax_n,x_n\rangle.$$

This formula implies $t\in\overline{W}(A)$. Stampfli in [6] says "Given an operator, how does one determine t? In general, there is no simple answer."

PROPOSITION 8. *Let the operator A have a large commutator. Then* $\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \|A\|$.

Proof. According to inequality (8)

$$\|A\|^2\leqslant\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\|^2.$$

As the inverse inequality is valid for any operator, the proof is complete. $\square$

Recall that the maximal numerical range $W_{\max}(A)$ of an operator A is defined as the set of all complex numbers λ for which there exists a sequence $\{x_n\}$ of unit vectors such that $\langle Ax_n,x_n\rangle\to\lambda$ and $\|Ax_n\|\to\|A\|$. Evidently $W_{\max}(A)\subset\overline{W}(A)$.

PROPOSITION 9. *Conditions* $\inf_{\lambda\in\mathbb{C}} \|A-\lambda I\| = \|A\|$ *and* $0 \in W_{\max}(A)$ *are equivalent.*

Proof. Let $\inf_{\lambda\in\mathbb{C}} \|A-\lambda I\| = \|A\|$, then

$$\|A\| = \sup_{\|x\|=1} \sqrt{\|Ax\|^2 - |\langle Ax, x\rangle|^2},$$

so there exists a sequence of unit vectors $\{x_n\}$ such that $\|Ax_n\| \to \|A\|$, $\langle Ax_n, x_n\rangle \to 0$, hence $0 \in W_{\max}(A)$. The inverse implication may be proved, reversing the above reasonings. □

COROLLARY. *Let A have a large self-commutator. Then* $0 \in W_{\max}(A)$.

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(Received July 8, 2009)

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