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JORDAN LEFT DERIVATIONS AND SOME LEFT DERIVABLE MAPS

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Abstract. Let $\mathscr{A}$ be an algebra and $\mathscr{M}$ be a left $\mathscr{A}$-module. We say that a linear mapping $\varphi : \mathscr{A} \to \mathscr{M}$ is a left derivable mapping at P if $\varphi(ST) = S\varphi(T) + T\varphi(S)$ for any $S,T \in \mathscr{A}$ with $ST = P$. In this paper, we show that Jordan left derivations or left derivable mappings at zero or unit on some algebras are zero under certain conditions.

1. Introduction

Suppose that $\mathscr{A}$ is an algebra over the complex field $\mathbb{C}$ and $\mathscr{M}$ is a left $\mathscr{A}$-module. A linear mapping δ is called a *left derivation* from $\mathscr{A}$ into $\mathscr{M}$ if $\delta(AB) = A\delta(B) + B\delta(A)$ for any $A,B \in \mathscr{A}$. δ is called a *Jordan left derivation* from $\mathscr{A}$ into $\mathscr{M}$ if $\delta(A^2) = 2A\delta(A)$ for any $A \in \mathscr{A}$. Clearly, every left derivation is a Jordan left derivation. One can easily prove that in a noncommutative prime ring, any left derivation is zero. In [3], Brešar and Vukman prove that the existence of a nonzero Jordan left derivation on prime ring R of $\operatorname{char} R \neq 2,3$ forces R to be commutative. More related results have been obtained in [1, 2, 6, 12].

In this paper, we study some propositions of linear left derivations on some Banach algebras.

In Section 2, we prove that if $\mathscr{L}$ is a CDCSL on H and $\mathscr{M}$ is a dual normal unital Banach left $\operatorname{alg}\mathscr{L}$-module, then every Jordan left derivation from $\operatorname{alg}\mathscr{L}$ into $\mathscr{M}$ is zero.

Let $\mathscr{A}$ be an algebra and $\mathscr{M}$ be a left $\mathscr{A}$-module. We say that a linear mapping $\delta : \mathscr{A} \to \mathscr{M}$ is a *left derivable mapping at A* if $\delta(ST) = S\delta(T) + T\delta(S)$ for any $S,T \in \mathscr{A}$ with $ST = A$.

In Sections 3 and 4, we show that every left derivable mapping at zero or unit is zero under certain conditions.

Let X be a complex Banach space and let $B(X)$ be the set of all bounded linear maps from X into itself. H denotes a complex separable Hilbert space.

A *subspace lattice* on X is a collection $\mathscr{L}$ of closed subspaces of X with $(0), X$ in $\mathscr{L}$ and such that for every family $\{M_r\}$ of elements of $\mathscr{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathscr{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. For a subspace lattice $\mathscr{L}$, $\operatorname{alg}\mathscr{L}$ denotes the algebra of all operators on X that leave invariant each element of $\mathscr{L}$.

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It is not difficult to show that $\mathrm{alg}\mathscr{L}$ is closed in operator-norm, and is a unital Banach algebra.

A subspace lattice $\mathscr{L}$ on H is called a *commutative subspace lattice (CSL)* if it consists of mutually commuting projections and $\mathrm{alg}\mathscr{L}$ is called a *CSL algebra*. A totally ordered subspace lattice $\mathscr{N}$ is called a *nest* and the associated algebra $\mathrm{alg}\mathscr{N}$ is called a *nest algebra*. If $\mathscr{L}$ is a *completely distributive commutative subspace lattice (CDCSL)*, then $\mathrm{alg}\mathscr{L}$ is called a *CDCSL algebra*. It is obvious that a nest algebra is a CDCSL algebra. Given a subspace lattice $\mathscr{L}$ on X, put

$$\mathscr{J}_{\mathscr{L}} = \{K \in \mathscr{L} : K \neq \{0\} \text{ and } K_- \neq X\},$$

where $K_- = \vee\{L \in \mathscr{L} : K \nsubseteq L\}$. Call $\mathscr{L}$ a $\mathscr{J}$*-subspace lattice* on X if it satisfies the following conditions:

(1) $\vee\{K : K \in \mathscr{J}_{\mathscr{L}}\} = X$;
(2) $\wedge\{K_- : K \in \mathscr{J}_{\mathscr{L}}\} = \{0\}$;
(3) $K \vee K_- = X$ for any $K \in \mathscr{J}_{\mathscr{L}}$;
(4) $K \wedge K_- = 0$ for any $K \in \mathscr{J}_{\mathscr{L}}$.

If $\mathscr{L}$ is a $\mathscr{J}$-subspace lattice, then $\mathrm{alg}\mathscr{L}$ is called a $\mathscr{J}$*-subspace lattice algebra*.

For $x \in X$ and $f \in X^*$, the operator $y \to f(y)x$ is denoted by $x \otimes f$. $\mathscr{F}(\mathscr{L})$ stands for the algebra of all finite rank operators in $\mathrm{alg}\mathscr{L}$.

The following lemmas will be used repeatedly.

LEMMA 1.1. [9, Lemma 3.1] *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. Then the rank one operator $x \otimes f \in \mathrm{alg}\mathscr{L}$ if and only if there exists a subspace $K \in \mathscr{J}(\mathscr{L})$ such that $x \in K$ and $f \in K_-^{\perp}$.*

The proof of the following lemma is analogous to that of [4, Lemma 2.10], we omit it.

LEMMA 1.2. *Suppose that $\mathscr{L}$ is a $\mathscr{J}$-subspace lattice on X. Then every rank one operator in $\mathrm{alg}\mathscr{L}$ is contained in the linear span of the idempotents in $\mathscr{F}(\mathscr{L})$.*

2. Jordan left derivations

In this section, we assume that $\mathscr{A}$ is a unital algebra and $\mathscr{M}$ is any left $\mathscr{A}$-module, unless stated otherwise.

Since the proof of the following lemma is analogous to that of [3, Proposition 1.1], we omit it.

LEMMA 2.1. *Let $\delta : \mathscr{A} \to \mathscr{M}$ be a Jordan left derivation. Then*
(i) $\delta(AB + BA) = 2A\delta(B) + 2B\delta(A)$;
(ii) $\delta(ABA) = A^2\delta(B) + 3AB\delta(A) - BA\delta(A)$.

LEMMA 2.2. *Let $\delta : \mathscr{A} \to \mathscr{M}$ be a Jordan left derivation. Then for every $A \in \mathscr{A}$ and every idempotent $P \in \mathscr{A}$,*
(i) $\delta(P) = 0$;
(ii) $\delta(PA) = \delta(AP) = P\delta(A)$.

Proof. (i) For any idempotent P in $\mathscr{A}$, $\delta(P)=\delta(P^2)=2P\delta(P)$. So $P\delta(P)=2P^2\delta(P)=2P\delta(P)$. We have that $P\delta(P)=0$. Thus

$$\delta(P)=2P\delta(P)=0. \tag{2.1}$$

(ii) By Lemma 2.1 and (2.1), for any $A\in\mathscr{A}$, $P=P^2\in\mathscr{A}$,

$$\begin{aligned}\delta(AP+PAP) &= \delta(APP+PAP)=2P\delta(AP),\\ \delta(AP+PAP) &= \delta(AP)+\delta(PAP)=\delta(AP)+P\delta(A).\end{aligned}$$

So $2P\delta(AP)=P\delta(A)+\delta(AP)$. Thus $P\delta(AP)=P\delta(A)$. We have that

$$\delta(AP)=P\delta(A). \tag{2.2}$$

Since $\delta(AP+PA)=2A\delta(P)+2P\delta(A)=2P\delta(A)$, by (2.2),

$$\delta(PA)=2P\delta(A)-\delta(AP)=P\delta(A). \tag{2.3}$$

By (2.2) and (2.3), $\delta(AP)=\delta(PA)=P\delta(A)$. □

By the introduction, it is easy to show the following result.

LEMMA 2.3. *If δ is a Jordan left derivation from $\mathscr{A}$ into $\mathscr{M}$, then for any idempotents $P_1,\dots,P_n$ in $\mathscr{A}$ and A in $\mathscr{A}$,*

$$\delta(P_1\dots P_nA)=\delta(AP_1\dots P_n)=P_1\dots P_n\delta(A).$$

We call a right ideal $\mathscr{I}$ of $\mathscr{A}$ a *right separating set* of $\mathscr{M}$, if for any m in $\mathscr{M}$, $\mathscr{I}m=0$ implies $m=0$.

THEOREM 2.4. *Let $\mathscr{I}$ be a right separating set of $\mathscr{M}$. Suppose that $\mathscr{I}$ is contained in the subalgebra of $\mathscr{A}$ generated by its idempotents. If δ is a Jordan left derivation from $\mathscr{A}$ into $\mathscr{M}$, then $\delta\equiv 0$. In particular, if δ is a left derivation from $\mathscr{A}$ into $\mathscr{M}$, then $\delta\equiv 0$.*

Proof. By Lemma 2.3, for any $S\in\mathscr{I}$ and any $A\in\mathscr{A}$,

$$\delta(AS)=\delta(SA)=S\delta(A). \tag{2.4}$$

Since $\mathscr{I}$ is a right ideal, $TA\in\mathscr{I}$ for any $T\in\mathscr{I}, A\in\mathscr{A}$. Thus for any $A\in\mathscr{A}$, $T\in\mathscr{I}$, by Lemma 2.2(i) and (2.4),

$$T\delta(A)=\delta(TA)=TA\delta(I)=0. \tag{2.5}$$

Since $\mathscr{I}$ is a right separating set, it follows from (2.5) that $\delta(A)=0$ for any $A\in\mathscr{A}$. □

Let $\mathscr{A}$ be an ultraweakly closed subalgebra of $B(H)$. The Banach space $\mathscr{M}$ is said to be a *dual normal Banach left $\mathscr{A}$-module* if $\mathscr{M}$ is a Banach left $\mathscr{A}$-module, $\mathscr{M}$ is a dual space, and for any $m\in\mathscr{M}$, the map $\mathscr{A}\ni a\to am$ is ultraweak to weak* continuous.

COROLLARY 2.5. *If $\mathscr{L}$ is a CDCSL on H and δ is a Jordan left derivation from* $\mathrm{alg}\mathscr{L}$ *into a dual normal unital Banach left* $\mathrm{alg}\mathscr{L}$*-module $\mathscr{M}$, then $\delta \equiv 0$. In particular, every Jordan left derivation from* $\mathrm{alg}\mathscr{L}$ *into itself is equal to zero.*

Proof. Let $\mathscr{I} = \mathrm{span}\{T : T \in \mathrm{alg}\mathscr{L}, \mathrm{rank}T = 1\}$. Then $\mathscr{I}$ is an ideal of $\mathrm{alg}\mathscr{L}$. By [4, Lemma 2.3], $\mathscr{I}$ is contained in the linear span of the idempotents in $\mathrm{alg}\mathscr{L}$. By [8, Theorem 3], we have that $\mathscr{I}$ is a right separating set of $\mathscr{M}$. Hence it follows from Theorem 2.4 that $\delta \equiv 0$. □

COROLLARY 2.6. *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. If δ is a Jordan left derivation from* $\mathrm{alg}\mathscr{L}$ *into itself, then $\delta \equiv 0$.*

Proof. Let $\mathscr{I} = \mathrm{span}\{T : T \in \mathrm{alg}\mathscr{L}, \mathrm{rank}T = 1\}$. Then $\mathscr{I}$ is an ideal of $\mathrm{alg}\mathscr{L}$. By Lemma 1.2, $\mathscr{I}$ is contained in the linear span of the idempotents in $\mathrm{alg}\mathscr{L}$. By [7, Lemma 2.3], $\mathscr{I}$ is a right separating set of $\mathrm{alg}\mathscr{L}$. Hence it follows from Theorem 2.4 that $\delta \equiv 0$. □

COROLLARY 2.7. *Suppose that $\mathscr{A}$ is a unital Banach subalgebra of $B(X)$ such that $\mathscr{A}$ contains $\{x_0 \otimes f,\ f \in X^*\}$, where $0 \neq x_0 \in X$. If $\delta : \mathscr{A} \to B(X)$ is a Jordan left derivation, then $\delta \equiv 0$.*

Proof. Let $\mathscr{I} = \{x_0 \otimes f,\ f \in X^*\}$. Then $\mathscr{I}$ is a right ideal of $\mathscr{A}$ and a right separating set of $B(X)$. For any $x_0 \otimes f$ in $\mathscr{A}$, if $f(x_0) \neq 0$, then $\frac{1}{f(x_0)} x_0 \otimes f$ is an idempotent in $\mathscr{I}$. If $f(x_0) = 0$, choose $f_1 \in X^*$, such that $f_1(x_0) = 1$, we have that $x_0 \otimes f = \frac{1}{2} x_0 \otimes (f + f_1) - \frac{1}{2} x_0 \otimes (f_1 - f)$, both $x_0 \otimes (f + f_1)$ and $x_0 \otimes (f_1 - f)$ are idempotents. By Theorem 2.4, we have that $\delta \equiv 0$. □

Let $\mathscr{A}$ be a weakly closed subalgebra of $B(H)$. If K is a complex separable Hilbert space, then the tensor product $\mathscr{A} \otimes B(K)$ is defined as the weak operator closure of the span of all elementary tensors $A \otimes B$ acting on $H \otimes K$, where $A \in \mathscr{A}$ and $B \in B(H)$. A weakly closed subalgebra $\mathscr{A}$ of $B(H)$ is said to be of *infinite multiplicity* if $\mathscr{A} \otimes B(K)$ is isomorphic to $\mathscr{A}$.

PROPOSITION 2.8. *Let $\mathscr{A}$ be a weakly closed unital subalgebra of $B(H)$ of infinite multiplicity. If δ is a Jordan left derivation from $\mathscr{A}$ into a left $\mathscr{A}$-module $\mathscr{M}$, then $\delta \equiv 0$.*

Proof. By [11, Theorem 4.3], every $A \in \mathscr{A}$ is a sum of eight idempotents in $\mathscr{A}$. Thus, it follows from Lemma 2.2 that $\delta(A) = 0$ for any $A \in \mathscr{A}$. □

PROPOSITION 2.9. *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. If δ is a linear mapping from $\mathscr{F}(\mathscr{L})$ into an algebra $\mathscr{B}$ such that $\delta(P) = 0$ for any idempotent $P \in \mathscr{F}(\mathscr{L})$, then $\delta \equiv 0$.*

Proof. For any $A,B \in \mathscr{F}(\mathscr{L})$, by [10, Proposition 3.2], we have that $A = A_1 + A_2 + \ldots + A_n$, where $A_i = x_i \otimes f_i$ are rank one operators in $\mathrm{alg}\mathscr{L}$. It follows from Lemma 1.2 and Lemma 2.2 that $\delta(A_i) = 0$, $i = 1,\ldots,n$. Thus $\delta(A) = 0$ for any $A \in \mathscr{F}(\mathscr{L})$. □

COROLLARY 2.10. *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. If δ is a Jordan left derivation from $\mathscr{F}(\mathscr{L})$ into a left $\mathrm{alg}\mathscr{L}$-module $\mathscr{M}$, then $\delta \equiv 0$.*

PROPOSITION 2.11. *Let $\mathscr{L}$ be a CSL on H. If δ is a bounded Jordan left derivation from $\mathrm{alg}\mathscr{L}$ into $B(H)$, then $\delta \equiv 0$.*

Proof. By Lemma 2.2(ii), for any $P = P^2 \in \mathrm{alg}\mathscr{L}$ and $A \in \mathrm{alg}\mathscr{L}$,

$$\delta(PA) = \delta(PPA) = P\delta(PA).$$

By [5, Theorem 2.20], $\delta(A) = A\delta(I)$, for any $A \in \mathrm{alg}\mathscr{L}$. It follows from Lemma 2.2(i) that $\delta(I) = 0$. Thus $\delta(A) = 0$ for any $A \in \mathrm{alg}\mathscr{L}$. □

DEFINITION 2.12. Let $\mathscr{M}$ be a Banach left $\mathscr{A}$-module. A linear mapping D from $\mathscr{A}$ into $\mathscr{M}$ is an *approximately local left derivation* if for each a in $\mathscr{A}$, there is a sequence of left derivations $\{D_{a,n}\}$ from $\mathscr{A}$ into $\mathscr{M}$ such that $D(a) = \lim_{n\to\infty} D_{a,n}(a)$. If, in addition, D is bounded, then we say that D is a *bounded approximately local derivation.*

Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{I}$ be the subalgebra of $\mathscr{A}$ generated by the idempotents in $\mathscr{A}$. We say that $\mathscr{A}$ is *topologically generated by idempotents* if $\mathscr{I}$ is dense in $\mathscr{A}$.

PROPOSITION 2.13. *Let $\mathscr{A}$ be a Banach algebra topologically generated by idempotents. Then every bounded approximately local left derivation from $\mathscr{A}$ into any Banach left $\mathscr{A}$-module $\mathscr{M}$ is zero.*

Proof. For any idempotents $e_1,\ldots,e_m$ in $\mathscr{A}$, there is a sequence of left derivations $\{D_n\}$ from $\mathscr{A}$ into $\mathscr{M}$ such that $D(e_1\ldots e_m) = \lim_{n\to\infty} D_n(e_1\ldots e_m)$. Since every left derivation is Jordan left derivation, it follows from Lemmas 2.2(i) and 2.3 that $D_n(e_1\ldots e_m) = e_1\ldots e_{m-1}D_n(e_m) = 0$. Thus $D(e_1\ldots e_m) = 0$ for any idempotents $e_1,\ldots,e_m$ in $\mathscr{A}$. Since $\mathscr{A}$ is generated by idempotents and D is bounded, we have that $D \equiv 0$. □

By the ideas in [3], we can use Theorem 2.4 to study the the following functional equations.

THEOREM 2.14. *Let $\mathscr{A}$ be a unital Banach algebra and $\mathscr{M}$ be a unital left $\mathscr{A}$-module. Suppose that $\mathscr{I}$ is a right separating set of $\mathscr{M}$ and $\mathscr{I}$ is contained in the subalgebra of $\mathscr{A}$ generated by idempotents. Let f, $g : \mathscr{A} \to \mathscr{M}$ be linear mappings. If*

$$f(A) = A^2 g(A^{-1}) \tag{2.6}$$

holds for any invertible element A in $\mathscr{A}$, then the following statements hold:

(i) $f(A)=g(A)$ for all $A\in\mathscr{A}$;

(ii) $f(A)=Af(I)$ for all $A\in\mathscr{A}$.

Proof. (i) By (2.6), we have that

$$g(A)=A^2f(A^{-1}) \tag{2.7}$$

Let $D=f-g$. It follows from (2.6) and (2.7) that $D(A)=-A^2D(A^{-1})$ holds for any invertible element $A\in\mathscr{A}$. Then $D(I)=0$. In the following, we prove that D is a Jordan left derivation. Since D is linear, we only need to show that

$$D(A^2)=2AD(A) \tag{2.8}$$

for any $A\in\mathscr{A}$. Let $A\in\mathscr{A}$ be arbitrary. Choose an integer n such that B^{-1} and $(I-B)^{-1}$ exist, where $B=nI+A$. Thus we have $B^2=B-(B^{-1}+(I-B)^{-1})^{-1}$. Then

$$\begin{aligned} D(B^2) &= D(B)-D((B^{-1}+(I-B)^{-1})^{-1}) \\ &= D(B)+(B^{-1}+(I-B)^{-1})^{-2}D(B^{-1}+(I-B)^{-1}) \\ &= D(B)-(I-B)^2B^2B^{-2}D(B)-B^2(I-B)^2(I-B)^{-2}D(I-B) \\ &= D(B)-(I-B)^2D(B)+B^2D(B)=2BD(B). \end{aligned}$$

Hence $D(B^2)=2BD(B)$ which implies (2.8) since D(I)=0. Thus D is a Jordan left derivation from $\mathscr{A}$ into $\mathscr{M}$. By Theorem 2.4, it follows that $D\equiv 0$. Hence $f(A)=g(A)$ for any $A\in\mathscr{A}$. The relation (2.6) can be written in the form

$$f(A)=A^2f(A^{-1}). \tag{2.9}$$

(ii) Let us first assume that $f(I)=0$. Our goal is to show that in this case, $f=0$.

For any $A\in\mathscr{A}$ and let us again choose an integer n such that B^{-1} and $(I-B)^{-1}$ exist, where $B=nI+A$. By (2.9), we have

$$\begin{aligned} f(B) &= B^2f(B^{-1})=B^2f(B^{-1}(I-B)) \\ &= B^2(B^{-1}(I-B))^2f((I-B)^{-1}B) \\ &= (I-B)^2f((I-B)^{-1}-I) \\ &= (I-B)^2((I-B)^{-1})^2f(I-B)=-f(B). \end{aligned}$$

Hence $f(B)=0$. Thus $f(A)=0$ for any $A\in\mathscr{A}$.

Now we assume that $f(I)\neq 0$. Let $h(A)=f(A)-Af(I)$. It is obvious that h is linear. A routine calculation shows that $h(A)=A^2h(A^{-1})$ holds for any invertible operator $A\in\mathscr{A}$. Since $h(I)=0$, we have that $h(A)=0$ for any $A\in\mathscr{A}$. Thus $f(A)=Af(I)$ for any $A\in\mathscr{A}$. □

COROLLARY 2.15. *Let $\mathscr{L}$ be a CDCSL or a $\mathscr{J}$-subspace lattice on H and let $f,g:\mathrm{alg}\mathscr{L}\to\mathrm{alg}\mathscr{L}$ be linear mappings. Suppose that $f(A)=A^2g(A^{-1})$ holds for any invertible element A in $\mathscr{A}$. Then the following statements hold:*

(i) $f(A)=g(A)$ for all $A\in\mathrm{alg}\mathscr{L}$;

(ii) $f(A)=Af(I)$ for all $A\in\mathrm{alg}\mathscr{L}$.

Similar to the proof of Theorem 2.14, by Proposition 2.11, we can get the following theorem.

THEOREM 2.16. *Let $\mathscr{L}$ be a CSL on H and let $f,g : \mathrm{alg}\mathscr{L} \to B(H)$ be bounded linear mappings. Suppose that $f(A) = A^2 g(A^{-1})$ holds for any invertible element A in $\mathscr{A}$. Then the following statements hold:*

(i) $f(A) = g(A)$ for all $A \in \mathrm{alg}\mathscr{L}$;

(ii) $f(A) = Af(I)$ for all $A \in \mathrm{alg}\mathscr{L}$.

3. Left derivable mappings at zero

In this section, we study some propositions of a left derivable mapping at zero for a class of algebras.

LEMMA 3.1. *If δ is a left derivable mapping at zero from a unital algebra $\mathscr{A}$ into a unital left $\mathscr{A}$-module $\mathscr{M}$, then for any $P = P^2 \in \mathscr{A}$, $A \in \mathscr{A}$,*

(i) $\delta(P) = P\delta(I) = P\delta(P)$;

(ii) $\delta(PA) = P\delta(A) + (AP - PA)\delta(I)$;

(iii) $\delta(AP) = P\delta(A)$.

Proof. (i) Since $P(I-P) = 0$, it follows that

$$0 = P\delta(I-P) + (I-P)\delta(P) = P\delta(I) - P\delta(P) + (I-P)\delta(P).$$

So $P\delta(I) = P\delta(P) = \delta(P)$.

(ii) Since $P(I-P)A = (I-P)PA = 0$, we have that

$$0 = P\delta((I-P)A) + (I-P)A\delta(P) = P\delta(A) - P\delta(PA) + A\delta(P) - PA\delta(P),$$
$$0 = (I-P)\delta(PA) + PA\delta(I-P) = \delta(PA) - P\delta(A) + PA\delta(I) - PA\delta(P).$$

Thus

$$\delta(PA) = P\delta(A) + A\delta(P) - PA\delta(I) = P\delta(A) + (AP - PA)\delta(I).$$

(iii) Since $AP(I-P) = A(I-P)P = 0$, we have that

$$0 = AP\delta(I-P) + (I-P)\delta(AP) = AP\delta(I) - AP\delta(P) + \delta(AP) - P\delta(AP),$$
$$0 = A(I-P)\delta(PA) + P\delta(A(I-P)) = A\delta(P) - AP\delta(P) + P\delta(A) - P\delta(AP).$$

Thus

$$\delta(AP) = A\delta(P) + P\delta(A) - AP\delta(I) = P\delta(A). \qquad \square$$

COROLLARY 3.2. *Let $\delta, \mathscr{A}$ and $\mathscr{M}$ be as in Lemma 3.1 with $\delta(I) = 0$. Suppose $\mathscr{B}$ is the subalgebra of $\mathscr{A}$ generated by all idempotents in $\mathscr{A}$. Then for any $S \in \mathscr{B}, A \in \mathscr{A}$, $\delta(SA) = \delta(AS) = S\delta(A)$.*

THEOREM 3.3. *Let $\delta, \mathscr{A}$ and $\mathscr{M}$ be as in Corollary 3.2. Suppose $\mathscr{A}$ contains a right separating set $\mathscr{I}$ of $\mathscr{M}$. If $\mathscr{I}$ is contained in the subalgebra of $\mathscr{A}$ generated by idempotents in $\mathscr{A}$, then $\delta \equiv 0$.*

Proof. By Corollary 3.2, for any $A, B \in \mathscr{A}$, $S \in \mathscr{I}$,

$$\delta(SAB) = S\delta(AB),\ \delta(SAB) = \delta((SA)B) = SA\delta(B).$$

Thus

$$S(\delta(AB) - A\delta(B)) = 0. \tag{3.1}$$

Since $\mathscr{I}$ is a right separating set of $\mathscr{M}$, by (3.1), it follows that $\delta(AB) = A\delta(B)$, for any $A, B \in \mathscr{A}$. Hence $\delta(A) = \delta(AI) = A\delta(I) = 0$, for any $A \in \mathscr{A}$. □

COROLLARY 3.4. *Suppose that $\mathscr{L}$ is a CDCSL algebra on H. If δ is a left derivable mapping at zero from* alg$\mathscr{L}$ *into a dual normal unital Banach left* alg$\mathscr{L}$*-module $\mathscr{M}$ and $\delta(I) = 0$, then $\delta \equiv 0$.*

COROLLARY 3.5. *Suppose that $\mathscr{L}$ is a $\mathscr{J}$-subspace lattice on X. If δ is a left derivable mapping at zero from* alg$\mathscr{L}$ *into itself and $\delta(I) = 0$, then $\delta \equiv 0$.*

COROLLARY 3.6. *Suppose that $\mathscr{A}$ is a unital Banach subalgebra of $B(X)$ such that $\mathscr{A}$ contains $\{x_0 \otimes f,\ f \in X^*\}$, where $0 \neq x_0 \in X$. If $\delta : \mathscr{A} \to B(X)$ is a left derivable mapping at zero and $\delta(I) = 0$, then $\delta \equiv 0$.*

PROPOSITION 3.7. *Let $\mathscr{A}$ be a weakly closed unital infinite multiplicity algebra of $B(H)$. If δ is a left derivable mapping at zero from $\mathscr{A}$ into a unital left $\mathscr{A}$-module $\mathscr{M}$ and $\delta(I) = 0$, then $\delta \equiv 0$.*

Proof. By [11, Theorem 4.3], every $A \in \mathscr{A}$ is a sum of eight idempotents in $\mathscr{A}$. Thus, it follows from Lemma 3.1(i) and $\delta(I) = 0$ that $\delta(A) = 0$ for any $A \in \mathscr{A}$. □

PROPOSITION 3.8. *Let $\mathscr{L}$ be a CSL on H. If $\delta :$ alg$\mathscr{L} \to B(H)$ is a bounded linear mapping such that $\delta(I) = 0$ and $A\delta(B) + B\delta(A) = 0$ for all $AB = 0$, then $\delta \equiv 0$.*

Proof. By Lemma 3.1 and $\delta(I) = 0$, for $P = P^2$ and A in alg$\mathscr{L}$,

$$\delta(PA) = \delta(PPA) = P\delta(PA).$$

By [5, Theorem 2.20], $\delta(A) = A\delta(I) = 0$, for any $A \in$ alg$\mathscr{L}$. □

4. Left derivable mappings at unit

LEMMA 4.1. *Let $\mathscr{A}$ be a unital algebra and $\mathscr{M}$ be a unital left $\mathscr{A}$-module. If δ is a left derivable mapping at I from $\mathscr{A}$ into $\mathscr{M}$, then*
(i) $\delta(P)=0$ for any $P=P^2\in\mathscr{A}$,
(ii) $P\delta(P)=0$ for any $P\in\mathscr{A}$ such that $P^2=0$.

Proof. (i) $\delta(I)=\delta(I\cdot I)=I\delta(I)+I\delta(I)=2\delta(I)$. So $\delta(I)=0$. For any idempotent $P\in\mathscr{A}$, we have that $I=(P-\frac{1-\sqrt{3}i}{2}I)(P-\frac{1+\sqrt{3}i}{2}I)$. It follows that

$$\begin{aligned}0=\delta(I)&=\delta\left(\left(P-\frac{1-\sqrt{3}i}{2}I\right)\left(P-\frac{1+\sqrt{3}i}{2}I\right)\right)\\&=\left(P-\frac{1-\sqrt{3}i}{2}I\right)\delta\left(P-\frac{1+\sqrt{3}i}{2}I\right)+\left(P-\frac{1+\sqrt{3}i}{2}I\right)\delta\left(P-\frac{1-\sqrt{3}i}{2}I\right)\\&=(2P-I)\delta(P)=2P\delta(P)-\delta(P).\end{aligned}$$

Thus $2P\delta(P)=P\delta(P)$. Hence $\delta(P)=2P\delta(P)=0$.

(ii) For any $P\in\mathscr{A}$ with $P^2=0$, we have that

$$\begin{aligned}0&=\delta(I)=\delta((I-P)(I+P))=(I-P)\delta(I+P)+(I+P)\delta(I-P)\\&=(I-P)\delta(P)-(I+P)\delta(P)=-2P\delta(P).\end{aligned}$$

Thus $P\delta(P)=0$. □

COROLLARY 4.2. *Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{M}$ be a unital normed left $\mathscr{A}$-module. If δ is a norm continuous left derivable mapping at I from $\mathscr{A}$ into $\mathscr{M}$, then $\delta\equiv 0$.*

Proof. For any orthogonal projections $P_1,\dots,P_n\in\mathscr{A}$ and for any $r_1,\dots,r_n\in\mathbb{R}$, let $Q=\sum_{i=1}^n r_iP_i$. By Lemma 4.1(i),

$$\delta(Q)=\delta(\sum_{i=1}^n r_iP_i)=\sum_{i=1}^n r_i\delta(P_i)=0.$$

Since δ is norm continuous, for any selfadjoint operator $S\in\mathscr{A}$, we have $\delta(S)=0$. For any $T\in\mathscr{A}$, there exist selfadjoint operators $T_1,T_2\in\mathscr{A}$ such that $T=T_1+iT_2$. Thus $\delta(T)=0$. □

COROLLARY 4.3. *Let $\mathscr{L}$ be a CDCSL on H. If δ is a strong operator topology continuous left derivable mapping at I from* alg$\mathscr{L}$ *into $B(H)$, then $\delta\equiv 0$.*

Proof. By Lemma 4.1(i), for any $P=P^2\in\text{alg}\mathscr{L}$, $\delta(P)=0$. Let $\mathscr{T}=\text{span}\{T\in\text{alg}\mathscr{L},\text{rank}(T)=1\}$. Then $\mathscr{T}$ is the subalgebra generated by rank one operators. By [4, Lemma 2.3], $\mathscr{T}$ is contained in the linear span of idempotents in alg$\mathscr{L}$. Thus for any $S \in \mathscr{T}$, $\delta(S) = 0$. By [8, Theorem 3], $\mathscr{T}$ is dense in alg$\mathscr{L}$ in strong operator topology. Since δ is strong operator topology continuous, it follows that $\delta(T) = 0$, for any $T \in$ alg$\mathscr{L}$. □

PROPOSITION 4.4. *Let $\mathscr{A}$ be a weakly closed unital algebra of $B(H)$ of infinite multiplicity. If δ is a left derivable mapping at I from $\mathscr{A}$ into a unial left $\mathscr{A}$-module $\mathscr{M}$, then $\delta \equiv 0$.*

Proof. By [11, Theorem 4.3], every $A \in \mathscr{A}$ is a sum of eight idempotents in $\mathscr{A}$. Thus, it follows from Lemma 4.1(i) that $\delta(A) = 0$ for any $A \in \mathscr{A}$. □

THEOREM 4.5. *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. If δ is a left derivable mapping at I from* alg$\mathscr{L}$ *into itself, then $\delta \equiv 0$.*

Proof. By Lemma 1.2 and Lemma 4.1(i), we have that $\delta(B) = 0$, where $B \in$ alg$\mathscr{L}$ and rank$B = 1$.

Let $T \in$ alg$\mathscr{L}$. For any $K \in \mathscr{J}(\mathscr{L})$, by $K_- \wedge K = 0$, we can choose $f \in (K_-)^{\perp}$ such that $f(t) \neq 0$ for any $0 \neq t \in K$.

For any $y \in K$. Let $x = \delta(T)y$. Thus $x \in K$. By Lemma 1.1, $x \otimes f \in$ alg$\mathscr{L}$. Take $\lambda \in \mathbb{C}$ such that $|\lambda| > \|T\|$ and $\|(\lambda I - T)^{-1}x\|\|f\| < 1$. Since $\lambda I - T$ and $\lambda I - T - x \otimes f = (\lambda I - T)(I - (\lambda I - T)^{-1}x \otimes f)$ are invertible and their inverses are still in alg$\mathscr{L}$. It is obvious that $(I - (\lambda I - T)^{-1}x \otimes f)^{-1} = I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f$, where $\alpha = f((\lambda I - T)^{-1}x)$. For any invertible A in alg$\mathscr{L}$, since δ is left derivable at I, we have that $\delta(A^{-1}) = -A^{-2}\delta(A)$. Hence

$$\begin{aligned} 0 &= (I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f)(\lambda I - T)^{-1}\delta(\lambda I - T - x \otimes f) \\ &\quad + (\lambda I - T - x \otimes f)\delta((I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f)(\lambda I - T)^{-1}) \\ &= -(I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f)(\lambda I - T)^{-1}\delta(T) \\ &\quad + (\lambda I - T - x \otimes f)\delta((\lambda I - T)^{-1}) \\ &= -(I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f)(\lambda I - T)^{-1}\delta(T) \\ &\quad - (\lambda I - T - x \otimes f)(\lambda I - T)^{-2}\delta(\lambda I - T) \\ &= -(I + (1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f)(\lambda I - T)^{-1}\delta(T) \\ &\quad + (\lambda I - T - x \otimes f)(\lambda I - T)^{-2}\delta(T) \\ &= -(1-\alpha)^{-1}(\lambda I - T)^{-1}x \otimes f(\lambda I - T)^{-1}\delta(T) \\ &\quad - x \otimes f(\lambda I - T)^{-2}\delta(T). \end{aligned} \tag{4.1}$$

By (4.1), for any $t \in K$,

$$(1-\alpha)^{-1}f((\lambda I - T)^{-1}\delta(T)(t))(\lambda I - T)^{-1}x = -f((\lambda I - T)^{-2}\delta(T)(t))x.$$

Case 1: When $(\lambda I - T)^{-1}\delta(T)(t) = 0$ or $(\lambda I - T)^{-2}\delta(T)(t) = 0$ for any $t \in K$, we have that $\delta(T)(t) = 0$ for any $t \in K$. Thus $\delta(T)y = 0$.

Case 2: Suppose that there exists $t_0 \in K$ such that $(\lambda I-T)^{-1}\delta(T)(t_0) \neq 0$ and $(\lambda I-T)^{-2}\delta(T)(t_0) \neq 0$, we have that $f((\lambda I-T)^{-1}\delta(T)(t_0)) \neq 0$ and $f((\lambda I-T)^{-2}\delta(T)(t_0)) \neq 0$. Thus there exists a scalar $\beta_\lambda \neq 0$ such that

$$(\lambda I-T)^{-1}x = \beta_\lambda x. \tag{4.2}$$

Since $x = \delta(T)y$, we have that

$$(\lambda I-T)^{-1}\delta(T)y = \beta_\lambda \delta(T)y, \tag{4.3}$$

$$(\lambda I-T)^{-2}\delta(T)y = \beta_\lambda^2 \delta(T)y. \tag{4.4}$$

It follows form (4.1), (4.2), (4.3) and (4.4) that

$$\begin{aligned} 0 &= (1-\alpha)^{-1}(\lambda I-T)^{-1}x \otimes f(\lambda I-T)^{-1}\delta(T)y + x \otimes f(\lambda I-T)^{-2}\delta(T)y \\ &= (1-\alpha)^{-1}\beta_\lambda x \otimes f\beta_\lambda \delta(T)y + x \otimes f\beta_\lambda^2 \delta(T)y \\ &= ((1-\alpha)^{-1}+1)\beta_\lambda^2 f(\delta(T)y)x. \end{aligned}$$

Since $|\alpha| < 1$ and $\beta_\lambda \neq 0$, we have that $f(x)x = 0$. If $x = 0$, then it is clear that $\delta(T)y = x = 0$. If $f(x) = 0$, it follows from $x \in K$ that $x = 0$. Thus $\delta(T)y = 0$.

Hence, by Cases 1 and 2, we have that for any $K \in \mathcal{J}(\mathcal{L})$, $\delta(T)|_K = 0$. Since $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$, it follows that $\delta(T) = 0$ for any $T \in \mathrm{alg}\mathcal{L}$. $\square$

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