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A MULTI–POINT DEGENERATE INTERPOLATION PROBLEM FOR GENERALIZED SCHUR FUNCTIONS

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Abstract. The Nevanlinna-Pick-Carathéodory-Fejér interpolation problem with finitely many interpolation conditions is considered in the class $\mathcal{S}_\kappa$ of meromorphic functions f with κ poles inside the unit disk $\mathbb{D}$ and with $\|f\|_{L^\infty(\mathbb{T})} \leqslant 1$. Necessary and sufficient conditions for the existence and for the uniqueness of a solution are given in terms of the Pick matrix P of the problem explicitly determined from interpolation data. In particular it is shown that the problem admits infinitely many solutions if and only if κ is not less than the number of nonpositive eigenvalues of P. For κ equal to the number of nonpositive eigenvalues of P, we describe the solution set of the problem. Also we present necessary and sufficient conditions for the existence of a meromorphic function with a given pole multiplicity satisfying interpolation conditions and having the minimal possible L^∞-norm on the unit circle $\mathbb{T}$.

1. Introduction

Let $\mathcal{S}$ denote the Schur class of analytic functions mapping the open unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$. The functions f from this class are characterized by the property that the associated kernel

$$K_f(z,\zeta) := \frac{1-f(z)\overline{f(\zeta)}}{1-z\overline{\zeta}} \tag{1.1}$$

is positive on $\mathbb{D}\times\mathbb{D}$ which implies that for every choice of an integer $k \geqslant 0$ and of two k-tuples $\mathbf{z} = (z_1,\dots,z_k) \in \mathbb{D}^k$ and $\mathbf{n} = (n_1,\dots,n_k) \in \mathbb{N}^k$, the *Schwarz-Pick matrix*

$$P^f_{\mathbf{n}}(\mathbf{z}) = \left[\left[\frac{1}{\ell!r!}\frac{\partial^{\ell+r}}{\partial z^\ell \partial\overline{\zeta}^r}\frac{1-f(z)\overline{f(\zeta)}}{1-z\overline{\zeta}}\Bigg|_{\substack{z=z_i\\ \zeta=z_j}}\right]_{\ell=0,\dots,n_i-1}^{r=0,\dots,n_j-1}\right]_{i,j=1}^{k} \tag{1.2}$$

is positive semidefinite. In fact, it is positive definite unless f is a Blaschke product of degree $\kappa < |\mathbf{n}| := n_1+\ldots+n_k$ in which case $\operatorname{rank} P^f_{\mathbf{n}}(\mathbf{z}) = \kappa$. It is convenient to

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represent $P^f_{\mathbf{n}}(\mathbf{z})$ in the block-matrix form $P^f_{\mathbf{n}}(\mathbf{z}) = \left[P^f_{ij}\right]_{i,j=1}^k$ and the straightforward differentiation in (1.2) gives

$$\left[P^f_{ij}\right]_{\ell,r} = \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!} \frac{z_i^{r-s}\bar{z}_j^{\ell-s}}{(1-z_i\bar{z}_j)^{\ell+r-s+1}} - \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!} \frac{z_i^{\beta-s}\bar{z}_j^{\alpha-s} f_{\ell-\alpha}(z_i)\overline{f_{r-\beta}(z_j)}}{(1-z_i\bar{z}_j)^{\alpha+\beta-s+1}} \tag{1.3}$$

where we have set $f_j(z_i) := f^{(j)}(z_i)/j!$. We then observe that the Schwarz-Pick matrix $P^f_{\mathbf{n}}(\mathbf{z})$ is the same for every function f analytic at $z_1,\ldots,z_k$ and such that

$$\frac{f^{(j)}(z_i)}{j!} = c_{i,j} \quad \text{for} \quad i=1,\ldots,k;\ j=0,\ldots,n_i-1 \tag{1.4}$$

for some preassigned complex numbers c_{ij}. Upon substituting (1.4) into (1.3) we conclude that this matrix can be written as

$$P_{\mathbf{n}} = [P_{ij}]_{i,j=1}^k \tag{1.5}$$

where the $n_i \times n_j$ blocks P_{ij} are defined entry-wise by

$$[P_{ij}]_{\ell,r} = \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!} \frac{z_i^{r-s}\bar{z}_j^{\ell-s}}{(1-z_i\bar{z}_j)^{\ell+r-s+1}} - \sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!} \frac{z_i^{\beta-s}\bar{z}_j^{\alpha-s} c_{i,\ell-\alpha}\bar{c}_{j,r-\beta}}{(1-z_i\bar{z}_j)^{\alpha+\beta-s+1}}. \tag{1.6}$$

The positivity of Schwarz-Pick matrices associated to Schur-class functions gives the necessity part in the following well-known Nevanlinna-Pick type theorem.

THEOREM 1.1. *There exists a Schur-class function f subject to interpolation conditions* (1.4) *if and only if the matrix $P_{\mathbf{n}}$ defined in* (1.5)*,* (1.6) *is positive semidefinite. Moreover:*

1. *If $P_{\mathbf{n}}$ is positive definite, then there are infinitely many functions $f \in \mathcal{S}$ subject to conditions* (1.4) *which can be parametrized by a linear fractional formula with free parameter running through the class $\mathcal{S}$.*

2. *If $P_{\mathbf{n}} \geqslant 0$ is singular, then there is a unique $f \in \mathcal{S}$ subject to* (1.4) *and this unique f is a Blaschke product of degree equal to the rank of $P_{\mathbf{n}}$.*

The matrix $P_{\mathbf{n}}$ completely determined by formulas (1.5), (1.6) from the data set

$$\{z_i,\ n_i,\ c_{i,j}:\ j=0,\ldots,n_i-1;\ i=1,\ldots,k\} \tag{1.7}$$

of interpolation problem (1.4) is called *the Pick matrix* of the problem. It is not hard to verify that it is Hermitian (by construction). Let us observe that the problem of finding a Schur-class function with preassigned Taylor coefficients at prescribed points is a combination of the classical Nevanlinna-Pick problem (where $n_i = 1$ for $i = 1,\dots,k$) and of the Carathéodory-Fejér problem (where $k = 1$).

NOTATION. Before to move on, we fix the following notation.

1. $\mathscr{B}_\kappa$ – the set of all Blaschke products of degree κ.
2. $\mathscr{B}_p/\mathscr{B}_q$ – the set of all coprime quotients $g = b/\theta$ with $b \in \mathscr{B}_p$ and $\theta \in \mathscr{B}_q$.
3. $\pi(P)$, $\nu(P)$, $\delta(P)$ – respectively the numbers of positive, negative and zero eigenvalues, counted with multiplicities, of a Hermitian matrix P.

In case $P_{\mathbf{n}}$ has negative eigenvalues, there are no Schur-class functions satisfying conditions (1.4); however the latter conditions can be satisfied by *generalized Schur functions*. Following [15], we define the *generalized Schur class* $\mathscr{S}_\kappa$ to be the set of all meromorphic functions of the form

$$f(z) = \frac{s(z)}{b(z)} \tag{1.8}$$

where the numerator $s \in \mathscr{S}$ and the denominator $b \in \mathscr{B}_\kappa$ have no common zeros. Formula (1.8) is called the Krein-Langer representation of a generalized Schur function f; the entries s and b are defined by f uniquely up to a unimodular constant. Via nontangential boundary limits, the $\mathscr{S}_\kappa$-functions can be identified with the functions from the unit ball of $L^\infty(\mathbb{T})$ which admit meromorphic continuation inside the unit disk with the total pole multiplicity equal κ. The problem of finding functions $f \in \mathscr{S}_\kappa$ analytic and with preassigned Taylor coefficients at prescribed points of the unit disk will be denoted by $\mathbf{IP}_\kappa$:

$\mathbf{IP}_\kappa$: *Given data as in* (1.7) *and given an integer* $\kappa \geqslant 0$*, find all functions* $f \in \mathscr{S}_\kappa$ *(if exist) which are analytic at* z_i *and satisfy interpolation conditions* (1.4).

The problem $\mathbf{IP}_0$ is handled in Theorem 1.1. The main objective of this paper is to get an analog of Theorem 1.1 (that is to obtain the existence and the uniqueness criteria for solutions in terms of the associated Pick matrix $P_{\mathbf{n}}$) in case $\kappa \geqslant 0$. An immediate necessary condition for the existence of a solution is that $\kappa \geqslant \nu(P_{\mathbf{n}})$. Indeed, the class $\mathscr{S}_\kappa$ can be alternatively defined as the class of functions f meromorphic on $\mathbb{D}$ and such that the associated kernel (1.1) has κ negative squares on $\rho(f)$, the domain of analyticity of f. Therefore, for every $f \in \mathscr{S}_\kappa$, the Schwarz-Pick matrix $P^f_{\mathbf{n}}(\mathbf{z})$ defined in (1.2) has at most κ negative eigenvalues and since $P^f_{\mathbf{n}}(\mathbf{z}) = P_{\mathbf{n}}$ for every solution f to the problem $\mathbf{IP}_\kappa$, condition $\kappa \geqslant \nu(P_{\mathbf{n}})$ follows.

To formulate our first result we need one more definition. For two tuples $\mathbf{n}$ and $\mathbf{m}$ in $\mathbb{Z}^k_+$, we will say that

$$\mathbf{m} = (m_1,\dots,m_k) \preceq (n_1,\dots,n_k) = \mathbf{n} \quad \text{if} \quad m_i \leqslant n_i \quad \text{for} \quad i = 1,\dots,k.$$

For a matrix $P_{\mathbf{n}}$ decomposed in blocks as in (1.5) and a tuple $\mathbf{m} \preceq \mathbf{n}$ as above, define the principal submatrix $P_{\mathbf{m}} = [(P_{\mathbf{m}})_{ij}]_{i,j=1}^{k}$ of $P_{\mathbf{n}}$ whose block entries $(P_{\mathbf{m}})_{ij}$'s are equal to the leading $m_i \times m_j$ submatrices of the corresponding blocks in $P_{\mathbf{n}}$ (the null-dimensional conventions for $m_i = 0$ are clear):

$$P_{\mathbf{m}} = [(P_{\mathbf{m}})_{ij}]_{i,j=1}^{k} \quad \text{where} \quad (P_{\mathbf{m}})_{ij} = \begin{bmatrix} I_{m_i} & 0 \end{bmatrix} P_{ij} \begin{bmatrix} I_{m_j} \\ 0 \end{bmatrix}. \tag{1.9}$$

The matrices $P_{\mathbf{m}}$ of the form (1.9) are quite special principal submatrices of P; however, we will call them just "principal submatrices", since principal submatrices of other types will never show up in this paper.

DEFINITION 1.2. We will say that $P_{\mathbf{n}}$ of the form (1.5), (1.6) is $\mathbf{n}$-*saturated* if for every $\mathbf{m} \preceq \mathbf{n}$ such that $|\mathbf{m}| := m_1 + \ldots + m_k = \operatorname{rank}(P_{\mathbf{n}})$, the submatrix $P_{\mathbf{m}}$ of $P_{\mathbf{n}}$ is invertible.

THEOREM 1.3. *Let $P_{\mathbf{n}}$ be the Pick matrix of the problem $\mathbf{IP}_{\kappa}$. Then*

1. *The problem has infinitely many solutions if and only if*

$$\kappa \geqslant \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}}). \tag{1.10}$$

2. *The problem has a unique solution if and only if $\kappa = \nu(P_{\mathbf{n}})$ and $P_{\mathbf{n}}$ is singular and $\mathbf{n}$-saturated. This unique solution belongs to $\mathscr{B}_{\pi(P_{\mathbf{n}})}/\mathscr{B}_{\nu(P_{\mathbf{n}})}$.*

3. *Otherwise, the problem has no solutions.*

The problem $\mathbf{IP}_{\kappa}$ and its matrix-valued generalizations have been studied extensively in recent years; the list of earlier major publications include [1, 3, 4, 5, 12, 16, 19]. Most of the authors considered the *nondegenerate case* where $P_{\mathbf{n}}$ is invertible (i.e., $\delta(P_{\mathbf{n}}) = 0$). In this case Theorem 1.3 reads:

THEOREM 1.4. *If $\delta(P_{\mathbf{n}}) = 0$, then the problem $\mathbf{IP}_{\kappa}$ has infinitely many solutions if $\kappa \geqslant \nu(P_{\mathbf{n}})$ and it has no solutions otherwise.*

The degenerate problem was studied so far for two particular cases. The case of simple interpolation nodes (i.e., $n_i = 1$ for $i = 1, \ldots, k$) appears in [9] and earlier in [20] (in the context of a related to $\mathscr{S}_{\kappa}$ class $\mathscr{N}_{\kappa}$ of generalized Nevanlinna functions). In this case $P_{\mathbf{n}}$ being saturated means that *every* $d \times d$ principal submatrix of $P_{\mathbf{n}}$ is invertible where $d = \operatorname{rank}(P_{\mathbf{n}})$. The single-point problem (i.e., $k = 1$) appears in [2] and earlier in [13]. In this context, $P_{\mathbf{n}}$ being saturated means that the $d \times d$ principal submatrix of $P_{\mathbf{n}}$ is invertible where $d = \operatorname{rank}(P_{\mathbf{n}})$; the latter is easily checked. The next theorem shows that even in the general multi-point case, there is a simple test to verify that a structured matrix of the form (1.5), (1.6) is $\mathbf{n}$-saturated.

THEOREM 1.5. *Let $P_{\mathbf{n}}$ be of the form* (1.5), (1.6) *and let $r = \operatorname{rank}(P_{\mathbf{n}})$. Then $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated if and only if at least one principal submatrix $P_{\mathbf{m}}$ of $P_{\mathbf{n}}$ with $|\mathbf{m}| = r + 1$ and $\operatorname{rank}(P_{\mathbf{m}}) = r$, is $\mathbf{m}$-saturated.*

According to this theorem, in order to verify that $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated, it suffices to pick any principal submatrix $P_{\mathbf{m}}$ of $P_{\mathbf{n}}$ with $|\mathbf{m}| = r+1$ and $\operatorname{rank}(P_{\mathbf{m}}) = r$ and to verify invertibility of its $r \times r$ principal submatrices $P_{\boldsymbol{\ell}}$ where $\boldsymbol{\ell} \preceq \mathbf{m}$ and $|\boldsymbol{\ell}| = |\mathbf{m}| - 1$ (the total number of such submatrices does not exceed k). Theorems 1.3 and 1.5 will be proved in Section 4.

Another issue discussed in the paper is the parametrization of all solutions of the problem $\mathbf{IP}_{\kappa}$ for each fixed κ subject to inequality (1.10). In the nodegenerate case such a parametrization can be given in terms of a linear fractional transformation with the parameter running through the generalized Schur class $\mathscr{S}_{\kappa-\nu(P_{\mathbf{n}})}$ and satisfying certain mild restrictions which will be recalled in Theorem 3.1 below. In fact all known methods working for the classical Schur-class interpolation problem $\mathbf{IP}_0$ can be modified to the indefinite setting of the nondegenerate $\mathbf{IP}_{\kappa}$ for $\kappa > 0$. In [14], a suitable degenerate modification of the Krein-Langer method of generalized resolvents was applied to show that the solution set of the Nevanlinna-Pick problem for generalized Nevanlinna functions $f \in \mathscr{N}_{\kappa}$ whose Pick matrix is singular and saturated (in our present terminology), can be parametrized by a linear fractional formula. This remarkable result has certain deficiencies from computational point of view. One of the entries (coefficients) in the linear fractional formula is given in terms of certain defect elements of a symmetric operator associated to the data set of the problem and the explicit formula is obtained only in case $\delta(P_{\mathbf{n}}) = 1$. On the other hand, the parameters in the parametrizing formula vary over certain subset of $\mathscr{N}_{\kappa-\nu(P_{\mathbf{n}})}$ and it is not clear how to generate effectively the functions from this subset.

In [9] we used the Schur reduction method to reduce the degenerate (not necessarily "saturated") Nevanlinna-Pick problem for generalized Schur functions to a similar problem with fewer interpolation conditions and with the Pick matrix equal to the zero matrix. This allowed us to parametrize the solution set of the original problem in terms of a *family* of linear fractional transformations with disjoint ranges. On this way, we got very explicit formulas for coefficient functions and parameters varying over the whole generalized Schur class $\mathscr{S}_{\kappa-\nu(P_{\mathbf{n}})-\delta(P_{\mathbf{n}})}$. It can be shown that the same approach handles the problem $\mathbf{IP}_{\kappa}$ with $n_i \leqslant 2$ for $i = 1, \ldots, k$. In Section 3 the Schur reduction will be applied to the general $\mathbf{IP}_{\kappa}$ (Theorem 3.5). Although this reduction is not that efficient in the general framework, the results obtained in Section 3 will imply parts (2) and (3) in Theorem 1.3. To prove part (1), it suffices to find numbers c_{ij} for $j = n_i, \ldots, n_i + \ell_i - 1$ and $i = 1, \ldots, k$ so that the Pick matrix $P_{\mathbf{n}+\boldsymbol{\ell}}$ of the extended interpolation problem with interpolation conditions

$$f^{(j)}(z_i) = j!\, c_{i,j} \qquad (i = 1, \ldots, k;\ j = 0, \ldots, n_i + \ell_i - 1) \tag{1.11}$$

is invertible and has $\nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$ negative eigenvalues. Then the desired statement will follow from the nondegenerate result in Theorem 1.4. In the next section we will show that such an extension always exists. Moreover, in Section 5 we will describe all extended interpolation problems of the form (1.11) with invertible Pick matrices having $\nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$ negative eigenvalues so that the disjoint union of their solution sets will be equal to the solution set of the original problem $\mathbf{IP}_{\kappa}$ for $\kappa = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$. Finally, in Section 6 we will discuss the existence of solutions of the problem $\mathbf{IP}_{\kappa}$ with the minimal possible L^∞-norm.

2. Structured extensions of Pick matrices

Let us associate with the given tuples $\mathbf{n}=(n_1,\dots,n_k)$ and $\mathbf{z}=(z_1,\dots,z_k)$ the matrices

$$T_{\mathbf{n}}=\begin{bmatrix} J_{n_1}(z_1) & & \\ & \ddots & \\ & & J_{n_k}(z_k)\end{bmatrix} \quad \text{and} \quad E_{\mathbf{n}}=\begin{bmatrix} E_{n_1} \\ \vdots \\ E_{n_k}\end{bmatrix}, \tag{2.1}$$

where we denote by $J_n(z)$ and E_n respectively the $n\times n$ Jordan block with $z\in\mathbb{C}$ on the main diagonal and the vector of the length n with the first coordinate equals one and other coordinates equal zero:

$$J_n(z)=\begin{bmatrix} z & 0 & \dots & 0 \\ 1 & z & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & z\end{bmatrix}, \quad E_n=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0\end{bmatrix}. \tag{2.2}$$

The numbers c_{ij} from (1.7) will be arranged in the column

$$C_{\mathbf{n}}=\begin{bmatrix} C_{1,n_1} \\ \vdots \\ C_{k,n_k}\end{bmatrix}, \quad \text{where} \quad C_{i,n_i}=\begin{bmatrix} c_{i,0} \\ \vdots \\ c_{i,n_i-1}\end{bmatrix} \tag{2.3}$$

for which we will use a more compact notation $C_{\mathbf{n}}=\operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant n_i-1} c_{i,j}$. The Pick matrix $P_{\mathbf{n}}$ defined in (1.5), (1.6) might be introduced in a more compact (but less explicit) way as a unique matrix satisfying the Stein identity

$$P_{\mathbf{n}}-T_{\mathbf{n}}P_{\mathbf{n}}T_{\mathbf{n}}^*=E_{\mathbf{n}}E_{\mathbf{n}}^*-C_{\mathbf{n}}C_{\mathbf{n}}^*. \tag{2.4}$$

Verification of (2.4) for the $P_{\mathbf{n}}$ of the form (1.5), (1.6) is straightforward, while the uniqueness follows since all the eigenvalues of $T_{\mathbf{n}}$ are in $\mathbb{D}$. The tuple $\mathbf{z}=(z_1,\dots,z_k)\in\mathbb{D}^k$ is fixed throughout the paper, so the matrix $T_{\mathbf{m}}$ and the vector $E_{\mathbf{m}}$ are defined by formulas (2.1), (2.2) for every tuple $\mathbf{m}\in\mathbb{Z}_+^k$. Furthermore, the spectrum of $T_{\mathbf{m}}$ is a subset of $\{z_1,\dots,z_k\}\subset\mathbb{D}$ and therefore, for every vector

$$B_{\mathbf{m}}=\operatorname{Col}_{1\leqslant i\leqslant k} B_{i,m_i}=\operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant m_i-1} b_{i,j}\in\mathbb{C}^{|\mathbf{m}|}, \tag{2.5}$$

the Stein equation

$$\mathbb{S}-T_{\mathbf{m}}\mathbb{S}T_{\mathbf{m}}^*=E_{\mathbf{m}}E_{\mathbf{m}}^*-B_{\mathbf{m}}B_{\mathbf{m}}^* \tag{2.6}$$

has a unique solution $\mathbb{S}$ which will be denoted by $\mathbb{S}(B_{\mathbf{m}})$ (notation makes sense, since the matrix is uniquely determined by $\mathbf{m}$ and $B_{\mathbf{m}}$). The Stein identity (2.6) (sometimes referred to as to the *displacement rank identity*) imposes certain structure on $\mathbb{S}$

for which reason such matrices are often called *the structured matrices*. The explicit formula for $\mathbb{S}(B_{\mathbf{m}})$ in terms of $B_{\mathbf{m}}$ and $\mathbf{z}$ is similar to (1.6). In what follows, the matrix $\mathbb{S} = \mathbb{S}(B_{\mathbf{m}})$ and the vector $B_{\mathbf{m}}$ will be called *associated* to each other. In this notation, the Pick matrix $P_{\mathbf{n}}$ of the problem $\mathbf{IP}_{\kappa}$ equals $\mathbb{S}(C_{\mathbf{n}})$ and the principal submatrix $P_{\mathbf{m}}$ defined in (1.9) is just $\mathbb{S}(C_{\mathbf{m}})$, where $C_{\mathbf{m}}$ is defined via formula (2.3) for a tuple $\mathbf{m} \preceq \mathbf{n}$. Structured extensions of $P_{\mathbf{n}}$ are defined as follows: given a tuple $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}_+^k$, let us extend the vector $C_{\mathbf{n}}$ to

$$C_{\mathbf{n}+\boldsymbol{\ell}} = \operatorname{Col}_{1 \leqslant i \leqslant k} \operatorname{Col}_{0 \leqslant j \leqslant n_i+\ell_i-1} c_{i,j} \tag{2.7}$$

and define $P_{\mathbf{n}+\boldsymbol{\ell}}$ to be $\mathbb{S}(C_{\mathbf{n}+\boldsymbol{\ell}})$. Clearly, $P_{\mathbf{n}}$ is a principal submatrix of $P_{\mathbf{n}+\boldsymbol{\ell}}$ which in turn, is the Pick matrix of the extended interpolation problem (1.11). For the reason explained in the previous section, we desire to show that given a matrix $P_{\mathbf{n}} = \mathbb{S}(C_{\mathbf{n}})$ always admits a structured invertible extension $P_{\mathbf{n}+\boldsymbol{\ell}}$ with $\nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$. This will be done in Theorem 2.13 below. To control inertia of structured extensions, it is also desired to have explicit formulas for inertia of the matrix $\mathbb{S}(B_{\mathbf{m}})$ in terms of the associated vector $B_{\mathbf{m}}$. Such formulas were obtained in [7] for quite special $B_{\mathbf{m}}$ and $\mathbb{S}(B_{\mathbf{m}})$.

DEFINITION 2.1. We will say that the matrix $P = \mathbb{S}(B_{\mathbf{m}})$ is $\mathbf{m}$-*singular* if for every $\mathbf{k} \preceq \mathbf{m}$ the submatrix $\mathbb{S}(B_{\mathbf{k}})$ of P is singular.

Characterization of $\mathbf{m}$-singular matrices in terms of the associated vectors is given below (see [7, Proposition 2.1] for the proof).

THEOREM 2.2. *Let $B_{\mathbf{m}}$ be of the form* (2.5). *Then the matrix $\mathbb{S}(B_{\mathbf{m}})$ is $\mathbf{m}$-singular if and only if*

$$b_{1,0} = b_{2,0} = \ldots = b_{k,0} = \gamma, \quad |\gamma| = 1 \tag{2.8}$$

and

$$b_{i,j} = 0 \quad \text{for every} \quad i = 1, \ldots, k \ \text{ and } \ 1 \leqslant j \leqslant \left[\frac{m_i}{2}\right],$$

where $[a]$ stands for the greatest integer less than or equal to $a \in \mathbb{R}$.

For $B_{\mathbf{m}}$ of the form (2.5), let $r_i(B_{\mathbf{m}})$ be the index of the first nonzero entry in the block B_{i,m_i} other than the top one:

$$r_i(B_{\mathbf{m}}) = \min\{j \in \{1, \ldots, m_i - 1\} : \ b_{i,j} \neq 0\}. \tag{2.9}$$

We extend this definition to the cases where $m_i \leqslant 1$ or where $b_{i,j} = 0$ for all $j = 1, \ldots, m_i - 1$ just by letting $r_i(B_{\mathbf{m}}) = m_i$. We then introduce the tuple

$$\mathbf{r}(B_{\mathbf{m}}) = (r_1(B_{\mathbf{m}}), \ldots, r_k(B_{\mathbf{m}})) \in \mathbb{Z}_+^k$$

and let as usual, $|\mathbf{r}(B_{\mathbf{m}})| := \sum_{i=1}^k r_1(B_{\mathbf{m}})$. Another result from [7] (see Theorem 2.2 and Corollary 2.3 there) relates the numbers (2.9) to inertia of $\mathbb{S}(B_{\mathbf{m}})$.

THEOREM 2.3. *Let* $\mathbf{m} = \{m_1, \ldots, m_k\} \in \mathbb{Z}_+^k$, *let the entries* $b_{i,0}$ *of the vector* $B_{\mathbf{m}}$ (2.5) *satisfy* (2.8) *and let us assume that*

$$r_i(B_{\mathbf{m}}) \geqslant \frac{m_i}{2} \quad for \quad i = 1, \ldots, k, \tag{2.10}$$

where the numbers $r_i(B_{\mathbf{m}})$ *are defined in* (2.9)*. Let* $\mathbb{S}(B_{\mathbf{m}})$ *be the matrix associated with* $B_{\mathbf{m}}$*. Then*

$$\pi(\mathbb{S}(B_{\mathbf{m}})) = \nu(\mathbb{S}(B_{\mathbf{m}})) = |\mathbf{m}| - |\mathbf{r}(B_{\mathbf{m}})| \quad and \quad \delta(\mathbb{S}(B_{\mathbf{m}})) = 2|\mathbf{r}(B_{\mathbf{m}})| - |\mathbf{m}|.$$

Furthermore, for each one of the $n_i \times n_i$ *diagonal blocks* $\mathbb{S}_{ii}$ *of* $\mathbb{S}(B_{\mathbf{m}}) = [\mathbb{S}_{ij}]_{i,j=1}^k$*, we have*

$$\pi(\mathbb{S}_{ii}) = \nu(\mathbb{S}_{ii}) = m_i - r_i(B_{\mathbf{m}}) \quad and \quad \delta(\mathbb{S}_{ii}) = 2r_i(B_{\mathbf{m}}) - m_i. \tag{2.11}$$

REMARK 2.4. Theorem 2.2 states that $\mathbb{S}(B_{\mathbf{m}})$ is $\mathbf{m}$-singular if and only if conditions (2.8) are satisfied and $r_i(B_{\mathbf{m}}) > \frac{m_i}{2}$ for every i such that $m_i > 0$. Therefore, Theorem 2.3 applies in particular, to $\mathbf{m}$-singular matrices.

In Proposition 2.12 below we will get an analog of Theorem 2.3 for structured matrices $\mathbb{S}(B_{\mathbf{m}})$ with no restrictions (2.8) and (2.10) imposed (to an extent fitting the objectives of this paper). As we will show, the general case reduces to the $\mathbf{m}$-singular one via standard Schur complement arguments. In the following subsection we discuss the Schur complements of structured matrices $\mathbb{S}(B_{\mathbf{m}})$ in some detail.

2.1. Schur complements

Let $P_{\mathbf{n}} = \mathbb{S}(C_{\mathbf{n}})$ be a unique solution of the Stein equation (2.4), let $\mathbf{d} = (d_1 \ldots, d_k) \in \mathbb{Z}_+^k$ be a tuple such that $\mathbf{d} \preceq \mathbf{n}$ and let $P_{\mathbf{d}}$ be a principal submatrix of $P_{\mathbf{n}}$ defined according to (1.9). Let U be the $|\mathbf{n}| \times |\mathbf{n}|$ permutation matrix defined by $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ where U_1 and U_2 are diagonal block matrices with the i-th diagonal blocks

$$U_{1,i} = \begin{bmatrix} I_{d_i} & 0 \end{bmatrix} \in \mathbb{C}^{d_i \times n_i} \quad \text{and} \quad U_{2,i} = \begin{bmatrix} 0 & I_{n_i - d_i} \end{bmatrix} \in \mathbb{C}^{(n_i - d_i) \times n_i}. \tag{2.12}$$

Then it is easily verified that

$$UP_{\mathbf{n}}U^* = \begin{bmatrix} P_{\mathbf{d}} & \Psi^* \\ \Psi & \Gamma \end{bmatrix} \quad \text{and} \quad UT_{\mathbf{n}}U^* = \begin{bmatrix} T_{\mathbf{d}} & 0 \\ R & T_{\mathbf{n}-\mathbf{d}} \end{bmatrix} \tag{2.13}$$

where the matrices $T_{\mathbf{d}}$ and $T_{\mathbf{n}-\mathbf{d}}$ are defined via formula (2.1) and where R is the diagonal block matrix with the diagonal blocks

$$R_i = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{(n_i - d_i) \times d_i}, \quad i = 1, \ldots, k.$$

The first formula in (2.13) means that the permutation U ensembles the submatrix $P_{\mathbf{d}}$ in the left upper corner of $P_{\mathbf{n}}$. If we decompose the vectors E_{n_i} and C_{i,n_i} from (2.1) and (2.3) conformally with (2.13) as

$$C_{i,n_i} = \begin{bmatrix} C_{i,d_i} \\ \widetilde{C}_i \end{bmatrix}, \quad E_{n_i} = \begin{bmatrix} E_{d_i} \\ \widetilde{E}_i \end{bmatrix} \quad \text{for} \quad i = 1, \ldots, k$$

and define the columns $\widetilde{C} = \text{Col}_{1 \leqslant i \leqslant k} \widetilde{C}_i$ and $\widetilde{E} = \text{Col}_{1 \leqslant i \leqslant k} \widetilde{E}_i$ then it is readily seen that

$$UC_{\mathbf{n}} = \begin{bmatrix} C_{\mathbf{d}} \\ \widetilde{C} \end{bmatrix}, \quad UE_{\mathbf{n}} = \begin{bmatrix} E_{\mathbf{d}} \\ \widetilde{E} \end{bmatrix}. \tag{2.14}$$

Multiplying both parts in (2.4) by U on the left and by U^* on the right and making use of (2.13), (2.14) we get

$$\begin{bmatrix} P_{\mathbf{d}} & \Psi^* \\ \Psi & \Gamma \end{bmatrix} - \begin{bmatrix} T_{\mathbf{d}} & 0 \\ R & T_{\mathbf{n-d}} \end{bmatrix} \begin{bmatrix} P_{\mathbf{d}} & \Psi^* \\ \Psi & \Gamma \end{bmatrix} \begin{bmatrix} T_{\mathbf{d}}^* & R^* \\ 0 & T_{\mathbf{n-d}}^* \end{bmatrix} = \begin{bmatrix} E_{\mathbf{d}} \\ \widetilde{E} \end{bmatrix} \begin{bmatrix} E_{\mathbf{d}}^* & \widetilde{E}^* \end{bmatrix} - \begin{bmatrix} C_{\mathbf{d}} \\ \widetilde{C} \end{bmatrix} \begin{bmatrix} C_{\mathbf{d}}^* & \widetilde{C}^* \end{bmatrix}$$

which amounts to

$$P_{\mathbf{d}} = \mathbb{S}(C_{\mathbf{d}}), \quad \Psi - T_{\mathbf{n-d}} \Psi T_{\mathbf{d}}^* - R P_{\mathbf{d}} T_{\mathbf{d}}^* = \widetilde{E} E_{\mathbf{d}}^* - \widetilde{C} C_{\mathbf{d}}^*, \tag{2.15}$$

$$\Gamma - T_{\mathbf{n-d}} \Gamma T_{\mathbf{d}}^* - R P_{\mathbf{d}} R^* - T_{\mathbf{n-d}} \Psi R^* - R \Psi T_{\mathbf{n-d}}^* = \widetilde{E}\widetilde{E}^* - \widetilde{C}\widetilde{C}^*. \tag{2.16}$$

If $P_{\mathbf{d}}$ is invertible, then the matrix

$$\mathbf{S}_{\mathbf{n-d}} := \Gamma - \Psi^* P_{\mathbf{d}}^{-1} \Psi$$

is called the *Schur complement* of $P_{\mathbf{d}}$ in $P_{\mathbf{n}}$. The matrix $\mathbf{S}_{\mathbf{n-d}}$ can be decomposed in blocks

$$\mathbf{S}_{\mathbf{n-d}} = [S_{ij}]_{i,j=1}^k \quad \text{with} \quad S_{ij} \in \mathbb{C}^{(n_i - d_i) \times (n_j - d_j)}. \tag{2.17}$$

If $d_i = n_i$, then the blocks S_{ij} and S_{ji} in (2.17) are null-dimensional; however, for the sake of uniformity, we will keep the $k \times k$-block structure of $\mathbf{S}_{\mathbf{n-d}}$ disregarding in what follows, equalities of null-dimensional matrices as trivial. Also it is convenient to break the index set $\mathscr{I} = \{1, \ldots, k\}$ into

$$\mathscr{I}_0 = \{i \in \mathscr{I} : d_i = n_i\} \quad \text{and} \quad \mathscr{I}_1 = \{i \in \mathscr{I} : d_i < n_i\}, \tag{2.18}$$

so that all positive-dimensional blocks S_{ij} in (2.17) correspond to $i, j \in \mathscr{I}_1$. The next proposition asserting that Schur complements preserve the structure imposed by Stein identities is well known.

PROPOSITION 2.5. *Let $P_{\mathbf{d}}$ be an invertible principal submatrix of $P_{\mathbf{n}}$. Then its Schur complement $\mathbf{S}_{\mathbf{n-d}}$ satisfies the Stein identity*

$$\mathbf{S}_{\mathbf{n-d}} - T_{\mathbf{n-d}} \mathbf{S}_{\mathbf{n-d}} T_{\mathbf{n-d}}^* = G_{\mathbf{n-d}} G_{\mathbf{n-d}}^* - Y_{\mathbf{n-d}} Y_{\mathbf{n-d}}^*, \tag{2.19}$$

where the vectors

$$G_{\mathbf{n}-\mathbf{d}} = \operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1} g_{i,j} \quad \textit{and} \quad Y_{\mathbf{n}-\mathbf{d}} = \operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1} y_{i,j} \tag{2.20}$$

are defined in terms of decompositions (2.13)*,* (2.14) *as*

$$\begin{aligned} G_{\mathbf{n}-\mathbf{d}} &= \widetilde{E} + \left(R - (I - T_{\mathbf{n}-\mathbf{d}})\Psi P_{\mathbf{d}}^{-1}\right)(I - T_{\mathbf{d}})^{-1} E_{\mathbf{d}}, \\ Y_{\mathbf{n}-\mathbf{d}} &= \widetilde{C} + \left(R - (I - T_{\mathbf{n}-\mathbf{d}})\Psi P_{\mathbf{d}}^{-1}\right)(I - T_{\mathbf{d}})^{-1} C_{\mathbf{d}}. \end{aligned} \tag{2.21}$$

For the proof, it suffices to multiply both parts of identity (2.4) by the matrix $\begin{bmatrix}-\Psi P_{\mathbf{d}}^{-1} & I\end{bmatrix}$ on the left, by its adjoint on the right and then to invoke relations (2.15), (2.16); computations are long but quite straightforward (see e.g., [11, Theorem 2.5] for details).

REMARK 2.6. Let us assume that $P_{\mathbf{d}}$ is invertible and apply formulas (2.21) to a tuple $\mathbf{m}$ such that $\mathbf{d} \preceq \mathbf{m} \preceq \mathbf{n}$ to define the vectors $G_{\mathbf{m}-\mathbf{d}}$ and $Y_{\mathbf{m}-\mathbf{d}}$ (note that in this formula, the entries $\widetilde{E}$, $\widetilde{C}$, Ψ and R depend on the tuple $\mathbf{m}$). Due to the Jordan structure of $T_{\mathbf{n}}$ and $T_{\mathbf{m}}$, it is readily seen from formulas (2.21) that $G_{\mathbf{m}-\mathbf{d}}$ and $Y_{\mathbf{m}-\mathbf{d}}$ are "structured subvectors" of $G_{\mathbf{n}-\mathbf{d}}$ and $Y_{\mathbf{n}-\mathbf{d}}$ respectively:

$$G_{\mathbf{m}-\mathbf{d}} = \operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant m_i-d_i-1} g_{i,j} = [G_{\mathbf{n}-\mathbf{d}}]_{\mathbf{m}-\mathbf{d}},$$

$$Y_{\mathbf{m}-\mathbf{d}} = \operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant m_i-d_i-1} y_{i,j} = [Y_{\mathbf{n}-\mathbf{d}}]_{\mathbf{m}-\mathbf{d}}.$$

The top entries $g_{i,0}$ and $y_{i,0}$ from the block entries of $G_{\mathbf{n}-\mathbf{d}}$ and $Y_{\mathbf{n}-\mathbf{d}}$ will play a special role in the subsequent analysis. It is helpful to have explicit formulas for them.

PROPOSITION 2.7. *The formulas for the entries* $g_{i,0}$ *and* $y_{i,0}$ *in* (2.20) *are:*

$$\begin{aligned} g_{i,0} &= \left(\widehat{E}_i P_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1 - z_i)c_{i,d_i}C_{\mathbf{d}}^*\right)\Gamma_i E_{\mathbf{d}}, \\ y_{i,0} &= c_{i,d_i} + \left(\widehat{E}_i P_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1 - z_i)c_{i,d_i}C_{\mathbf{d}}^*\right)\Gamma_i C_{\mathbf{d}}, \end{aligned} \quad \textit{if } 0 < d_i < n_i, \tag{2.22}$$

and

$$\begin{aligned} g_{i,0} &= 1 + (1 - z_i)(c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i E_{\mathbf{d}}, \\ y_{i,0} &= c_{i,0} + (1 - z_i)(c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i C_{\mathbf{d}}, \end{aligned} \quad \textit{if } d_i = 0, \tag{2.23}$$

where

$$\widehat{E}_i = \begin{bmatrix}0 & \cdots & 0 & 1 & 0 & \cdots & 0\end{bmatrix} \tag{2.24}$$

is the row of the length $|\mathbf{d}|$ *with* 1 *in the* $(d_1 + \ldots + d_i)$*-th slot and where*

$$\Gamma_i = (I - z_i T_{\mathbf{d}}^*)^{-1} P_{\mathbf{d}}^{-1} (I - T_{\mathbf{d}})^{-1}. \tag{2.25}$$

Proof. Formulas (2.21) hold for any tuple $\mathbf{n}\succeq\mathbf{d}$. Let us apply it to $\mathbf{n}=\mathbf{d}+\mathbf{e}_i$, where $\mathbf{e}_i$ stands for the k-tuple with the i-th entry equals one and all other entries equal zero. In this case,

$$G_{\mathbf{n}-\mathbf{d}}=g_{i,0},\quad Y_{\mathbf{n}-\mathbf{d}}=y_{i,0},\quad T_{\mathbf{n}-\mathbf{d}}=z_i,\quad \widetilde{C}=c_{i,d_i} \tag{2.26}$$

and we can solve the second equality in (2.15) for Ψ to get

$$\Psi=(RP_{\mathbf{d}}T_{\mathbf{d}}^*+\widetilde{E}E_{\mathbf{d}}^*-c_{i,d_i}C_{\mathbf{d}}^*)(I-z_iT_{\mathbf{d}}^*)^{-1}. \tag{2.27}$$

If $d_i>0$, then $\widetilde{E}=0$ (see (2.14)) and $R=\widehat{E}_i$ which being substituted together with (2.26) and (2.27) into (2.21) gives (2.22). Formulas (2.23) follow from (2.21) in much the same way once we observe that if $d_i=0$, then $\widetilde{E}=1$ and $R=0$. $\square$

Now we let $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$ to be the block diagonal matrix with block entries equal to lower triangular toeplitz matrices defined in terms of the entries $g_{i,j}$ of the vector $G_{\mathbf{n}-\mathbf{d}}$ as

$$\mathbf{G}_{\mathbf{n}-\mathbf{d}}=\begin{bmatrix}\mathbf{G}_1 & & 0\\ & \ddots & \\ 0 & & \mathbf{G}_k\end{bmatrix},\quad \mathbf{G}_i=\begin{bmatrix} g_{i,0} & 0 & \dots & 0\\ g_{i,1} & g_{i,0} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ g_{i,n_i-d_i-1} & \cdots & g_{i,1} & g_{i,0}\end{bmatrix} \tag{2.28}$$

and observe that

$$\mathbf{G}_{\mathbf{n}-\mathbf{d}}T_{\mathbf{n}-\mathbf{d}}=T_{\mathbf{n}-\mathbf{d}}\mathbf{G}_{\mathbf{n}-\mathbf{d}}\quad\text{and}\quad \mathbf{G}_{\mathbf{n}-\mathbf{d}}E_{\mathbf{n}-\mathbf{d}}=G_{\mathbf{n}-\mathbf{d}}, \tag{2.29}$$

where the commutation relation follows due to the toeplitz structure of the diagonal blocks $\mathbf{G}_i$ of $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$ and the second relation is readily seen from definitions (2.1) and (2.20) of $E_{\mathbf{n}-\mathbf{d}}$ and $G_{\mathbf{n}-\mathbf{d}}$, respectively. If $g_{i,0}\neq 0$ for every $i\in\mathscr{I}_1$, then the matrix $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$ is invertible, in which case we let

$$X_{\mathbf{n}-\mathbf{d}}:=\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-1}Y_{\mathbf{n}-\mathbf{d}}=\operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1}x_{i,j}\in\mathbb{C}^{|\mathbf{n}-\mathbf{d}|} \tag{2.30}$$

and

$$\widetilde{P}=\left[\widetilde{P}_{ij}\right]_{i,j=1}^{k}:=\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-1}\mathbf{S}_{\mathbf{n}-\mathbf{d}}\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-*}=\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-1}(\Gamma-\Psi P_{\mathbf{d}}^{-1}\Psi^*)\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-*}. \tag{2.31}$$

The matrix $\widetilde{P}$ is decomposed in blocks conformally with (2.17) so that $\widetilde{P}_{ij}\in\mathbb{C}^{(n_i-d_i)\times(n_j-d_j)}$. With a slight abuse of classical notation, we will call the matrix $\widetilde{P}$ (along with the matrix $\mathbf{S}_{\mathbf{n}-\mathbf{d}}$) *the Schur complement of* $P_{\mathbf{d}}$ *in* $P_{\mathbf{n}}$ and we will use notation $\widetilde{P}:=P_{\mathbf{n}}/P_{\mathbf{d}}$.

By construction, the matrix $P_{\mathbf{n}}/P_{\mathbf{d}}$ exists if and only if $P_{\mathbf{d}}$ and $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$ are invertible. By the structure (2.28) of $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$, its invertibility is equivalent to the numbers $g_{i,0}$ defined in (2.22) and (2.23) be nonzero for every $i\in\mathscr{I}_1$. The formulas (2.22) and (2.23) are too heavy to verify inequalities $g_{i,0}\neq 0$ directly. The next proposition provides simple sufficient conditions for these inequalities to hold.

PROPOSITION 2.8. *Let $P_{\mathbf{d}}$ be invertible and let us assume that*

$$\det(P_{\mathbf{d}+\mathbf{e}_i}) = 0 \tag{2.32}$$

where $\mathbf{e}_i$ is the i-th "coordinate" tuple in $\mathbb{Z}_+^k$. Then $|g_{i,0}| = |y_{i,0}| \neq 0$.

Proof. As in the proof of Proposition 2.7, we apply Proposition 2.5 to the tuple $\mathbf{n} = \mathbf{d} + \mathbf{e}_i$. Identity (2.19) then takes the form

$$\mathbf{S}_{\mathbf{e}_i} - |z_i|^2 \mathbf{S}_{\mathbf{e}_i} = |g_{i,0}|^2 - |y_{i,0}|^2, \tag{2.33}$$

where $\mathbf{S}_{\mathbf{e}_i}$ is the Schur complement of $P_{\mathbf{d}}$ in $P_{\mathbf{d}+\mathbf{e}_i}$. It follows from (2.32) that $\mathbf{S}_{\mathbf{e}_i} = 0$ which together with (2.33) implies $|g_{i,0}| = |y_{i,0}|$. Let us assume that

$$g_{i,0} = y_{i,0} = 0. \tag{2.34}$$

If $d_i = 0$, then we have from (2.23) and on account of (2.34), (2.15) and (2.25) that

$$\begin{aligned} 0 &= g_{i,0}E_{\mathbf{d}}^* - y_{i,0}C_{\mathbf{d}}^* \\ &= E_{\mathbf{d}}^* - c_{i,0}C_{\mathbf{d}}^* + (1-z_i)(c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i(E_{\mathbf{d}}E_{\mathbf{d}}^* - C_{\mathbf{d}}C_{\mathbf{d}}^*) \\ &= (c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\left[I + (1-z_i)\Gamma_i(P_{\mathbf{d}} - T_{\mathbf{d}}P_{\mathbf{d}}T_{\mathbf{d}}^*)\right] \\ &= (c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i(z_iI - T_{\mathbf{d}})P(I - T_{\mathbf{d}}^*). \end{aligned} \tag{2.35}$$

Since $d_i = 0$, it follows that $z_i \notin \operatorname{spec}(T_{\mathbf{d}})$. Therefore the matrix $\Gamma_i(z_iI - T_{\mathbf{d}})P(I - T_{\mathbf{d}}^*)$ is invertible and we conclude from (2.35) that

$$c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^* = 0.$$

But then the first formula in (2.23) gives $g_{i,0} = 1$ which contradicts (2.34). The case where $d_i > 0$ is handled similarly: assuming (2.34) to be in force we get from (2.22)

$$\begin{aligned} 0 &= g_{i,0}E_{\mathbf{d}}^* - y_{i,0}C_{\mathbf{d}}^* \\ &= -c_{i,d_i}C_{\mathbf{d}}^* + \left(\widehat{E}_iP_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1-z_i)c_{i,d_i}C_{\mathbf{d}}^*\right)\Gamma_i(E_{\mathbf{d}}E_{\mathbf{d}}^* - C_{\mathbf{d}}C_{\mathbf{d}}^*) \\ &= -c_{i,d_i}C_{\mathbf{d}}^* + \left(\widehat{E}_iP_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1-z_i)c_{i,d_i}C_{\mathbf{d}}^*\right)\Gamma_i(P_{\mathbf{d}} - T_{\mathbf{d}}P_{\mathbf{d}}T_{\mathbf{d}}^*) \\ &= -c_{i,d_i}C_{\mathbf{d}}^*\Gamma_i(z_iI - T_{\mathbf{d}})P_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + \widehat{E}_iP_{\mathbf{d}}\Gamma_i(P_{\mathbf{d}} - T_{\mathbf{d}}P_{\mathbf{d}}T_{\mathbf{d}}^*). \end{aligned}$$

Multiplying both parts of the latter equality by $(I - T_{\mathbf{d}}^*)^{-1}P_{\mathbf{d}}^{-1}(I - T_{\mathbf{d}})^{-1}$ on the right we get

$$\begin{aligned} 0 &= -c_{i,d_i}C_{\mathbf{d}}^*(I - z_iT_{\mathbf{d}}^*)^{-1}P_{\mathbf{d}}^{-1}(z_iI - T_{\mathbf{d}}) \\ &\quad + \widehat{E}_iP_{\mathbf{d}}(I - z_iT_{\mathbf{d}}^*)^{-1}\left[P_{\mathbf{d}}^{-1} - T_{\mathbf{d}}^*P_{\mathbf{d}}^{-1}T_{\mathbf{d}}\right] \\ &= (\widehat{E}_iP_{\mathbf{d}}T_{\mathbf{d}}^* - c_{i,d_i}C_{\mathbf{d}}^*)(I - z_iT_{\mathbf{d}}^*)^{-1}P_{\mathbf{d}}^{-1}(z_iI - T_{\mathbf{d}}) + \widehat{E}_i. \end{aligned} \tag{2.36}$$

Since $d_i > 0$, it follows that $T_{\mathbf{d}}$ contains the nontrivial block $J_{d_i}(z_i)$ and thus, all the entries in the $(d_1 + \ldots + d_i)$-th column of the matrix $z_iI - T_{\mathbf{d}}$ equal zero. Then by definition (2.24) of $\widehat{E}_i$, the $(d_1+\ldots+d_i)$-th entry in the row vector on the right hand side of (2.36) equals one which contradicts (2.36). Thus, equalities (2.34) cannot hold and since $|g_{i,0}|=|y_{i,0}|$, the desired statement follows. □

For the rest of the section, we fix a tuple $\mathbf{d}=(d_1,\ldots,d_k)\preceq\mathbf{n}$ such that the corresponding $P_{\mathbf{d}}$ is an invertible *maximal* submatrix of $P_{\mathbf{n}}$ in the sense that

$$\det P_{\mathbf{d}}\neq 0 \quad \text{and} \quad \det P_{\mathbf{m}}=0 \quad \text{for every} \quad \mathbf{d}\precneq\mathbf{m}\preceq\mathbf{n}. \tag{2.37}$$

The matrix $P_{\mathbf{n}}$ for which the only tuple $\mathbf{d}$ satisfying (2.37) is the zero tuple, is $\mathbf{n}$-singular (see Definition 2.1). It is clear that a maximal invertible principal submatrix is not unique in general. We leave open the two following questions: *Do all maximal invertible principal submatrices have the same dimension (that is, does the property* (2.37) *imply that* $\det P_{\mathbf{m}}=0$ *for every* $\mathbf{m}\preceq\mathbf{n}$ *with* $|\mathbf{m}|>|\mathbf{d}|$*) and if yes, do they have the same inertia?*

PROPOSITION 2.9. *Let $P_{\mathbf{d}}$ be a maximal invertible principal submatrix of $P_{\mathbf{n}}$. Then*

(1) The Schur complement $\widetilde{P}=P_{\mathbf{n}}/P_{\mathbf{d}}$ exists and is $(\mathbf{n}-\mathbf{d})$-singular.

(2) The matrix $\widetilde{P}$ satisfies the Stein identity

$$\widetilde{P}-T_{\mathbf{n}-\mathbf{d}}\widetilde{P}T^*_{\mathbf{n}-\mathbf{d}}=E_{\mathbf{n}-\mathbf{d}}E^*_{\mathbf{n}-\mathbf{d}}-X_{\mathbf{n}-\mathbf{d}}X^*_{\mathbf{n}-\mathbf{d}} \quad (\textit{i.e., } \widetilde{P}=\mathbb{S}(X_{\mathbf{n}-\mathbf{d}})), \tag{2.38}$$

where $X_{\mathbf{n}-\mathbf{d}}$ is the vector given by (2.30)*. Furthermore,*

$$\pi(\widetilde{P})=\pi(P_{\mathbf{n}})-\pi(P_{\mathbf{d}}), \quad \nu(\widetilde{P})=\nu(P_{\mathbf{n}})-\nu(P_{\mathbf{d}}), \quad \delta(\widetilde{P})=\delta(P_{\mathbf{n}}). \tag{2.39}$$

(3) The entries $x_{i,0}$ in $X_{\mathbf{n}-\mathbf{d}}$ are subject to

$$x_{i,0}=\gamma \ \ (|\gamma|=1) \quad \textit{for every} \quad i\in\mathscr{I}_1. \tag{2.40}$$

Proof. By the maximality assumption (2.37), condition (2.32) is satisfied for every $i\in\mathscr{I}_1$ and therefore $g_{i,0}\neq 0$ for every $i\in\mathscr{I}_1$, by Proposition 2.8. Therefore the matrix $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$ given in (2.28) is invertible, and the matrix $\widetilde{P}=P_{\mathbf{n}}/P_{\mathbf{d}}$ exists (by formula (2.31)). Let us assume that $\widetilde{P}$ is not $(\mathbf{n}-\mathbf{d})$-singular, i.e., that there exists a tuple $\mathbf{m}\preceq\mathbf{n}-\mathbf{d}$ such that the principal submatrix $\widetilde{P}_{\mathbf{m}}$ of $\widetilde{P}$ is invertible. Since $\mathbf{G}^{-1}_{\mathbf{n}-\mathbf{d}}$ is block diagonal with lower triangular diagonal blocks, it follows from (2.39) that $\widetilde{P}_{\mathbf{m}}$ is congruent to the principal submatrix $[\mathbf{S}_{\mathbf{n}-\mathbf{d}}]_{\mathbf{m}}$ of $\mathbf{S}_{\mathbf{n}-\mathbf{d}}$ which therefore, is also invertible. Since $[\mathbf{S}_{\mathbf{n}-\mathbf{d}}]_{\mathbf{m}}$ is the Schur complement of the block $P_{\mathbf{d}}$ in the matrix $P_{\mathbf{d}+\mathbf{m}}$, it then follows that the principal submatrix $P_{\mathbf{d}+\mathbf{m}}$ of $P_{\mathbf{n}}$ is invertible which contradicts the maximality assumption (2.37). This completes the proof of part (1).

Multiplying the Stein identity (2.19) by $\mathbf{G}^{-1}_{\mathbf{n}-\mathbf{d}}$ on the left and by its adjoint on the right we get (2.38), due to relations (2.29) and definitions (2.30) and (2.31). Since the matrices $\widetilde{P}$ and $\mathbf{S}_{\mathbf{n}-\mathbf{d}}$ are congruent, their inertia coincide and then relations (2.39) follow from well known formulas for inertia of the Schur complement $\mathbf{S}_{\mathbf{n}-\mathbf{d}}$. This completes the proof of part (2). The last statement follows from Theorem 2.2 and Remark 2.4, since $\widetilde{P}=\mathbb{S}(X_{\mathbf{n}-\mathbf{d}})$ is $(\mathbf{n}-\mathbf{d})$-singular. □

REMARK 2.10. *The number* γ *from* (2.40) *can be defined explicitly as follows:*

$$\gamma = \begin{cases} \dfrac{c_{i,0} + (1-z_i)(c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i C_{\mathbf{d}}}{1+(1-z_i)(c_{i,0}C_{\mathbf{d}}^* - E_{\mathbf{d}}^*)\Gamma_i E_{\mathbf{d}}} & \text{if } d_i = 0, \\[2ex] \dfrac{c_{i,d_i} + (\widehat{E}_i P_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1-z_i)c_{i,d_i}C_{\mathbf{d}}^*)\Gamma_i C_{\mathbf{d}}}{(\widehat{E}_i P_{\mathbf{d}}(I - T_{\mathbf{d}}^*) + (1-z_i)c_{i,d_i}C_{\mathbf{d}}^*)\Gamma_i E_{\mathbf{d}}} & \text{if } d_i \geqslant 1, \end{cases} \tag{2.41}$$

where $\widehat{E}_i$ *and* Γ_i *are given in* (2.24), (2.25), *and where* i *is any index from* $\mathscr{I}_1$.

Indeed, by definition (2.30), $x_{i,0} = y_{i,0}/g_{i,0}$ and formulas (2.41) follow from (2.22) and (2.23). We next justify the existence of certain Schur complements which will appear below.

PROPOSITION 2.11. *Let* $P_{\mathbf{d}}$ *be a maximal invertible principal submatrix of* $P_{\mathbf{n}}$, *let* $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbb{Z}_+^k$ *be a tuple such that* $s_i = 0$ *for every* $i \in \mathscr{I}_0$ *and let* $P_{\mathbf{n}+\mathbf{s}}$ *be a structured extension of* $P_{\mathbf{n}}$. *Then*

(1) The Schur complement $P_{\mathbf{d}+\mathbf{s}}/P_{\mathbf{d}}$ *exists.*

(2) The Schur complement $P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}}$ *exists and extends* $P_{\mathbf{n}}/P_{\mathbf{d}}$.

Proof. By Proposition 2.9, the Schur complement $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$ exists and therefore, $g_{i,0} \neq 0$ for every $i \in \mathscr{I}_1$. Since we have assumed that $s_i = 0$ for every $i \in \mathscr{I}_0$, it follows that $g_{i,0} \neq 0$ for every i such that $s_i > 0$, that is, for every i such that $d_i < d_i + s_i$. But the latter condition is necessary and sufficient for the matrix $P_{\mathbf{d}+\mathbf{s}}/P_{\mathbf{d}}$ to exist.

To prove the second statement, note that the tuple $\mathbf{s}' = (s_1', \ldots, s_k') := \mathbf{n} - \mathbf{d} + \mathbf{s}$ also satisfies $s_i' = 0$ for every $i \in \mathscr{I}_0$ (by definition (2.18) of $\mathscr{I}_0$). Therefore, the matrix $P_{\mathbf{d}+\mathbf{s}'}/P_{\mathbf{d}} = P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}}$ exists, by part (1). To show that $P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}}$ extends $P_{\mathbf{n}}/P_{\mathbf{d}}$, we first apply formula (2.21) to the tuples $\mathbf{d}$ and $\mathbf{n}+\mathbf{r}$ to get the vectors $G_{\mathbf{n}+\mathbf{r}-\mathbf{d}}$ and $Y_{\mathbf{n}+\mathbf{r}-\mathbf{d}}$. By Remark 2.6 applied to the tuples $\mathbf{d} \preceq \mathbf{n} \preceq \mathbf{n}+\mathbf{s}$ we conclude that $G_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ and $Y_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ extend the vectors $G_{\mathbf{n}}$ and $Y_{\mathbf{n}}$ respectively:

$$G_{\mathbf{n}-\mathbf{d}} = [G_{\mathbf{n}+\mathbf{s}-\mathbf{d}}]_{\mathbf{n}-\mathbf{d}}, \quad Y_{\mathbf{n}-\mathbf{d}} = [Y_{\mathbf{n}+\mathbf{s}-\mathbf{d}}]_{\mathbf{n}-\mathbf{d}}.$$

Therefore, the block diagonal matrix $\mathbf{G}_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ constructed from the vector $G_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ via formulas (2.28) extends the matrix $\mathbf{G}_{\mathbf{n}-\mathbf{d}}$:

$$\mathbf{G}_{\mathbf{n}-\mathbf{d}} = [\mathbf{G}_{\mathbf{n}+\mathbf{s}-\mathbf{d}}]_{\mathbf{n}-\mathbf{d}}. \tag{2.42}$$

Since the Schur complement $P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}}$ exists, the matrix $G_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ is invertible and we can define the vector

$$X_{\mathbf{n}+\mathbf{s}-\mathbf{d}} := \mathbf{G}_{\mathbf{n}+\mathbf{s}-\mathbf{d}}^{-1} Y_{\mathbf{n}+\mathbf{s}-\mathbf{d}} \tag{2.43}$$

associated to $P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}}$. By the lower triangular structure of diagonal blocks in $\mathbf{G}_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ and also by (2.42), we have $\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-1} = [\mathbf{G}_{\mathbf{n}+\mathbf{s}-\mathbf{d}}^{-1}]_{\mathbf{n}-\mathbf{d}}$. The latter formula and (2.43) imply that $X_{\mathbf{n}+\mathbf{s}-\mathbf{d}}$ extends the vector $X_{\mathbf{n}-\mathbf{d}}$ associated with $P_{\mathbf{n}}/P_{\mathbf{d}}$ and therefore $P_{\mathbf{n}+\mathbf{s}}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}+\mathbf{r}-\mathbf{d}})$ extends the matrix $P_{\mathbf{n}}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}-\mathbf{d}})$. $\square$

For a maximal invertible principal submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$, we define the tuple

$$\boldsymbol{\ell} = (\ell_1, \dots, \ell_k), \quad \text{where} \quad \ell_i := \delta(P_{\mathbf{d}+(n_i-d_i)\mathbf{e}_i}) \tag{2.44}$$

and where by definition of the "coordinate" tuple $\mathbf{e}_i$,

$$\mathbf{d} + (n_i - d_i)\mathbf{e}_i = (d_1, \dots, d_{i-1}, n_i, d_{i+1}, \dots, d_k).$$

The tuple $\mathbf{s} = \mathbf{d} + (n_i - d_i)\mathbf{e}_i$ meets the conditions of Proposition 2.11 by part (2) of which we then conclude that the Schur complement $P_{\mathbf{d}+(n_i-d_i)\mathbf{e}_i}/P_{\mathbf{d}}$ exists. It is not hard to see from (2.31), that this Schur complement equals the i-th diagonal block $\widetilde{P}_{ii}$ of $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$. Thus, $P_{\mathbf{d}+(n_i-d_i)\mathbf{e}_i}/P_{\mathbf{d}} = \widetilde{P}_{ii}$ and therefore,

$$\delta(\widetilde{P}_{ii}) = \ell_i \quad \text{for} \quad i = 1, \dots, k, \tag{2.45}$$

by (2.44) and Proposition 2.9. In particular,

$$\ell_i \leqslant n_i - d_i \quad (i = 1, \dots, k). \tag{2.46}$$

We now break the set $\mathscr{I}_1$ from (2.18) into two parts:

$$\mathscr{I}_1' = \{i \in \mathscr{I}_1 : \ \ell_i < n_i - d_i\} \quad \text{and} \quad \mathscr{I}_1'' = \{i \in \mathscr{I}_1 : \ \ell_i = n_i - d_i\}, \tag{2.47}$$

assigning to $\mathscr{I}_1'$ ($\mathscr{I}_1''$) all indices i for which $\widetilde{P}_{ii}$ is a nonzero matrix (the zero matrix).

PROPOSITION 2.12. *Let $P_{\mathbf{d}}$ be a maximal invertible principal submatrix of $P_{\mathbf{n}} = \mathbb{S}(C_{\mathbf{n}})$, so that the Schur complement $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$ exists (by Proposition* 2.11*) and equals $\mathbb{S}(X_{\mathbf{n}-\mathbf{d}})$ (by Proposition* 2.9*) where $X_{\mathbf{n}-\mathbf{d}}$ is defined in* (2.30)*. Let $\boldsymbol{\ell} \in \mathbb{Z}_+^k$ be the tuple defined in* (2.44)*. Then*

$$\pi(\widetilde{P}) = \nu(\widetilde{P}) = |\mathbf{n} - \mathbf{d} - \ell|/2, \qquad \delta(\widetilde{P}) = \delta(P) = |\boldsymbol{\ell}| \tag{2.48}$$

and

$$r_i(X_{\mathbf{n}-\mathbf{d}}) = \frac{n_i - d_i + \ell_i}{2} \quad (i = 1, \dots, k), \tag{2.49}$$

where the integers $r_i(X_{\mathbf{n}-\mathbf{d}})$ are defined via formula (2.9)*. Consequently,*

$$\nu(P_{\mathbf{n}}) = \nu(P_{\mathbf{d}}) + |\mathbf{n} - \mathbf{d} - \ell|/2 \quad \textit{and} \quad \pi(P_{\mathbf{n}}) = \pi(P_{\mathbf{d}}) + |\mathbf{n} - \mathbf{d} - \ell|/2. \tag{2.50}$$

Proof. By Proposition 2.9, the matrix $\widetilde{P} = \mathbb{S}(X_{\mathbf{n}-\mathbf{d}})$ is $(\mathbf{n} - \mathbf{d})$-singular. Then it follows by Remark 2.4 that Theorem 2.3 applies to $B_{\mathbf{m}} = X_{\mathbf{n}-\mathbf{d}}$ and gives in particular (see the second equality in (2.11)),

$$\delta(\widetilde{P}_{ii}) = 2r_i(X_{\mathbf{n}-\mathbf{d}}) - (n_i - d_i) \quad \text{for} \quad i = 1, \dots, k.$$

Combining the latter formulas with (2.45) gives (2.49). Now equalities (2.48) follow from (2.49) again by Theorem 2.3 applied to $B_{\mathbf{m}} = X_{\mathbf{n}-\mathbf{d}}$. Finally, equalities (2.50) follow from (2.48) and (2.39). □

2.2. Structured extensions of the Pick matrix

We now apply the preceding results to estimate the inertia of some special structured extensions of the Pick matrix $P_{\mathbf{n}}$.

THEOREM 2.13. *Let $P_{\mathbf{d}}$ be a maximal invertible principal submatrix of $P_{\mathbf{n}} = \mathbb{S}(C_{\mathbf{n}})$, let $\boldsymbol{\ell} \in \mathbb{Z}_+^k$ be the tuple defined in* (2.44) *and let $\mathscr{I}_1'$ and $\mathscr{I}_1''$ be defined as in* (2.47)*. Let $C_{\mathbf{n}+\boldsymbol{\ell}}$ of the form* (2.7) *be an extension of $C_{\mathbf{n}}$ and let $P_{\mathbf{n}+\boldsymbol{\ell}} = \mathbb{S}(C_{\mathbf{n}+\boldsymbol{\ell}})$ be the corresponding extension of $P_{\mathbf{n}}$. Then*

1. $\delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}) = \ell_i - 1$ *for every* $i \in \mathscr{I}_1'$.
2. *For every $i \in \mathscr{I}_1''$, the number $\delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i})$ equals either $\ell_i - 1$ or $\ell_i + 1$.*
3. *The following are equivalent:*
   - *(a) $P_{\mathbf{n}+\boldsymbol{\ell}}$ is invertible, i.e., $\delta(P_{\mathbf{n}+\boldsymbol{\ell}}) = 0$.*
   - *(b) $\nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$.*
   - *(c) For every $i \in \mathscr{I}_1''$, the extending number c_{i,n_i} is such that*

$$\delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}) = \ell_i - 1. \tag{2.51}$$

Proof. We will prove the statements of the theorem in the reversed order, thus starting with part (3). By Proposition 2.9, the Schur complement $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$ exists and equalities (2.39) hold. The tuple $\boldsymbol{\ell}$ defined in (2.44) meets the assumptions of Proposition 2.11 and therefore, the Schur complement $P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}$ exists and

$$\nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = \nu(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) + \nu(P_{\mathbf{d}}), \quad \delta(P_{\mathbf{n}+\boldsymbol{\ell}}) = \delta(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}). \tag{2.52}$$

Making subsequent use of the first relation in (2.52) and then of (2.39) we get

$$\begin{aligned} \nu(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) - \nu(\widetilde{P}) - \delta(\widetilde{P}) &= \nu(P_{\mathbf{n}+\boldsymbol{\ell}}) - \nu(P_{\mathbf{d}}) - \nu(\widetilde{P}) - \delta(\widetilde{P}) \\ &= \nu(P_{\mathbf{n}+\boldsymbol{\ell}}) - \nu(P_{\mathbf{n}}) - \delta(P_{\mathbf{n}}). \end{aligned} \tag{2.53}$$

In order to prove the equivalence $(a) \Leftrightarrow (b)$, it suffices to show that

$$(a') \;\; \delta(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) = 0 \quad \Longleftrightarrow \quad (b') \;\; \nu(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) = \nu(\widetilde{P}) + \delta(\widetilde{P}), \tag{2.54}$$

since statements (a) and (b) are equivalent respectively to statements (a') and (b'), by (2.52) and (2.53). Let $X_{\mathbf{n}-\mathbf{d}}$ and $X_{\mathbf{n}+\boldsymbol{\ell}-\mathbf{d}}$ be the vectors corresponding to $\widetilde{P}$ and $P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}$, respectively, so that

$$\widetilde{P} = \mathbb{S}(X_{\mathbf{n}-\mathbf{d}}), \quad P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}+\boldsymbol{\ell}-\mathbf{d}}). \tag{2.55}$$

We know from Proposition 2.11 that $X_{\mathbf{n}+\boldsymbol{\ell}-\mathbf{d}}$ extends $X_{\mathbf{n}-\mathbf{d}}$ and $P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}$ extends $\widetilde{P}$. Equalities (2.40), (2.49) and (2.48) hold by Propositions 2.9 and 2.12. For the extended vector $X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}}$ we have (by definition (2.9)), $r_i(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}}) \geqslant r_i(X_{\mathbf{n}-\mathbf{d}})$ with equality holding if and only if either $i \in \mathscr{I}_0 \cup \mathscr{I}_1'$ or $i \in \mathscr{I}_1''$ and $x_{i,n_i-d_i} \neq 0$. Then it follows from (2.49) that

$$|\mathbf{r}(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}})| \geqslant |\mathbf{r}(X_{\mathbf{n}})| = |\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}|/2 \tag{2.56}$$

with equality holding if and only $x_{i,n_i-d_i} \neq 0$ for every $i \in \mathscr{I}_1''$. Now we get by Theorem 2.3 (applied to $B_{\mathbf{m}} = X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}}$) and in view of (2.55), (2.56) and (2.48) that

$$\begin{aligned} \nu(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) &= |\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}| - |\mathbf{r}(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}})| \\ &\leqslant |\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}|/2 = \nu(\widetilde{P}) + \delta(\widetilde{P}) \end{aligned} \tag{2.57}$$

and

$$\delta(P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}) = 2|\mathbf{r}(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}})| - |\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}| \geqslant 0 \tag{2.58}$$

with equalities holding in (2.57) and (2.58) if and only $x_{i,n_i-d_i} \neq 0$ for every $i \in \mathscr{I}_1''$. This proves (2.54) and therefore, the equivalence $(a) \Leftrightarrow (b)$ in part (3) of the theorem. To complete the proof of part (3), it suffices to verify that condition (2.51) holds if and only if $x_{i,n_i-d_i} \neq 0$. To this end, note that $P_{\mathbf{d}}$ is the maximal invertible submatrix of $P_{\mathbf{d}+(n_i-d_i)\mathbf{e}_i}$ and by Proposition 2.11 (which now applies to the tuple $\mathbf{s} = (n_i-d_i)\mathbf{e}_i$), the Schur complement

$$P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}/P_{\mathbf{d}} = \mathbb{S}(X_{(n_i-d_i+1)\mathbf{e}_i}) \tag{2.59}$$

exists and extends $P_{\mathbf{d}+(n_i-d_i)\mathbf{e}_i}/P_{\mathbf{d}} = \widetilde{P}_{ii} = \mathbb{S}(X_{(n_i-d_i)\mathbf{e}_i})$. Since $i \in \mathscr{I}_1''$, we have $\widetilde{P}_{ii} = 0$ and therefore, the entries of the associated vector $X_{(n_i-d_i)\mathbf{e}_i} = \operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1} x_{i,j}$ are subject to

$$|x_{i,0}| = 1, \quad x_{i,1} = \ldots = x_{i,n_i-d_i-1} = 0. \tag{2.60}$$

The vector $X_{(n_i-d_i+1)\mathbf{e}_i}$ extends $X_{(n_i-d_i)\mathbf{e}_i}$ by the bottom entry x_{i,n_i-d_i} which is either zero or nonzero. By definition (2.9), we have

$$r_i(X_{(n_i-d_i+1)\mathbf{e}_i}) = \begin{cases} n_i-d_i+1, & \text{if } x_{i,n_i-d_i} = 0, \\ n_i-d_i, & \text{if } x_{i,n_i-d_i} \neq 0. \end{cases}$$

Now we apply Theorem 2.3 to the matrix (2.59) and make use of equalities $n_i - d_i = \ell_i$ (by definition (2.47) of $\mathscr{I}_1''$) to conclude that

$$\begin{aligned} \delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}) &= \delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}/P_{\mathbf{d}}) = \delta(\mathbb{S}(X_{(n_i-d_i+1)\mathbf{e}_i})) \\ &= 2r_i(X_{(n_i-d_i+1)\mathbf{e}_i}) - (n_i-d_i+1) \\ &= \begin{cases} \ell_i+1, & \text{if } x_{i,n_i-d_i} = 0, \\ \ell_i-1, & \text{if } x_{i,n_i-d_i} \neq 0, \end{cases} \quad \text{for} \quad i \in \mathscr{I}_1''. \end{aligned} \tag{2.61}$$

Thus, condition (2.51) holds if and only if $x_{i,n_i-d_i} \neq 0$ which completes the proof of statement (3) of the theorem. Statement (2) also follows from (2.61). Finally, if $i \in \mathscr{I}'$ so that $\ell_i < n_1 - d_i$, then the vector $X_{(n_i-d_i)\mathbf{e}_i}$ already contains a nonzero entry (besides $x_{i,0}$) and therefore

$$r_i(X_{(n_i-d_i+1)\mathbf{e}_i} = r_i(X_{(n_i-d_i)\mathbf{e}_i} = r_i(X_{\mathbf{n}-\mathbf{d}}) = (n_i-d_i+\ell_i)/2 \tag{2.62}$$

where the last equality follows from (2.49). Making use of (2.62) we repeat calculation (2.61):

$$\begin{aligned}\delta(P_{\mathbf{d}+(n_i-d_i+1)\mathbf{e}_i}) &= 2r_i(X_{(n_i-d_i+1)\mathbf{e}_i}) - (n_i-d_i+1)\\ &= (n_i-d_i+\ell_i)-(n_i-d_i+1)=\ell_i-1\end{aligned}$$

which proves statement (1) and completes the proof of the theorem. □

We now use the numbers ℓ_j from (2.44) to define the tuples

$$\boldsymbol{\ell}' = \sum_{i\in\mathscr{I}_1'} \ell_i\mathbf{e}_i \quad \text{and} \quad \boldsymbol{\ell}'' = \sum_{i\in\mathscr{I}_1''} \ell_i\mathbf{e}_i, \tag{2.63}$$

that is, the tuples $\boldsymbol{\ell}'=(\ell_1',\dots,\ell_k')$ and $\boldsymbol{\ell}''=(\ell_1'',\dots,\ell_k'')$ where

$$\ell_i' = \begin{cases} \ell_i, \text{ if } i\in\mathscr{I}_1', \\ 0, \text{ otherwise,}\end{cases} \qquad \ell_i'' = \begin{cases} \ell_i, \text{ if } i\in\mathscr{I}_1'', \\ 0, \text{ otherwise.}\end{cases} \tag{2.64}$$

THEOREM 2.14. *Let $P_{\mathbf{d}}$ be a maximal invertible principal submatrix of $P_{\mathbf{n}}$, let the tuples $\boldsymbol{\ell}$, $\boldsymbol{\ell}'$ and $\boldsymbol{\ell}''$ be defined as in* (2.44) *and* (2.64)*. Then*

1. *For every extension $P_{\mathbf{n}+\boldsymbol{\ell}'}$ of $P_{\mathbf{n}}$,*

$$\nu(P_{\mathbf{n}+\boldsymbol{\ell}'}) = \nu(P_{\mathbf{d}}) + |\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'-\boldsymbol{\ell}''|/2, \qquad \delta(P_{\mathbf{n}+\boldsymbol{\ell}'}) = |\boldsymbol{\ell}''|. \tag{2.65}$$

2. *The matrix $P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}$ is an invertible principal submatrix of $P_{\mathbf{n}+\boldsymbol{\ell}'}$ and*

$$P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''} = 0. \tag{2.66}$$

Proof. Define analogously to (2.63),

$$\mathbf{n}' = \sum_{i\in\mathscr{I}_1'} n_i\mathbf{e}_i, \quad \mathbf{n}'' = \sum_{i\in\mathscr{I}_1''} n_i\mathbf{e}_i, \quad \mathbf{d}' = \sum_{i\in\mathscr{I}_1'} d_i\mathbf{e}_i, \quad \mathbf{d}'' = \sum_{i\in\mathscr{I}_1''} d_i\mathbf{e}_i,$$

and observe that by definitions (2.47) and (2.64), $\boldsymbol{\ell}''=\mathbf{n}''-\mathbf{d}''$ so that

$$\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}'' = \mathbf{d}+\mathbf{n}'-\mathbf{d}'+\boldsymbol{\ell}'.$$

By Proposition 2.11, the Schur complements

$$P_{\mathbf{n}}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}-\mathbf{d}}), \quad P_{\mathbf{d}+\mathbf{n}'-\mathbf{d}'}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}'-\mathbf{d}'}), \quad P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'})$$

and

$$P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}/P_{\mathbf{d}} = P_{\mathbf{d}+\mathbf{n}'-\mathbf{d}'+\boldsymbol{\ell}'}/P_{\mathbf{d}} = \mathbb{S}(X_{\mathbf{n}'-\mathbf{d}'+\boldsymbol{\ell}'})$$

exist and relations (2.40) hold (by the proof of Theorem 2.13). By formula (2.49), $r_i(X_{\mathbf{n}-\mathbf{d}}) < n_i-d_i$ for every $i\in\mathscr{I}_1'$. Since $X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'}$ is obtained from $X_{\mathbf{n}-\mathbf{d}}$ by extending the blocks $\operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1} x_{i,j}$ which already contain nonzero entries (besides $x_{i,0}$), the positions of these nonzero entries in the extended vector $X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'}$ (counted from the top) remain the same. Therefore, $\mathbf{r}(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'}) = \mathbf{r}(X_{\mathbf{n}-\mathbf{d}})$. By formulas (2.49),

$$r_i(X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'}) = \frac{n_i - d_i + \ell_i}{2} \quad \text{for} \quad i \in \mathscr{I}_1,$$

and in view of (2.40), Theorem 2.3 applies to $B_{\mathbf{m}} = X_{\mathbf{n}-\mathbf{d}+\boldsymbol{\ell}'}$ producing equalities

$$\nu(P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{d}}) = |\mathbf{n}-\mathbf{d}+\ell'| - |\mathbf{n}-\mathbf{d}+\ell|/2 = |\mathbf{n}-\mathbf{d}+\ell' - \boldsymbol{\ell}''|/2,$$

which in turn, imply (2.65) since

$$\nu(P_{\mathbf{n}+\boldsymbol{\ell}'}) = \nu(P_{\mathbf{d}}) + \nu(P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{d}}) \quad \text{and} \quad \delta(P_{\mathbf{n}+\boldsymbol{\ell}'}) = |\mathbf{n}+\boldsymbol{\ell}'| - \pi(P_{\mathbf{n}+\boldsymbol{\ell}'}) - \nu(P_{\mathbf{n}+\boldsymbol{\ell}'}).$$

The same arguments apply to the extension $X_{\mathbf{n}'-\mathbf{d}'+\boldsymbol{\ell}'}$ of $X_{\mathbf{n}'-\mathbf{d}'}$:

$$r_i(X_{\mathbf{n}'-\mathbf{d}'+\boldsymbol{\ell}'}) = r_i(X_{\mathbf{n}'-\mathbf{d}'}) = \frac{n_i - d_i + \ell_i}{2} \quad \text{for} \quad i \in \mathscr{I}_1',$$

and by Theorem 2.3, $\delta(P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}/P_{\mathbf{d}}) = 0$. Therefore, $\delta(P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}) = 0$ so that $P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}$ is invertible. The Schur complement $P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}$ exists by Proposition 2.11 and we conclude from the second relation in (2.65) that

$$\delta(P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}) = \delta(P_{\mathbf{n}+\boldsymbol{\ell}'}) = |\boldsymbol{\ell}''|.$$

Since $P_{\mathbf{n}+\boldsymbol{\ell}'}/P_{\mathbf{n}+\boldsymbol{\ell}'-\boldsymbol{\ell}''}$ is an $|\boldsymbol{\ell}''| \times |\boldsymbol{\ell}''|$ matrix, the latter equality implies (2.66). □

REMARK 2.15. $|\boldsymbol{\ell}'| = 0$ *(i.e.,* $\boldsymbol{\ell}'$ *is the zero tuple) if and only if* $\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}})$*.*

Proof. Equality $\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}})$ is equivalent to the Schur complement $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$ be the $|\mathbf{n}-\mathbf{d}| \times |\mathbf{n}-\mathbf{d}|$ zero matrix, so that $\delta(\widetilde{P}) = |\mathbf{n}-\mathbf{d}|$. Comparing this equality with the second equality in (2.48) gives $|\boldsymbol{\ell}| = |\mathbf{n}-\mathbf{d}|$ which in view of (2.46), is equivalent to $\ell_i = n_i - d_i$ for every $i \in \mathscr{I}_1$. The latter means that the set $\mathscr{I}_1'$ defined in (2.47) is empty , so that $\boldsymbol{\ell}'$ is the zero tuple, by definition (2.64). □

REMARK 2.16. It is obvious that in case $n_i = 1$ for $i = 1,\ldots,k$ (i.e., when the problem $\mathbf{IP}_\kappa$ amonts to the Nevanlinn-Pick problem), $\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}})$ for every maximal invertible submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$. It is somewhat less obvious that the same holds in case $n_i \leqslant 1$ for $i = 1,\ldots,k$. Indeed, for each $i \in \mathscr{I}_1'$ we have $d_i < n_1$ and $\ell_i < n_i - d_i$. Since $n_i \leqslant 2$, then we have $d_i = \ell_i = 0$. Then $\boldsymbol{\ell}'$ is the zero tuple and $\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}})$ by Remark 2.15.

3. Reduction to an $\mathbf{n}$-singular case

In case the Pick matrix $P_{\mathbf{n}}$ is invertible and $\kappa \geqslant \nu(P_{\mathbf{n}})$, all solutions of the problem $\mathbf{IP}_{\kappa}$ can be parametrized by a linear fractional formula. If $P_{\mathbf{n}}$ is singular and not $\mathbf{n}$-singular, we can pick an invertible principal submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$ (for some tuple $\mathbf{d} = (d_1,\dots,d_k) \preceq \mathbf{n}$) which is the Pick matrix of the nondegenerate subproblem

$$f^{(j)}(z_i) = j!\,c_{i,j} \qquad (i=1,\dots,k;\ j=0,\dots,d_i-1). \tag{3.1}$$

It is convenient to let d_i to be nonnegative integers with the understanding that if $d_i = 0$, there are no interpolation conditions in (3.1) at the corresponding point z_i. With the subproblem (3.1), we associate the 2×2 matrix valued function

$$\begin{aligned}\Theta(z) &= \begin{bmatrix}\Theta_{11}(z) & \Theta_{12}(z)\\ \Theta_{21}(z) & \Theta_{22}(z)\end{bmatrix} \\ &= I_2 + (z-1)\begin{bmatrix}E_{\mathbf{d}}^*\\ C_{\mathbf{d}}^*\end{bmatrix}(I - zT_{\mathbf{d}}^*)^{-1}P_{\mathbf{d}}^{-1}(I-T_{\mathbf{d}})^{-1}\begin{bmatrix}E_{\mathbf{d}} & -C_{\mathbf{d}}\end{bmatrix}\end{aligned} \tag{3.2}$$

where $T_{\mathbf{d}}$, $E_{\mathbf{d}}$ and $C_{\mathbf{d}}$ are defined via formulas (2.1) and (2.3). The following result can be found in [8] (see also [3]–[5] for the case $\kappa = \nu(P_{\mathbf{n}})$).

THEOREM 3.1. *If $P_{\mathbf{d}}$ is invertible and $\kappa \geqslant \nu(P_{\mathbf{d}})$, then all functions $f \in \mathcal{S}_\kappa$ satisfying interpolation conditions* (3.1) *are parametrized by the formula*

$$f = \frac{\Theta_{11}S + \Theta_{12}B}{\Theta_{21}S + \Theta_{22}B}, \tag{3.3}$$

where the parameters $S \in \mathcal{S}$ and $B \in \mathcal{B}_{\kappa-\nu(P_{\mathbf{d}})}$ do not have common zeros and satisfy conditions

$$\Theta_{21}(z_i)S(z_i) + \Theta_{22}(z_i)B(z_i) \neq 0 \tag{3.4}$$

for every $i \in \{1,\dots,k\}$ such that $d_i > 0$. Furthermore, the correspondence $f \mapsto \frac{S}{B}$ is one-to-one and f is unimodular on $\mathbb{T}$ if and only if S is.

Also we will need some results on zero cancellation in linear fractional formula (3.3). Let $U_{S,B}$ and $V_{S,B}$ denote respectively the numerator and the denominator on the right hand side of (3.3):

$$U_{S,B} = \Theta_{11}S + \Theta_{12}B, \quad V_{S,B}(z) = \Theta_{21}S + \Theta_{22}B \tag{3.5}$$

and let $N\{g\}$ stand for the total number of zeroes of a function g that fall inside $\mathbb{D}$.

THEOREM 3.2. *Let $P_{\mathbf{d}}$ be invertible, let $S \in \mathcal{S}$, $B \in \mathcal{B}_\kappa$, let Θ, $U_{S,B}$ and $V_{S,B}$ be given as in* (3.2) *and* (3.5)*, and let $\mathcal{Z} = \{z_i :\ d_i > 0\}$. Then*

1. *$N\{V_{S,B}\} = \nu(P_{\mathbf{d}}) + \kappa$. If, in addition, S is a finite Blaschke product of degree m (i.e., if $S \in \mathcal{B}_m$), then $N\{U_{S,B}\} = \pi(P_{\mathbf{d}}) + m$.*

2. *$U_{S,B}$ and $V_{S,B}$ do not have common zeros in $\mathbb{D}\backslash\mathcal{Z}$.*

3. *If $V_{S,B}$ has zero of multiplicity $m_i < d_i$ at $z_i \in \mathscr{Z}$, then $U_{S,B}$ has zero of multiplicity at least m_i at z_i. In this case the function f of the form* (3.3) *still satisfies interpolation conditions*

$$f^{(j)}(z_i) = j!\, c_{i,j} \quad \text{for} \quad j = 0, \dots, d_i - m_i - 1.$$

4. *If $V_{S,B}$ has the zero of multiplicity $m_i \geqslant d_i$ at $z_i \in \mathscr{Z}$, then $U_{S,B}$ has the zero of multiplicity d_i at z_i.*

REMARK 3.3. Parametrization (3.3) from Theorem 3.1 can be reformulated in terms of the ratio

$$\mathscr{E} = \frac{S}{B} \in \mathscr{S}_{\kappa - \nu(P_{\mathbf{d}})} \tag{3.6}$$

as follows: *f must be of the form*

$$f = \mathbf{T}_{\Theta}[\mathscr{E}] := \frac{\Theta_{11}\mathscr{E} + \Theta_{12}}{\Theta_{21}\mathscr{E} + \Theta_{22}}, \tag{3.7}$$

where the parameter $\mathscr{E} \in \mathscr{S}_{\kappa - \nu(P)}$ either satisfies condition

$$\Theta_{21}(z_i)\mathscr{E}(z_i) + \Theta_{22}(z_i) \neq 0 \tag{3.8}$$

or has a pole at z_i in case $\Theta_{21}(z_i) \neq \Theta_{22}(z_i) = 0$. The possibility of $\mathscr{E}$ to have a pole at an interpolation node (see [9] for an example) makes the projective description (3.3) more convenient.

In order for f of the form (3.3) to satisfy the remaining $|\mathbf{n} - \mathbf{d}|$ conditions

$$f^{(j)}(z_i) = j!\, c_{i,j} \qquad (i = 1, \dots, k;\ j = d_i, \dots, n_i - 1) \tag{3.9}$$

from (1.4) (with understanding that if $d_i = n_i$, then there are no conditions at z_i in (3.9)), the parameter $\mathscr{E}$ must satisfy certain conditions which are specified in Theorem 3.5 below. For the proof, it is convenient to write interpolation conditions (1.4) in the residue form [18], [4].

REMARK 3.4. For $T_{\mathbf{n}}$ and $E_{\mathbf{n}}$ of the form (2.1), it follows by residue calculus that for every function f analytic at $z_1, \dots, z_k$,

$$\sum_{i=1}^{k} \operatorname{Res}_{\zeta = z_i} (\zeta I - T_{\mathbf{n}})^{-1} E_{\mathbf{n}} f(\zeta) = \operatorname{Col}_{1 \leqslant i \leqslant k} \operatorname{Col}_{0 \leqslant j \leqslant n_i - 1} \frac{f^{(j)}(z_i)}{j!}.$$

Thus, the problem $\mathbf{IP}_{\kappa}$ can be formulated in the residue form as follows: *Find all functions $f \in \mathscr{S}_{\kappa}$ analytic at $z_1, \dots, z_k$ and such that*

$$\sum_{i=1}^{k} \operatorname{Res}_{\zeta = z_i} (\zeta I - T_{\mathbf{n}})^{-1} E_{\mathbf{n}} f(\zeta) = C_{\mathbf{n}}.$$

THEOREM 3.5. *Let $P_{\mathbf{d}}$ be a maximal (in the sense of* (2.37)*) invertible principal submatrix of $P_{\mathbf{n}}$ and let the index set $\mathscr{I}_1$ be defined as in* (2.18)*. A function f of the form* (3.3) *is a solution of the problem $\mathbf{IP}_\kappa$ if and only if parameters $S\in\mathscr{S}$ and $B\in\mathscr{B}_{\kappa-\nu(P_{\mathbf{d}})}$ meet conditions* (3.4) *for every $i=1,\dots,k$ and the ratio $\mathscr{E}=S/B$ satisfies interpolation conditions*

$$\mathscr{E}^{(j)}(z_i)=j!x_{i,j}\qquad (i\in\mathscr{I}_1;\ j=0,\dots,n_i-d_i-1) \tag{3.10}$$

where the vector $X=\operatorname{Col}_{1\leqslant i\leqslant k}\operatorname{Col}_{0\leqslant j\leqslant n_i-d_i-1}x_{i,j}\in\mathbb{C}^{|\mathbf{n}-\mathbf{d}|}$ is defined in (2.30)*.*

Proof. Let us assume that f of the form (3.7) (or equivalently, of the form (3.3)) is a solution of the problem $\mathbf{IP}_\kappa$. Then conditions (3.4) are met for every $i=1,\dots,k$. Indeed, if $d_i>0$, this follows from Theorem 3.1. If $d_i=0$, then $z_i\notin\operatorname{spec}(T_{\mathbf{d}})$ and since

$$\det\Theta(z)=\prod_{z_j\in\operatorname{spec}(T_{\mathbf{d}})}\left(\frac{(z-z_j)(1-\bar{z}_j)}{(1-z\bar{z}_j)(1-z_j)}\right)^{n_i}$$

(see e.g., [6] for the proof), it follows that

$$\det\Theta(z_i)\neq 0\quad\text{if}\quad d_i=0. \tag{3.11}$$

Assuming that (3.4) fails, i.e., that the denominator in (3.3) vanishes at z_i we conclude that the numerator vanishes at z_i as well, since f is analytic at z_i. But then $\Theta(z_i)\begin{bmatrix}S(z_i)\\ B(z_i)\end{bmatrix}=0$ and since S and B do not share common zeroes, it follows that $\Theta(z_i)$ is singular which contradicts (3.11).

By Remark 3.4, interpolation conditions (1.4) can be written as

$$\sum_{i=1}^k \operatorname{Res}_{\zeta=z_i}(\zeta I-T_{\mathbf{n}})^{-1}(E_{\mathbf{n}}f(\zeta)-C_{\mathbf{n}})=0, \tag{3.12}$$

where $T_{\mathbf{n}}$, $E_{\mathbf{n}}$ and $C_{\mathbf{n}}$ are given by (2.1) and (2.3). Since conditions (3.4) are met, equality (3.12) is equivalent to

$$\sum_{i=1}^k \operatorname{Res}_{\zeta=z_i}(\zeta I-T_{\mathbf{n}})^{-1}(E_{\mathbf{n}}f(\zeta)-C_{\mathbf{n}})(\Theta_{21}(\zeta)S(\zeta)+\Theta_{22}(\zeta)B(\zeta))=0. \tag{3.13}$$

It is readily checked that for an f of the form (3.3),

$$(E_{\mathbf{n}}f-C_{\mathbf{n}})(\Theta_{21}S+\Theta_{22}B)=\begin{bmatrix}E_{\mathbf{n}} & -C_{\mathbf{n}}\end{bmatrix}\Theta\begin{bmatrix}S\\ B\end{bmatrix} \tag{3.14}$$

which allows us to rewrite (3.13) as

$$\sum_{i=1}^k \operatorname{Res}_{\zeta=z_i}(\zeta I-T_{\mathbf{n}})^{-1}\begin{bmatrix}E_{\mathbf{n}} & -C_{\mathbf{n}}\end{bmatrix}\Theta(\zeta)\begin{bmatrix}S(\zeta)\\ B(\zeta)\end{bmatrix}=0. \tag{3.15}$$

Now we observe the identity

$$U(\zeta I - T_{\mathbf{n}})^{-1}\begin{bmatrix} E_{\mathbf{n}} & -C_{\mathbf{n}} \end{bmatrix}\Theta(\zeta) = \begin{bmatrix} 0 \\ (\zeta I - T_{\mathbf{n}-\mathbf{d}})^{-1}\begin{bmatrix} G_{\mathbf{n}-\mathbf{d}} & -Y_{\mathbf{n}-\mathbf{d}} \end{bmatrix} \end{bmatrix} + \phi(z), \tag{3.16}$$

holding for every $\zeta \in \mathbb{C}\backslash\{z_1,\dots,z_k\}$, where U is the permutation matrix constructed in (2.12), $G_{\mathbf{n}-\mathbf{d}}$ and $Y_{\mathbf{n}-\mathbf{d}}$ are the vectors given by (2.21), $T_{\mathbf{n}-\mathbf{d}}$ is defined as in (2.1) and where

$$\phi(z) = \begin{bmatrix} P_{\mathbf{d}} \\ \Psi \end{bmatrix}(I - zT_{\mathbf{d}}^*)^{-1}(I - T_{\mathbf{d}}^*)P_{\mathbf{d}}^{-1}(I - T_{\mathbf{d}})^{-1}\begin{bmatrix} E_{\mathbf{d}} & -C_{\mathbf{d}} \end{bmatrix}. \tag{3.17}$$

To verify (3.16), it suffices to plug in the formula (3.3) for Θ, decompositions (2.13), (2.14) and

$$U(zI - T_{\mathbf{n}})^{-1}U^* = \begin{bmatrix} (zI - T_{\mathbf{d}})^{-1} & 0 \\ (zI - T_{\mathbf{n}-\mathbf{d}})^{-1}R(zI - T_{\mathbf{d}})^{-1} & (zI - T_{\mathbf{n}-\mathbf{d}})^{-1} \end{bmatrix}$$

into the left hand side part of (3.16) and then to invoke equalities (2.15). Note that ϕ defined in (3.17) is a rational matrix function analytic on $\mathbb{D}$ and therefore

$$\sum_{i=1}^{k} \operatorname{Res}_{\zeta=z_i}\phi(z)\begin{bmatrix} S(\zeta) \\ B(\zeta) \end{bmatrix} = 0. \tag{3.18}$$

Multiplying both parts in (3.16) by $\begin{bmatrix} S(\zeta) \\ B(\zeta) \end{bmatrix}$ on the right and taking the residues we obtain, on account of (3.15) and (3.18),

$$\sum_{z_i \in \operatorname{spec}(T_{\mathbf{n}-\mathbf{d}})} \operatorname{Res}_{\zeta=z_i}(\zeta I - T_{\mathbf{n}-\mathbf{d}})^{-1}(G_{\mathbf{n}-\mathbf{d}}S(\zeta) - Y_{\mathbf{n}-\mathbf{d}}B(\zeta)) = 0. \tag{3.19}$$

The latter equality implies in particular, that

$$B(z_i) \neq 0 \quad \text{for every} \quad z_i \in \operatorname{spec}(T_{\mathbf{n}-\mathbf{d}}). \tag{3.20}$$

Indeed, by the Jordan structure of $T_{\mathbf{n}-\mathbf{d}}$, it follows that

$$\operatorname{Res}_{\zeta=z_i}(\zeta - z_i)^{-1}(g_{i,0}S(\zeta) - y_{i,0}B(\zeta)) = 0$$

for every $z_i \in \operatorname{spec}(T_{\mathbf{n}-\mathbf{d}})$ which is equivalent to $g_{i,0}S(z_i) = y_{i,0}B(z_i)$. Assuming that $B(z_i) = 0$ we get from the latter equality and (2.37) that $S(z_i) = 0$ which is impossible since S and B have no common zeros. On account of (3.20), equality (3.19) is equivalent to

$$\sum_{z_i \in \operatorname{spec}(T_{\mathbf{n}-\mathbf{d}})} \operatorname{Res}_{\zeta=z_i}(\zeta I - T_{\mathbf{n}-\mathbf{d}})^{-1}(G_{\mathbf{n}-\mathbf{d}}\mathscr{E}(\zeta) - Y_{\mathbf{n}-\mathbf{d}}) = 0$$

where $\mathscr{E} = S/B$. Multiplying the last equality by the matrix $\mathbf{G}_{\mathbf{n}-\mathbf{d}}^{-1}$ on the left and making use of (2.29), (2.30), we arrive at

$$\sum_{z_i \in \operatorname{spec}(T_{\mathbf{n}-\mathbf{d}})} \operatorname{Res}_{\zeta=z_i}(\zeta I - T_{\mathbf{n}-\mathbf{d}})^{-1}(E_{\mathbf{n}-\mathbf{d}}\mathscr{E}(\zeta) - X_{\mathbf{n}-\mathbf{d}}) = 0 \tag{3.21}$$

which is equivalent to (3.10).

To prove the converse statement, we reverse the preceding arguments. Let us assume that $\mathscr{E}$ satisfies interpolation conditions (3.10) which can be written in the residue form (3.21). Multiplying both parts in (3.21) by the matrix $\mathbf{G_{n-d}}$ and making use of (2.29) and (2.30) we get (3.19). Equality (3.15) now follows from (3.16), (3.18) and (3.19). If f is defined by (3.7) (or equivalently, by (3.3)) we use formula (3.14) to rewrite (3.15) in the form (3.13). Since conditions (3.4) are satisfied for every $i=1,\dots,k$, condition (3.13) is equivalent to (3.12) which means that f is a solution of the problem $\mathbf{IP}_\kappa$. □

COROLLARY 3.6. *Let $P_{\mathbf{d}}$ be an invertible principal submatrix of $P_{\mathbf{n}}$ such that*

$$\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}}). \tag{3.22}$$

A function f of the form (3.3) *is a solution of the problem $\mathbf{IP}_\kappa$ if and only if parameters $S\in\mathscr{S}$ and $B\in\mathscr{B}_{\kappa-\nu(P_{\mathbf{d}})}$ meet conditions* (3.4) *for every $i=1,\dots,k$ and satisfy interpolation conditions*

$$S^{(j)}(z_i)=\gamma B^{(j)}(z_i) \quad \textit{for all} \quad i\in\mathscr{I}_1 \ \textit{and} \ j=0,\dots,n_i-d_i-1, \tag{3.23}$$

where γ is the unimodular number defined in (2.41).

Proof. Conditions (3.23) can be written in terms of the function $\mathscr{E}=S/B$ as follows:

$$\mathscr{E}(z_i)=\gamma \quad \text{and} \quad \mathscr{E}^{(j)}(z_i)=0 \quad \text{for } i\in\mathscr{I}_1 \text{ and } j=1,\dots,n_i-d_i-1. \tag{3.24}$$

Observe that condition (3.22) guarantees that $P_{\mathbf{d}}$ is a maximal invertible principal submatrix of P (in the sense of (2.37)). Thus, the statement will follow from Theorem 3.5 once we will show that $x_{i,0}=\gamma$ and $x_{i,j}=0$ for every $i\in\mathscr{I}_1$ and $j=1,\dots,n_i-d_i-1$ or equivalently, that $X_{\mathbf{n-d}}=\gamma E_{\mathbf{n-d}}$. To this end, we observe that by assumption (3.22), the Schur complement $\widetilde{P}=P_{\mathbf{n}}/P_{\mathbf{d}}$ equals the zero matrix. Then it follows from (2.38) that $E_{\mathbf{n-d}}E_{\mathbf{n-d}}^*=X_{\mathbf{n-d}}X_{\mathbf{n-d}}^*$ and therefore, $X_{\mathbf{n-d}}=\gamma E_{\mathbf{n-d}}$ for some unimodular number γ. Comparing the nonzero entries in the latter equality we conclude that γ equals $x_{i,0}$ (for every $i\in\mathscr{I}_1$) and therefore, it is defined by formula (2.41). □

4. Proof of Theorem 1.3

The matrix $\widetilde{P}$ defined in (2.31) is the Pick matrix of the interpolation problem (3.10). Indeed, the Pick matrix of this problem can be defined as a unique solution of the Stein equation (2.38) and the matrix (2.31) satisfies this equation by Proposition 2.9. Let us say that a function $\mathscr{E}$ is a solution of the associated (with the problem $\mathbf{IP}_\kappa$ and a particular choice of the maximal invertible submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$) problem $\mathbf{AP}_\kappa$ if $\mathscr{E}\in\mathscr{S}_{\kappa-\nu(P_{\mathbf{d}})}$,

(1) $\mathscr{E}^{(j)}(z_i) = j!x_{i,j}$ for $i \in \mathscr{I}_1$ and $j = 0, \ldots, n_i - d_i - 1$. (4.1)

(2) Condition (3.8) holds for every $i \in \mathscr{I}_1$.

(3) For every $i \in \mathscr{I}_0$, either condition (3.8) holds or $\mathscr{E}$ has a pole at z_i in case $\Theta_{21}(z_i) \neq \Theta_{22}(z_i) = 0$.

By Theorem 3.5, the problem $\mathbf{IP}_\kappa$ has a solution if and only if the associated problem $\mathbf{AP}_\kappa$ does. In this section we will explore this result to prove Theorem 1.3. As was mentioned in introduction, the problem $\mathbf{IP}_\kappa$ has no solutions if the Pick matrix $P_{\mathbf{n}}$ has more than κ negative eigenvalues. The following proposition completes the proof of part (3) in Theorem 1.3.

PROPOSITION 4.1. *If $\nu(P_{\mathbf{n}}) < \kappa < \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$, then the problem $\mathbf{IP}_\kappa$ has no solutions.*

Proof. Let us assume that the problem $\mathbf{IP}_\kappa$ has a solution f which is necessarily of the form (3.7) for some solution $\mathscr{E} \in \mathscr{S}_{\kappa-\nu(P_{\mathbf{d}})}$ of the problem $\mathbf{AP}_\kappa$. Since $\kappa - \nu(P_{\mathbf{d}}) > 0$, $\mathscr{E}$ is not a constant function. By (2.40) and definition (2.9) of integers $r_i(X_{\mathbf{n}-\mathbf{d}})$, the function $\mathscr{E}$ satisfies interpolation conditions

$$\mathscr{E}(z_i) = \gamma \quad \text{and} \quad \mathscr{E}^{(j)}(z_i) = 0 \quad \text{for} \quad i \in \mathscr{I}_1 \quad \text{and} \quad j = 1, \ldots, r_i(X_{\mathbf{n}-\mathbf{d}}) - 1, \tag{4.2}$$

which can be equivalently written in terms of the Krein-Langer representation (3.6) for $\mathscr{E}$ as

$$S^{(j)}(z_i) = \gamma B^{(j)}(z_i) \quad \text{for} \quad i \in \mathscr{I}_1 \quad \text{and} \quad j = 0, \ldots, r_i(X_{\mathbf{n}-\mathbf{d}}) - 1. \tag{4.3}$$

Thus, the Schur class function $\bar{\gamma}S$ (recall that $|\gamma| = 1$) coincides with $B \in \mathscr{B}_{\kappa-\nu(P_{\mathbf{d}})}$ (which is not constant since $\kappa - \nu(P_{\mathbf{d}}) > 0$) at $|\mathbf{r}(X_{\mathbf{n}-\mathbf{d}})|$ points counted with multiplicities. Since by (2.48), (2.49), (2.39) and the assumption that $\kappa < \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$,

$$\begin{aligned} |\mathbf{r}(X_{\mathbf{n}-\mathbf{d}})| &= |\mathbf{n} - \mathbf{d} + \ell|/2 \\ &= \nu(\widetilde{P}) + \delta(P_{\mathbf{n}}) = \nu(P_{\mathbf{n}}) - \nu(P_{\mathbf{d}}) + \delta(P_{\mathbf{n}}) > \kappa - \nu(P_{\mathbf{d}}) = \deg B, \end{aligned}$$

it follows that $\bar{\gamma}S$ is equal to B identically. Therefore $\mathscr{E} = S/B$ is a constant which is a contradiction. □

The next proposition supplements Theorem 1.3 showing that the number $\pi(P_{\mathbf{n}})$ controls the degree of numerators of unimodular rational solutions to the problem $\mathbf{IP}_\kappa$ (for the Nevanlinna-Pick problem this result was established in [20]).

PROPOSITION 4.2. *The problem $\mathbf{IP}_\kappa$ has no solutions in $\mathscr{B}_m/\mathscr{B}_\kappa$ for $\pi(P_{\mathbf{n}}) < m < \pi(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$.*

Proof. Let us assume that the problem $\mathbf{IP}_\kappa$ has a solution $f \in \mathscr{B}_m/\mathscr{B}_\kappa$ which is necessarily of the form (3.7) for some solution $\mathscr{E} \in \mathscr{S}_{\kappa-\nu(P_{\mathbf{d}})}$ of the problem $\mathbf{AP}_\kappa$.

Since f is rational and unimodular on $\mathbb{T}$, the function $\mathscr{E}$ has the same properties and therefore it is a ratio of two finite Blaschke products, i.e., the numerator S in (3.6) is a finite Blaschke product. As in the proof of Proposition 4.2, we conclude that S and B satisfy conditions (4.3). Now we take f in the form (3.3) (which is equivalent to (3.7)) and note that the denominator in this representation does not vanish at $z_1,\dots,z_k$ by (3.4) (which is equivalent to constraints (2) and (3) in the problem $\mathbf{AP}_\kappa$). Since the numerator and the denominator in this representation do not have common zeros outside $\{z_1,\dots,z_k\}$ (by part (2) of Theorem 3.2), they do not have common zeros at all. By part (1) in Theorem 3.2, the numerator in (3.3) has $\deg S+\pi(P_{\mathbf{d}})$ zeroes. Since there is no zero cancellation in (3.3) and since $f\in\mathscr{B}_m/\mathscr{B}_\kappa$, we have $m=\deg S+\pi(P_{\mathbf{d}})$. By (4.3), the Schur-class function γB coincides with $S\in\mathscr{B}_m$ at $|\mathbf{r}(X_{\mathbf{n}-\mathbf{d}})|$ points counted with multiplicities. Since by (2.48), (2.49), (2.39) and the assumption that $m<\pi(P_{\mathbf{n}})+\delta(P_{\mathbf{n}})$,

$$\begin{aligned}|\mathbf{r}(X_{\mathbf{n}-\mathbf{d}})| &= |\mathbf{n}-\mathbf{d}+\ell|/2\\ &= \pi(\widetilde{P})+\delta(P_{\mathbf{n}})=\pi(P_{\mathbf{n}})-\pi(P_{\mathbf{d}})+\delta(P_{\mathbf{n}})>m-\nu(P_{\mathbf{d}})=\deg S,\end{aligned}$$

it follows that S is equal to γB identically. Therefore $\mathscr{E}=S/B$ is a constant and therefore S is constant which is a contradiction since $\deg S=m-\pi(P_{\mathbf{n}})>0$. □

In order to proceed, we need the following result concerning Schwarz-Pick matrices (see formula (1.2) for definition) which perhaps is known. For the reader's convenience, we include the proof in Section 6.

THEOREM 4.3. *Let $f\in\mathscr{B}_m/\mathscr{B}_\kappa$ be a ratio of two finite Blaschke products, let $z_1,\dots,z_k$ be distinct points in $\mathbb{D}\cap\rho(f)$, and let $\mathbf{n}=(n_1,\dots,n_k)\in\mathbb{N}^k$. If $|\mathbf{n}|=\kappa+m$, then the Schwarz-Pick matrix $P^f_{\mathbf{n}}(\mathbf{z})$ is invertible. Moreover, if $|\mathbf{n}|\geqslant\kappa+m$, then*

$$\nu(P^f_{\mathbf{n}}(\mathbf{z}))=\kappa \quad \textit{and} \quad \pi(P^f_{\mathbf{n}}(\mathbf{z}))=m.$$

COROLLARY 4.4. *If f belongs to $\mathscr{B}_m/\mathscr{B}_\kappa$, then the Schwarz-Pick matrix $P^f_{\mathbf{n}}(\mathbf{z})$ is $\mathbf{n}$-saturated for every $\mathbf{n}\in\mathbb{N}^k$ such that $|\mathbf{n}|>\kappa+m$ and any points $z_1,\dots,z_n\in\mathbb{D}\cap\rho(f)$.*

Proof. By Theorem 4.3, $r:=\operatorname{rank}(P^f_{\mathbf{n}}(\mathbf{z}))=\kappa+m$. On the other hand, any principal submatrix $[P^f_{\mathbf{n}}(\mathbf{z})]_{\mathbf{m}}=P^f_{\mathbf{m}}(\mathbf{z})$ of $P^f_{\mathbf{n}}(\mathbf{z})$ (with $|\mathbf{m}|=r$) is itself a Schwarz-Pick matrix for f and then again by Theorem 4.3, $\operatorname{rank}(P^f_{\mathbf{m}}(\mathbf{z}))=r$, so that $P^f_{\mathbf{m}}(\mathbf{z})$ is invertible. Therefore, $P^f_{\mathbf{n}}(\mathbf{z})$ is $\mathbf{n}$-saturated by Definition 1.2. □

PROPOSITION 4.5. *If $\kappa=\nu(P_{\mathbf{n}})$, and $P_{\mathbf{n}}$ is singular, then the problem $\mathbf{IP}_\kappa$ has at most one solution.*

Proof. We fix the maximal invertiple principal submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$ and let $\mathscr{I}_1'$ to be the set defined in (2.47). We first show that if $\kappa=\nu(P_{\mathbf{n}})$ and there exists an $f\in\mathscr{S}_\kappa$ satisfying conditions (1.4), then the set $\mathscr{I}_1'$ is empty. To this end, observe that the

Schwarz-Pick matrix $P_{\mathbf{n}+\boldsymbol{\ell}'} := P^f_{\mathbf{n}+\boldsymbol{\ell}'}(\mathbf{z})$ has at most κ negative eigenvalues (being an extension of $P_{\mathbf{n}}$, it has in fact exactly κ negative eigenvalues). On the other hand, it follows from the first equalities in (2.65) and (2.50) that

$$\begin{aligned}\nu(P_{\mathbf{n}+\boldsymbol{\ell}'}) - \kappa &= \nu(P_{\mathbf{d}}) - \nu(P_{\mathbf{n}}) + |\mathbf{n}-\mathbf{d}+\ell'-\boldsymbol{\ell}''|/2 \\ &= |\mathbf{n}-\mathbf{d}+\ell'-\boldsymbol{\ell}''|/2 - |\mathbf{n}-\mathbf{d}-\ell|/2 = |\boldsymbol{\ell}'|,\end{aligned}$$

where the last equality holds since $\boldsymbol{\ell} = \boldsymbol{\ell}' + \boldsymbol{\ell}''$. Thus $\nu(P^f_{\mathbf{n}+\boldsymbol{\ell}'}(\mathbf{z})) > \kappa$ (which is a contradiction) unless $|\boldsymbol{\ell}'| = 0$. In this case $\operatorname{rank}(P_{\mathbf{d}}) = \operatorname{rank}(P_{\mathbf{n}})$, by Remark 2.16, and in particular, $\nu(P_{\mathbf{d}}) = \nu(P_{\mathbf{n}}) = \kappa$. Then by Corollary 3.6, every solution f of the problem $\mathbf{IP}_\kappa$ is necessarily of the form (3.7) with a function $\mathscr{E} \in \mathscr{S}_{\kappa-\nu(P_{\mathbf{d}})} = \mathscr{S}$ satisfying conditions (3.24). Since $|\gamma| = 1$, it follows by the maximum modulus principle that the only $\mathscr{E} \in \mathscr{S}$ satisfying (3.24) is the constant function $\mathscr{E} \equiv \gamma$. Substituting this $\mathscr{E}$ into (3.24) leads us to the function

$$f^0(z) = \frac{\Theta_{11}(z)\gamma + \Theta_{12}(z)}{\Theta_{21}(z)\gamma + \Theta_{22}(z)} \tag{4.4}$$

which is the only possible solution of the problem $\mathbf{IP}_\kappa$, by Theorem 3.5. □

REMARK 4.6. *f^0 of the form* (4.4) *is a ratio of two finite Blaschke products.*

Indeed, since $|\gamma| = 1$ and Θ is rational, it follows that f^0 is rational and unimodular on $\mathbb{T}$. Therefore f^0 is a ratio of two finite Blaschke products.

REMARK 4.7. *If f^0 of the form* (4.4) *is a solution to the problem $\mathbf{IP}_\kappa$, then $f^0 \in \mathscr{B}_{\pi(P_{\mathbf{n}})}/\mathscr{B}_{\nu(P_{\mathbf{n}})}$.*

Proof. If f^0 is a solution to the problem $\mathbf{IP}_\kappa$, then the function $\mathscr{E} \equiv \gamma$ is a solution of the problem $\mathbf{AP}_\kappa$ (by Theorem 3.1). Constraints (2) and (3) in $\mathbf{AP}_\kappa$ guarantee that $\Theta_{21}\gamma + \Theta_{22}$ does not vanish at $z_1, \ldots, z_k$. Therefore, by Theorem 3.2 (part (2)), there is no zero cancellation in (4.4). By Theorem 3.2 (part (1)) applied to $S \equiv \gamma$ and $B \equiv 1$,

$$N\{\Theta_{11}\gamma + \Theta_{12}\} = \pi(P_{\mathbf{d}}) \quad \text{and} \quad N\{\Theta_{21}\gamma + \Theta_{22}\} = \nu(P_{\mathbf{d}}) \tag{4.5}$$

and therefore, $f^0 \in \mathscr{B}_{\pi(P_{\mathbf{d}})}/\mathscr{B}_{\nu(P_{\mathbf{d}})}$. Since $\mathscr{E} \equiv \gamma$ satisfies conditions (4.1), all the numbers $x_{i,j}$ ($j \geqslant 1$) are zeros. Then $X_{\mathbf{n}-\mathbf{d}} = \gamma E_{\mathbf{n}-\mathbf{d}}$ and the unique solution $\widetilde{P}$ of the Stein equation (2.38) is the zero matrix. Then it follows from (2.39) that $\pi(P_{\mathbf{n}}) = \pi(P_{\mathbf{d}})$ and $\nu(P_{\mathbf{n}}) = \nu(P_{\mathbf{d}})$ and the statement follows. □

PROPOSITION 4.8. *f^0 of the form* (4.4) *satisfies conditions* (1.4) *for every $i \in \mathscr{I}_1$.*

Proof. By Theorem 3.1 (or by part (3) in Theorem 3.2), it suffices to show that the function $\mathscr{E} \equiv \gamma$ satisfies conditions (3.8) for every $i \in \mathscr{I}_1$. We will prove this for a fixed $i \in \mathscr{I}_1$ separating the cases where $d_i > 0$ and where $d_i = 0$.

If $d_i > 0$, we have relations (4.5) by Theorem 3.2 (part (1)). Let us assume that

$$\Theta_{21}(z_i)\gamma + \Theta_{22}(z_i) = 0.$$

Then $\Theta_{11}(z_i)\gamma + \Theta_{12}(z_i) = 0$ by Theorem 3.2 (parts (3) and (4)) and after zero cancellation at z_i in (4.4) we conclude that $f^0 \in \mathscr{B}_m/\mathscr{B}_r$ for some $m < \pi(P_{\mathbf{d}})$ and $r < \nu(P_{\mathbf{d}})$. Then the Schwarz-Pick matrix $P_{\mathbf{d}}^{f^0}(\mathbf{z})$ defined via formula (1.2) is singular by Theorem 4.3. On the other hand, since f^0 satisfies interpolation conditions (4.1) (by Theorem 3.1), the matrix $P_{\mathbf{d}}^{f^0}(\mathbf{z})$ is equal to the Pick matrix of the problem (4.1) which is $P_{\mathbf{d}}$ and therefore, it is invertible. This contradiction completes the proof of the first case.

For the case $d_i = 0$, we will use formulas (2.23) which can be written in terms of Θ as

$$\begin{bmatrix} g_{i,0} & -y_{i,0} \end{bmatrix} = \begin{bmatrix} 1 & -c_{i,0} \end{bmatrix} \Theta(z_i). \tag{4.6}$$

By (3.11), the matrix $\Theta(z_i)$ is invertible. Since $\gamma = x_{i,0} = \frac{y_{i,0}}{g_{i,0}}$, we have by (4.6),

$$\begin{aligned} \Theta_{21}(z_i)\gamma + \Theta_{22}(z_i) &= \frac{1}{g_{i,0}} \cdot \begin{bmatrix} g_{i,0} & -y_{i,0} \end{bmatrix} \begin{bmatrix} \Theta_{22}(z_i) \\ -\Theta_{21}(z_i) \end{bmatrix} \\ &= \frac{\det\Theta(z_i)}{g_{i,0}} \cdot \begin{bmatrix} 1 & -c_{i,0} \end{bmatrix} \Theta(z_i)\Theta(z_i)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{\det\Theta(z_i)}{g_{i,0}} \neq 0 \end{aligned}$$

which completes the proof. □

COROLLARY 4.9. *The function f^0 is a solution of the problem $\mathbf{IP}_\kappa$ if and only if*

$$\Theta_{21}(z_i)\gamma + \Theta_{22}(z_i) \neq 0 \quad \textit{for every} \quad i \in \mathscr{I}_0.$$

The statement follows immediately from Theorem 3.1, Proposition 4.8 and the observation that the constant function $\mathscr{E} \equiv \gamma$ has no poles.

THEOREM 4.10. *The function f^0 is a solution of the problem $\mathbf{IP}_\kappa$ (with $\kappa = \nu(P_{\mathbf{n}})$) if and only if the Pick matrix $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated.*

Proof. If $\kappa = \nu(P_{\mathbf{n}})$ and $P_{\mathbf{n}}$ is singular, then the problem $\mathbf{IP}_\kappa$ has at most one solution by Proposition 4.5. The unique candidate is the function f^0 defined in (4.4). If f^0 is a solution to problem $\mathbf{IP}_\kappa$, then it belongs to $\mathscr{B}_{\pi(P_{\mathbf{n}})}/\mathscr{B}_{\nu(P_{\mathbf{n}})}$ by Remark 4.6. Since f^0 is a solution to the problem $\mathbf{IP}_\kappa$, we have $P_{\mathbf{n}}^{f^0}(\mathbf{z}) = P_{\mathbf{n}}$ and therefore, $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated by Corollary 4.4.

Now we assume that $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated and show that the function f^0 defined by (4.4) solves the problem $\mathbf{IP}_\kappa$. We first observe that since $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated and $P_{\mathbf{d}}$ is a maximal invertible principal submatrix of $P_{\mathbf{n}}$, it follows that $\operatorname{rank}(P_{\mathbf{n}}) = \operatorname{rank}(P_{\mathbf{d}}) = |\mathbf{d}|$.

Then we will argue via contradiction as follows. Let the function $\Theta_{21}\gamma+\Theta_{22}$ have zero of multiplicity $m_i \geqslant 0$ at z_i:

$$(\Theta_{21}(z_i)\gamma+\Theta_{22}(z_i))^{(j)}=0 \quad \text{for} \quad j=0,\dots,m_i-1;\ i=1,\dots,k.$$

Since the case where $m_i=0$ is not excluded, the latter assumption is not restrictive. By Theorem 4.8, $m_i=0$ for every $i\in\mathscr{I}_1$. Thus, if f^0 is not a solution to the problem $\mathbf{IP}_\kappa$, then $m_i>0$ for some $i\in\mathscr{I}_0$. With the tuple $\mathbf{m}=(m_1,\dots,m_k)\in\mathbb{Z}_+^k$, we associate the positive integer

$$\mu_{\mathbf{m}}=\sum_{i=1}^k \min\{m_i,n_i\}=\sum_{i=1}^k \min\{m_i,d_i\}>0 \tag{4.7}$$

where the second equality follows since $\min\{m_i,n_i\}=\min\{m_i,d_i\}=m_i=0$ for every $i\in\mathscr{I}_1$ and on the other hand, since $n_i=d_i$ for every $i\in\mathscr{I}_0$ by the very definition (2.18) of $\mathscr{I}_0$. Inequality $\mu_{\mathbf{m}}>0$ follows since at least one of the m_i's is positive. Let us define the tuple $\mathbf{r}=(r_1,\dots,r_k)$ by letting

$$r_i=n_i\ \ (i\in\mathscr{I}_1),\quad r_i=n_i-m_i\ \ (i\in\mathscr{I}_0,\ m_i<n_i),\quad r_i=0\ (i\in\mathscr{I}_0,\ m_i\geqslant n_i)$$

and observe that

$$|\mathbf{r}|=|\mathbf{n}|-\mu_{\mathbf{m}}. \tag{4.8}$$

By Theorem 3.2, there are exactly $\mu_{\mathbf{m}}$ zero cancellations in (4.4); therefore, and in view of (4.5), $f^0\in\mathscr{B}_{\pi(P_{\mathbf{d}})-\mu_{\mathbf{m}}}/\mathscr{B}_{\nu(P_{\mathbf{d}})-\mu_{\mathbf{m}}}$. On the other hand f^0 still satisfies interpolation conditions

$$f^{(j)}(z_i)=j!\,c_{i,j} \qquad (i=1,\dots,k;\ j=0,\dots,r_i-1)$$

by part (3) in Theorem 3.2. Therefore, the Schwarz-Pick matrix $P_{\mathbf{r}}^{f^0}(\mathbf{z})$ defined via formula (1.2) is equal to the principal submatrix $P_{\mathbf{r}}$ of $P_{\mathbf{n}}$. Since by (4.8),

$$|\mathbf{r}|=|\mathbf{n}|-\mu_{\mathbf{m}}>|\mathbf{d}|-2\mu_{\mathbf{m}}=\pi(P_{\mathbf{d}})-\mu_{\mathbf{m}}+\nu(P_{\mathbf{d}})-\mu_{\mathbf{m}},$$

and since $f^0\in\mathscr{B}_{\pi(P_{\mathbf{d}})-\mu_{\mathbf{m}}}/\mathscr{B}_{\nu(P_{\mathbf{d}})-\mu_{\mathbf{m}}}$, it follows by Theorem 4.3, that

$$\operatorname{rank}(P_{\mathbf{r}})=\operatorname{rank}(P_{\mathbf{r}}^{f^0}(\mathbf{z}))=\pi(P_{\mathbf{d}})-\mu_{\mathbf{m}}+\nu(P_{\mathbf{d}})-\mu_{\mathbf{m}}=|\mathbf{d}|-2\mu_{\mathbf{m}}. \tag{4.9}$$

If $|\mathbf{r}|>|\mathbf{d}|$, then for every tuple $\widetilde{\mathbf{r}}\preceq\mathbf{r}$ such that $|\widetilde{\mathbf{r}}|=|\mathbf{d}|$, we have

$$\operatorname{rank}(P_{\widetilde{\mathbf{r}}})\leqslant\operatorname{rank}(P_{\mathbf{r}})=|\mathbf{d}|-2\mu_{\mathbf{m}}<|\mathbf{d}|. \tag{4.10}$$

On the other hand, if $|\mathbf{r}|\leqslant|\mathbf{d}|$, then for every tuple $\widetilde{\mathbf{r}}\succeq\mathbf{r}$ such that $|\widetilde{\mathbf{r}}|=|\mathbf{d}|$, we have inequality

$$\operatorname{rank}(P_{\widetilde{\mathbf{r}}})\leqslant\operatorname{rank}(P_{\mathbf{r}})+2(|\mathbf{d}|-|\mathbf{r}|),$$

since the dimension of $P_{\widetilde{\mathbf{r}}}$ exceeds the dimension of $P_{\mathbf{r}}$ by $|\mathbf{d}|-|\mathbf{r}|$. Now we substitute (4.9) and (4.8) into the last inequality to get

$$\operatorname{rank}(P_{\widetilde{\mathbf{r}}})\leqslant|\mathbf{d}|-2\mu_{\mathbf{m}}+2(|\mathbf{d}|-|\mathbf{n}|+\mu_{\mathbf{m}})=3|\mathbf{d}|-2|\mathbf{n}|<|\mathbf{d}|. \tag{4.11}$$

Recall that $\operatorname{rank}(P_{\mathbf{n}}) = |\mathbf{d}|$. Inequalities (4.10) and (4.11) show that in any case there exists a singular $|\mathbf{d}| \times |\mathbf{d}|$ principal submatrix $P_{\tilde{\mathbf{r}}}$ of $P_{\mathbf{n}}$ which contradicts the assumption that $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated. The obtained contradiction completes the proof of the theorem. □

Now we are ready to prove Theorem 1.3 and Theorem 1.5.

Proof of Theorem 1.3. Since the nodegenerate case is known, it suffices to consider the case where the Pick matrix $P_{\mathbf{n}}$ of the problem $\mathbf{IP}_\kappa$ is singular. If $\kappa < \nu(P_{\mathbf{n}})$ or if $\nu(P_{\mathbf{n}}) < \kappa < \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$, then the problem $\mathbf{IP}_\kappa$ has no solutions by the explanation given in introduction and by Proposition 4.1. If $\kappa = \nu(P_{\mathbf{n}})$, then the problem has a solution if and only if $P_{\mathbf{n}}$ is $\mathbf{n}$-saturated by Theorem 4.10. If this is the case, the unique solution is the ratio of two Blaschke products of desired degrees, by Remark 4.7. Furthermore, by Theorem 2.13, the problem $\mathbf{IP}_\kappa$ can be extended to a problem (1.11), whose Pick matrix $P_{\mathbf{n}+\boldsymbol{\ell}}$ is invertible and has $\nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$ negative eigenvalues. By virtue of Theorem 3.1, there are infinitely many functions $f \in \mathscr{S}_\kappa$ satisfying conditions (1.11) for every $\kappa \geqslant \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$. All these functions are solutions to the problem $\mathbf{IP}_\kappa$. Thus, if $\kappa \geqslant \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$, then the problem $\mathbf{IP}_\kappa$ has infinitely many solutions. The "only if" parts in statements (1) and (2) of the theorem are now obvious. □

Proof of Theorem 1.5. The "only if" part is trivial. To prove the "if" part, let us assume that $P_{\mathbf{m}}$ is an $\mathbf{m}$-saturated principal submatrix of $P_{\mathbf{n}}$ with $|\mathbf{m}| = r+1$ and such that $\operatorname{rank}(P_{\mathbf{m}}) = \operatorname{rank}(P_{\mathbf{n}}) = r$. Pick any index i such that $m_i > 0$ and let $\mathbf{d} = \mathbf{m} - \mathbf{e}_i$. The matrix $P_{\mathbf{d}}$ is a maximal invertible principal submatrix of $P_{\mathbf{m}}$. Since $P_{\mathbf{m}}$ is the Pick matrix of the subproblem of $\mathbf{IP}_\kappa$ with interpolation conditions

$$f^{(j)}(z_i) = c_{i,j} \qquad (i = 1, \dots, k;\ j = 0, \dots, m_i - 1)$$

and is singular and $\mathbf{m}$-saturated, it follows by Theorem 1.3, that there exists a unique function f^0 in the class $\mathscr{S}_{\kappa - \nu(P_{\mathbf{d}})}$ satisfying (4.7) and this function f^0 is given by formula (4.4). By Theorem 4.10, f^0 also satisfies conditions (1.4) for every i such that $d_i < n_i$. On the other hand, all conditions in (1.4) corresponding to i's such that $d_i = n_i$ are contained in (4.7). Thus, f^0 is (a unique) solution of the "whole" problem $\mathbf{IP}_\kappa$ with $\kappa = \nu(P_{\mathbf{d}})$. Therefore $P_{\mathbf{n}}$ is equal to the Schwarz-Pick matrix $P^{f^0}_{\mathbf{d}}(\mathbf{z})$ and therefore, it is $\mathbf{n}$-saturated by Corollary 4.4. □

5. Description of all solutions in case $\kappa = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$

In this section we present a description of all solutions of an indeterminate problem $\mathbf{IP}_\kappa$ for the minimally possible κ. The description is obtained by combining Theorems 2.13, 3.1 and the following auxiliary result.

THEOREM 5.1. *Let f be a solution of the problem $\mathbf{IP}_\kappa$ with $\kappa = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$ and $\delta(P_{\mathbf{n}}) > 0$, let $P_{\mathbf{d}}$ be a fixed maximal invertible principal submatrix of $P_{\mathbf{n}}$ and let $\boldsymbol{\ell} \in \mathbb{Z}_+^k$ be the tuple given in* (2.44)*. Then the Schwarz-Pick matrix $P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z})$ is invertible.*

Proof. Since f is a solution of the problem $\mathbf{IP}_\kappa$, we have $P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z}) = P_{\mathbf{n}}$ and therefore, the matrix $P_{\mathbf{n}+\boldsymbol{\ell}} := P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z})$ is a structured extension of $P_{\mathbf{n}}$. Let $X_{\mathbf{n}-\mathbf{d}}$ and $X_{\mathbf{n}+\boldsymbol{\ell}-\mathbf{d}}$ be the vectors associated to the Schur complements $\widetilde{P} = P_{\mathbf{n}}/P_{\mathbf{d}}$ and $P_{\mathbf{n}+\boldsymbol{\ell}}/P_{\mathbf{d}}$. Relations (2.40) hold by Proposition 2.9. By Theorem 3.5, f is of the form (3.7) for some $\mathscr{E} \in \mathscr{S}_{\kappa-\nu(P_{\mathbf{d}})}$ satisfying conditions (3.10). By (2.39), we have

$$\kappa - \nu(P_{\mathbf{d}}) = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}}) - \nu(P_{\mathbf{d}}) = \nu(\widetilde{P}) + \delta(\widetilde{P}) > 0. \tag{5.1}$$

Note also that since $\mathscr{E}$ is analytic at z_i for every $i \in \mathscr{I}_1$, we can define the numbers

$$x_{i,j} := \frac{\mathscr{E}^{(j)}(z_i)}{j!} \tag{5.2}$$

for every $i \in \mathscr{I}_1$ and $j \geqslant 1$. If $x_{i,j} = 0$ for some $i \in \mathscr{I}_1$ and every $j \geqslant 1$, then $\mathscr{E} \equiv x_{i,0} = \gamma$ and thus, $\mathscr{E} \in \mathscr{S}_0$ which contradicts (5.1). Therefore, for every $i \in \mathscr{I}_1$, there exists an $m_i \geqslant 1$ such that $x_{i,m_i} \neq 0$. Define the tuple $\widetilde{\mathbf{n}} = (\widetilde{n}_1, \ldots, \widetilde{n}_k)$ by

$$\widetilde{n}_i = \begin{cases} n_i & \text{if } i \in \mathscr{I}_0 \cup \mathscr{I}_1', \\ \min\{m \in \mathbb{N} : \ x_{i,m-d_i} \neq 0\} & \text{if } i \in \mathscr{I}_1''. \end{cases} \tag{5.3}$$

Recall that for every $i \in \mathscr{I}''$, conditions (2.60) hold and thus, $\widetilde{n}_i \geqslant n_i$ for every $i \in \mathscr{I}''$. Therefore, $\widetilde{\mathbf{n}} \succeq \mathbf{n}$.

Let us assume that $P_{\mathbf{n}+\boldsymbol{\ell}}$ is singular. Combining part (3) in Theorem 2.13 with formula (2.61) we then conclude that $x_{i,n_i-d_i} = 0$ for some $i \in \mathscr{I}_1''$ so that for this particular i, we have $\widetilde{n}_i > n_i$. Therefore $|\widetilde{\mathbf{n}}| > |\mathbf{n}|$. Let us consider the column

$$X_{2\widetilde{\mathbf{n}}-2\mathbf{d}} = \operatorname{Col}_{1\leqslant i\leqslant k} \operatorname{Col}_{0\leqslant j\leqslant 2\widetilde{n}_i-d_i-1} x_{ij}$$

constructed from the numbers (5.2) and let $\mathbb{S}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})$ be the associated structured matrix. By definition (2.9) of $r_i(\cdot)$ and by (5.3) we have

$$r_i(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}}) = \widetilde{n}_i - d_i.$$

Therefore (and in view of (2.40)), Theorem 2.3 applies to $B_{\mathbf{m}} = X_{2\widetilde{\mathbf{n}}-2\mathbf{d}}$ and gives

$$\nu(\mathbb{S}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})) = |2\widetilde{\mathbf{n}} - 2\mathbf{d}| - |\mathbf{r}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})| = |\widetilde{\mathbf{n}} - \mathbf{d}| > |\mathbf{n} - \mathbf{d}|.$$

Since $|\boldsymbol{\ell}| \leqslant |\mathbf{n} - \mathbf{d}|$, we further get from (2.48)

$$|\mathbf{n} - \mathbf{d}| \geqslant |\mathbf{n} - \mathbf{d} + \ell|/2 = \nu(\widetilde{P}) + \delta(\widetilde{P})$$

and eventually, it follows that

$$\nu(\mathbb{S}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})) > \nu(\widetilde{P}) + \delta(\widetilde{P}). \tag{5.4}$$

By (5.2), the matrix $\mathbb{S}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})$ is equal to the Schwarz-Pick matrix $P^{\mathscr{E}}_{2\widetilde{\mathbf{n}}-2\mathbf{d}}(\mathbf{z})$ and since by (5.1), $\mathscr{E} \in \mathscr{S}_{\nu(\widetilde{P})+\delta(\widetilde{P})}$, it follows that

$$\nu(\mathbb{S}(X_{2\widetilde{\mathbf{n}}-2\mathbf{d}})) = \nu(P^{\mathscr{E}}_{2\widetilde{\mathbf{n}}-2\mathbf{d}}(\mathbf{z})) \leqslant \nu(\widetilde{P}) + \delta(\widetilde{P})$$

which contradicts (5.4) and completes the proof of the theorem. □

Now we will present a procedure describing the solution set to the problem $\mathbf{IP}_\kappa$ in case $\kappa=\nu(P_{\mathbf{n}})+\delta(P_{\mathbf{n}})$ and $\delta(P_{\mathbf{n}})>0$, where $P_{\mathbf{n}}$ is the Pick matrix of the problem.

Step 1: *Find and fix a maximal invertible principal submatrix* $P_{\mathbf{d}}$ *of* $P_{\mathbf{n}}$.

We do not discuss here a computationally efficient way to find an invertible principal submatrix $P_{\mathbf{d}}$ of $P_{\mathbf{n}}$ which is maximal in the sence of (2.37). An unefficient (but finite) procedure to get one is the following: compute $\det P_{\mathbf{d}}$ for *all* tuples $\mathbf{d}\preceq\mathbf{n}$ and pick one of the maximal size, i.e., such that

$$\det P_{\mathbf{d}}\neq 0 \quad \text{and} \quad \det P_{\mathbf{r}}=0 \quad \text{for every} \quad \mathbf{r}\preceq\mathbf{n}: \ |\mathbf{r}|>|\mathbf{d}|. \tag{5.5}$$

It is clear that every such $P_{\mathbf{d}}$ is also maximal in the sense of (2.37).

Step 2: *For the chosen* $P_{\mathbf{d}}$, *compute the tuple* $\boldsymbol{\ell}\in\mathbb{Z}_+$ *via* (2.44) *and then the sets* $\mathcal{I}_1'$ *and* $\mathcal{I}_1''$ *as in* (2.47).

DEFINITION 5.2. An extension $C_{\mathbf{n}+\boldsymbol{\ell}}$ of $C_{\mathbf{n}}$ will be called $P_{\mathbf{d}}$*-admissible* if the extending numbers c_{i,n_i} satisfy condition (2.51) for every $i\in\mathcal{I}_1''$.

For every $P_{\mathbf{d}}$-admissible extension $C_{\mathbf{n}+\boldsymbol{\ell}}$, the matrix $P_{\mathbf{n}+\boldsymbol{\ell}}:=\mathbb{S}(C_{\mathbf{n}+\boldsymbol{\ell}})$ is invertible (by part (3) in Theorem 2.13) and its entries can be computed from the formula similar to (1.6). Then we define the function

$$\begin{aligned}\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}(z)&=\begin{bmatrix}\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},11}(z) & \Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},12}(z)\\ \Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},21}(z) & \Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},22}(z)\end{bmatrix}\\ &=I_2+(z-1)\begin{bmatrix}E^*_{\mathbf{n}+\boldsymbol{\ell}}\\ C^*_{\mathbf{n}+\boldsymbol{\ell}}\end{bmatrix}(I-zT^*_{\mathbf{n}+\boldsymbol{\ell}})^{-1}P^{-1}_{\mathbf{n}+\boldsymbol{\ell}}(I-T_{\mathbf{d}})^{-1}\begin{bmatrix}E_{\mathbf{n}+\boldsymbol{\ell}} & -C_{\mathbf{n}+\boldsymbol{\ell}}\end{bmatrix},\end{aligned} \tag{5.6}$$

where $T_{\mathbf{n}+\boldsymbol{\ell}}$ and $E_{\mathbf{n}+\boldsymbol{\ell}}$ are defined via formulas (2.1).

Step 3: *Take any Schur-class function* $\mathcal{E}$ *such that*

$$\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},21}(z_i)\mathcal{E}(z_i)+\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},22}(z_i)\neq 0 \quad \textit{for every} \quad i=1,\ldots,k \tag{5.7}$$

and compute the function

$$f=\mathbf{T}_{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}}[\mathcal{E}]:=\frac{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},11}\mathcal{E}+\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},12}}{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},21}\mathcal{E}+\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}},22}}. \tag{5.8}$$

This function f solves the problem $\mathbf{IP}_\kappa$ and all solutions to the problem can be obtained in this way. The next theorem justifies this and shows that in fact, the solution set of the problem $\mathbf{IP}_\kappa$ is the disjoint union of the ranges of linear fractional fransfornmations (5.8) where the union is taken over all $P_{\mathbf{d}}$-admisssible extensions $C_{\mathbf{n}+\boldsymbol{\ell}}$ of $C_{\mathbf{n}}$.

THEOREM 5.3. *Let $P_{\mathbf{d}}$ be a fixed maximal invertible principal submatrix of the Pick matrix $P_{\mathbf{n}}$ of the problem $\mathbf{IP}_{\kappa}$ and let $\kappa = \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}}) > \nu(P_{\mathbf{n}})$. Let $\Omega_{\mathbf{d}}$ be the set of all pairs $\{C_{\mathbf{n}+\boldsymbol{\ell}}, \mathscr{E}\}$ consisting of a $P_{\mathbf{d}}$-admissible extension $C_{\mathbf{n}+\boldsymbol{\ell}}$ of $C_{\mathbf{n}}$ and of a function $\mathscr{E} \in \mathscr{S}$ satisfying conditions* (5.7)*. Then the correspondence*

$$\{C_{\mathbf{n}+\boldsymbol{\ell}}, \mathscr{E}\} \mapsto f = \mathbf{T}_{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}}[\mathscr{E}] \tag{5.9}$$

establishes a bijection between $\Omega_{\mathbf{d}}$ and the solution set of the problem $\mathbf{IP}_{\kappa}$.

Proof. If $C_{\mathbf{n}+\boldsymbol{\ell}}$ is a $P_{\mathbf{d}}$-admissible extension of $C_{\mathbf{n}}$, then the matrix $P_{\mathbf{n}+\boldsymbol{\ell}} := \mathbb{S}(C_{\mathbf{n}+\boldsymbol{\ell}})$ satisfies

$$\delta(P_{\mathbf{n}+\boldsymbol{\ell}}) \neq 0 \quad \text{and} \quad \nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = \nu(P) + \delta(P),$$

by Theorem 2.13 (part (3)). Thus, $\kappa - \nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = 0$. Now it follows from Remark 3.3 that for every $\mathscr{E} \in \mathscr{S}_{\kappa-\nu(P_{\mathbf{n}+\boldsymbol{\ell}})} = \mathscr{S}$ subject to constraints (5.7), the function $f = \mathbf{T}_{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}}[\mathscr{E}]$ belongs to the class $\mathscr{S}_{\nu(P_{\mathbf{n}+\boldsymbol{\ell}})} = \mathscr{S}_{\kappa}$ and satisfies interpolation conditions (1.11). In particular, f is a solution of the problem $\mathbf{IP}_{\kappa}$.

To show that transformation (5.9) is one-to-one, take two pairs $\{C_{\mathbf{n}+\boldsymbol{\ell}}, \mathscr{E}\}$ and $\{\widetilde{C}_{\mathbf{n}+\boldsymbol{\ell}}, \widetilde{\mathscr{E}}\}$ in $\Omega_{\mathbf{d}}$ and let $f := \mathbf{T}_{\Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}}[\mathscr{E}]$ and $\widetilde{f} := \mathbf{T}_{\Theta_{\widetilde{C}_{\mathbf{n}+\boldsymbol{\ell}}}}[\widetilde{\mathscr{E}}]$. If $C_{\mathbf{n}+\boldsymbol{\ell}} \neq \widetilde{C}_{\mathbf{n}+\boldsymbol{\ell}}$, then $c_{i,j} \neq \widetilde{c}_{i,j}$ for some $i \in \mathscr{I}_1$ and $j \geqslant n_i$. Since $f^{(j)}(z_i) = j!\,c_{i,j}$ and $\widetilde{f}^{(j)}(z_i) = j!\,\widetilde{c}_{i,j}$ by Remark 3.3, it follows that $f \neq \widetilde{f}$. If $C_{\mathbf{n}+\boldsymbol{\ell}} = \widetilde{C}_{\mathbf{n}+\boldsymbol{\ell}}$ but $\mathscr{E} \neq \widetilde{\mathscr{E}}$, then $f \neq \widetilde{f}$ since transformation $\mathscr{E} \mapsto \Theta_{C_{\mathbf{n}+\boldsymbol{\ell}}}[\mathscr{E}]$ is invertible and therefore, it is one-to-one. Thus, different pairs in $\Omega_{\mathbf{d}}$ correspond to different solutions of the problem $\mathbf{IP}_{\kappa}$.

To show that transformation (5.9) is onto, let f be a solution of the problem $\mathbf{IP}_{\kappa}$. Then the vector

$$C_{\mathbf{n}+\boldsymbol{\ell}} := \operatorname{Col}_{1\leqslant i\leqslant k} \operatorname{Col}_{0\leqslant j\leqslant n_i-1} \frac{f^{(j)}(z_i)}{j!}$$

extends $C_{\mathbf{n}}$. By Theorem 5.1, the matrix

$$P_{\mathbf{n}+\boldsymbol{\ell}} := \mathbb{S}(C_{\mathbf{n}+\boldsymbol{\ell}}) = P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z})$$

is invertible. Since $P_{\mathbf{n}+\boldsymbol{\ell}}$ is a structured extension of $P = P_{\mathbf{n}}$, we conclude by Theorem 2.13 (part (3)) that $\nu(P_{\mathbf{n}+\boldsymbol{\ell}}) = \nu(P) + \delta(P) = \kappa$ and that the extension $C_{\mathbf{n}+\boldsymbol{\ell}}$ of $C_{\mathbf{n}}$ is $P_{\mathbf{d}}$-admissible. The matrix $P_{\mathbf{n}+\boldsymbol{\ell}}$ is the Pick matrix of the extended problem (1.11) and by Remark 3.3, f is of the form (5.8) for some $\mathscr{E} \in \mathscr{S}$ satisfying conditions (5.7). An additional possibility for $\mathscr{E}$ to have a pole at interpolation nodes is not realizable in the present context since $\mathscr{E}$ is a Schur class function. $\square$

In case $\kappa > \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$, one can modify Theorem 5.3 upon letting the parameter $\mathscr{E}$ to run through the class $\mathscr{S}_{\kappa-\nu(P_{\mathbf{n}})-\delta(P_{\mathbf{n}})}$ (and still satisfying conditions (5.7)). Then description (5.8) provides infinitely many solutions of the problem $\mathbf{IP}_{\kappa}$, but not all of them. It is not hard to see from the preceding analysis that formula (5.8) gives all solutions f of the problem $\mathbf{IP}_{\kappa}$ for which the Schwarz-Pick matrix $P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z})$ is invertible. Also it is easily seen that for $\kappa > \nu(P_{\mathbf{n}}) + \delta(P_{\mathbf{n}})$, there are solutions f of $\mathbf{IP}_{\kappa}$ with singular Schwarz-Pick matrix $P^f_{\mathbf{n}+\boldsymbol{\ell}}(\mathbf{z})$.

6. Extremal functions

Let H^∞ be the space of bounded analytic functions on $\mathbb{D}$ and let H_k^∞ be the set of all functions f of the form (1.8) where $s \in H^\infty$ and $b \in \mathscr{B}_k$ may have common zeros. From this definition it follows that $\mathscr{S}_k = (H_k^\infty \backslash H_{k-1}^\infty) \cap \mathscr{B}L^\infty$ where $\mathscr{B}L^\infty$ denotes the unit ball of $L^\infty(\mathbb{T})$. Let

$$\mathbf{G} := \left\{ g : \frac{g^{(j)}(z_i)}{j!} = c_{ij} \ \text{ for } \ j = 0, \dots, n_i - 1; \ i = 1, \dots, k \right\} \tag{6.1}$$

be the set of all functions g satisfying interpolation conditions (1.4). Below we consider the two related questions which originate to [1]:

1. Find the value of $\mu_k := \inf_{g \in \mathbf{G} \cap H_k^\infty} \|g\|_\infty$ in terms of interpolation data (1.7).
2. Find necessary and sufficient condition for the existence of an extremal function $g_{k,\min} \in \mathbf{G} \cap H_k^\infty$ such that $\|g_{k,\min}\|_\infty = \mu_k$.

In Theorem 6.1 below, the answers for these questions will be given in terms of the matrix pencil $P_{\mathbf{n}}(\lambda) = \lambda^2 \cdot \mathbb{S}(\lambda^{-1} \cdot C_{\mathbf{n}})$ defined as a unique solution of the Stein equation

$$P_{\mathbf{n}}(\lambda) - T_{\mathbf{n}} P_{\mathbf{n}}(\lambda) T_{\mathbf{n}}^* = \lambda^2 E_{\mathbf{n}} E_{\mathbf{n}}^* - C_{\mathbf{n}} C_{\mathbf{n}}^*$$

for every fixed $\lambda \in \mathbb{R}$ so that the Pick matrix $P_{\mathbf{n}}$ defined in (1.5), (1.6) equals $P_{\mathbf{n}}(1)$. The explicit formulas for $P_{\mathbf{n}}(\lambda)$ are similar to (1.5), (1.6):

$$P_{\mathbf{n}}(\lambda) = [P_{ij}(\lambda)]_{i,j=1}^k \tag{6.2}$$

where the $n_i \times n_j$ blocks $P_{ij}(\lambda)$ are defined entry-wise by

$$\begin{aligned} [P_{ij}(\lambda)]_{\ell,r} = \lambda^2 \cdot \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!} \frac{z_i^{r-s} \overline{z}_j^{\ell-s}}{(1-z_i\overline{z}_j)^{\ell+r-s+1}} \\ - \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{r} \sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!} \frac{z_i^{\beta-s} \overline{z}_j^{\alpha-s} c_{i,\ell-\alpha} \overline{c}_{j,r-\beta}}{(1-z_i\overline{z}_j)^{\alpha+\beta-s+1}}. \end{aligned} \tag{6.3}$$

THEOREM 6.1. *Let $\lambda_0 \geqslant \lambda_1 \geqslant \dots \geqslant \lambda_m > 0$ be all positive solutions of the equation $\det P_{\mathbf{n}}(\lambda) = 0$. Let $\mathbf{G}$ be the set defined in (6.1) and let $\widetilde{\lambda} \in \mathbb{R}_+ \backslash \{\lambda_0, \lambda_1, \dots, \lambda_m\}$. Then*

1. *If $\widetilde{\lambda} > \lambda_k$, then there exists $g \in \mathbf{G} \cap (H_k^\infty \backslash H_{k-1}^\infty)$ with $\|g\|_\infty \leqslant \widetilde{\lambda}$.*
2. *If $\widetilde{\lambda} < \lambda_k$, then $\|h\|_\infty > \widetilde{\lambda}$ for every $h \in \mathbf{G} \cap H_k^\infty$.*
3. *For every $k \geqslant 0$,*

$$\mu_k := \inf_{g \in \mathbf{G} \cap H_k^\infty} \|g\|_\infty = \inf_{g \in \mathbf{G} \cap (H_k^\infty \backslash H_{k-1}^\infty)} \|g\|_\infty = \begin{cases} \lambda_k & \text{if } k \leqslant m, \\ 0 & \text{if } k > m. \end{cases} \tag{6.4}$$

4. *For every $k \geqslant 0$, there exists a (unique) function $g \in \mathbf{G} \cap H_k^\infty$ with $\|g\|_\infty = \mu_k = \lambda_k$ if and only if the matrix $P^{(\lambda_k)}$ defined via formulas* (6.2), (6.3) *is* $\mathbf{n}$*-saturated. This extremal function belongs to $H_k^\infty \backslash H_{k-1}^\infty$ if and only if either $k = 0$ or $\lambda_k < \lambda_{k-1}$.*

Proof. The scaled matrix $\widetilde{\lambda}^{-2} \cdot P_{\mathbf{n}}(\lambda)$ is the Pick matrix of the problem with interpolation conditions

$$\frac{f^{(j)}(z_i)}{j!} = \widetilde{\lambda}^{-1} c_{ij} \quad \text{for} \quad j = 0, \dots, n_i - 1;\ i = 1, \dots, k, \tag{6.5}$$

and this matrix is invertible, since $\widetilde{\lambda}$ is not equal to any of λ_i's. If $\widetilde{\lambda} > \lambda_k$, then $\nu(P_{\mathbf{n}}(\widetilde{\lambda})) \leqslant k$ and then by Theorem 1.4, there is a function $f \in \mathscr{S}_k$ satisfying conditions (6.5). Then the function $g = \widetilde{\lambda} f$ satisfies conditions in (1.4) and belongs to $H_k^\infty \backslash H_{k-1}^\infty$ with $\|g\|_\infty \leqslant \widetilde{\lambda}$. This proves part (1). On the other hand, if $\widetilde{\lambda} < \lambda_k$, then

$$\nu(P_{\mathbf{n}}(\widetilde{\lambda})) \geqslant k + 1. \tag{6.6}$$

Assuming that there exists an $h \in \mathbf{G} \cap H_k^\infty$ with $\|h\|_\infty \leqslant \widetilde{\lambda}$, we conclude that the function $f = \widetilde{\lambda}^{-1} h$ satisfies conditions (6.5) and belongs $\mathscr{S}_\kappa$ for some $\kappa \leqslant k$. But then the Pick matrix $\widetilde{\lambda}^{-2} \cdot P_{\mathbf{n}}(\widetilde{\lambda})$ of the problem (6.5) has at most κ negative eigenvalues. This contradicts to (6.6) and completes the proof of part (2). For every $k \leqslant m$ we have

$$\lambda_k \leqslant \inf_{g \in \mathbf{G} \cap H_k^\infty} \|g\|_\infty \leqslant \inf_{g \in \mathbf{G} \cap (H_k^\infty \backslash H_{k-1}^\infty)} \|g\|_\infty \leqslant \lambda_k, \tag{6.7}$$

where the second inequality is obvious while the first and the third follow by parts (2) and (1) respectively. Equalities (6.4) (for $k \leqslant m$) follow from (6.7). The case $k > m$ is proved in much the same way.

We start the proof of the last statement with the case $k > m$: the function $g \equiv 0$ (the only function in H_k^∞ with $\|g\| = \mu_k = 0$) belongs to $\mathbf{G}$ if and only if all interpolation conditions are homogeneous, in which case the equation $\det P_{\mathbf{n}}(\lambda) = 0$ has no positive solutions).

Let us now assume that $k \leqslant m$. We seek a function $g \in \mathbf{G} \cap H_k^\infty$ such that $\|g\|_\infty = \lambda_k$ or equivalently, such that $\|g\|_\infty \leqslant \lambda_k$; this equivalence follows by part (2) according to which $\|g\|_\infty \geqslant \lambda_k$ for every $g \in \mathbf{G} \cap H_k^\infty$. It is convenient to seek g in the form $g = \lambda_k f$ where

$$f \in H_k^\infty \cap \mathscr{B}L^\infty \quad \text{and} \quad \frac{f^{(j)}(z_i)}{j!} = \lambda_k^{-1} c_{ij} \quad (j = 0, \dots, n_i - 1 :\ i = 1, \dots, k). \tag{6.8}$$

Thus, the extremal function $g_{k,\min}$ with $\|g_{k,\min}\|_\infty = \lambda_k$ exists if and only if the interpolation problem (6.8) has a solution in $\mathscr{S}_\kappa$ for some $\kappa \leqslant k$. The extremal function belongs to $\mathbf{G} \cap H_k^\infty \backslash H_{k-1}^\infty$ if and only if the problem (6.8) has a solution in $\mathscr{S}_k$. The Pick matrix of this problem equals $\lambda_k^{-2} \cdot P_{\mathbf{n}}(\lambda_k)$ and is singular by the definition of λ_i's. We will consider separately two cases.

Case 1: Let $\lambda_0 = \ldots = \lambda_k$. Then $\nu(P_{\mathbf{n}}(\lambda_k)) = 0$ and $\delta(P_{\mathbf{n}}(\lambda_k)) \geqslant k+1$. By Theorem 1.3, the problem (6.8) has a unique solution $f \in \mathscr{S}_0$ and does not have solutions in $\mathscr{S}_\kappa$ for $\kappa = 1,\ldots,k$. Thus, the problem has a solution in $\mathscr{S}_k$ if and only if $k = 0$.

Case 2: Let $\lambda_{\ell-1} > \lambda_\ell = \ldots = \lambda_k$. Then $P_{\mathbf{n}}(\lambda_k) = P_{\mathbf{n}}(\lambda_\ell)$,

$$\nu(P_{\mathbf{n}}(\lambda_k)) = \ell \quad \text{and} \quad \delta(P_{\mathbf{n}}(\lambda_k)) \geqslant k - \ell + 1.$$

By Theorem 1.3, the problem (6.8) has a (unique) solution in $\mathscr{S}_\ell$ if and only if the matrix $P_{\mathbf{n}}(\lambda_k)$ is $\mathbf{n}$-saturated and it does not have solutions in $\mathscr{S}_\kappa$ for $\kappa = \ell+1,\ldots,k$. Thus, the problem has a solution in $\mathscr{S}_k$ if and only if $k = \ell$ in which case we have $\lambda_{k-1} > \lambda_k$. $\square$

7. Appendix: The proof of Theorem 4.3

In this section we prove Theorem 4.3, showing that saturated Pick matrices (see Definition 1.2) arise as Schwarz-Pick matrices of ratios of finite Blaschke products of low degrees. The proof needs some preliminaries. We note that the case where $n_i = 1$ for $i = 1,\ldots,k$ was considered in [10, Theorem 3.4]. The present proof is based on pretty much the same arguments.

Let H^2 be the Hardy space of square integrable functions on the unit circle $\mathbb{T}$ that admit analytic continuation inside the unit disk. The functions

$$\mathbf{k}_{z,j}(t) = \frac{1}{j!}\frac{\partial^j}{\partial \bar{z}^j}\left(\frac{1}{1-t\bar{z}}\right) = \frac{t^j}{(1-t\bar{z})^{j+1}} \quad (j \geqslant 0;\ z \in \mathbb{D}) \tag{7.1}$$

belong to H^2 for every fixed $j \geqslant 0$ and $z \in \mathbb{D}$ and

$$\langle h, \mathbf{k}_{z,j}\rangle_{H^2} = \frac{1}{j!}h^{(j)}(z) \quad \text{for every} \quad h \in H^2. \tag{7.2}$$

LEMMA 7.1. *Let* $b(z) = \prod_{i=1}^{\ell}\left(\frac{z-\lambda_i}{1-z\bar{\lambda}_i}\right)^{r_i}$ *be a finite Blaschke product of degree* κ *(*$\lambda_1,\ldots,\lambda_\ell \in \mathbb{D}$ *are distinct) and let* $\mathscr{K}_b := H^2 \ominus bH^2$ *be the model space. Then*

1. *The functions*
$$\{\mathbf{k}_{\lambda_i,j} :\ i = 1,\ldots,\ell,\ j = 0,\ldots,r_j - 1\} \tag{7.3}$$
defined via (7.1)*, form a basis for* $\mathscr{K}_b$*; therefore,* $\dim K_b = \kappa$.

2. *A function* g *belongs to* $\mathscr{K}_b$ *if and only if it admits a representation*
$$g(t) = \frac{q(t)}{\prod_{j=1}^{\ell}(1-t\bar{\lambda}_j)^{r_j}} \tag{7.4}$$
for some polynomial q *of degree* $\deg q \leqslant \kappa - 1$.

3. Let $\mathscr{P}_b$ denote the orthogonal projection of H^2 onto $\mathscr{K}_b$. Then

$$\left(\mathscr{P}_b\mathbf{k}_{\zeta,j}\right)(t)=\frac{1}{j!}\frac{\partial^j}{\partial\bar{\zeta}^j}\left(\frac{1-b(t)\overline{b(\zeta)}}{1-t\bar{\zeta}}\right)\quad \textit{for every}\ \ \zeta\in\mathbb{D}. \tag{7.5}$$

To verify the first statement one can check that the functions (7.3) belong to $\mathscr{K}_b$ and that any H^2 function orthogonal to them has a zero of multiplicity at least r_i at λ_i. It follows directly from (7.1) that a function g is a linear combination of the functions (7.3) if and only if it is of the form (7.4), which proves the second statement. The last statement follows from the observation that $g_1+g_2=\mathbf{k}_{\zeta,j}$ for every fixed $j\geqslant 0$ and $\zeta\in\mathbb{D}$, where the functions

$$g_1(z)=\frac{1}{j!}\frac{\partial^j}{\partial\bar{\zeta}^j}\frac{b(t)\overline{b(\zeta)}}{1-t\bar{\zeta}}\quad\text{and}\quad g_2(z)=\frac{1}{j!}\frac{\partial^j}{\partial\bar{\zeta}^j}\frac{1-b(t)\overline{b(\zeta)}}{1-t\bar{\zeta}}$$

belong to bH^2 and $\mathscr{K}_B$, respectively.

LEMMA 7.2. *Let $b\in\mathscr{B}_\kappa$ and $\theta\in\mathscr{B}_m$ be finite Blaschke products*

$$b(z)=\prod_{j=1}^{\ell}\left(\frac{z-\lambda_j}{1-z\bar{\lambda}_j}\right)^{r_j}\quad \textit{and}\quad \theta(z)=\prod_{j=1}^{\widetilde{\ell}}\left(\frac{z-w_j}{1-z\bar{w}_j}\right)^{\widetilde{r}_j}$$

with no common zeros, let $z_1,\dots,z_k$ be distinct points in $\mathbb{D}$ and let $\mathbf{n}=(n_1,\dots,n_k)\in\mathbb{N}^k$. If $|\mathbf{n}|=\kappa+m$, then the difference of the Schwarz-Pick matrices

$$\mathbb{P}_{\mathbf{n}}:=P^{\theta}_{\mathbf{n}}(\mathbf{z})-P^{b}_{\mathbf{n}}(\mathbf{z}) \tag{7.6}$$

is invertible. Furthermore, if $|\mathbf{n}|\geqslant\kappa+m$, then

$$\nu(\mathbb{P}_{\mathbf{n}})=\kappa\quad \textit{and}\quad \pi(\mathbb{P}_{\mathbf{n}})=m. \tag{7.7}$$

Proof. Let $\mathscr{K}$ be the subspace of H^2 defined as

$$\mathscr{K}=\operatorname{span}\{\mathbf{k}_{z_j,i}:\ \ i=0,\dots,n_j-1,\ j=1,\dots,k\}. \tag{7.8}$$

By (7.5) and by the reproducing property (7.2),

$$\begin{aligned}\left\langle\mathscr{P}_b\mathbf{k}_{z_j,\beta},\,\mathscr{P}_b\mathbf{k}_{z_i,\alpha}\right\rangle_{H^2}&=\left\langle\mathscr{P}_b\mathbf{k}_{z_j,\beta},\,\mathbf{k}_{z_i,\alpha}\right\rangle_{H^2}\\ &=\frac{1}{\alpha!\beta!}\,\frac{\partial^{\alpha+\beta}}{\partial z^\alpha\partial\bar{\zeta}^\beta}\,\frac{1-b(z)\overline{b(\zeta)}}{1-z\bar{\zeta}}\Bigg|_{\substack{z=z_i\\ \zeta=z_j}}\end{aligned}$$

and thus, by definition of the Schwarz-Pick matrix,

$$P^{B}_{\mathbf{n}}(\mathbf{z})=\left[\left[\left\langle\mathscr{P}_b\mathbf{k}_{z_j,\beta},\,\mathscr{P}_b\mathbf{k}_{z_i,\alpha}\right\rangle_{H^2}\right]_{\alpha=0,\dots,n_i-1}^{\beta=0,\dots,n_j-1}\right]_{i,j=1}^{k}. \tag{7.9}$$

Similarly,

$$P_{\mathbf{n}}^{\theta}(\mathbf{z})=\left[\left[\left\langle \mathscr{P}_{\theta}\mathbf{k}_{z_j,\beta},\ \mathscr{P}_{\theta}\mathbf{k}_{z_i,\alpha}\right\rangle_{H^2}\right]_{\alpha=0,\dots,n_i-1}^{\beta=0,\dots,n_j-1}\right]_{i,j=1}^{k}, \tag{7.10}$$

where $\mathscr{P}_{\theta}$ stands for the orthogonal projection of H^2 onto the subspace $\mathscr{K}_{\theta}=H^2\ominus\theta H^2$ of H^2. Therefore, the matrix $\mathbb{P}_{\mathbf{n}}$ defined as in (7.6) is nonsingular if and only if the quadratic form

$$D(x,y):=\langle \mathscr{P}_{\theta}x,\ \mathscr{P}_{\theta}y\rangle_{H^2}-\langle \mathscr{P}_{b}x,\ \mathscr{P}_{b}y\rangle_{H^2} \tag{7.11}$$

is not degenerate on $\mathscr{K}$. Since the spaces $\mathscr{K}_b$ and $\mathscr{K}_{\theta}$ are of dimensions κ and m respectively (by Statement 1 in Lemma 7.2), it follows from (7.9) and (7.10) that for every n,

$$\operatorname{rank}(P_{\mathbf{n}}^{b}(\mathbf{z}))\leqslant\kappa\quad\text{and}\quad\operatorname{rank}(P_{\mathbf{n}}^{\theta}(\mathbf{z}))\leqslant m$$

and, since the latter Schwarz-Pick matrices are positive semidefinite, we have for their difference

$$\nu(\mathbb{P}_{\mathbf{n}})\leqslant\kappa\quad\text{and}\quad\pi(\mathbb{P}_{\mathbf{n}})\leqslant m. \tag{7.12}$$

Let us assume that $|\mathbf{n}|=\kappa+m$ and that the form D is degenerate, i.e., that there exists $x\in\mathscr{K}$ such that $D(x,y)=0$ for every $y\in\mathscr{K}$. Then we have, by (7.11),

$$\begin{aligned}0&=\langle \mathscr{P}_{\theta}x,\ \mathscr{P}_{\theta}y\rangle_{H^2}-\langle \mathscr{P}_{b}x,\ \mathscr{P}_{b}y\rangle_{H^2}\\&=\langle \mathscr{P}_{\theta}x,\ y\rangle_{H^2}-\langle \mathscr{P}_{b}x,\ y\rangle_{H^2}=\langle (\mathscr{P}_{\theta}-\mathscr{P}_{b})x,\ y\rangle_{H^2}\end{aligned}$$

for every $y\in\mathscr{K}$. Upon letting $y=\mathbf{k}_{z_j,i}$ in the latter equality, we conclude, by the reproducing property (7.2), that

$$(\mathscr{P}_{\theta}x-\mathscr{P}_{b}x)^{(i)}(z_j)=0\quad\text{for}\quad i=0,\dots,n_j-1\quad\text{and}\quad j=1,\dots,m. \tag{7.13}$$

The functions $\mathscr{P}_{b}x$ and $\mathscr{P}_{\theta}x$ belong to the spaces $\mathscr{K}_b$ and $\mathscr{K}_{\theta}$, respectively, and therefore, by the second statement in Lemma 7.2, they are of the form

$$(\mathscr{P}_{b}x)(t)=\frac{q(t)}{\prod_{j=1}^{\ell}(1-t\bar{\lambda}_j)^{r_j}}\quad\text{and}\quad(\mathscr{P}_{\theta}x)(t)=\frac{\widetilde{q}(t)}{\prod_{j=1}^{\widetilde{\ell}}(1-t\bar{w}_j)^{\widetilde{r}_j}} \tag{7.14}$$

for some polynomials q and $\widetilde{q}$ with

$$\deg q\leqslant\kappa-1\quad\text{and}\quad\deg\widetilde{q}\leqslant m-1. \tag{7.15}$$

The denominators in (7.14) are polynomials of degree κ and $\widetilde{\kappa}$, respectively, and thus, it follows readily from (7.14) and (7.15)

$$(\mathscr{P}_{\theta}x)(t)-(\mathscr{P}_{b}x)(t)=\frac{r(t)}{p(t)}, \tag{7.16}$$

where p and r are the polynomials whose degrees satisfy

$$\deg p=\kappa+m=|\mathbf{n}|\quad\text{and}\quad\deg r\leqslant\kappa+m-1=|\mathbf{n}|-1. \tag{7.17}$$

By (7.13), the rational function $\frac{r}{p}$ has k distinct zeros of total multiplicity $|\mathbf{n}|$ which together with (7.17) implies that $r \equiv 0$. Thus, we have from (7.16)

$$\mathscr{P}_b x \equiv \mathscr{P}_\theta x. \tag{7.18}$$

By the first statement in Lemma 7.2, the spaces $\mathscr{K}_b$ and $\mathscr{K}_\theta$ are spanned by functions (7.3) and by functions

$$\{\mathbf{k}_{w_j,i}: \ j = 1,\dots,\widetilde{\ell}, \ i = 0,\dots,\widetilde{r}_j - 1\},$$

respectively. Since all the functions in (7.3) are linearly independent and since $\lambda_i \neq w_j$ for $i = 1,\dots,\ell$ and $j = 1,\dots,\widetilde{\ell}$, it follows that $\mathscr{K}_b \cap \mathscr{K}_\theta = \{0\}$. Thus, relation (7.18) implies

$$\mathscr{P}_b x \equiv 0 \quad \text{and} \quad \mathscr{P}_\theta x \equiv 0.$$

Therefore, x is orthogonal to both of $\mathscr{K}_b$ and $\mathscr{K}_\theta$ and thus, $x \in bH^2 \cap \theta H^2$. Since b and θ have no common zeros, it follows that $x \in (b\theta)H^2$. In particular, x has at least $\kappa + m = |\mathbf{n}|$ zeros (counted with multiplicities). On the other hand, x belongs to $\mathscr{K}$ and therefore, by definition (7.8),

$$x(t) = \sum_{j=1}^{m} \sum_{i=0}^{n_j - 1} \frac{\gamma_{ji} t^i}{(1 - t\bar{z}_j)^{i+1}} = \frac{q(t)}{\prod_{j=1}^{m}(1 - t\bar{z}_j)^{n_j}},$$

where q is a polynomial with $\deg q \leqslant |\mathbf{n}| - 1$. Therefore, if $x \neq 0$, it cannot have more than $|\mathbf{n}| - 1$ zeros. Therefore, $x = 0$ and the form D is nondegenerate on $\mathscr{K}$. Therefore, the matrix $\mathbb{P}_{\mathbf{n}}$ is invertible. Moreover, we have

$$\kappa + m = |\mathbf{n}| = \operatorname{rank}(\mathbb{P}_{\mathbf{n}}) = \nu(\mathbb{P}_{\mathbf{n}}) + \pi(\mathbb{P}_{\mathbf{n}}),$$

which together with bounds (7.12) implies (7.7) for the case when $\mathbf{n} = \kappa + m$.

Let $|\mathbf{n}| > \kappa + m$, let $\widetilde{\mathbf{n}} = (\widetilde{n}_1,\dots,\widetilde{n}_m) \in \mathbb{N}^m$ be any tuple such that $|\widetilde{\mathbf{n}}| = \kappa + m$ and $\widetilde{n}_j \leqslant n_j$ for $j = 1,\dots,m$. By the preceding analysis, the matrix

$$\mathbb{P}_{\widetilde{\mathbf{n}}} := P^{\theta}_{\widetilde{\mathbf{n}}}(\mathbf{z}) - P^{b}_{\widetilde{\mathbf{n}}}(\mathbf{z})$$

is invertible and $\nu(\mathbb{P}_{\widetilde{\mathbf{n}}}) = \kappa$ and $\pi(\mathbb{P}_{\widetilde{\mathbf{n}}}) = m$. On the other hand, since $\mathbb{P}_{\widetilde{\mathbf{n}}}$ is a principal submatrix of $\mathbb{P}_{\mathbf{n}}$, we have

$$\nu(\mathbb{P}_{\mathbf{n}}) \geqslant \nu(\mathbb{P}_{\widetilde{\mathbf{n}}}) = \kappa, \quad \pi(\mathbb{P}_{\mathbf{n}}) \geqslant \pi(\mathbb{P}_{\widetilde{\mathbf{n}}}) = m$$

which together with bounds (7.12) complete the proof of (7.7). □

Proof of Theorem 4.3. Since f belongs to $\mathscr{B}_m/\mathscr{B}_\kappa$, it is of the form

$$f(z) = \frac{\theta(z)}{b(z)}, \quad \deg b = \kappa, \ \deg \theta = \widetilde{\kappa}. \tag{7.19}$$

Since $z_1,\dots,z_n\in\rho(f)$, it follows that $b(z_i)\neq 0$ for $i=1,\dots,\kappa$. We will show that

$$P_{\mathbf{n}}^{\theta}(\mathbf{z})-P_{\mathbf{n}}^{b}(\mathbf{z})=\mathbf{M}_{\mathbf{n}}^{b}P_{\mathbf{n}}^{f}(\mathbf{z})(\mathbf{M}_{\mathbf{n}}^{b})^{*} \tag{7.20}$$

where $\mathbf{M}_{\mathbf{n}}^{b}$ is the block diagonal matrix with lower triangular toeplitz diagonal blocks

$$\mathbf{M}_{\mathbf{n}}^{b}=\begin{bmatrix}\mathbf{M}_1^b & & 0\\ & \ddots & \\ 0 & & \mathbf{M}_k^b\end{bmatrix},\quad \text{where}\quad \mathbf{M}_i^b=\begin{bmatrix}b_{i,0} & 0 & \dots & 0\\ b_{i,1} & b_{i,0} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ b_{i,n_i-1} & \dots & b_{i,1} & b_{i,0}\end{bmatrix}, \tag{7.21}$$

where we have set $g_{i,j}=\frac{b^{(j)}(z_i)}{j!}$. Indeed, the self-evident identity

$$\frac{1-\theta(z)\theta(\zeta)^*}{1-z\overline{\zeta}}-\frac{1-b(z)b(\zeta)^*}{1-z\overline{\zeta}}=b(z)\frac{1-f(z)\overline{f(\zeta)}}{1-z\overline{\zeta}}\overline{b(\zeta)}$$

can be written in terms of the associated kernels as

$$K_b(z,\zeta)-K_\theta(z,\zeta)=b(z)K_f(z,\zeta)\overline{b(\zeta)}. \tag{7.22}$$

Upon applying $\frac{1}{\ell!r!}\frac{\partial^{\ell+r}}{\partial z^\ell\partial\overline{\zeta}^r}$ to both parts of (7.22) and evaluating the obtained equality at $z=z_i$ and $\zeta=z_j$ for all needed values if i,j,ℓ and r, we get equalities between the corresponding entries in the matrix equality (7.21). The matrix $\mathbf{M}_{\mathbf{n}}^{b}$ is invertible, since it is lower triangular and its diagonal entries $B(z_i)$ are all nonzero. Now all the statements in Theorem 4.3 follow from (7.20) by the corresponding statements in Lemma 7.2. □

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