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SOME QUADRATIC CORRECT EXTENSIONS OF MINIMAL OPERATORS IN BANACH SPACES

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Abstract. Let A_0 be a minimal operator from a complex Banach space X into X with finite defect, def $A_0 = m$, and $\widehat{A}$ is a linear correct extension of A_0. Let $E_c(A_0,\widehat{A})$ $\left(\text{resp. } E_c(A_0^2,\widehat{A}^2)\right)$ denote the set of all correct extensions B of A_0 with domain $D(B) = D(\widehat{A})$ $\left(\text{resp. } B_1 \text{ of } A_0^2\right.$ with $\left.D(B_1) = D(\widehat{A}^2)\right)$ and let $E_c^m(A_0,\widehat{A})$ $\left(\text{resp. } E_c^{m+k}(A_0^2,\widehat{A}^2), k \leqslant m, k,m \in \mathbf{N}\right)$ denote the subset of $E_c(A_0,\widehat{A})$ $\left(\text{resp. } E_c(A_0^2,\widehat{A}^2)\right)$ consisting of all $B \in E_c(A_0,\widehat{A})$ $\left(\text{resp. } E_c(A_0^2,\widehat{A}^2)\right)$ such that $\dim R(B-\widehat{A}) = m$ $\left(\text{resp. } \dim R(B_1-\widehat{A}^2) = m+k\right)$. In this paper:

1. we characterize the set of all operators $B_1 \in E_c^{m+k}(A_0^2,\widehat{A}^2)$ with the help of $\widehat{A}$ and some vectors S and G and give the solution of the problem $B_1x = f$,
2. we describe the subset $E_{2c}^{2m}(A_0^2,\widehat{A}^2)$ of all operators $B_2 \in E_c^{2m}(A_0^2,\widehat{A}^2)$ such that $B_2 = B^2$, where B is an operator of $E_c^m(A_0,\widehat{A})$ corresponding to B_2,
3. we give the solution of problems $B_2x = f$.

1. Introduction

An important tool in creating correct operators and solving boundary value problems containing differential or integro-differential equations is the correct extensions of minimal operators. Correct extensions of densely defined minimal operators in Banach and Hilbert spaces have been investigated by M. I. Vishik [6], A. A. Dezin [5], M. Otelbaev [13], R. Oinarov [1] and many others. Self-adjoint extensions of a densely defined minimal symmetric operator A_0 have been studied by a number of authors as Neumann J. Von [2], E. A. Coddington, A. Dijksma [8], A. N. Kochubei [10], V. A. Mikhailets [12], V. I. Gorbachuk and M. L. Gorbachuk [3]. Often they described the extensions as restrictions of some operators, usually of the adjoint operator A_0^* of A_0. In [7] and [11] have been studied extensions of nondensely defined symmetric operators. The correct restrictions B of some maximal operator A, when B is a product of correct restrictions B_1, B_2 of A, have been investigated by Shynibekov [14]. Our correct extensions are not, generally, restrictions of some maximal operator. The essential ingredient in our approach is the extension of the main idea in [1].

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The paper is organized as follows. In Section 2 we recall some basic terminology and notation about operators. In Sections 3, 4 we prove the main general results. Finally, in Section 5 we discuss some examples of integro-differential equations which are of mathematical interest and show the usefulness of our results.

2. Terminology and notation

Let X be a complex Banach space and X^* its adjoint space, i.e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(x)$ the value of f on x or in scalar product form $(f,x)_X$, where $f\in X^*, x\in X$. So we have $(f,x)_X = f(x)$ as in [16, p.191]. We consider f to be linear on x and x to be anti-linear on f, i.e. we have

$$(f,a_1x_1+a_2x_2)_X = a_1(f,x_1)_X + a_2(f,x_2)_X = a_1f(x_1)+a_2f(x_2),$$
$$(b_1f_1+b_2f_2,x)_X = \bar{b}_1(f_1,x)_X+\bar{b}_2(f_2,x)_X = \bar{b}_1f_1(x)+\bar{b}_2f_2(x),$$

where a_1,a_2,b_1,b_2 are complex numbers and $\bar{b}_1,\bar{b}_2$ are complex conjugates.

We note that in [9, p.11] (f,x) is defined by $(f,x)=\overline{f(x)}$.

We write $D(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a restriction of A_2, in symbol $A_1\subset A_2$, if $D(A_2)\supseteq D(A_1)$ and $A_1x=A_2x$, for all $x\in D(A_1)$. An operator $A: X\to X$ is called *closed* if for every sequence x_n in $D(A)$ converging to x_0 with $Ax_n\to f_0$, it follows that $x_0\in D(A)$ and $Ax_0=f_0$. A closed operator $A_0: X\to X$ is called *minimal* if $R(A_0)\neq X$ and the inverse A_0^{-1} exists on $R(A_0)$ and is continuous. A is called *maximal* if $R(A)=X$ and $\ker A\neq\{0\}$. An operator $\widehat{A}$ is called *correct* if $R(\widehat{A})=X$ and the inverse $\widehat{A}^{-1}$ exists and is continuous. An operator $\widehat{A}$ is called a *correct extension* (resp. restriction) of the minimal (resp. maximal) operator A_0 (resp. A) if it is a correct operator and $A_0\subset\widehat{A}$ (resp. $\widehat{A}\subset A$).

Let A be an operator from X into X with domain $D(A)$ dense in X. The *adjoint* operator $A^*: X^*\longrightarrow X^*$ of A with domain $D(A^*)$ is defined by the equation $(A^*y,x)_X=(y,Ax)_X$ for every $x\in D(A)$ and every $y\in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y\in X^*$ for which the functional $x\longmapsto (y,Ax)_X$ is continuous on $D(A)$. The *defect*, $\operatorname{def}A_0$, of an operator A_0 is the dimension of the annihilator $R(A_0)^\perp\subset X^*$ of its range $R(A_0)$.

If $\Phi_i\in X^*, i=1,\dots,m$, then we will write $\Phi=(\Phi_1,\dots,\Phi_m)$, $\Phi^k=(\Phi_1,\dots,\Phi_k)$, $k\leqslant m$, $(\Phi^m=\Phi)$, $\mathscr{F}_1=(\widehat{A}^{*-1}\Phi^k,\Phi)=(\widehat{A}^{*-1}\Phi_1,\dots,\widehat{A}^{*-1}\Phi_k,\Phi_1,\dots,\Phi_m)$, $k\leqslant m$ and $\widehat{A}^{-2}=(\widehat{A}^{-1})^2$. We will also write Φ^t and $(\Phi^t,Ax)_{X^m}$ for the column vectors $col(\Phi_1,\dots,\Phi_m)$ and $col((\Phi_1,Ax)_X,\dots,(\Phi_m,Ax)_X)$, respectively. Let $G=(G_1,\dots,G_m)$ be a vector of X^m. We will denote by M^t the transpose matrix of M and by $(\Phi^t,G)_{X^m}$ the $m\times m$ matrix whose i,j-th entry is the value of functional Φ_i on element G_j and by $(\Phi^{k^t},G)_{X^{k,m}}$ the $k\times m$ matrix whose i,j-th entry is the value of Φ_i on G_j. We will also denote by I_m and $[0]_m$ the identity $m\times m$ and the zero $m\times m$ matrices, respectively. By $\vec{0}$ we will denote the zero vector.

It is evident that for $m\times m$ matrix C holds $(\Phi^t,GC)_{X^m}=(\Phi^t,G)_{X^m}C$.

3. Some correct extensions of minimal operators in Banach spaces

We begin with the following lemma

LEMMA 3.1. *Let X be a complex Banach space and $\widehat{A}: X \to X$ a correct, densely defined operator.*

(i) The operator $A_0 \subset \widehat{A}$ is minimal with def $A_0 = \dim R(A_0)^{\perp} = m$ if and only if there exist linearly independent elements $\Phi_1, \ldots, \Phi_m$ of X^* such that

$$D(A_0) = \{x \in D(\widehat{A}) : \quad (\Phi^t, \widehat{A}x)_{X^m} = \vec{0}\}, \tag{3.1}$$

where $\Phi = (\Phi_1, \ldots, \Phi_m)$.

(ii) The adjoint operator $\widehat{A}^* : X^* \to X^*$ is correct.

Proof. (i) First we show the "if" part of the theorem. From $A_0 \subset \widehat{A}$, $\ker \widehat{A} = \{0\}$, $R(\widehat{A}) = X$ and (3.1) it follows easily that $\ker A_0 = \{0\}$, the inverse operator A_0^{-1} is a restriction of $\widehat{A}^{-1}$ and

$$R(A_0) = \{f \in X : (\Phi^t, f)_{X^m} = \vec{0}\} \quad \text{or} \quad R(A_0) = \bigcap_{i=1}^{m} \ker \Phi_i. \tag{3.2}$$

From (3.2) and the linear independence of $\Phi_1, \ldots, \Phi_m$ it follows that $\operatorname{def} A_0 = \dim R(A_0)^{\perp} = m$ and $\Phi_1, \ldots, \Phi_m$ is a basis of $R(A_0)^{\perp}$. Now we show that A_0 is a closed operator. Let $x_n \in D(A_0)$, $x_n \to x_0$ and $A_0 x_n \to f_0$. We put $f_n = A_0 x_n$. Then $f_n \to f_0$, $x_n = A_0^{-1} f_n = \widehat{A}^{-1} f_n \to x_0$. The operator $\widehat{A}^{-1}$ is closed because $\widehat{A}$ is correct. So $\widehat{A}^{-1} f_0 = x_0$ i.e. $f_0 = \widehat{A} x_0$. From (3.1) and $A_0 x_n \to f_0 = \widehat{A} x_0$ it follows that $(\Phi^t, \widehat{A} x_n)_{X^m} \to (\Phi^t, \widehat{A} x_0)_{X^m} = \vec{0}$, so $x_0 \in D(A_0)$. Hence $A_0 x_0 = f_0$ and so A_0 is a closed operator. The boundedness of $\widehat{A}^{-1}$ and that $A_0^{-1} \subset \widehat{A}^{-1}$ imply the boundedness of A_0^{-1}. Hence A_0 is minimal.

Now we show the "only if" part of the theorem. Let $\Phi_1, \ldots, \Phi_m$ a basis of $R(A_0)^{\perp}$. We will show (3.1). Let $x \in D(A_0)$. Then $(\Phi^t, A_0 x)_{X^m} = \vec{0}$ and, since $A_0 \subset \widehat{A}$, $(\Phi^t, \widehat{A} x)_{X^m} = \vec{0}$. Let now $x \in D(\widehat{A})$ such that $(\Phi^t, \widehat{A} x)_{X^m} = \vec{0}$. Then $\widehat{A} x \in R(A_0)^{\perp\perp}$. By the Bipolar theorem $R(A_0)^{\perp\perp} = \overline{R(A_0)} = R(A_0)$, since $R(A_0)$ is closed. Hence $x \in D(A_0)$.

(ii) It has been proved in [4, Theorem 5.3]

Throughout this paper $\widehat{A}$ will denote a correct densely defined operator on a complex Banach space X and A_0 a minimal restriction of $\widehat{A}$ with finite defect, def $A_0 = m$.

Our first Theorem 3.6 is implied from the following two Theorems 3.2, 3.4 proved in [1] [Theorem 2 and Theorem 3,i=2 respectively], which we present hear without proof. At first few words about the notation in these theorems. X, Y are complex Banach spaces, $A_0, A_p, A : X \to Y$ stand for, respectively, a minimal, a correct and a maximal operator. It holds that $A_0 \subset A_p \subset A$ and

$$Y = R(A_0) \dot{+} M, \qquad R(A_0) \bigcap M = \{0\}, \tag{3.3}$$

where $\dim M = n$. For the operator A_p holds

$$D(A_p) = \{x : x \in D(A), \Gamma x = 0\}, \tag{3.4}$$

where Γ is a closed linear operator from $D(\Gamma) \subset X$ into the boundary values space Z with $D(\Gamma) \supseteq D(A)$. The elements $F_1, \dots, F_n \in Y^*$ is a biorthogonal system to a basis $\phi_1, \dots, \phi_n$ of M. The symbol $R_r(A_0, A)$ denotes the set of all correct extensions of A_0 with domain in $D(A)$ and Ψ is the set of the vectors $\psi = (\psi_1, \dots, \psi_n) \in D(A)^n$ satisfying the condition α :

$$\Gamma \psi F(f) = 0. \tag{3.5}$$

$$[I_n + F(A\psi)]F(f) = \vec{0} \quad \text{or} \quad dF(f) = \vec{0} \tag{3.6}$$

implies $F(f) = \vec{0}$, where $F = col(F_1, \dots, F_n)$.

THEOREM 3.2. *Let A_0 be a minimal operator satisfying* (3.3)*. Then:*
(i) For every $A_\psi \in R_r(A_0, A)$, there exists a vector $\psi \in \Psi$ such that

$$A_\psi^{-1} f = A_p^{-1} f + \psi F(f), \quad f \in Y, \tag{3.7}$$

where $F = col(F_1, \dots, F_n)$, $F_i \in Y^*$, $\bigcap_{i=1}^n \ker F_i = R(A_0)$.
(ii) Conversely, for every $\psi \in \Psi$, there exists an operator $A_\psi \in R_r(A_0, A)$ such that (3.7) *holds.*

REMARK 3.3. The vector $\psi \in \Psi$ in Theorem 3.2 (i) is unique. Indeed, suppose that for an $A_\psi \in R_r(A_0, A)$ and two vectors $\psi^{(1)}$, $\psi^{(2)} \in \Psi$ holds $A_\psi^{-1} f = A_p^{-1} f + \psi^{(1)} F(f) = A_p^{-1} f + \psi^{(2)} F(f)$, $\forall f \in Y$. Then $(\psi^{(1)} - \psi^{(2)}) F(f) = 0$, for every $f \in Y$. The last, since the components of the vector F are linearly independent, implies $\psi^{(1)} = \psi^{(2)}$.

By virtue of the previous theorem the following set is defined $R_r^2(A_0, A) \stackrel{\text{def}}{=} \{A_\psi \in R_r(A_0, A) : A_\psi^{-1} f \stackrel{\text{Th. 3.2}}{=} A_p^{-1} f + \psi F(f), \psi \in \Psi_2, f \in Y\}$, where $\Psi_2 = \{\psi \in \Psi : \det[I_n + F(A\psi)] = \det d \neq 0\}$ [1, p.44].

Next theorem is Theorem 3 of [1] for i=2.

THEOREM 3.4. *(i) For every $A_\psi \in R_r^2(A_0, A)$, there exists a vector $\psi \in \Psi_2$ such that*

$$A_\psi x = Ax - A\psi d^{-1} F(Ax), \quad x \in D(A_\psi), \tag{3.8}$$

$$D(A_\psi) = \{x \in D(A) : \Gamma x = \Gamma \psi d^{-1} F(Ax)\}. \tag{3.9}$$

(ii) Conversely, for every $\psi \in \Psi_2$, the operator A_ψ defined by (3.9) *and* (3.8) *belongs to $R_r^2(A_0, A)$.*

We define

$$R_r^2(A_0, A_p) \stackrel{\text{def}}{=} \{A_\psi \in R_r^2(A_0, A) : D(A_\psi) = D(A_p)\}$$

and

$$\Psi_2(A_p) \stackrel{\text{def}}{=} \{\psi \in \Psi_2 : \psi \in D(A_p)^n\}.$$

It is evident that

$$R_r^2(A_0,A_p) \subset R_r^2(A_0,A) \subset R_r(A_0,A) \text{ and } \Psi_2(A_p) \subset \Psi_2 \subset \Psi. \tag{3.10}$$

From Theorem 3.4 follows the next corollary

COROLLARY 3.5. *(i) For every $A_\psi \in R_r^2(A_0,A_p)$, there exists a vector $\psi \in \Psi_2(A_p)$ such that*

$$A_\psi x = A_p x - A_p \psi d^{-1} F(A_p x) = f, \quad D(A_\psi) = D(A_p). \tag{3.11}$$

(ii) Conversely, for every $\psi \in \Psi_2(A_p)$, the operator A_ψ defined by (3.11) *belongs to $R_r^2(A_0,A_p)$.*

(iii) The unique solution of (3.11)*, when B is correct, is given by*

$$x = A_p^{-1} f + \psi F(f), \quad f \in Y. \tag{3.12}$$

Proof. (i) Let $A_\psi \in R_r^2(A_0,A_p)$. Then $D(A_\psi) = D(A_p)$ and, since (3.10), $A_\psi \in R_r^2(A_0,A)$. By Theorem 3.4, there exists the vector $\psi \in \Psi_2$ such that (3.8), (3.9) hold. From (3.9), since $D(A_\psi) = D(A_p)$, and (3.4), it follows that $\Gamma\psi d^{-1}F(Ax) = 0$. This, since the components of the vector F are linearly independent elements and $R(A) = Y$, implies $\Gamma\psi = \vec{0}$. Hence $\psi \in D(A_p)^n$ and so $\psi \in \Psi_2(A_p)$. From (3.8) follows easily (3.11).

(ii) Let $\psi \in \Psi_2(A_p)$. Then $\psi \in \Psi_2 \bigcap D(A_p)^n$, $\Gamma\psi = \vec{0}$. By Theorem 3.4, the corresponding operator A_ψ defined by (3.8), (3.9) belongs to $R_r^2(A_0,A)$. Also (3.9) implies $D(A_\psi) = D(A_p)$. This equality and (3.8) imply (3.11) and so $A_\psi \in R_r^2(A_0,A_p)$.

(iii) Let $A_\psi \in R_r^2(A_0,A_p)$ and $A_\psi x = f$. Then $x = A_\psi^{-1} f$. By Theorem 3.2 and Remark 3.3, there exists the unique vector $\psi \in \Psi$ such that (3.7) holds. So the unique solution of (3.11) is given by (3.12).

If in the above corollary instead of A_ψ, A_p, F, Y and $R_r^2(A_0,A_p)$ we use the symbols $B, \widehat{A}, \Phi, X$ and $E_c(A_0,\widehat{A})$ respectively, then we get the next theorem.

THEOREM 3.6. *Suppose that $\Phi, A_0, \widehat{A}$ are as in Lemma 3.1. Then:*

(i) For every $B \in E_c(A_0,\widehat{A})$, there exists a vector $\psi = (\psi_1,\ldots,\psi_m)$ with $\psi_i \in D(\widehat{A}), i = 1,\ldots,m$ such that

$$det\, d = det\left[I_m + (\Phi^t, \widehat{A}\psi)_{X^m}\right] \neq 0, \tag{3.13}$$

$$Bx = \widehat{A}x - \widehat{A}\psi d^{-1}(\Phi^t, \widehat{A}x)_{X^m} = f, \quad D(B) = D(\widehat{A}), f \in X. \tag{3.14}$$

(ii) Conversely, for every vector $\psi = (\psi_1,\ldots,\psi_m)$ with $\psi_i \in D(\widehat{A}), i = 1,\ldots,m$, which satisfies (3.13)*, the operator B defined by* (3.14) *belongs to $E_c(A_0,\widehat{A})$.*

(iii) If B is correct, then the unique solution of (3.14) *is given by*

$$x = B^{-1} f = \widehat{A}^{-1} f + \psi(\Phi^t, f)_{X^m}. \tag{3.15}$$

From this theorem follows the next one:

THEOREM 3.7. *We suppose that* $\Phi, A_0, \widehat{A}$ *are as in Lemma* 3.1. *Then:*

(i) For every $B \in E_c^m(A_0,\widehat{A})$, *there exists a unique vector* $G = (g_1,\dots,g_m)$, *where* $g_1,\dots,g_m$ *are linearly independent elements of* X, *such that*

$$det\, W = det\left[I_m - (\Phi^t, G)_{X^m}\right] \neq 0, \tag{3.16}$$

$$Bx = \widehat{A}x - G(\Phi^t, \widehat{A}x)_{X^m} = f, \quad D(B) = D(\widehat{A}),\, f \in X. \tag{3.17}$$

(ii) Conversely, for every vector $G = (g_1,\dots,g_m)$, $g_1,\dots,g_m \in X$ *which satisfies* (3.16) *and has exactly* n *linearly independent components (* $n \leqslant m$*), the operator* B *defined by* (3.17) *belongs to* $E_c^n(A_0,\widehat{A})$.

(iii) The unique solution of (3.17), *when* B *is correct, is given by*

$$x = B^{-1}f = \widehat{A}^{-1}f + (\widehat{A}^{-1}G)\left[I_m - (\Phi^t, G)_{X^m}\right]^{-1}(\Phi^t, f)_{X^m}. \tag{3.18}$$

Proof. (i) Let $B \in E_c^m(A_0,\widehat{A})$. Then, by Theorem 3.6, there exists a vector ψ such that (3.13) and (3.14) hold true. We put $G = \widehat{A}\psi d^{-1}$. Then $(\Phi^t, G)_{X^m} = (\Phi^t, \widehat{A}\psi)_{X^m} d^{-1}$ $= \left[\left((\Phi^t, \widehat{A}\psi)_{X^m} + I_m\right) - I_m\right] d^{-1} = (d - I_m)d^{-1} = I_m - d^{-1}$. Then $d^{-1} = I_m - (\Phi^t, G)_{X^m}$ $= W$ and $\det W \neq 0$. From (3.14), by putting $\widehat{A}\psi d^{-1} = G$, we obtain (3.17) or $(B - \widehat{A})x = -G(\Phi^t, \widehat{A}x)_{X^m}$ for all $x \in D(\widehat{A})$. Since $\dim R(B - \widehat{A}) = m$, the elements $\Phi_1,\dots,\Phi_m$ are linearly independent and $\widehat{A}$ is correct, it follows that the elements $g_1,\dots,g_m$ are linearly independent. Suppose now there exist two vectors G_1 and G_2 such that $Bx = \widehat{A}x - G_1(\Phi^t, \widehat{A}x)_{X^m} = \widehat{A}x - G_2(\Phi^t, \widehat{A}x)_{X^m}$. Then $(G_1 - G_2)(\Phi^t, \widehat{A}x)_{X^m} = 0$ for all $x \in D(\widehat{A})$, which implies, since the vector Φ has m linearly independent components and $R(\widehat{A}) = X$, $G_1 = G_2$.

(ii) Conversely, let G be a vector defined as in (ii) such that $\det W \neq 0$. Since $R(\widehat{A}) = X$, there exists a vector $\psi = (\psi_1,\dots,\psi_m)$ with $\psi_i \in D(\widehat{A})$, $i = 1,\dots,m$ such that $\widehat{A}\psi = GW^{-1}$. Then $d = I_m + (\Phi^t, \widehat{A}\psi)_{X^m} = I_m + (\Phi^t, G)_{X^m} W^{-1} = I_m + (I_m - W)W^{-1} = W^{-1}$. Hence $d = W^{-1}$ and $\det d \neq 0$. Then $G = \widehat{A}\psi W = \widehat{A}\psi d^{-1}$. If we substitute G in (3.17) we take (3.14) and, by Theorem 3.6, the operator B is correct. Now using the proof of (i) it is easy to see that $\dim R(B - \widehat{A}) = n$.

(iii) From $G = \widehat{A}\psi W$ we get $\psi = (\widehat{A}^{-1}G)W^{-1}$ and if substitute this in (3.15) we get (3.18). The theorem has been proved.

Next theorem gives a criterion of correctness and is useful in applications.

THEOREM 3.8. *Let* $\widehat{A}$ *be a correct operator on* X, *the components of the vector* $\Phi = (\Phi_1,\dots,\Phi_m)$ *be linearly independent elements of* X^* *and* $G = (g_1,\dots,g_m) \in X^m$. *Then:*

(i) The operator B *defined by* (3.17) *is correct if and only if (*3.16*) holds true.*

(ii) If B *is correct, then* $\dim R(B - \widehat{A}) = n \leqslant m$ *iff the vector* G *has exactly* n *linearly independent components (* $n \leqslant m$ *).*

(iii) If B *is correct, then the unique solution of* (3.17) *is given by* (3.18).

Proof. (i) Let the operator B defined by (3.17) is correct. We define for the problem (3.17) the minimal operator A_0 by (3.1).

If $n = m$, then the theorem is true by Theorem 3.7.

If $n < m$, then by using (3.17) we have

$$\begin{aligned}(\Phi^t, f)_{X^m} &= (\Phi^t, \widehat{A}x)_{X^m} - (\Phi^t, G)_{X^m}(\Phi^t, \widehat{A}x)_{X^m} \\ &= \left[I_m - (\Phi^t, G)_{X^m}\right](\Phi^t, \widehat{A}x)_{X^m}\end{aligned}$$

or

$$\left[I_m - (\Phi^t, G)_{X^m}\right](\Phi^t, \widehat{A}x)_{X^m} = (\Phi^t, f)_{X^m}, \quad \text{for all} \quad f \in X.$$

Let $z_1, \ldots, z_m$ biorthogonal to $\Phi_1, \ldots, \Phi_m$, i.e. $(\Phi_i, z_j) = \delta_{i,j}$, $i, j = 1, \ldots, m$ and $W = I_m - (\Phi^t, G)_{X^m}$. Suppose that rank $W = k < m$ and that the first k lines of the matrix W are linearly independent. Then for $f = z_{k+1}$ the system $W(\Phi^t, \widehat{A}x)_{X^m} = (\Phi^t, f)_{X^m}$ has no solution since the rank of the augmented matrix is $k+1 \neq k$. Then $Bx = z_{k+1}$ has no solution and $R(B) \neq X$. Consequently B is not a correct operator. So (3.16) holds true.

Conversely, let $\det W \neq 0$ and that G has n linearly independent components, $n \leqslant m$. Then, by Theorem 3.7, $B \in E_c^n(A_0, \widehat{A})$.

The cases (ii) and (iii) are proved as in Theorem 3.7.

If the elements $\Phi_1, \ldots, \Phi_m$ are not linearly independent, then we have the following theorem.

THEOREM 3.9. *Let the operator B be defined by* (3.17)*, where $\Phi \in X^{*m}, G \in X^m$. We suppose that the components of $\Phi^k = (\Phi_1, \ldots, \Phi_k)\,(k < m)$ are linearly independent elements and the components of $\mathbf{\Phi}_{m-k} = (\Phi_{k+1}, \ldots, \Phi_m)$ are linear combinations of $\Phi_1, \ldots, \Phi_k$. Let $\Phi = (\Phi^k, \mathbf{\Phi}_{m-k})$, $G = (G^k, G_{m-k})$ and $M_{m-k,k}$ the matrix such that $\mathbf{\Phi}^t_{m-k} = M_{m-k,k}\Phi^{k^t}$. Then:*

$$(i) \qquad Bx = \widehat{A}x - G_M^k(\Phi^{k^t}, \widehat{A}x)_{X^k} = f, \qquad D(B) = D(\widehat{A}), \tag{3.19}$$

where $G_M^k = G^k + G_{m-k}\overline{M}_{m-k,k}$.

(ii) B is correct if and only if (3.16) *holds true, or equivalently*

$$det\, W_k = det\left[I_k - (\Phi^{k^t}, G_M^k)_{X^k}\right] \neq 0. \tag{3.20}$$

(iii) If B is correct, then the unique solution of (3.17) *or* (3.19) *is given by* (3.18)*, or by*

$$x = B^{-1}f = \widehat{A}^{-1}f + (\widehat{A}^{-1}G_M^k)\left[I_k - (\Phi^{k^t}, G_M^k)_{X^k}\right]^{-1}(\Phi^{k^t}, f)_{X^k}. \tag{3.21}$$

Proof. (i) Using the symbolism $\Phi = (\Phi^k, \mathbf{\Phi}_{m-k})$, $G = (G^k, G_{m-k})$ we obtain

$$\begin{aligned}G(\Phi^t, \widehat{A}x)_{X^m} &= G^k(\Phi^{k^t}, \widehat{A}x)_{X^k} + G_{m-k}\overline{M}_{m-k,k}(\Phi^{k^t}, \widehat{A}x)_{X^k} \\ &= (G^k + G_{m-k}\overline{M}_{m-k,k})(\Phi^{k^t}, \widehat{A}x)_{X^k} = G_M^k(\Phi^{k^t}, \widehat{A}x)_{X^k}.\end{aligned}$$

Hence, by substituting in (3.17) $G(\Phi^t,\widehat{A}x)_{X^m}$ by $G^k_M(\Phi^{k^t},\widehat{A}x)_{X^k}$, we get (3.19).

(ii) Using again the symbolism $\Phi=(\Phi^k,\mathbf{\Phi}_{m-k})$, $G=(G^k,G_{m-k})$, we find:

$$\begin{aligned}\det W &= \det\left[I_m-(\Phi^t,G)_{X^m}\right]\\ &=(-1)^m\det\begin{pmatrix}(\Phi^{k^t},G^k)_{X^k}-I_k & (\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\\ (\mathbf{\Phi}^t_{m-k},G^k)_{X^{m-k,k}} & (\mathbf{\Phi}^t_{m-k},G_{m-k})_{X^{m-k}}-I_{m-k}\end{pmatrix}\\ &=(-1)^m\det\begin{pmatrix}(\Phi^{k^t},G^k)_{X^k}-I_k & (\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\\ \overline{M}_{m-k,k}(\Phi^{k^t},G^k)_{X^{m-k,k}} & \overline{M}_{m-k,k}(\Phi^{k^t},G_{m-k})_{X^{m-k}}-I_{m-k}\end{pmatrix}.\end{aligned}$$

Multiplying from the left the first line of the last determinant by the matrix $-\overline{M}_{m-k,k}$ and adding to the second line of the determinant, we take

$$\begin{aligned}\det W &= (-1)^m\det\begin{pmatrix}(\Phi^{k^t},G^k)_{X^k}-I_k & (\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\\ \overline{M}_{m-k,k} & -I_{m-k}\end{pmatrix}\\ &=(-1)^m\det\begin{pmatrix}(\Phi^{k^t},G^k)_{X^k}-I_k+(\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\overline{M}_{m-k,k} & (\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\\ [0]_{m-k,k} & -I_{m-k}\end{pmatrix}\\ &=\det\left[I_k-(\Phi^{k^t},G^k)_{X^k}-(\Phi^{k^t},G_{m-k})_{X^{k,m-k}}\overline{M}_{m-k,k}\right]\\ &=\det\left[I_k-(\Phi^{k^t},G^k+G_{m-k}\overline{M}_{m-k,k})_{X^k}\right]=\det\left[I_k-(\Phi^{k^t},G^k_M)_{X^k}\right]=\det W_k.\end{aligned}$$

By Theorem 3.8 (i), since $\Phi_1,\dots,\Phi_k$ are linearly independent and $G(\Phi^t,\widehat{A}x)_{X^m}=G^k_M(\Phi^{k^t},\widehat{A}x)_{X^k}$, $\det W=\det W_k$, the operator B defined by (3.19) is correct iff $\det W_k\neq 0$ and

(iii) then the unique solution of (3.19) is given by (3.21).

REMARK 3.10.

1. If $\Phi_1,\dots,\Phi_m$ are linearly dependent, then the operator B, as we saw in the previous theorem, can be defined either by (3.17) or by (3.19). Since the solution of $Bx=f$ is unique, it follows, by comparing (3.18) and (3.21), that

$$(\widehat{A}^{-1}G)W^{-1}(\Phi^t,f)_{X^m}=(\widehat{A}^{-1}G^k_M)W_k^{-1}(\Phi^{k^t},f)_{X^k}.$$

2. The previous theorem shows that the correctness of the operator B and the solution of $Bx=f$ do not depend on the linear independence of the elements $\Phi_1,\dots,\Phi_m$. The correctness condition of $Bx=f$ is $\det W\neq 0$ or $\det W_k\neq 0$. The linear independence of $\Phi_1,\dots,\Phi_m$ is needed to determine the $\dim R(B-\widehat{A})$ and to prove the existence of the unique vector G for every operator $B\in E^m_c(A_0,\widehat{A})$.

3. The determinant $\det W_k$ and the solution (3.21) are simpler than $\det W$ and the solution (3.18) respectively.

From Theorems 3.8, 3.9, since $\det W=\det W_k$ and $G(\Phi^t,\widehat{A}x)_{X^m}=G^k_M(\Phi^{k^t},\widehat{A}x)_{X^k}$, it follows (see Remark 3.10 (2)) next corollary, where the components of the vectors Φ and G are arbitrary elements of X^* and X respectively.

COROLLARY 3.11. *Let $\widehat{A}$ be a correct operator on X and the components of the vectors $\Phi = (\Phi_1, \ldots, \Phi_m)$, $G = (g_1, \ldots, g_m)$ are arbitrary elements of X^* and X respectively. Then the operator B defined by* (3.17) *is correct if and only if* (3.16) *holds true. If B is correct, then the unique solution of* (3.17) *is given by* (3.18).

4. Some quadratic correct extensions of minimal operators in Banach spaces

Next lemma holds true for any minimal operator A_0 and its correct extension $\widehat{A}$.

LEMMA 4.1. *Let $A_0 : X \to X$ be a minimal operator and $\widehat{A}$ a correct extension of A_0. Then:*
(i) A_0^2 is a minimal operator on X.
(ii) $\widehat{A}^2$ is a correct extension of A_0^2 on X.

Proof. (i) First we show that A_0^2 is a closed operator. Suppose that $x_n \to x$ and $A_0^2 x_n = f_n \to f$, where $x_n \in D(A_0^2)$, $f_n \in R(A_0^2)$ and $x, f \in X$, $n \in N$. We denote by $y_n = A_0 x_n = A_0^{-1} f_n$, where $y_n \in D(A_0)$. Since A_0^{-1} is bounded and $(f_n)_{n=1}^{\infty}$ is a convergent sequence, y_n converges to some $y \in X$. But A_0 is closed, therefore $x \in D(A_0)$ and $A_0 x = y$. Then we have $y_n \to y$, $A_0^2 x_n = A_0 y_n \to f$. Since A_0 is closed, it follows $y \in D(A_0)$ and $A_0 y = f$ or $x \in D(A_0^2)$ and $A_0^2 x = f$. Hence A_0^2 is a closed operator. Now we show that $R(A_0^2) \neq X$ and that there exists the inverse operator $(A_0^2)^{-1}$, denoted by A_0^{-2}, and that this is a bounded operator. From the evident inclusion $R(A_0^2) \subseteq R(A_0)$ and $R(A_0) \neq X$ it follows that $R(A_0^2) \neq X$. From $A_0^2 x = f$, where $x \in D(A_0^2)$, $f \in R(A_0^2)$, we have $A_0 x = A_0^{-1} f$ and $x = (A_0^{-1})^2 f$, which is the unique solution of $A_0^2 x = f$. Hence, there exists the operator $(A_0^2)^{-1}$ on $R(A_0^2)$ and is equal $(A_0^{-1})^2$. The operator A_0^{-2} is bounded since $(A_0^{-1})^2$ is bounded and so A_0^2 is minimal.

(ii) Since $\widehat{A}$ is a correct operator, the equation $\widehat{A}^2 u = f$, for each $f \in X$, has the unique solution $u = (\widehat{A}^{-1})^2 f = \widehat{A}^{-2} f$. Then $R(\widehat{A}^2) = X$ and $\widehat{A}^{-2}$ is bounded on X. Hence $\widehat{A}^2$ is correct. Let $x \in D(A_0^2)$. Then $x \in D(\widehat{A}^2)$ and since $A_0 \subset \widehat{A}$ we obtain $A_0^2 x = \widehat{A}^2 x$. Hence $A_0^2 \subset \widehat{A}^2$. So the lemma has been proved.

REMARK 4.2. From the proof of (ii) it is evident that if $\widehat{A}$ is correct on X, then $\widehat{A}^2$ is also correct.

Let the operators $\widehat{A}$ and A_0 and vector Φ be defined as in Lemma 3.1, $k \leqslant m$ and the elements

$$\widehat{A}^{*-1}\Phi_{k+1}, \ldots, \widehat{A}^{*-1}\Phi_m \in \mathscr{L}(\Phi_1, \ldots, \Phi_m, \widehat{A}^{*-1}\Phi_1, \ldots, \widehat{A}^{*-1}\Phi_k). \tag{4.1}$$

In the sequel we will make use of the following condition (LI): the components of the vector

$$\mathscr{F}_1 = (\widehat{A}^{*-1}\Phi^k, \Phi) = (\widehat{A}^{*-1}\Phi_1, \ldots, \widehat{A}^{*-1}\Phi_k, \Phi_1, \ldots, \Phi_m), k \leqslant m$$

are linearly independent elements of $R(A_0^2)^{\perp} \subset X^*$.

From (3.1) and (4.1) it follows that

$$D(A_0^2) = \{x \in D(\widehat{A}^2) : \quad (\Phi^{kt}, \widehat{A}x)_{X^k} = \vec{0}, \quad (\Phi^t, \widehat{A}^2 x)_{X^m} = \vec{0}\} \tag{4.2}$$

or

$$D(A_0^2) = \{x \in D(\widehat{A}^2) : \quad (\mathscr{F}_1^t, \widehat{A}^2 x)_{X^{k+m}} = \vec{0}\}. \tag{4.3}$$

Then $R(A_0^2) = \{f \in X : (\mathscr{F}_1^t, f)_{X^{k+m}} = \vec{0}\}$. It is evident that $\operatorname{def} A_0^2 = \dim R(A_0^2)^{\perp} = m+k$ and that the components of the vector $\mathscr{F}_1$ is a basis of $R(A_0^2)^{\perp} \subset X^*$. From Lemma 4.1 it follows that the operator A_0^2 is a minimal restriction of the correct operator $\widehat{A}^2$. Then, by Theorem 3.7, we can easily describe the set $E_c^{k+m}(A_0^2, \widehat{A}^2)$ of all correct extensions B_1 of the minimal operator A_0^2 using its correct extension $\widehat{A}^2$. We have the following theorem.

THEOREM 4.3. *We suppose that* $A_0, \widehat{A}$ *are as in Lemma* 3.1, A_0^2 *is defined by* (4.3) *and* $\mathscr{F}_1$ *satisfies condition (LI). Then:*

(i) For every $B_1 \in E_c^{k+m}(A_0^2, \widehat{A}^2)$, *there exists a unique vector* $\mathscr{G} \in X^{k+m}$ *with linearly independent components of* X, *such that*

$$det\, W_1 = det\left[I_{k+m} - (\mathscr{F}_1^t, \mathscr{G})_{X^{k+m}}\right] \neq 0, \tag{4.4}$$

$$B_1 x = \widehat{A}^2 x - \mathscr{G}(\mathscr{F}_1^t, \widehat{A}^2 x)_{X^{k+m}} = f, \quad D(B_1) = D(\widehat{A}^2),\, f \in X. \tag{4.5}$$

(ii) Conversely, for every vector $\mathscr{G} \in X^{k+m}$ *which satisfies* (4.4) *and has exactly* n *linearly independent components* ($n \leqslant k+m$), *the operator* B_1 *defined by* (4.5) *belongs to* $E_c^n(A_0^2, \widehat{A}^2)$.

(iii) If B_1 *is correct, then the unique solution of* (4.5) *is given by*

$$x = B_1^{-1} f = \widehat{A}^{-2} f + (\widehat{A}^{-2}\mathscr{G})\left[I_{k+m} - (\mathscr{F}_1^t, \mathscr{G})_{X^{k+m}}\right]^{-1} (\mathscr{F}_1^t, f)_{X^{k+m}}. \tag{4.6}$$

From the above theorem it follows the next one which shows that every operator $B_1 \in E_c^{k+m}(A_0^2, \widehat{A}^2)$ can be uniquely determined by two vectors S and G of length k and m respectively. The solution of $B_1 x = f$ is also obtained.

THEOREM 4.4. *We suppose that* $A_0, \widehat{A}$ *are as in Lemma* 3.1, A_0^2 *is defined by* (4.3) *and* $\mathscr{F}_1$ *satisfies condition (LI). Then:*

(i) For every $B_1 \in E_c^{k+m}(A_0^2, \widehat{A}^2)$, *there exists a unique pair of vectors* $S = (s_1, \ldots, s_k)$, $G = (g_1, \ldots, g_m)$, *with* $s_1, \ldots, s_k, g_1, \ldots, g_m$ ($k \leqslant m$) *linearly independent elements of* X *such that*

$$\det W_1 = \det \begin{pmatrix} (\Phi^{kt}, \widehat{A}^{-1} S)_{X^k} - I_k & (\Phi^{kt}, \widehat{A}^{-1} G)_{X^{km}} \\ (\Phi^t, S)_{X^{mk}} & (\Phi^t, G)_{X^m} - I_m \end{pmatrix} \neq 0 \tag{4.7}$$

and for all $x \in D(B_1) = D(\widehat{A}^2)$ *we have*

$$B_1 x = \widehat{A}^2 x - S(\Phi^{kt}, \widehat{A}x)_{X^k} - G(\Phi^t, \widehat{A}^2 x)_{X^m} = f. \tag{4.8}$$

(ii) Conversely, for every pair of vectors $S=(s_1,\ldots,s_k)$, $G=(g_1,\ldots,g_m)$ with components from X and such that the vector $\mathscr{G}=(S,G)=(s_1,\ldots,s_k,g_1,\ldots,g_m)$ satisfies (4.7) and has exactly n linearly independent elements, $n\leqslant k+m$, the operator B_1 defined by (4.8) belongs to $E_c^n(A_0^2,\widehat{A}^2)$.

(iii) If B_1 is correct, then the unique solution of (4.8), for every $f\in X$, is given by

$$x=\widehat{A}^{-2}f-\widehat{A}^{-2}(S,G)\cdot \cdot\begin{pmatrix}(\Phi^{k^t},\widehat{A}^{-1}S)_{X^k}-I_k & (\Phi^{k^t},\widehat{A}^{-1}G)_{X^{km}}\\ (\Phi^t,S)_{X^{mk}} & (\Phi^t,G)_{X^m}-I_m\end{pmatrix}^{-1}\begin{pmatrix}(\Phi^{k^t},\widehat{A}^{-1}f)_{X^k}\\ (\Phi^t,f)_{X^m}\end{pmatrix}. \tag{4.9}$$

Proof. (i) From Theorem 4.3, there exists a unique vector $\mathscr{G}\in X^{k+m}$ with linearly independent components such that (4.4) and (4.5) hold true. If we put $\mathscr{G}=(S,G)=(s_1,\ldots,s_k,g_1,\ldots,g_m)$ we obtain for the matrix W_1 in (4.4)

$$I_{k+m}-(\mathscr{F}_1^t,\mathscr{G})_{X^{k+m}}=-\begin{pmatrix}(\widehat{A}^{*-1}\Phi^{k^t},S)_{X^k}-I_k & (\widehat{A}^{*-1}\Phi^{k^t},G)_{X^{km}}\\ (\Phi^t,S)_{X^{mk}} & (\Phi^t,G)_{X^m}-I_m\end{pmatrix}.$$

Since $\widehat{A}$ is a correct operator, we have $\widehat{A}^{*-1}=\widehat{A}^{-1^*}$ [15]. Taking this into account the above equality is written in the form

$$I_{k+m}-(\mathscr{F}_1^t,\mathscr{G})_{X^{k+m}}=-\begin{pmatrix}(\Phi^{k^t},\widehat{A}^{-1}S)_{X^k}-I_k & (\Phi^{k^t},\widehat{A}^{-1}G)_{X^{km}}\\ (\Phi^t,S)_{X^{mk}} & (\Phi^t,G)_{X^m}-I_m\end{pmatrix}, \tag{4.10}$$

which shows that (4.4) is equivalent to (4.7). Since $\mathscr{G}(\mathscr{F}_1^t,\widehat{A}^2x)_{X^{k+m}}=S(\Phi^{k^t},\widehat{A}x)_{X^k}+G(\Phi^t,\widehat{A}^2x)_{X^m}$, (4.5) implies (4.8) and conversely. The uniqueness of the vectors S, G follows immediately from the uniqueness of the vector $\mathscr{G}$ (Theorem 4.3).

(ii) If the vector $\mathscr{G}=(S,G)$ has n linearly independent components, then, from the previous theorem, the operator B_1 defined by (4.5) belongs to $E_c^n(A_0^2,\widehat{A}^2)$.

(iii) Using (4.10), from (4.6) we get (4.9) .

Below by B_G and B_{SG} we will denote the operators defined by the vector G and the pair of vectors (S,G), respectively, by

$$B_Gx=\widehat{A}x-G(\Phi^t,\widehat{A}x)_{X^m}=f, \qquad D(B_G)=D(\widehat{A}), \tag{4.11}$$

$$B_{SG}x=\widehat{A}^2x-S(\Phi^t,\widehat{A}x)_{X^m}-G(\Phi^t,\widehat{A}^2x)_{X^m}=f, \quad D(B_{SG})=D(\widehat{A}^2), \tag{4.12}$$

where $S=(s_1,\ldots,s_m)$, $G=(g_1,\ldots,g_m)\in X^m$, the components of the vector $\Phi=(\Phi_1,\ldots,\Phi_m)$ are linearly independent elements of X^* and $\widehat{A}$ is a correct densely defined operator on X. We note that the operator B_G (resp. B_{SG}) is an extension of the minimal operator A_0 (resp. A_0^2), where

$$A_0\subset\widehat{A}, \quad D(A_0)=\{x\in D(\widehat{A}): \quad (\Phi^t,\widehat{A}x)_{X^m}=\vec{0}\}, \tag{4.13}$$

$$A_0^2\subset\widehat{A}^2, D(A_0^2)=\{x\in D(\widehat{A}^2): (\Phi^t,\widehat{A}x)_{X^m}=\vec{0}, (\Phi^t,\widehat{A}^2x)_{X^m}=\vec{0}\}. \tag{4.14}$$

We define the set

$$E_{2c}(A_0^2,\widehat{A}^2)=\{B_2\in E_c(A_0^2,\widehat{A}^2): \text{there exists an operator} B\in E_c(A_0,\widehat{A}) \\ \text{such that} B_2=B^2\}. \tag{4.15}$$

LEMMA 4.5. *For the operator B_G, defined by (4.11), hold true the statements:*
(i) $D(B_G^2)=D(\widehat{A}^2)$ *if and only if* $G\in D(\widehat{A})^m$.
(ii) *If* $G\in D(\widehat{A})^m$ *then the operator B_G^2 is defined by*

$$B_G^2x=\widehat{A}^2x-\big[\widehat{A}G-G(\Phi^t,\widehat{A}G)_{X^m}\big](\Phi^t,\widehat{A}x)_{X^m}-G(\Phi^t,\widehat{A}^2x)_{X^m}. \tag{4.16}$$

$$or \qquad B_G^2x=\widehat{A}^2x-B_GG(\Phi^t,\widehat{A}x)_{X^m}-G(\Phi^t,\widehat{A}^2x)_{X^m}. \tag{4.17}$$

Proof. (i) Let $x\in D(B_G^2)=D(\widehat{A}^2)$. Then $B_Gx=\widehat{A}x-G(\Phi^t,\widehat{A}x)_{X^m}\in D(\widehat{A})$ and since the operator $\widehat{A}$ is correct, it follows that $G\in D(\widehat{A})^m$.

Conversely, let $G\in D(\widehat{A})^m$. If $x\in D(B_G^2)$, then $x\in D(\widehat{A})$ and $B_Gx=\widehat{A}x-G(\Phi^t,\widehat{A}x)_{X^m}\in D(\widehat{A})$, which implies $x\in D(\widehat{A}^2)$.

If $x\in D(\widehat{A}^2)$, then $B_Gx\in D(\widehat{A})=D(B_G)$. So $x\in D(B_G^2)$.

(ii) We find the formula of the operator B_G^2. Let $x\in D(B_G^2)$, $y=B_Gx$. Then since (4.11) and the statement (i) we have $D(B_G^2)=D(\widehat{A}^2)$ and

$$\begin{aligned}B_G^2x&=B_Gy=\widehat{A}y-G(\Phi^t,\widehat{A}y)_{X^m}=\widehat{A}B_Gx-G(\Phi^t,\widehat{A}B_Gx)_{X^m}\\&=\widehat{A}[\widehat{A}x-G(\Phi^t,\widehat{A}x)_{X^m}]-G\big(\Phi^t,\widehat{A}[\widehat{A}x-G(\Phi^t,\widehat{A}x)_{X^m}]\big)_{X^m}\\&=\widehat{A}^2x-\widehat{A}G(\Phi^t,\widehat{A}x)_{X^m}-G(\Phi^t,\widehat{A}^2x)_{X^m}+G(\Phi^t,\widehat{A}G)_{X^m}(\Phi^t,\widehat{A}x)_{X^m},\end{aligned}$$

which gives (4.16). It is easy to verify, by using (4.11), that $B_GG=\widehat{A}G-G(\Phi^t,\widehat{A}G)_{X^m}$. From this and (4.16) immediately follows (4.17).

In the next theorem we investigate the relation between B_G and B_{SG} defined by (4.11) and (4.12) respectively.

THEOREM 4.6. *We consider the operators $\widehat{A},B_G,B_{SG}:X\to X$, where $\widehat{A}$ is correct and densely defined and B_G,B_{SG} are defined by (4.11), (4.12) respectively. Then:*
(i) $B_{SG}=B_G^2$ *if and only if* $G\in D(\widehat{A})^m$ *and* $S=\widehat{A}G-G(\Phi^t,\widehat{A}G)_{X^m}$.
(ii) For each $G\in D(\widehat{A})^m$ *and* $S=\widehat{A}G-G(\Phi^t,\widehat{A}G)_{X^m}$,

$$B_{SG} \text{ is correct iff } B_G \text{ is correct iff } \det W=\det\big[I_m-(\Phi^t,G)_{X^m}\big]\neq 0.$$

Proof. (i) $B_{SG}=B_G^2$ if and only if $D(B_{SG})=D(\widehat{A}^2)=D(B_G^2)$ and $B_{SG}x=B_G^2x$ for each $x\in D(\widehat{A}^2)$. By Lemma 4.5, the first relation holds true if and only if $G\in D(\widehat{A})^m$. By comparing (4.12) with (4.16), it is easy to verify that $B_{SG}x=B_G^2x$ for each $x\in D(\widehat{A}^2)$ if and only if $G\in D(\widehat{A})^m$ and $S=\widehat{A}G-G(\Phi^t,\widehat{A}G)_{X^m}$, since the elements $\Phi_1,\dots,\Phi_m$ are linearly independent and $\widehat{A}$ is correct.

(ii) The operator B_{SG} can be written in the form

$$B_{SG}x = \widehat{\mathscr{A}x} - \mathscr{G}(\mathscr{F}_2^t, \widehat{\mathscr{A}x})_{X^{2m}} = f, \quad D(B_{SG}) = D(\widehat{\mathscr{A}}), \tag{4.18}$$

where $\widehat{\mathscr{A}} = \widehat{A}^2$, $\mathscr{G} = (S,G)$, $\mathscr{F}_2 = (\widehat{A}^{*-1}\Phi, \Phi)$. By Corollary 3.11 the operator B_{SG} is correct iff

$$det\, W_2 = det\left[I_{2m} - (\mathscr{F}_2^t, \mathscr{G})_{X^{2m}}\right] \neq 0. \tag{4.19}$$

By substituting in (4.19) $\mathscr{G} = (S,G)$, $\mathscr{F}_2 = (\widehat{A}^{*-1}\Phi, \Phi)$, $S = \widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}$ and using the formula $\det\begin{pmatrix} A & B \\ G & D \end{pmatrix} = \det\begin{pmatrix} A+BC & B \\ G+DC & D \end{pmatrix}$, where A,B,G,D,C are a $m \times m$ matrices and $C = (\Phi^t, \widehat{A}G)_{X^m}$, we take

$$\begin{aligned}
\det W_2 &= \det\begin{pmatrix} \left(\Phi^t, G - \widehat{A}^{-1}G(\Phi^t, \widehat{A}G)_{X^m}\right)_{X^m} - I_m & (\Phi^t, \widehat{A}^{-1}G)_{X^m} \\ (\Phi^t, \widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m})_{X^m} & (\Phi^t, G)_{X^m} - I_m \end{pmatrix} \\
&= \det\begin{pmatrix} (\Phi^t, G)_{X^m} - I_m & (\Phi^t, \widehat{A}^{-1}G)_{X^m} \\ [0]_m & (\Phi^t, G)_{X^m} - I_m \end{pmatrix} \\
&= \left(\det\left[I_m - (\Phi^t, G)_{X^m}\right]\right)^2.
\end{aligned}$$

So $\det W_2 = (\det W)^2$, since, from (3.16), $W = I_m - (\Phi^t, G)_{X^m}$. Now by Theorem 3.8 B_G is correct iff $\det W \neq 0$ iff $\det W_2 \neq 0$ iff B_{SG} is correct.

COROLLARY 4.7. *Let $\widehat{A}$ be a correct and densely defined operator on X and A_0, A_0^2, B_G, B_{SG} are defined by* (4.13), (4.14), (4.11), (4.12) *respectively. Then, for each $G \in D(\widehat{A})^m$ and $S = \widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}$, it holds*

$$B_{SG} = B_2 \in E_{2c}(A_0^2, \widehat{A}^2) \ \textit{if and only if} \quad B_G \in E_c(A_0, \widehat{A}).$$

Proof. It is evident that B_G (resp. B_{SG}) is an extension of A_0 (resp. A_0^2). So, by the previous result, we have $B_{SG} = B_2 \in E_{2c}(A_0^2, \widehat{A}^2)$ if and only if $B_G \in E_c(A_0, \widehat{A})$.

The next theorem follows from Theorem 4.4 and corollary 4.7 and shows that every operator B_2 of $E_{2c}^{2m}(A_0^2, \widehat{A}^2)$ can be uniquely determined by only one vector G. It also gives the solution of $B_2x = f$.

THEOREM 4.8. *We suppose that $\widehat{A}$ is as usually, A_0, A_0^2 are defined by* (4.13), (4.14), *respectively, and the components of the vector $\mathscr{F}_2 = (\widehat{A}^{*-1}\Phi, \Phi)$ are linearly independent. Then:*

(i) For every $B_2 \in E_{2c}^{2m}(A_0^2, \widehat{A}^2)$, there exists a unique vector $G = (g_1, \dots, g_m) \in D(\widehat{A})^m$ such that the $2m$ components of the vector $\left(\widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}, G\right)$ are linearly independent and hold

$$det\, W = det\left[I_m - (\Phi^t, G)_{X^m}\right] \neq 0 \tag{4.20}$$

and

$$B_2x = \widehat{A}^2x - \left[\widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}\right](\Phi^t, \widehat{A}x)_{X^m} - G(\Phi^t, \widehat{A}^2x)_{X^m} = f. \tag{4.21}$$

(ii) Conversely, for every vector $G \in D(\widehat{A})^m$*, such that the vector* $\big(\widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}, G\big)$ *has* n $(n \leqslant 2m)$ *exactly linearly independent components and* (4.20) *holds true, the operator* B_2 *with* $D(B_2) = D(\widehat{A^2})$ *defined by* (4.21) *belongs to* $E^n_{2c}(A^2_0, \widehat{A}^2)$.

(iii) If B_2 *is correct, then the unique solution of* (4.21) *is given by*

$$x = B_2^{-1} f = \widehat{A}^{-2} f + \Big[\widehat{A}^{-2} G + (\widehat{A}^{-1} G) W^{-1} (\Phi^t, \widehat{A}^{-1} G)_{X^m}\Big] W^{-1} (\Phi^t, f)_{X^m} + (\widehat{A}^{-1} G) W^{-1} (\Phi^t, \widehat{A}^{-1} f)_{X^m}. \tag{4.22}$$

Proof. (i) Let $B_2 \in E^{2m}_{2c}(A^2_0, \widehat{A}^2)$. Then, since (4.15), $B_2 \in E^{2m}_c(A^2_0, \widehat{A}^2)$ and there exists an operator $B \in E_c(A_0, \widehat{A})$ such that $B_2 = B^2$. By Theorem 4.4 there exists a pair of vectors $S, G \in X^m$ such that the components of the vector (S, G) are linearly independent elements of X and $B_2 = B_{SG}$. By Theorem 3.7 there exists $G_1 \in X^m$ such that $B = B_{G_1}$ with $D(B_{G_1}) = D(\widehat{A})$. Hence $B_{SG} = B^2_{G_1}$ and so $D(B^2_{G_1}) = D(\widehat{A}^2)$. The last equation implies that $G_1 \in D(\widehat{A})^m$. Then since (4.16) we have $B^2_{G_1} x = \widehat{A}^2 x - S_1 (\Phi^t, \widehat{A} x)_{X^m} - G_1 (\Phi^t, \widehat{A}^2 x)_{X^m}$, where $S_1 = \widehat{A} G_1 - G_1 (\Phi^t, \widehat{A} G_1)_{X^m}$. From $B_{SG} = B^2_{G_1}$ we take $(S - S_1)(\Phi^t, \widehat{A} x)_{X^m} + (G - G_1)(\Phi^t, \widehat{A}^2 x)_{X^m} = 0$ or $(S - S_1, G - G_1)(\mathscr{F}_2, \widehat{A}^2 x)_{X^m} = 0$ for all $x \in D(\widehat{A}^2)$. By definition, $\widehat{A}$ is correct, the components of $\mathscr{F}_2$ are linearly independent and this implies $S = S_1$, $G = G_1$. Hence $B_G = B_{G_1}$, $B_{SG} = B^2_G$, $S = \widehat{A} G - G(\Phi^t, \widehat{A} G)_{X^m}$ and by Theorem 4.6, $\det W \neq 0$.

(ii) Conversely, let $G \in D(\widehat{A})^m$, and $S = \widehat{A} G - G(\Phi^t, \widehat{A} G)_{X^m}$. The vectors S, G define the operators B_G, $B_{SG} = B_2$ by (4.11), (4.12) respectively and by Theorem 4.6 $B_{SG} = B^2_G$ and B_{SG} is correct. It is evident that $A^2_0 \subset B_2$. Now we show that $\dim R(B_2 - \widehat{A}^2) = n$. The equation (4.21) can be written as $(B_2 - \widehat{A}^2) x = -(S, G)(\mathscr{F}^t_2, \widehat{A}^2 x)_{X^{2m}}$, which since the dimension of (S, G) equal n, $R(\widehat{A}^2) = X$ and the components of the vector $\mathscr{F}_2$ are linearly independent elements of X^* implies $\dim R(B_2 - \widehat{A}^2) = n$. So $B_2 \in E^n_{2c}(A^2_0, \widehat{A}^2)$.

(iii) Finally we find the solution of (4.21) by using Theorem 4.4. If we substitute in the matrix W_1 (with $k = m$) of (4.7) $S = \widehat{A} G - G(\Phi^t, \widehat{A} G)_{X^m}$ we take

$$W_1 = \begin{pmatrix} (\Phi^t, G)_{X^m} - (\Phi^t, \widehat{A}^{-1} G)_{X^m} (\Phi^t, \widehat{A} G)_{X^m} - I_m & (\Phi^t, \widehat{A}^{-1} G)_{X^m} \\ (\Phi^t, \widehat{A} G)_{X^m} - (\Phi^t, G)_{X^m} (\Phi^t, \widehat{A} G)_{X^m} & (\Phi^t, G)_{X^m} - I_m \end{pmatrix}.$$

We put $M = (\Phi^t, \widehat{A}^{-1} G)_{X^m}$, $N = (\Phi^t, \widehat{A} G)_{X^m}$ and recall that (Theorem 3.7) $W = I_m - (\Phi^t, G)_{X^m}$. Then $\widehat{A}^{-2}(S, G) = (\widehat{A}^{-1} G - \widehat{A}^{-2} G N, \widehat{A}^{-2} G)$. We rewrite the matrix W_1 and find its inverse W_1^{-1} in terms of W, M, N.

$$W_1 = \begin{pmatrix} -W - MN & M \\ WN & -W \end{pmatrix}, \; W_1^{-1} = -\begin{pmatrix} W^{-1} & W^{-1} M W^{-1} \\ N W^{-1} & N W^{-1} M W^{-1} + W^{-1} \end{pmatrix}.$$

It follows that $\widehat{A}^{-2}(S,G)W_1^{-1} = -(Y,U)$, where

$$\begin{aligned} Y &= (\widehat{A}^{-1}G - \widehat{A}^{-2}GN)W^{-1} + \widehat{A}^{-2}GNW^{-1} = \widehat{A}^{-1}GW^{-1}, \\ U &= (\widehat{A}^{-1}G - \widehat{A}^{-2}GN)W^{-1}MW^{-1} + \widehat{A}^{-2}G(NW^{-1}MW^{-1} + W^{-1}) \\ &= \widehat{A}^{-1}GW^{-1}MW^{-1} + \widehat{A}^{-2}GW^{-1}. \end{aligned}$$

Hence $\widehat{A}^{-2}(S,G)W_1^{-1} = -\left(\widehat{A}^{-1}GW^{-1}, \widehat{A}^{-1}GW^{-1}MW^{-1} + \widehat{A}^{-2}GW^{-1}\right)$ and substituting this into (4.9) we obtain (4.22). This completes the proof.

The following corollary contains some of the facts proved in the last theorem in the case when the components of vector $\mathscr{F}_2 = (\widehat{A}^{*-1}\Phi, \Phi)$ are not linearly independent.

COROLLARY 4.9. *Let the operator $B_{SG} : X \to X$ be defined by*

$$B_{SG}x = \widehat{A}^2x - S(\Phi^t, \widehat{A}x)_{X^m} - G(\Phi^t, \widehat{A}^2x)_{X^m} = f, \quad D(B_{SG}) = D(\widehat{A}^2), \tag{4.23}$$

where $\widehat{A}$ is a correct, densely defined operator on X, $S = (s_1,\dots,s_m) \in X^m$, $G = (g_1,\dots,g_m) \in D(\widehat{A})^m$, $S = \widehat{A}G - G(\Phi^t, \widehat{A}G)_{X^m}$ and the components of the vector Φ are linearly independent elements of X^*. Then:

(i) B_{SG} is a correct operator if and only if (4.20) *holds true.*

(ii) If B_{SG} is correct, then the unique solution of (4.23) *is given by* (4.22).

5. Examples

By $V^0[a,b]$[[16, page 372] we denote the subspace of all functions of bounded variation on [a,b] which satisfy the conditions that they are zero at $x = a$ and continuous from the right everywhere on $(a,b]$.

It is easy to see that the operator $\widehat{A} : C[0,1] \to C[0,1]$, defined by

$$\widehat{A}u = u' = f, \quad D(\widehat{A}) = \{u(t) \in C^1[0,1] : u(0) = ku(1), \text{where constant} \, k \neq 1\} \tag{5.1}$$

is correct and densely defined and the unique solution of the problem (5.1) is given by the formula

$$u(t,k) = \widehat{A}^{-1}f = \int_0^t f(x)dx + k_1\int_0^1 f(x)dx \quad \text{for all} \quad f \in C[0,1], \tag{5.2}$$

where $k_1 = k/(1-k)$. Then by the Remark 4.2 the operator $\widehat{A}^2$ defined by

$$\widehat{A}^2u = u'' = f, \quad D(\widehat{A}^2) = \{u \in C^2[0,1] : u(0) = ku(1), u'(0) = ku'(1)\} \tag{5.3}$$

is correct too and the reader can verify that for every $f \in C[0,1]$ the unique solution of the problem (5.3) is given by the formula

$$u(t,k) = \widehat{A}^{-2}f = \int_0^t (t-x)f(x)dx + k_1\int_0^1 (t-x+k_1+1)f(x)dx. \tag{5.4}$$

EXAMPLE 5.1. The operator $B_1 : C[0,1] \to C[0,1]$ with $D(B_1) = D(\widehat{A}^2)$ from (5.3), which corresponds to the problem:

$$B_1 u = u'' - \Big(\pi \cos \pi t + \frac{2 \sin \pi t}{\pi}\Big) \int_0^1 x u'(x) dx - \sin \pi t \int_0^1 x u''(x) dx = f(t) \tag{5.5}$$

is correct, $\dim R(B_1 - \widehat{A}^2) = 2$ and the unique solution of (5.5) for every $f \in C[0,1]$ is given by the formula

$$\begin{aligned} u(t,k) = &\int_0^t (t-x) f(x) dx + k_1 \int_0^1 (t - x + k_1 + 1) f(x) dx \\ &+ \Big[\frac{\pi t - \sin \pi t + \pi k_1 (2t + 2k_1 + 1)}{\pi(\pi - 1)} + \frac{(2k_1 + 1 - \cos \pi t)(2\pi^2 k_1 + \pi^2 + 4)}{2\pi^2 (\pi - 1)^2}\Big] \cdot \\ &\cdot \int_0^1 x f(x) dx + \frac{2k_1 + 1 - \cos \pi t}{2(\pi - 1)} \int_0^1 (1 + k_1 - x^2) f(x) dx. \end{aligned} \tag{5.6}$$

Proof. We refer to Theorem 4.8 (ii). If we compare equation (5.5) with equation (4.21), it is natural to take the operator $\widehat{A}^2$ as in (5.3), $m = 1, G = \sin \pi t$. Then $\widehat{A}$ can be defined by (5.1), $(\Phi^t, \widehat{A} u)_C = \int_0^1 x u'(x) dx$, $(\Phi^t, \widehat{A}^2 u)_C = \int_0^1 x u''(x) dx$ and the functional Φ, for every $u(x) \in C[0,1]$, to be defined by $(\Phi, u)_C = \int_0^1 x u(x) dx = \int_0^1 u(x) d\big(\frac{x^2}{2}\big) = \int_0^1 u(x) dw_1(x)$. From the last relation we take $(\Phi, \widehat{A} u)_C = \int_0^1 x u'(x) dx = u(1) + \int_0^1 u(x) d(-x) = \int_0^1 u(x) dw_2(x) = (F, u)_C$, where $w_2(x) = \begin{cases} -x, & if\, x \in [0,1) \\ 0, & if\, x = 1 \end{cases}$. It is clear that $G \in D(\widehat{A})$ and $w_1, w_2 \in V^0[0,1]$. Then, by Theorem [16, page 373] $\Phi, F \in (C[0,1])^*$ and $F = \widehat{A}^* \Phi$. Since w_1, w_2 are linearly independent elements of $V^0[0,1]$, the components of the vector $\widehat{A}^* \mathscr{F}_2 = (\Phi, \widehat{A}^* \Phi)$ are linearly independent in $(C[0,1])^*$. With simple calculations we find $\widehat{A} G - G(\Phi^t, \widehat{A} G)_{C^m} = \pi \cos \pi t + \frac{1}{\pi}(2 \sin \pi t)$. This show that the operator $B_1 = B_2$ where B_2 is defined by (4.21). Also we find $(\Phi^t, G)_C = \frac{1}{\pi}$, $\det W = \det[I_m - (\Phi^t, G)_{C^m}] = \frac{\pi - 1}{\pi} \neq 0$, $W^{-1} = \frac{\pi}{\pi - 1}$. Then, by Theorem 4.8 (ii), the operator B_1 is correct and $\dim R(B_1 - \widehat{A}^2) = 2$, because $\widehat{A} G - G(\Phi^t, \widehat{A} G)_{X^m}, G$ linearly independent. Now we find $\widehat{A}^{-1} G = \frac{1}{\pi}(2k_1 + 1 - \cos \pi t)$, $\widehat{A}^{-2} G = \frac{1}{\pi^2}[\pi t - \sin \pi t + \pi k_1 (2t + 2k_1 + 1)]$, $(\Phi^t, \widehat{A}^{-1} G)_C = \frac{1}{2\pi^3}(2\pi^2 k_1 + \pi^2 + 4)$, $(\Phi^t, f)_C = \int_0^1 x f(x) dx$, $(\Phi^t, \widehat{A}^{-1} f)_C = \frac{1}{2} \int_0^1 (1 + k_1 - x^2) f(x) dx$. From the above and (4.22) follows the solution (5.6).

Let $\overline{\Pi} = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y \leqslant 1\}$. It is easy to verify that the operator $\widehat{A} : C(\overline{\Pi}) \to C(\overline{\Pi})$, defined by

$$\begin{aligned} \widehat{A} u = u_{xy} = f, \quad D(\widehat{A}) = \{&u \in C(\overline{\Pi}) : u_x \in C(\overline{\Pi}), \\ &u_{xy} \in C(\overline{\Pi}), \quad u_x(x,0) = 0, \quad u(0,y) = v(y) u(1,1)\} \end{aligned} \tag{5.7}$$

is correct for each $v(y) \in C[0,1]$, $v(1) = 0$ and the unique solution of the problem (5.7) is given by the formula

$$u=\widehat{A}^{-1}f=\int_0^x\int_0^y f(t,s)dsdt+v(y)\int_0^1\int_0^1 f(t,s)dsdt \quad \text{for all } f\in C(\overline{\Pi}). \tag{5.8}$$

Also by Remark 4.2 the operator $\widehat{A}^2$ defined by

$$\begin{aligned}&\widehat{A}^2u=u_{xyxy}=f(x,y),\quad D(\widehat{A}^2)=\{u\in D(\widehat{A}):u_{xyx}\in C(\overline{\Pi}),\\ &u_{xyxy}\in C(\overline{\Pi}),\quad u_{xyx}(x,0)=0,\quad u_{xy}(0,y)=v(y)u_{xy}(1,1)\}\end{aligned} \tag{5.9}$$

is correct too and the reader can verify that for every $f\in C(\overline{\Pi})$ the unique solution of the problem (5.9), for each $v(y)\in C[0,1]$, $v(1)=0$, is given by the formula $u=\widehat{A}^{-2}f$, i.e.

$$\begin{aligned}u=&\int_0^x(x-t)dt\int_0^y(y-s)f(t,s)ds+v(y)\int_0^1(1-t)dt\int_0^1(1-s)f(t,s)ds\\ &+\left[x\int_0^y v(s)ds+v(y)\int_0^1 v(s)ds\right]\int_0^1\int_0^1 f(t,s)dsdt.\end{aligned} \tag{5.10}$$

EXAMPLE 5.2. The operator $B_1:C(\overline{\Pi})\to C(\overline{\Pi})$ with $D(B_1)=D(\widehat{A}^2)$ from (5.9) which corresponds to the problem

$$\begin{aligned}B_1u=u_{xyxy}&-\left(\pi\cos\pi x+\frac{2}{\pi}y\sin\pi x\right)\int_0^1\int_0^1 tu_{ts}(t,s)dsdt\\ &-y\sin\pi x\int_0^1 tu_{tst}(t,1)dt=f(x,y),\end{aligned} \tag{5.11}$$

is correct for each $v(y)\in C[0,1]$, $v(1)=0$ and the unique solution of (5.11), for every $f\in C(\overline{\Pi})$, is given by the formula

$$\begin{aligned}u(x,y)=&\widehat{A}^{-2}f+\frac{2}{2\pi-1}\Big\{\frac{1}{6\pi}[y^3(\pi x-\sin\pi x)+\pi v(y)]+x\int_0^y v(s)ds\\ &+v(y)\int_0^1 v(s)ds+\frac{1}{2(2\pi-1)}[y^2(1-\cos\pi x)+2v(y)]\Big[\frac{4+\pi^2}{6\pi^2}+\int_0^1 v(s)ds\Big]\Big\}\cdot\\ &\cdot\int_0^1\int_0^1 tf(t,s)dsdt+\frac{1}{2(2\pi-1)}[y^2(1-\cos\pi x)\\ &+2v(y)]\Big[\int_0^1(1-t^2)dt\int_0^1(1-s)f(t,s))ds+\int_0^1 v(s)ds\int_0^1\int_0^1 f(t,s)dsdt\Big].\end{aligned} \tag{5.12}$$

Proof. We refer to corollary 4.9. If we compare equation (5.11) with equation (4.23), we are led to take the operator $\widehat{A}^2$ as in (5.9), $m=1$, $\Phi^t=\Phi$, $(\Phi,\widehat{A}u)_C=\int_0^1\int_0^1 tu_{ts}(t,s)dsdt$. So $\widehat{A}$ can be defined by (5.7) and the functional Φ for every $u(x)\in C(\overline{\Pi})$ by $(\Phi,u)_C=\int_0^1\int_0^1 tu(t,s)dsdt$. Then with integration by parts and (5.9) we obtain $(\Phi,\widehat{A}^2u)_C=\int_0^1\int_0^1 tu_{tsts}(t,s)dsdt=\int_0^1 tu_{tst}(t,1)dt$ and so we take $S=\pi\cos\pi x+$

$\frac{2}{\pi}y\sin\pi x$, $G = y\sin\pi x$. It is clear that $G\in D(\widehat{A})$. By simple calculations we find $\widehat{A}G - G(\Phi,\widehat{A}G)_{C^m} = \pi\cos\pi x + \frac{2}{\pi}y\sin\pi x = S$, $(\Phi,G)_C = \frac{1}{2\pi}$, $\det W = \det[I_m - (\Phi,G)_{C^m}] = \frac{2\pi-1}{2\pi} \neq 0$, $W^{-1} = \frac{2\pi}{2\pi-1}$. Then, by corollary 4.9, the operator B_1 is correct. Now using (5.8) and (5.10) we find respectively $\widehat{A}^{-1}G = \frac{1}{2\pi}[y^2(1-\cos\pi x)+2v(y)]$ and $\widehat{A}^{-2}G = \frac{1}{6\pi^2}[y^3(\pi x - \sin\pi x) + \pi v(y)] + \frac{1}{\pi}\Big[x\int_0^y v(s)ds + v(y)\int_0^1 v(s)ds\Big]$. Then $(\Phi,\widehat{A}^{-1}G)_C = \frac{1}{12\pi^3}(4+\pi^2) + \frac{1}{2\pi}\int_0^1 v(s)ds$, $(\Phi,f)_C = \int_0^1\int_0^1 tf(t,s)dsdt$, $(\Phi,\widehat{A}^{-1}f)_C = \frac{1}{2}\Big[\int_0^1(1-t^2)dt\int_0^1(1-s)f(t,s))ds + \int_0^1 v(s)ds\int_0^1\int_0^1 f(t,s)dsdt\Big]$. In the last relation we have used the simple formula

$$\int_0^1\int_0^1 x\int_0^x\int_0^y f(t,s)dsdtdydx = \frac{1}{2}\int_0^1(1-t^2)dt\int_0^1(1-y)f(t,y)dy$$

From the above and (4.22) follows that

$$\begin{aligned}u(x,y) = \widehat{A}^{-2}f + \Big\{&\frac{1}{6\pi^2}[y^3(\pi x - \sin\pi x) + \pi v(y)] + \frac{1}{\pi}\Big[x\int_0^y v(s)ds + v(y)\int_0^1 v(s)ds\Big]\\ &+\frac{1}{2\pi}[y^2(1-\cos\pi x)+2v(y)]\frac{2\pi}{2\pi-1}\Big[\frac{1}{12\pi^3}(4+\pi^2)+\frac{1}{2\pi}\int_0^1 v(s)ds\Big]\Big\}\\ &\cdot\frac{2\pi}{2\pi-1}\int_0^1\int_0^1 tf(t,s)dsdt + \frac{1}{2\pi}[y^2(1-\cos\pi x)+2v(y)]\frac{2\pi}{2\pi-1}\\ &\cdot\frac{1}{2}\Big[\int_0^1(1-t^2)dt\int_0^1(1-s)f(t,s)ds + \int_0^1 v(s)ds\int_0^1\int_0^1 f(t,s)dsdt\Big]\end{aligned}$$

which gives the solution (5.12).

A comment from the first author: The second author passed away from a heart attack in the Fall of 2009, at the age of 64. I would like to express my deepest sorry for his sudden death.

REFERENCES

[1] R. O. OINAROV, I. N. PARASIDI, *Correct extensions of operators with finite defect in Banach spases*, Izv. Akad. Kaz. SSR, **5** (1988), 42–46 [Russian].

[2] J. VON NEUMANN, *Allgemeine Eigenwerttheorie Hermitescher Functional operatoren,* Math. Ann., **102** (1929-1930), 49–131.

[3] V. I. GORBACHUK AND M. L. GORBACHUK, *Boundary Value Problems for Operator Differential equations,* Dordrecht, Kluwer, 1991.

[4] S. G. KREIN, *Linear equations in Banach spase*, Nauka, Moskva, 1971 [Russian].

[5] A. A. DEZIN, *General aspects of Boundary Value problems*, Moskva “Nauka”, 1980 [Russian].

[6] M. I. VISHIK, *About Boundary Value Problems for Elliptical Differential equations,* Mosk. Math. Sosity, **1** (1952), 187–246 [Russian].

[7] E. A. CODDINGTON, *Self-adjoint subspace extensions of nondensely defined symmetric operators*, Adv. Math., **14** (1974), 309–332.

[8] E. A. CODDINGTON, A. DIJKSMA, *Self-adjoint subspaces and eigenfunction expansions for ordinary differential subspaces,* J. Diff. Equat., **20** (1976), 473–526.

[9] E. A. CODDINGTON, A. DIJKSMA, *Adjoint subspaces in Banach spaces with Applications to ordinary differential subspaces,* J. Diff. Equat., **20** (1976), 473–526.
[10] A. N. KOCHUBEI, *Extensions of positive definite symmetric operators*, D.A.N. USSR, A, **3** (1979), 168–171 [Russian].
[11] A. N. KOCHUBEI, *Extensions of a nondensely defined symmetric operator,* Sib. Math. J., **18** (1977), 225–229.
[12] B. A. MIKHAILETS, *The spectrum of operators and boundary value problems, spectral analysis of differential operators,* Kiev, Inst. of Math. Ukrain. Akadems (1980), 106–131 [Russian].
[13] B. K. KOKEBAEV, M. OTELBAEV, A. N. SHYNIBEKOV, *About Restrictions and Extensions of operators,* D. A. N. SSSR, **271**, 6 (1983), 1307–1310 [Russian].
[14] A. N. SHYNIBEKOV, *On correct extensions and restrictions for some differential operators*, Summary of Ph. D. thesis, Alma-Ata, 1983, 1–16 [Russian].
[15] E. KREYSZIG, *Introductory Functional Analysis with Applications*, 532. John Wiley and Sons, 1989.
[16] A. N. KOLMOGOROV, S. V. FOMIN, *The elements of theory of funtions and functional analysis*, "Nauka", Moskva, 1976 [Russian, there is a translation in English].

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