LINEAR MAPS PRESERVING THE MINIMUM MODULUS

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Abstract. We characterize surjective linear maps that preserve the minimum modulus between unital semisimple Banach algebras, one of them is a unital C^*-algebra having either real rank zero or essential socle. We also describe surjective linear maps on $\mathscr{L}(H)$, with H an infinite-dimensional Hilbert space, preserving the essential minimum modulus. Results concerning surjectivity and maximum modulus are also obtained.

1. Introduction

Let $\mathscr{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X. For an operator $T \in \mathscr{L}(X)$, let $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$ and $r(T)$ denote the spectrum, the approximate point spectrum, the surjectivity spectrum and the spectral radius respectively of T. The *minimum modulus* of T is $m(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}$, and the *surjectivity modulus* of T is defined by $q(T) := \sup\{\varepsilon \geqslant 0 : \varepsilon B_X \subseteq T(B_X)\}$, where as usual B_X denotes the unit ball of X. Note that $m(T) > 0$ if and only if T is injective and has closed range, and that $q(T) > 0$ if and only if T is surjective. Moreover, $m(T) = \inf\{\|TS\| : S \in \mathscr{L}(X), \|S\| = 1\}$ and $q(T) = \inf\{\|ST\| : S \in \mathscr{L}(X), \|S\| = 1\}$; see [17, Theorem II.9.11].

In [16], Mbekhta described unital surjective linear maps that preserve the minimum and surjectivity moduli of Hilbert space operators. He showed that if H is an infinite-dimensional complex Hilbert space, then a unital surjective linear map Φ on $\mathscr{L}(H)$ satisfies $m(\Phi(T)) = m(T)$ for all $T \in \mathscr{L}(H)$ (resp. $q(\Phi(T)) = q(T)$ for all $T \in \mathscr{L}(H)$) if and only if there exists a unitary operator $U \in \mathscr{L}(H)$ such that $\Phi(T) = UTU^*$ for all $T \in \mathscr{L}(H)$. His argument does not work without the fact that Φ is unital and extra efforts are needed to deal with the natural question left by him regarding the description of non necessarily unital linear maps preserving the minimum and the surjectivity moduli. The aim of this paper is not only to extend Mbekhta's result to the more general setting of surjective linear maps between unital semisimple Banach algebras $\mathscr{A}$ and $\mathscr{B}$ such that one of them is either a C^*-algebra with real rank zero or has an essential socle but mainly is to answer this question in a more general context. In Section 2, we introduce the minimum, surjectivity and maximum moduli

of an element in a Banach algebra, and characterize surjective linear maps between $\mathscr{A}$ and $\mathscr{B}$ preserving a ∂-spectrum. Relying on these results, we show in Section 3 that if $\Phi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map preserving either the minimum, surjectivity, or maximum modulus, then Φ is an isometry. In the last section, we characterize those surjective linear maps preserving the essential minimum modulus of Hilbert space operators.

2. Preliminaries

In this section, we first introduce some notation and definitions. Next, we establish some results from the theory of invertibility preservers that will be need in the sequel. We finally define the minimum, surjectivity and maximum moduli of an element in a Banach algebra, and review some of their elementary properties.

2.1. Notation

Throughout this paper, the term Banach algebra means a unital complex associative Banach algebra, with unit $\mathbf{1}$, and a C^*-algebra means a unital complex associative C^*-algebra. Let $\mathscr{A}$ be a Banach algebra, and $\text{Inv}(\mathscr{A})$ be the group of all invertible elements of $\mathscr{A}$. For an element a in $\mathscr{A}$, let $\sigma(a)$, $\partial\sigma(a)$ and $\text{r}(a)$ denote the spectrum, the boundary of the spectrum and the spectral radius of a, respectively. According to [16], a map Λ from $\mathscr{A}$ to the closed subsets of $\mathbb{C}$ is called a ∂-*spectrum* if

$$\partial\sigma(a) \subseteq \Lambda(a) \subseteq \sigma(a)$$

for all $a \in \mathscr{A}$. Purely topological arguments show that if Λ is a ∂-spectrum in $\mathscr{A}$, then the polynomial convex hull of $\Lambda(a)$ coincides with the polynomial convex hull of $\sigma(a)$ for all $a \in \mathscr{A}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras. A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ is called *unital* if $\Phi(\mathbf{1}) = \mathbf{1}$, and it is said to be a *Jordan homomorphism* if $\Phi(a^2) = \Phi(a)^2$ for all $a \in \mathscr{A}$. Equivalently, the map Φ is a Jordan homomorphism if and only if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$ for all a and b in $\mathscr{A}$. It is called a *Jordan isomorphism* provided that it is a bijective Jordan homomorphism. Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. It is well known that if $\Phi : \mathscr{A} \to \mathscr{B}$ is a Jordan homomorphism, then

$$\Phi(aba) = \Phi(a)\Phi(b)\Phi(a) \tag{1}$$

for all $a, b \in \mathscr{A}$. Moreover, if Φ is a Jordan isomorphism, then Φ strongly preserves invertibility, that is

$$\Phi(a^{-1}) = \Phi(a)^{-1} \tag{2}$$

for every invertible element a in $\mathscr{A}$. We finally recall that if $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras, then the map Φ is said to be *selfadjoint* provided that $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathscr{A}$.

2.2. Linear maps preserving ∂-spectra

Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras. Let Λ_1 and Λ_2 be ∂-spectra in $\mathscr{A}$, and let Λ be a ∂-spectrum in $\mathscr{B}$. By the Zemánek spectral characterization of the radical, [2, Theorem 5.3.1], it is easy to show that if $\Phi : A \to B$ is a surjective linear map such that $\Lambda_1(a) \subset \Lambda(\Phi(a)) \subset \Lambda_2(a)$ for all $a \in A$, then Φ maps the radical of $\mathscr{A}$ onto the radical of $\mathscr{B}$; i.e., $\Phi(\mathrm{Rad}(\mathscr{A})) = \mathrm{Rad}(\mathscr{B})$. Thus if $\mathscr{A}$ is semisimple, then $\mathscr{B}$ is semisimple and Φ is injective.

The following two results show, in particular, that such maps are Jordan isomorphisms provided that one of the Banach algebras either has large socle or it is a C^*-algebra of real rank zero. Recall that an ideal I of $\mathscr{A}$ is said to be *essential* if it has nonzero intersection with every nonzero ideal of A. If $\mathscr{A}$ is semisimple, then I is essential if and only if the only element $a \in \mathscr{A}$ for which $aI = 0$ is zero. The *socle* of $\mathscr{A}$, denoted by $\mathrm{Soc}(\mathscr{A})$, is defined as the sum of all minimal left (or right) ideals of $\mathscr{A}$. In particular, for a complex Banach space X, $\mathrm{Soc}(\mathscr{L}(X))$ coincides with the ideal of all finite rank operators, and it is essential. We refer the reader to [1, Chapter 5] for more information on the socle of a Banach algebra.

THEOREM 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple Banach algebras such that the socle of one of them is essential. Let Λ_1 and Λ_2 be ∂-spectra in $\mathscr{A}$, and let Λ be a ∂-spectrum in $\mathscr{B}$. If $\Phi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map for which $\Lambda_1(a) \subset \Lambda(\Phi(a)) \subset \Lambda_2(a)$ for all $a \in \mathscr{A}$, then Φ is a Jordan isomorphism.*

Proof. An adaptation of the arguments given in the proof of either [3, Theorem 3.1] or [6, Theorem 1.1] yields this result. □

A C^*-algebra $\mathscr{A}$ has *real rank zero* if the set of all real linear combinations of orthogonal projections is dense in the set of all hermitian elements of $\mathscr{A}$; see [8]. Notice that every von Neumann algebra, and in particular the algebra $\mathscr{L}(H)$ of all bounded linear operators on a complex Hilbert space H, has real rank zero. Other examples of this kind of algebra include Bunce-Deddens algebras, Cuntz algebras, AF-algebras, and irrational rotation algebras; see [10].

THEOREM 2.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple Banach algebras such that one of them is a C^*-algebra of real rank zero. Let Λ_1 and Λ_2 be ∂-spectra in $\mathscr{A}$, and let Λ be a ∂-spectrum in $\mathscr{B}$. If $\Phi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map for which $\Lambda_1(a) \subset \Lambda(\Phi(a)) \subset \Lambda_2(a)$ for all $a \in \mathscr{A}$, then Φ is a Jordan isomorphism.*

Proof. Similar arguments to the ones given by Aupetit in [4] show that Φ is unital and maps idempotent elements of $\mathscr{A}$ into idempotent elements of $\mathscr{B}$; see also [16, Proposition 3.4]. Since one of the algebras $\mathscr{A}$ and $\mathscr{B}$ is a C^*-algebra of real rank zero, [15, Lemma 1] implies that Φ is a Jordan homomorphism. □

2.3. Minimum, surjectivity and maximum moduli in Banach algebras

For an an element a of a Banach algebra $\mathscr{A}$, the *minimum modulus* and the *surjectivity modulus* are defined respectively by

$$\mathrm{m}(a) := \mathrm{m}(L_a) = \inf\{\|ax\| : x \in A, \|x\| = 1\}$$

and

$$\mathrm{q}(a) := \mathrm{m}(R_a) = \inf\{\|xa\| : x \in A, \|x\| = 1\},$$

where L_a and R_a are the left and right multiplication operators by a. The *maximum modulus* of a is defined by

$$\mathrm{M}(a) := \max\{\mathrm{m}(a), \mathrm{q}(a)\}.$$

Obviously, $\mathrm{m}(a) = 0$ (respectively $\mathrm{q}(a) = 0$) if and only if a is a left (respectively right) topological divisor of zero. Also $\mathrm{M}(a) = 0$ if and only if a is a topological divisor of zero. Note also that $\mathrm{M}(a) = \mathrm{m}(a) = \mathrm{q}(a) = \|a^{-1}\|^{-1}$ for all invertible elements $a \in \mathscr{A}$, and that

$$\mathrm{m}(a)\mathrm{m}(b) \leqslant \mathrm{m}(ab) \leqslant \|a\|\mathrm{m}(b), \text{ and } \quad \mathrm{q}(a)\mathrm{q}(b) \leqslant \mathrm{q}(ab) \leqslant \mathrm{q}(a)\|b\| \qquad (3)$$

for all a, $b \in \mathscr{A}$. Moreover, if a is an element of a C^*-algebra $\mathscr{A}$, then $\mathrm{m}(a) > 0$ (respectively $\mathrm{q}(a) > 0$) if and only if a is left (respectively right) invertible. Furthermore, we always have $\mathrm{m}(a) = \mathrm{m}(uav)$, $\mathrm{q}(a) = \mathrm{q}(uav)$, and $\mathrm{M}(a) = \mathrm{M}(uav)$ for all unitary elements u, v of $\mathscr{A}$.

We close this section by noticing that, for a Banach algebra $\mathscr{A}$, the approximate point spectrum, $\sigma_{ap}(.)$, the surjective spectrum, $\sigma_s(.)$, and their intersection, $\sigma_{ap,s}(.)$, given by

$$\sigma_{ap}(a) := \{\lambda \in \mathbb{C} : \mathrm{m}(a - \lambda) = 0\},$$
$$\sigma_s(a) := \{\lambda \in \mathbb{C} : \mathrm{q}(a - \lambda) = 0\},$$
$$\sigma_{ap,s}(a) := \{\lambda \in \mathbb{C} : \mathrm{M}(a - \lambda) = 0\},$$

are all ∂-spectra of $\mathscr{A}$.

3. Linear maps preserving the minimum modulus in Banach algebras

As pointed in the previous section, the reader should keep in mind that throughout this paper, the term algebra means a unital complex associative algebra with unit **1**.

Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a linear map. We say that Φ *preserves the minimum modulus* if $\mathrm{m}(\Phi(a)) = \mathrm{m}(a)$ for all $a \in \mathscr{A}$. Similarly, we say that Φ *preserves the surjectivity* (respectively *maximum*) *modulus* provided that $\mathrm{q}(\Phi(a)) = \mathrm{q}(a)$ (respectively $\mathrm{M}(\Phi(a)) = \mathrm{M}(a)$) for all $a \in \mathscr{A}$.

We begin by stating the main result of this paper which answers, in a more general setting, the question left by Mbekhta regarding the description of non necessarily unital linear maps preserving the minimum and the surjectivity moduli.

THEOREM 3.1. *Let $\mathscr{A}$ be a semisimple Banach algebra and let $\mathscr{B}$ be a C^*-algebra. Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a surjective linear map preserving either the minimum modulus or the surjectivity modulus or the maximum modulus. Assume that either $\mathscr{A}$ (or $\mathscr{B}$) has essential socle or $\mathscr{B}$ has real rank zero. Then $\mathscr{A}$ (for its norm and some involution) is a C^*-algebra, and Φ is an isometric selfadjoint Jordan isomorphism multiplied by an unitary element of $\mathscr{B}$.*

Before proving this theorem, we shall make some comments. Assume that $\mathscr{A}$ and $\mathscr{B}$ are two C^*-algebras, and that Φ is a selfadjoint Jordan isomorphism φ multiplied on the left by an unitary element u of $\mathscr{B}$. The map Φ can be written as a selfadjoint Jordan isomorphism multiplied on the right by an unitary element of $\mathscr{B}$. Indeed, we have

$$\Phi(.) = u\varphi(.) = \left(u\varphi(.)u^*\right)u,$$

and $u\varphi(.)u^*$ is a selfadjoint Jordan isomorphism.

In Theorem 3.1, the role of $\mathscr{A}$ or $\mathscr{B}$ being a C^*-algebra is symmetrical since a surjective linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between semisimple Banach algebras is bijective provided that it preserves either the minimum, surjectivity, or maximum modulus. Indeed, assume for instance that Φ preserves the minimum modulus, and that $\Phi(a_0) = 0$ for some $a_0 \in \mathscr{A}$. For all $a \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, we have

$$\mathrm{m}(a_0 + a - \lambda) = \mathrm{m}(\Phi(a_0 + a - \lambda)) = \mathrm{m}(\Phi(a - \lambda)) = \mathrm{m}(a - \lambda).$$

This gives $\sigma_{ap}(a_0 + a) = \sigma_{ap}(a)$ for all $a \in \mathscr{A}$, and implies that $\mathrm{r}(a_0 + a) = \mathrm{r}(a)$ for all $a \in \mathscr{A}$. By the spectral characterization of the radical, [2, Theorem 5.3.1], $a_0 = 0$ and thus Φ is injective.

Note also that a linear isometry between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ need not preserve the minimum or surjectivity modulus. The map $\Phi : \mathscr{L}(H) \to \mathscr{L}(H)$ defined by $\Phi(T) = T^{tr}$ $(T \in \mathscr{L}(H))$, is a linear isometry that preserves neither the minimum modulus nor the surjectivity modulus. Here, T^{tr} denotes the transpose of T with respect to an arbitrary but fixed orthonormal basis in H.

Finally, we need to recall some concepts from non-associative algebras. Following [21], we define *Jordan algebras* as those commutative algebras J satisfying the *Jordan identity* $(xy)x^2 = x(yx^2)$ for all $x, y \in J$. For an element x in a Jordan algebra J, denote by U_x the mapping given by $U_x(y) := 2x(xy) - x^2y$ for all $y \in J$. If $\mathscr{A}$ is an associative algebra, then the algebra $\mathscr{A}^+$, consisting on the underlying vector space of $\mathscr{A}$ and the product

$$x \circ y := \frac{1}{2}(xy + yx), \quad (x, y \in \mathscr{A}),$$

becomes a Jordan algebra. Clearly a linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between Banach algebras is a Jordan homomorphism if and only if $\Phi : \mathscr{A}^+ \to \mathscr{B}^+$ is a homomorphism.

By a *JB*-algebra* we mean a complete normed complex Jordan algebra (say J) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every $x \in J$. It is easy to prove that, if $\mathscr{A}$ is a C^*-algebra, then $\mathscr{A}^+$, with the norm and involution of $\mathscr{A}$, becomes a JB*-algebra. This well known fact was the precursor of the theory of JB^*-algebras; see [22, 23, 19, 12]. In [19, Theorem 2], Rodríguez showed

that the converse is also true, that is, if $\mathscr{A}$ is an associative complex algebra such that $\mathscr{A}^+$ is a JB^*-algebra for some norm and involution, then $\mathscr{A}$ with the same norm and involution is a C^*-algebra.

All the necessary ingredients are collected and we are therefore in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We only consider the case when Φ preserves the minimum modulus. Since $\mathrm{m}(\Phi(\mathbf{1})) = \mathrm{m}(\mathbf{1}) = 1 > 0$, there is $b \in \mathscr{B}$ such that $b\Phi(\mathbf{1}) = \mathbf{1}$. Let us consider the unital surjective linear map $\varphi : \mathscr{A} \to \mathscr{B}$ defined by

$$\varphi(x) := b\Phi(x)$$

for all $x \in \mathscr{A}$. By (3), it is clear that

$$\mathrm{m}(b)\mathrm{m}(x) = \mathrm{m}(b)\mathrm{m}(\Phi(x)) \leqslant \mathrm{m}(\varphi(x)) \leqslant \|b\|\mathrm{m}(\Phi(x)) = \|b\|\mathrm{m}(x), \qquad (4)$$

for all $x \in \mathscr{A}$. Thus, $\mathrm{m}(x)$ is positive whenever so is $\mathrm{m}(\varphi(x))$, and since φ is unital, this shows that $\sigma_{ap}(x) \subseteq \sigma_{ap}(\varphi(x))$ for all $x \in \mathscr{A}$. Hence $\mathrm{r}(x) \leqslant \mathrm{r}(\varphi(x))$ for all $x \in \mathscr{A}$. We claim that φ is injective. Let $a_0 \in \mathscr{A}$ be such that $\varphi(a_0) = 0$, and pick $a \in \mathscr{A}$. For every $\lambda \in \mathbb{C}$, we have

$$\mathrm{r}(\lambda a_0 + a) \leqslant \mathrm{r}(\varphi(\lambda a_0 + a)) = \mathrm{r}(\varphi(a)).$$

As $\lambda \mapsto \mathrm{r}(\lambda a_0 + a)$ is a subharmonic function on $\mathbb{C}$, Liouville's Theorem implies that $\mathrm{r}(\lambda a_0 + a) = \mathrm{r}(a)$ for all $\lambda \in \mathbb{C}$. Because a is an arbitrary element of $\mathscr{A}$, the spectral characterization of the radical, together with the semisimplicity of $\mathscr{A}$ imply that $a_0 = 0$, and hence φ is injective. Next, let us show that b, equivalently $\Phi(\mathbf{1})$, is invertible. Since Φ is surjective, there exists $c \in A$ such that $\Phi(c) = \mathbf{1} - \Phi(\mathbf{1})b$. Hence,

$$\varphi(c) = b\Phi(c) = b(\mathbf{1} - \Phi(\mathbf{1})b) = 0.$$

This shows that $c = 0$, and implies that $\mathbf{1} = \Phi(\mathbf{1})b$. Thus b is invertible in $\mathscr{B}$.

As $\mathrm{m}(b) = \|\Phi(\mathbf{1})^{-1}\| > 0$, from (4) we, in fact, deduce that

$$\sigma_{ap}(x) = \sigma_{ap}(\varphi(x)) \qquad (5)$$

for all $x \in \mathscr{A}$. By applying again Theorem 2.1 (if $\mathrm{Soc}(\mathscr{A})$ or $\mathrm{Soc}(\mathscr{B})$ is essential) or Theorem 2.2 (if $\mathscr{B}$ has real rank zero) we get that φ is a Jordan isomorphism.

Now, let us show that φ is isometric. First, note that $\|b\| = \mathrm{m}(\Phi(\mathbf{1}))^{-1} = \mathrm{m}(\mathbf{1})^{-1} = 1$, and hence, by (4), $\mathrm{m}(\varphi(x)) \leqslant \mathrm{m}(x)$ for all $x \in \mathscr{A}$. As by (2), $\varphi(u^{-1}) = \varphi(u)^{-1}$ for every $u \in \mathrm{Inv}(\mathscr{A})$, it is clear that

$$\|u^{-1}\| = \mathrm{m}(u)^{-1} \leqslant \mathrm{m}(\varphi(u))^{-1} = \|\varphi(u^{-1})\|$$

for all $u \in \mathrm{Inv}(\mathscr{A})$. Equivalently, $\|\varphi^{-1}(u)\| \leqslant \|u\|$ for all $u \in \mathrm{Inv}(\mathscr{B})$. In view of [20, Corollary 1], we have $\|\varphi^{-1}\| = 1$ and $\|\varphi^{-1}y\| = \leqslant \|y\|$ for all $y \in \mathscr{B}$. As the mapping $y \mapsto \|\varphi^{-1}y\|$ is an algebra norm on the JB^*-algebra $\mathscr{B}^+$, and every JB^*-algebra has

minimality of the norm (see [18, Proposition 11]), we deduce that φ^{-1} (and hence φ) is in fact an isometry. As φ^{-1} is a Jordan isomorphism, we have

$$\varphi^{-1}((y \circ z)^*) = \varphi^{-1}(y^* \circ z^*) = \varphi^{-1}(y^*) \circ \varphi^{-1}(z^*),$$

and having in mind (1)

$$\|U_{\varphi^{-1}(y)}(\varphi^{-1}(y^*))\| = \|\varphi^{-1}(y)\varphi^{-1}(y^*)\varphi^{-1}(y)\| = \|\varphi^{-1}(yy^*y)\|$$
$$= \|yy^*y\| = \|y\|^3 = \|\varphi^{-1}(y)\|^3 \quad \text{for all } y, \, z \in \mathscr{B}.$$

Hence, the mapping $\varphi^{-1}(y) \mapsto \varphi^{-1}(y^*)$ defines a JB^*-involution on $\mathscr{A}^+$. By [19, Theorem 2], $\mathscr{A}$ with its norm and this involution is a C^*-algebra. Clearly, $\varphi : \mathscr{A} \to \mathscr{B}$ is selfadjoint. This proves that φ is an isometric selfadjoint Jordan isomorphism; as desired.

In order to conclude the proof, we need to see that b is unitary. For every $x \in \text{Inv}(\mathscr{A})$, we have $\Phi(x)^{-1} = \varphi(x^{-1})b$. Since Φ preserves the minimum modulus and φ is an isometry, it follows that

$$\|\varphi(x^{-1})b\| = \|\Phi(x)^{-1}\| = \|x^{-1}\| = \|\varphi(x^{-1})\|$$

for all $x \in \text{Inv}(\mathscr{A})$. Thus, $\|yb\| = \|y\|$ for all $y \in \text{Inv}(\mathscr{B})$, and consequently b is unitary. $\square$

REMARK 3.2. The proof of Theorem 3.1 runs in a similar way if the map Φ is supposed to preserve the other quantities instead of the minimum modulus. A slight difference occurs when Φ preserves the maximum modulus. In this case, one gets two containments instead of a similar equality to (5). But of course these two containments are sufficient to apply Theorem 2.1 and Theorem 2.2. Indeed, assume that Φ preserves the maximum modulus. Since $M(\Phi(\mathbf{1})) = M(\mathbf{1}) = 1 > 0$, we may and shall assume that there is $b \in \mathscr{B}$ such that $b\Phi(\mathbf{1}) = \mathbf{1}$. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be the unital surjective linear map given by $\varphi(x) := b\Phi(x)$ for all $x \in \mathscr{A}$. By (3), we have

$$m(\varphi(x)) = m(b\Phi(x)) \leqslant \|b\|m(\Phi(x)) \leqslant \|b\|M(\Phi(x)) = \|b\|M(x) \quad (6)$$

for all $x \in \mathscr{A}$. It therefore follows that $\sigma_{ap,s}(x) \subseteq \sigma_{ap}(\varphi(x))$ for all $x \in \mathscr{A}$. Hence, we also have $r(x) \leqslant r(\varphi(x))$ for all $x \in \mathscr{A}$. From this, we infer that φ is bijective, b is invertible, and

$$\|\Phi(\mathbf{1})\|^{-1}M(x) = M(b)M(\Phi(x)) \leqslant M(b\Phi(x)) = M(\varphi(x)) \quad (7)$$

for all $x \in \mathscr{A}$. From (6) and (7), it follows that

$$\sigma_{ap,s}(\varphi(x)) \subset \sigma_{ap,s}(x) \subset \sigma_{ap}(\varphi(x)) \quad (8)$$

for all $x \in \mathscr{A}$; as promised.

Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a linear map. It is easy to check that if Φ is an isometric anti-isomorphism, then Φ preserves either the minimum modulus or the surjectivity modulus if and only if every left invertible element in $\mathscr{A}$ and $\mathscr{B}$ is invertible. The next corollary follows directly from this fact together with the above theorem and the well known result of Herstein, [11, Theorem H], that asserts that every Jordan homomorphism from an algebra onto a prime algebra is either a homomorphism or an anti-homomorphism.

COROLLARY 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. Suppose that either $\mathscr{A}$ is of real rank zero or $\mathrm{Soc}(\mathscr{A})$ is an essential ideal, and that $\mathscr{B}$ is prime. Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a surjective linear map. If $\mathscr{B}$ contains a non invertible element that is left invertible, then Φ preserves the minimum modulus (resp. the surjectivity modulus) if and only if Φ is an isometric isomorphism multiplied by a unitary element of $\mathscr{B}$.*

We also get the following corollary.

COROLLARY 3.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras, and $\Phi : \mathscr{A} \to \mathscr{B}$ be a surjective linear map. Assume that $\mathscr{A}$ has real rank zero or $\mathrm{Soc}(\mathscr{A})$ is essential, and that $\mathscr{B}$ is prime. Then Φ preserves the maximum modulus if and only if Φ is either an isometric isomorphism or an isometric anti-isomorphism multiplied by a unitary element of $\mathscr{B}$.*

By particularizing Corollary 3.3 to the setting of standard operator algebras, we obtain the following result that generalizes [16, Theorem 3.1].

COROLLARY 3.5. *Let $\mathfrak{A}$ be a unital C^*-subalgebra of $\mathscr{L}(H)$ containing the ideal of compact operators, and let Φ be a surjective linear map on $\mathfrak{A}$. If $\mathfrak{A}$ contains a non invertible operator that is left invertible, then Φ preserves the minimum modulus (resp. the surjectivity modulus) if and only if there are unitary operators $U, V \in \mathfrak{A}$ such that $\Phi(T) = UTV$ for all $T \in \mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ contains the ideal of compact operators, it is straightforward to check that the algebra $\mathfrak{A}$ is prime, and that its socle is essential. So, the desired result holds by applying Corollary 3.3 together with [9, Corollary 3.2]. □

We conclude this section by describing the inner automorphisms preserving the minimum, surjectivity or maximum modulus on C^*-algebras. The proof of the following theorem is inspired in that of [16, Theorem 3.8].

THEOREM 3.6. *Let $\mathscr{A}$ be a C^*-algebra, and let $a, b \in \mathrm{Inv}(\mathscr{A})$. The following statements are equivalent.*

 (i) *ab is a unitary element, and $|a|$ is central in $\mathscr{A}$.*

 (ii) *$\mathrm{m}(x) = \mathrm{m}(axb)$ for all $x \in \mathscr{A}$.*

 (iii) *$\mathrm{q}(x) = \mathrm{q}(axb)$ for all $x \in \mathscr{A}$.*

(iv) $M(x) = M(axb)$ *for all* $x \in \mathscr{A}$.

Proof. Assume that ab is a unitary element and that $|a|$ is central in $\mathscr{A}$. Since a^*a is as well central in $\mathscr{A}$, we have

$$|axa^{-1}|^2 = (axa^{-1})^* axa^{-1} = (a^{-1})^* x^* a^* axa^{-1}$$
$$= (a^{-1})^* a^* ax^* xa^{-1} = a|x|^2 a^{-1}$$

for all $x \in A$. This gives that $\sigma(|axa^{-1}|) = \sigma(|x|)$, and in particular $m(x) = m(axa^{-1})$ for all $x \in \mathscr{A}$. Since ab is unitary, we have

$$m(axb) = m((axa^{-1})ab) = m(axa^{-1}) = m(x)$$

for all $x \in \mathscr{A}$ and the implication (i)$\Longrightarrow$(ii) is established.

Conversely, suppose that the statement (ii) holds and consider the linear maps Φ and φ on A, given by

$$\Phi(x) := axb, \text{ and } \quad \varphi(x) := axa^{-1} \quad (x \in \mathscr{A}).$$

Obviously, $\Phi(x) = \varphi(x)ab$ for all $x \in \mathscr{A}$, and

$$m(\varphi(x))m(ab) = m(\varphi(x)) \leqslant m(\Phi(x)) = m(x)$$

for all $x \in \mathscr{A}$. Arguing as in the proof of Theorem 3.1, it follows that ab is unitary and that φ is an isometry. Thus φ is selfadjoint, and

$$axa^{-1} = \varphi(x) = \varphi(x^*)^* = a^{*-1}xa^*$$

for all $x \in \mathscr{A}$. This shows that $|a|^2 x = x|a|^2$ for all $x \in A$. As $|a|$ can be approximated by polynomials in $|a|^2$, it follows that $|a|x = x|a|$ for all $x \in A$. This establishes the reverse implication (ii)$\Longrightarrow$(i).

In the same way, one proves that the equivalences (i)$\Longleftrightarrow$(iii) and (i)$\Longleftrightarrow$(iv) hold. $\square$

As an immediate consequence of the previous theorem we obtain the following characterization of unitary elements in C^*-algebras with trivial center.

COROLLARY 3.7. *Let $\mathscr{A}$ be a C^*-algebra with trivial center (in particular, $\mathscr{A}$ can be prime). Let $a \in \mathrm{Inv}(\mathscr{A})$. The following are equivalent.*

(i) The element a is a scalar multiple of a unitary element of $\mathscr{A}$.

(ii) $m(x) = m(axa^{-1})$ *for all* $x \in \mathscr{A}$.

(iii) $q(x) = q(axa^{-1})$ *for all* $x \in \mathscr{A}$.

(iv) $M(x) = M(axa^{-1})$ *for all* $x \in \mathscr{A}$.

4. Linear maps preserving the essential minimum modulus

In what follows, let H be an infinite dimensional complex Hilbert space. The closed ideal of all compact operators on H is denoted by $\mathcal{K}(H)$, and the Calkin algebra is denoted, as usual, by $\mathcal{C}(H) := \mathcal{L}(H)/\mathcal{K}(H)$. For an operator $T \in \mathcal{L}(H)$, let $\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$ and $\sigma_{le}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left Fredholm}\}$ denote the essential and the left essential spectrum respectively of T. The essential norm of T is given by $\|T\|_e := \operatorname{dist}(T, \mathcal{K}(H))$ and the essential spectral radius of T, denoted by $\mathrm{r}_e(T)$, is the limit of the convergent sequence $(\|T^n\|_e^{1/n})_{n \geqslant 1}$. It coincides with $\mathrm{r}(\pi(T))$ the classical spectral radius of $\pi(T)$, where π denotes the canonical quotient map from $\mathcal{L}(H)$ onto $\mathcal{C}(H)$. While, the *essential minimum modulus* of T is given by

$$\mathrm{m}_e(T) := \mathrm{m}(\pi(T)) = \inf\{\|TS\|_e : S \in \mathcal{L}(H), \|S\|_e = 1\} = \inf\{\lambda : \lambda \in \sigma_e(|T|)\}.$$

In a similar way, the *essential surjectivity modulus* and the *essential maximum modulus* of T are defined respectively by

$$\mathrm{q}_e(T) := \mathrm{q}(\pi(T)) = \inf\{\|ST\|_e : S \in \mathcal{L}(H), \|S\|_e = 1\} = \inf\{\lambda : \lambda \in \sigma_e(|T^*|)\}$$

and

$$\mathrm{M}_e(T) := \mathrm{M}(\pi(T)) = \max\{\mathrm{m}_e(T), \mathrm{q}_e(T)\}.$$

The interested reader is refered to [5] and [24], where the essential minimum modulus for operators on a Hilbert space was deeply studied.

The following result characterizes the surjective up to compact operators linear maps on $\mathcal{L}(H)$ preserving the essential minimum modulus and the essential maximum modulus.

THEOREM 4.1. *Let* $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ *be a linear map such that* $\mathcal{L}(H) = \Phi(\mathcal{L}(H)) + \mathcal{K}(H)$. *The following statements are equivalent.*

(i) $\mathrm{m}_e(\Phi(T)) = \mathrm{m}_e(T)$ *for all* $T \in \mathcal{L}(H)$.

(ii) $\mathrm{q}_e(\Phi(T)) = \mathrm{q}_e(T)$ *for all* $T \in \mathcal{L}(H)$.

(iii) $\Phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$, *and the induced mapping* $\hat{\Phi} : \mathcal{C}(H) \to \mathcal{C}(H)$, *defined by* $\hat{\Phi}(T + \mathcal{K}(H)) := \Phi(T) + \mathcal{K}(H)$, $(T \in \mathcal{L}(H))$, *is an isometric selfadjoint isomorphism multiplied by a unitary element in* $\mathcal{C}(H)$.

Proof. It is easy to see that the implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) always hold.

Now, assume that $\mathrm{m}_e(\Phi(T)) = \mathrm{m}_e(T)$ for all $T \in \mathcal{L}(H)$ and let us show that $\Phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$. Pick up a compact operator $K \in \mathcal{K}(H)$, and let us prove that $\Phi(K)$ is a compact operator as well. For every $S \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$, there are $K_1, K_2 \in \mathcal{K}(H)$ and $T, R \in \mathcal{L}(H)$ such that $\lambda = \Phi(R) + K_1$ and $S = \Phi(T) + K_2$, and thus

$$\begin{aligned} \mathrm{m}_e(\Phi(K) + S - \lambda) &= \mathrm{m}_e(\Phi(K + T - R) - K_1 + K_2) = \mathrm{m}_e(\Phi(K + T - R)) \\ &= \mathrm{m}_e(K + T - R) = \mathrm{m}_e(T - R) = \mathrm{m}_e(\Phi(T - R)) \\ &= \mathrm{m}_e(S - \lambda + K_1 - K_2) = \mathrm{m}_e(S - \lambda). \end{aligned}$$

Hence, $\sigma_{le}(\Phi(K)+S) = \sigma_{le}(S)$ and $r(\pi(\Phi(K)+S)) = r_e(\Phi(K)+S) = r_e(S) = r(\pi(S))$ for all $S \in \mathscr{L}(H)$. By the the Zemánek spectral characterization of the radical, [2, Theorem 5.3.1], and the semisimplicity of $\mathscr{C}(H)$, we get that $\pi(\Phi(K)) = 0$ and $\Phi(K)$ is a compact operator.

We proved that $\Phi(\mathscr{K}(H)) \subseteq \mathscr{K}(H)$, and thus Φ induces a surjective linear map $\hat{\Phi} : \mathscr{C}(H) \to \mathscr{C}(H)$, defined by

$$\hat{\Phi}(\pi(T)) := \pi(\Phi(T)), \ (T \in \mathscr{L}(H)),$$

that preserves the minimum modulus; i.e., $m(\hat{\Phi}(\pi(T))) = m(\pi(T))$ for all $T \in \mathscr{L}(H)$. As $\mathscr{C}(H)$ is prime and has real rank zero, Corollary 3.3 tell us that $\hat{\Phi}$ is an isometric selfadjoint isomorphism multiplied by an unitary element of $\mathscr{C}(H)$. This establishes the implication (i) $\Rightarrow$ (iii) and finishes the proof since the implication (ii) $\Rightarrow$ (iii) can be established in a similar way. $\quad\square$

We close this paper with the following result which characterizes the surjective up to compact operators linear maps on $\mathscr{L}(H)$ preserving the essential maximum modulus. Its proof is omitted as it proceeds along the same lines as the one of the above theorem.

THEOREM 4.2. *Let* $\Phi : \mathscr{L}(H) \to \mathscr{L}(H)$ *be a linear map such that* $\mathscr{L}(H) = \Phi(\mathscr{L}(H)) + \mathscr{K}(H)$. *The following statements are equivalent.*

(i) $M_e(\Phi(T)) = M_e(T)$ *for all* $T \in \mathscr{L}(H)$.

(ii) $\Phi(\mathscr{K}(H)) \subseteq \mathscr{K}(H)$, *and the induced mapping* $\hat{\Phi} : \mathscr{C}(H) \to \mathscr{C}(H)$, *is either an isometric selfadjoint isomorphism or an isometric selfadjoint anti-isomorphism multiplied by a unitary element in* $\mathscr{C}(H)$.

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