Operators and Matrices
Volume 4, Number 2 (2010), 273–281

OPERATOR RADII AND UNITARY OPERATORS

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(*Communicated by J. Pečarić*)

Abstract. Let $\rho \geqslant 1$ and $w_\rho(A)$ be the operator radius of a linear operator A. Suppose m is a positive integer. It is shown that for a given invertible linear operator A acting on a Hilbert space, one has $w_\rho(A^{-m}) \geqslant w_\rho(A)^{-m}$. The equality holds if and only if A is a multiple of a unitary operator.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$. Denote by $B(\mathscr{H})$ the algebra of bounded linear operators acting on $\mathscr{H}$ with the *operator norm* defined by

$$\|A\| = \sup\{\|Ax\| : x \in \mathscr{H}, \|x\| = 1\} \qquad \text{for } A \in B(\mathscr{H}).$$

It is easy to see that $A \in B(\mathscr{H})$ is *unitary* if and only if it is invertible and

$$\|A\| \leqslant 1 \quad \text{and} \quad \|A^{-1}\| \leqslant 1. \tag{1.1}$$

If the requirement (1.1) is weakened as

$$\|A^n\| \leqslant \rho \quad \text{and} \quad \|A^{-n}\| \leqslant \rho \quad (n = 1,2,\ldots) \text{ for some } \rho \geqslant 1, \tag{1.2}$$

then, by a theorem of Sz.-Nagy [8], the operator A is *similar* to a unitary operator, that is,

$$A = S^{-1}US \quad \text{for some invertible } S \text{ and unitary } U,$$

and consequently its *spectrum* $\sigma(A)$ is included in the unit circle of the complex plane.

Recall that the *numerical radius* of $A \in B(\mathscr{H})$ is defined by

$$w(A) = \sup\{|\langle x, Ax\rangle| \,:\, x \in \mathscr{H}, \|x\| \leqslant 1\}.$$

In [7, Corollary 1] (see also [6]), it was shown that in (1.1) the operator norm $\|\cdot\|$ can be replaced by the numerical radius $w(\cdot)$, namely, that an invertible operator A is

Mathematics subject classification (2010): 47A20, 47A12, 15A60.
Keywords and phrases: Unitary ρ-dilation, operator radius, numerical radius, unitary operator.
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unitary if $w(A) \leqslant 1$ and $w(A^{-1}) \leqslant 1$. Notice that the map $A \longmapsto w(A)$ is *convex* and (1.2) is guaranteed by the known property of the numerical radius (see [9]), namely, for any $A \in B(\mathscr{H})$

$$w(A) \leqslant \|A\| \leqslant 2 \cdot w(A) \quad \text{and} \quad w(A^n) \leqslant w(A)^n \quad \text{for } n = 1, 2, \ldots.$$

Very recently, Choi and Li [2, Theorem 3.9] showed that for a positive integer m and an invertible operator $A \in B(\mathscr{H})$, we have

$$w(A^{-m}) \geqslant w(A)^{-m}; \tag{1.3}$$

the equality holds if and only if A is a multiple of a unitary operator. Clearly, the same result holds if one replaces the numerical radius by the operator norm. (A short proof of this case is included in Section 3).

In [9], Sz.-Nagy and Foiaş considered the class $\mathscr{C}_\rho$ of operators $T \in B(\mathscr{H})$ which admits a *unitary* ρ*-dilation*, that is, there is a unitary operator U on a superspace $\mathscr{K} \supset \mathscr{H}$ such that

$$T^n = \rho P U^n|_{\mathscr{H}} \quad \text{for } n = 1, 2, \ldots,$$

where P is the orthoprojection from $\mathscr{K}$ to $\mathscr{H}$. In connection with this, one can define the ρ*-radius* of $A \in B(\mathscr{H})$ by

$$w_\rho(A) = \inf\{\lambda > 0 : \lambda^{-1} A \in \mathscr{C}_\rho\}.$$

When $\rho = 1$ and $\rho = 2$, this definition reduces to the operator norm and the numerical radius, respectively. The operator radii have the following properties (see [9], [4] and [5]):

(i) For each ρ, the functional $A \longmapsto w_\rho(A)$ is strictly positive, and (non-linear) positive-homogeneous and $w_\rho(A) = w_\rho(A^*)$.

(ii) Let $r(A)$ be the spectral radius of $A \in B(\mathscr{H})$. Then $\lim_{\rho \to \infty} w_\rho(A) = r(A)$, and the function $\rho \longmapsto w_\rho(A)$ is *non-increasing*. Consequently, we have

$$r(A) \leqslant w_\rho(A) \leqslant \|A\|.$$

(iii) For each $A \in B(\mathscr{H})$, we have

$$w_\rho(A) \leqslant \|A\| \leqslant \rho \cdot w_\rho(A) \quad \text{and} \quad w_\rho(A^n) \leqslant w_\rho(A)^n \quad \text{for } n = 1, 2, \ldots.$$

(iv) For $A \in B(\mathscr{H})$, the function $A \longmapsto w_\rho(A)$ is *convex* (only) when $1 \leqslant \rho \leqslant 2$.

In this paper, we show that the inequality (1.3) and the condition for equality are valid if we replace the numerical radius by the ρ-radius for any $\rho \geqslant 1$. Specifically, we have the following.

THEOREM 1.1. *Let $\rho \geqslant 1$, and m be a positive integer. If $A \in B(\mathscr{H})$ is invertible, then*

$$w_\rho(A^{-m}) \geqslant w_\rho(A)^{-m}.$$

The equality holds if and only if A is a multiple of a unitary operator.

We will characterize those invertible $A \in B(\mathscr{H})$ satisfying $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$ in Section 2, and prove Theorem 1.1 in Section 3. Our proof depends on the following characterization of $A \in B(\mathscr{H})$ satisfying $w_\rho(A) \leqslant 1$ obtained by Ando [1] for the case $\rho = 2$ and by Durszt [3] for the general case (see also [5]).

LEMMA 1.2. *For an operator A and $\rho > 1$, the condition $w_\rho(A) \leqslant 1$ is valid if and only if there is $0 \leqslant C \leqslant I$ and a contraction W, that is, $\|W\| \leqslant 1$, such that*

$$A = \rho(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}WC^{1/2}.$$

If A is invertible, $0 < C < I$ and W can be chosen as unitary.

Here, as usual, the order relation $S \leqslant T$ between two selfadjoint operators S, T means that $T-S$ is *positive semi-definite*, or equivalently

$$\langle x, Sx\rangle \;\leqslant\; \langle x, Tx\rangle \quad (x \in \mathscr{H}),$$

and $S < T$ means that $T-S$ is invertible in addition.

2. Auxiliary results

In this section, we characterize those invertible $A \in B(\mathscr{H})$ such that $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$.

We first consider the case when the Hilbert space $\mathscr{H}$ has a *finite dimension*, say N. Therefore each A is considered as a matrix, and we can use the *determinant* function.

THEOREM 2.1. *Suppose $\rho \geqslant 1$. An invertible matrix A is unitary if and only if $w_\rho(A) \leqslant 1$ and its spectrum $\sigma(A)$ is included in the unit circle.*

Proof. The implication ($\Rightarrow$) is clear. We consider the converse. Suppose $\rho = 1$. Then A is unitarily similar to a lower triangular matrix T so that each diagonal entry is an eigenvalue lying in the unit circle. Since $w_1(A) = w_1(T) = \|T\| = 1$, we see that all off diagonal entries of T are zero. Next, assume that $\rho > 1$. Since $w_\rho(A) \leqslant 1$ and A is invertible, by Lemma 1.2 there is $0 < C < I$ and unitary W such that

$$A = \rho(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}WC^{1/2}. \tag{2.1}$$

Then since the determinant of a matrix is the product of all its eigenvalues (counting multiplicities) and since $\det(XY) = \det(X)\det(Y)$ for any matrices X, Y,

$$\begin{aligned} 1 &= |\det(A)|^2 = \det(A^*A) \\ &= \det\left(\rho^2 C^{1/2} W^*(I-C)\{I+\rho(\rho-2)C\}^{-1} W C^{1/2}\right) \\ &= \det\left(\rho^2 C(I-C)\{I+\rho(\rho-2)C\}^{-1}\right) \\ &= \prod_{j=1}^{N} \frac{\rho^2\lambda_j(1-\lambda_j)}{1+\rho(\rho-2)\lambda_j}, \end{aligned}$$

where λ_j $(j = 1,2,\dots,N)$ are the eigenvalues of C with multiplicities counted. It is easy to see that

$$f(t) \equiv f_\rho(t) := \frac{\rho^2 t(1-t)}{1+\rho(\rho-2)t} \leqslant 1 \quad (0 \leqslant t \leqslant 1)$$

and the maximum value 1 is attained only at $t = \rho^{-1}$. We conclude $\lambda_j = \rho^{-1}$ $(j = 1,2,\dots,N)$, that is, $C = \rho^{-1}I$. Then by (2.1) we have

$$A = \frac{\rho\sqrt{1-\rho^{-1}}}{\sqrt{\rho\{1+\rho(\rho-2)\rho^{-1}\}}} W = W.$$

Therefore A is unitary. $\square$

If $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$, then by the property (iii) of the ρ-radii and the theorem of Sz.-Nagy [8], the spectrum $\sigma(A)$ is included in the unit circle. Therefore we can conclude from Theorem 2.1 that an invertible matrix is unitary if $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$ for any ρ.

Now we turn to the infinite dimensional case. The following example shows that the extension of Theorem 2.1 to the infinite dimensional case is not possible even for $\rho = 1$.

EXAMPLE 2.2. *There is a non-unitary contraction which is similar to a unitary operator.*

Construction. Let $\mathscr{H} = L^2(-\infty,\infty)$ with respect to the Lebesgue measure on $(-\infty,\infty)$. Let $\varphi(t)$ is a *strictly increasing* continuous function on $(-\infty,\infty)$ such that

$$\lim_{t\to-\infty} \varphi(t) = \alpha > 0 \quad \text{and} \quad \lim_{t\to\infty} \varphi(t) = \beta < \infty.$$

Let T be the *multiplication* operator by the function $\varphi(t)$ and U the *right-shift* operator by unit one, that is,

$$(Tf)(t) = \varphi(t)f(t) \quad \text{and} \quad (Uf)(t) = f(t+1) \quad (-\infty < t < \infty).$$

Let C be the multiplication operator by the function $\frac{\varphi(t-1)}{\varphi(t)}$. Then $T > 0$ and U is unitary while C is a non-unitary contraction because of the strict-increasingness of $\varphi(t)$. Now it is easy to see that $TU = UCT$, which implies that the non-unitary contraction $A = UC$ is similar to the unitary operator U. $\square$

The following theorem generalizes a result of Stampfli [7, Corollary 1] (see also [2] and [6]) to general operator radii $w_\rho(\cdot)$.

THEOREM 2.3. *Suppose $\rho \geqslant 1$. An invertible operator $A \in B(\mathscr{H})$ is unitary if and only if $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$.*

Proof. The implication ($\Rightarrow$) is clear. We consider the converse. Suppose $\rho = 1$. If $\|Ax\| < 1$ for any unit vector $x \in \mathscr{H}$, then $\|x\| = \|A^{-1}(Ax)\| < 1$, which is a contradiction. Thus, $\|Ax\| = 1$ for all unit vector $x \in \mathscr{H}$. Since A is invertible, A is unitary. Next, assume $\rho > 1$. Consider again the function

$$f(t) \equiv f_\rho(t) \;:=\; \frac{\rho^2(1-t)}{1+\rho(\rho-2)t} \qquad (0 \leqslant t \leqslant 1). \tag{2.2}$$

Then simple computations will show the following relations:

$$\frac{1}{t} = f(t) + \frac{(1-\rho t)^2}{t\{1+\rho(\rho-2)t\}} \quad \text{and} \quad t = \frac{1}{f(t)} - \frac{(1-\rho t)^2}{\rho^2(1-t)} \quad (0 < t < 1). \tag{2.3}$$

Since by assumption $w_\rho(A) \leqslant 1$ and $w_\rho\left((A^{-1})^*\right) = w_\rho(A^{-1}) \leqslant 1$, by Lemma 1.2 there are $0 < X,\ Y < I$ and unitary $U,\ V$ such that

$$A \;=\; f(X)^{1/2}UX^{1/2} \quad \text{and} \quad (A^{-1})^* = f(Y)^{1/2}VY^{1/2}. \tag{2.4}$$

Then it follows from (2.4) that

$$I = A\cdot A^{-1} = f(X)^{1/2}UX^{1/2}\cdot Y^{1/2}V^*f(Y)^{1/2},$$

which implies

$$UX^{1/2}Y^{1/2} = f(X)^{-1/2}f(Y)^{-1/2}V,$$

and hence

$$Y^{1/2}XY^{1/2} = V^*f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}V.$$

This means that $Y^{1/2}XY^{1/2}$ and $f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}$ are *unitarily similar*. Therefore they have the same spectrum

$$\sigma\left(Y^{1/2}XY^{1/2}\right) \;=\; \sigma\left(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\right),$$

which implies obviously

$$\lambda_{\max}\left(Y^{1/2}XY^{1/2}\right) \;=\; \lambda_{\max}\left(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\right) \tag{2.5}$$

and

$$\lambda_{\min}\left(Y^{1/2}XY^{1/2}\right) \;=\; \lambda_{\min}\left(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\right), \tag{2.6}$$

where, for a selfadjoint operator Z, the symbol $\lambda_{\max}(Z)$ (resp. $\lambda_{\min}(Z)$) denotes the *maximum* (resp. *minimum*) of the spectrum $\sigma(Z)$.

Now write, according to (2.5),

$$\gamma := \lambda_{\max}(Y^{1/2}XY^{1/2}) = \lambda_{\max}\left(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\right). \tag{2.7}$$

This γ is characterized as a positive number such that $\gamma Y^{-1} - X \geqslant 0$ and

$$\lim_{n\to\infty}\langle a_n, (\gamma Y^{-1} - X)a_n\rangle = 0 \quad \text{for some } a_n \text{ with } \|a_n\| = 1. \tag{2.8}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}\gamma Y^{-1} - X &= \gamma f(Y) - f(X)^{-1} + \gamma Y^{-1}\{I + \rho(\rho-2)Y\}^{-1}(I-\rho Y)^2 \\ &\quad + \rho^{-2}(I-X)^{-1}(I-\rho X)^2.\end{aligned}$$

Then since $0 < X,\ Y < I$, there is $\varepsilon > 0$ such that

$$\gamma Y^{-1} - X \geqslant \gamma f(Y) - f(X)^{-1} + \varepsilon(I-\rho Y)^2 + \varepsilon(I-\rho X)^2. \tag{2.9}$$

Since by (2.5) $\gamma Y^{-1} - X \geqslant 0$ implies $\gamma f(Y) - f(X)^{-1} \geqslant 0$, it follows from (2.8) and (2.9) that

$$\lim_{n\to\infty}\{\|(I-\rho Y)a_n\|^2 + \|(I-\rho X)a_n\|^2\} = 0. \tag{2.10}$$

Finally we have from (2.8) and (2.10) that

$$0 = \lim_{n\to\infty}\langle a_n, (\gamma Y^{-1} - X)a_n\rangle = \rho\gamma - \rho^{-1}$$

that is,

$$\lambda_{\max}\left(Y^{1/2}XY^{1/2}\right) = \gamma = \rho^{-2}. \tag{2.11}$$

Incidentally we have shown that, with $\gamma = \rho^{-2}$,

$$\ker\left(\rho^{-2}Y^{-1} - X\right) \;\subset\; \ker(I - \rho X). \tag{2.12}$$

Next write, according to (2.6),

$$\kappa := \lambda_{\min}\left(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\right) = \lambda_{\min}\left(Y^{1/2}XY^{1/2}\right). \tag{2.13}$$

This κ is characterized as a positive number such that $f(X)^{-1} - \kappa f(Y) \geqslant 0$ and

$$\lim_{n\to\infty}\langle b_n, \{f(X)^{-1} - \kappa f(Y)\}b_n\rangle = 0 \quad \text{for some } b_n \text{ with } \|b_n\| = 1. \tag{2.14}$$

Now we have by (2.3)

$$f(X)^{-1}-\kappa f(Y)=X-\kappa Y^{-1}+\rho^{-2}(I-X)^{-1}(I-\rho X)^2 \\ +\kappa Y^{-1}\{I+\rho(\rho-2)Y\}^{-1}(I-\rho Y)^2.$$

Since by (2.6) $f(X)^{-1}-\kappa f(Y)\geqslant 0$ implies $X-\kappa Y^{-1}\geqslant 0$, as in the foregoing arguments we can conclude that

$$f(X)^{-1}-\kappa f(Y)\ \geqslant\ X-\kappa Y^{-1}\geqslant 0 \tag{2.15}$$

and for some $\varepsilon>0$

$$f(X)^{-1}-\kappa f(Y)\ \geqslant\ \varepsilon(I-\rho X)^2+\varepsilon(I-\rho Y)^2. \tag{2.16}$$

Then by (2.14) and (2.16) we have

$$\lim_{n\to\infty}\{\|b_n-\rho Xb_n\|^2+\|b_n-\rho Yb_n\|^2\}=0,$$

and by (2.14) and (2.15)

$$\lim_{n\to\infty}\langle b_n,(X-\kappa Y^{-1})b_n\rangle=0.$$

From the above we can conclude that $\kappa=\rho^{-2}$, hence $\kappa=\gamma$ by (2.11). This means that $Y^{1/2}XY^{1/2}=\rho^{-2}I$, so that $\rho^{-2}Y^{-1}-X=0$, that is, $\ker(\rho^{-2}Y^{-1}-X)=\mathscr{H}$. Finally by (2.12) this implies $\ker(I-\rho X)=\mathscr{H}$, or equivalently $X=\rho^{-1}I$. Now we can conclude by (2.2) and (2.4)

$$A=f(\rho^{-1})^{1/2}\rho^{-1/2}U=U,$$

that is, A is unitary. This completes the proof. $\square$

3. Proof of the main theorem

We use the fact that for any $T\in B(H)$

$$r(T)\leqslant w_\rho(T)\leqslant\|T\|,$$

where $r(T)$ is the spectral radius of T, and

$$w_\rho(T^k)\leqslant w_\rho(T)^k,\quad k=1,2,\dots.$$

If $A\in B(\mathscr{H})$ is invertible, then

$$w_\rho(A^{-1})\geqslant r(A^{-1})=1/\inf\{|\mu|:\mu\in\sigma(A)\}\geqslant r(A)^{-1}\geqslant w_\rho(A)^{-1}.$$

Replacing A by A^m, we have $w_\rho(A^{-m})\geqslant w_\rho(A^m)^{-1}$. Since $w_\rho(A^m)\leqslant w_\rho(A)^m$, we have

$$w_\rho(A^m)^{-1}\geqslant w_\rho(A)^{-m}.$$

If γA is unitary for some positive number γ, then $w_\rho(A^{-m}) = \gamma^{-m} = w_\rho(A)^{-m}$. Conversely, suppose $w_\rho(A^{-m}) = w_\rho(A)^{-m}$. We may replace A by γA for a suitable positive number γ and assume that $w_\rho(A^{-m}) = w_\rho(A)^{-m} = 1$. Thus,

$$1 = w_\rho(A^{-m}) \geqslant w_\rho(A^m)^{-1} \geqslant w_\rho(A)^{-m} = 1. \tag{3.1}$$

So, $1 = w_\rho(A^m) = w_\rho(A^{-m})$. By Theorem 2.3, A^m is unitary. By (3.1), we also have $w_\rho(A) = 1$. If $\rho = 1$, then for any unit vector $x \in \mathscr{H}$, $\|A\| = 1 = \|A^m x\|$ implies that $x, Ax, \ldots, A^m x$ are all unit vectors. Thus, $1 = \|Ax\|$ for all x. Since A is invertible, A is unitary. Suppose $\rho > 1$. By Lemma 1.2,

$$A = \rho(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}WC^{1/2}$$

for some $0 < C < I$ and a unitary W.

Let

$$\tilde{C} = \rho C^{1/2}(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}.$$

As shown in the proof of Theorem 2.1, we have

$$f(t) \equiv f_\rho(t) := \frac{\rho^2 t(1-t)}{1+\rho(\rho-2)t} \leqslant 1 \quad (0 \leqslant t \leqslant 1).$$

Thus $\tilde{C}^2 = f(C) \leqslant I$, and $\tilde{C}$ is a contraction. As a result,

$$A^m = \rho(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}\widetilde{W}C^{1/2}$$

such that

$$\widetilde{W} = W(\widetilde{C}W)^{m-1}$$

is a contraction. Since A^m is unitary, we see that

$$\begin{aligned} I &= (A^m)(A^m)^* \\ &= \rho^2(I-C)^{1/2}\{I+\rho(\rho-2)C\}^{-1/2}\widetilde{W}C^{1/2}C^{1/2}\widetilde{W}^*\{I+\rho(\rho-2)C\}^{-1/2}(I-C)^{1/2}. \end{aligned}$$

Thus,

$$\{I+\rho(\rho-2)C\}(I-C)^{-1} = \rho^2\widetilde{W}C\widetilde{W}^*. \tag{3.2}$$

When both $\widetilde{W}$ and C are invertible, we know

$$\sigma\left(\widetilde{W}C\widetilde{W}^*\right) = \sigma\left(C^{1/2}\widetilde{W}^*\widetilde{W}C^{1/2}\right). \tag{3.3}$$

Since in general

$$C^{1/2}\widetilde{W}^*\widetilde{W}C^{1/2} \leqslant C \quad \text{for } \|\widetilde{W}\| \leqslant 1,$$

we have

$$\lambda_{\max}(\widetilde{W}C\widetilde{W}^*) \leqslant \lambda_{\max}(C) \quad \text{and} \quad \lambda_{\min}(\widetilde{W}C\widetilde{W}^*) \leqslant \lambda_{\min}(C). \tag{3.4}$$

Since the function

$$g(t) := \frac{1+\rho(\rho-2)t}{1-t}$$

is increasing for $0 \leqslant t < 1$, we have

$$\lambda_{\max}\Big(g(C)\Big) = g\Big(\lambda_{\max}(C)\Big) \quad \text{and} \quad \lambda_{\min}\Big(g(C)\Big) = g\Big(\lambda_{\min}(C)\Big). \tag{3.5}$$

Then it follows from (3.2), (3.4) and (3.5)

$$g(t) \leqslant \rho^2 t \quad \text{for } t = \lambda_{\max}(C),\ \lambda_{\min}(C). \tag{3.6}$$

Since

$$g(t) - \rho^2 t = \frac{(1-\rho t)^2}{1-t} \geqslant 0 \quad (0 \leqslant t < 1),$$

(3.6) is possible only when

$$\lambda_{\max}(C) = \lambda_{\min}(C) = \frac{1}{\rho},$$

and hence $C = \frac{1}{\rho} I$. Consequently

$$A = \rho(I-C)^{1/2}\{I + \rho(\rho-2)C\}^{-1/2} W C^{1/2} = W$$

is unitary as asserted. $\square$

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(Received August 23, 2009)

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