Operators and Matrices
Volume 4, Number 2 (2010), 283–292

ON COMMUTATORS IN MATRIX THEORY

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(*Communicated by I. M. Spitkovsky*)

Abstract. We investigate intertwining relations arising from commutators such as $AB - BA = D$ when $AD = DA$, and $AB - BA^T = D$ when $AD = DA^T$, where A, B and D are n-by-n matrices. Depending on the properties of A, such equations often force D to be zero or at least nilpotent, and it is the properties of D that we investigate. We briefly discuss the situation when $AB + BA = D$, $AD = DA^T$ for A normal.

1. Introduction

Let A, B and D be n-by-n matrices. Equations such as $AB - BA = D$ and $AB - BA^T = D$ are important in matrix theory (the bracket notation $[A, B] = AB - BA$ is often used, see Zhang [10] for elementary properties of $[A, B]$). We survey a number of well known properties of the *commutator* $AB - BA$ and give some new properties. We also look at the commutator-type expressions $AB - BA^T$ and $AB + BA$.

The space of all n-by-m complex matrices will be denoted by $M_{n,m}(\mathbb{C})$ (or just M_n when $m = n$), and the corresponding space of real matrices will be denoted $M_{n,m}(\mathbb{R})$. Our vectors are in $\mathbb{C}^n = M_{n,1}(\mathbb{C})$, the space of n-by-1 complex matrices. Our notation will follow [3]. The transpose of the matrix A will be denoted by A^T and A^* will denote the conjugate transpose $\bar{A}^T$.

Recall that a matrix $A \in M_n$ is *nonderogatory* if every eigenvalue is of geometric multiplicity equal to one. In this case, each eigenvalue has exactly one Jordan block in which it appears.

The *commutant* of $A \in M_n$ is the set

$$C(A) = \{B \in M_n : AB = BA\}.$$

It is known that $C(A)$ is Abelian if and only if A is a nonderogatory matrix (see [5], Theorem 4.4.19/Corollary 4.4.18). Notice that $C(A) \subseteq C(A^2)$, so if A^2 is nonderogatory then A is also nonderogatory.

We will make repeated use of Sylvester's Theorem: if $A \in M_n$ and $B \in M_m$ have no eigenvalues in common, then the matrix equation $AX - XB = C$, has a unique solution $X \in M_{n,m}(\mathbb{C})$. When $C = 0$, this solution is $X = 0$. (see problem 9 in (2.4) of [3] for a proof).

Mathematics subject classification (2010): 15A18, 15A27.
Keywords and phrases: Matrix commutator, nonderogatory matrix, normal matrix.

Paper OaM-04-15

2. Basic Results About Commutators

A result of Jacobson [6] says that if $AB - BA = D$ and $AD = DA$, then D is a nilpotent matrix [3], page 98. Putnam [7] showed that for bounded normal operators A, B and D on a Hilbert space, with $AB - BA = D$ and $AD = DA$, necessarily $D = 0$. This result was improved by H. Shapiro [8], in the matrix setting, to show that if A is diagonalizable, then $D = 0$. It is well known (see the American Mathematical Monthly, March 2002, Problem 10930) that for $A, B \in M_2$, with $AB - BA = D$ and $AD = DA$, under the additional condition that $BD = DB$, necessarily $D = 0$. We prove these results and give various generalizations. Our first theorem starts with proofs of Jacobson's Lemma and results of Shapiro, and continues with some new properties of commutators. The proof of Jacobson's lemma is close to that in [3], page 98, but [4], 2.4 Problem 12 gives a new proof which is possibly more elegant. Theorem 1(b) and (c) are due to Shapiro [8], but the proof of (c) is new and the results in (d), (e) and (f) are new.

THEOREM 1. *Let $A, B, D \in M_n$ with $AB - BA = D$:*

(a) *If $AD = DA$ then D is a nilpotent matrix (Jacobson's Lemma).*

(b) *If $AD = DA$ where A is diagonalizable, then $D = 0$ (Shapiro [8]).*

(c) *If $AD = DA$ where A is nonderogatory, then A and B are simultaneously triangularizable (Shapiro [8]).*

(d) *If $AD = DA$, $BD = DB$, and A is nonderogatory then $D^2 = 0$. The number of distinct eigenvalues of B is less than or equal to the number of distinct eigenvalues of A and $\operatorname{rank}(D) < n/2$.*

(e) *If $AD = DA$, $BD = DB$ and the algebraic multiplicity of every eigenvalue of A is less than or equal to 2, then $D = 0$.*

(f) *If $AD = -DA$ and A^2 is a nonderogatory matrix, then $D = 0$.*

Proof. (a) Set $D = PJP^{-1}$ where J is the Jordan canonical form of D and P is a nonsingular matrix. We can assume that J is the direct sum of the form:

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k,$$

where each J_i is the direct sum of Jordan blocks corresponding to the same eigenvalue λ_i, so the spectrum of J_i, $\sigma(J_i) = \{\lambda_i\}$ is a singleton set, and each of the λ_i's is distinct, $i = 1, 2, \ldots, k$.

Now $AD = DA$ gives $(P^{-1}AP)J = J(P^{-1}AP)$ or $\tilde{A}J = J\tilde{A}$, where $\tilde{A} = P^{-1}AP$. Partition $\tilde{A}$ conformally with J, then since the eigenvalues of each J_i are distinct, using Sylvester's Theorem we can write

$$\tilde{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_k, \quad \text{where} \quad A_iJ_i = J_iA_i, \; i = 1, \ldots, k.$$

The equation $AB - BA = D = PJP^{-1}$ can then be written as

$$(P^{-1}AP)(P^{-1}BP) - (P^{-1}BP)(P^{-1}AP) = J,$$

or

$$\tilde{A}\tilde{B}-\tilde{B}\tilde{A}=J \quad \text{where} \quad \tilde{B}=P^{-1}BP.$$

Writing $\tilde{B}=[B_{ij}]$ conformal to $\tilde{A}$ and J gives $A_iB_{ij}-B_{ij}A_j=J_i$ when $i=j$ and zero otherwise. Thus the trace of J_i is $\operatorname{tr}(J_i)=\operatorname{tr}(A_iB_{ii}-B_{ii}A_i)=0$. This implies that $\lambda_i=0$, $i=1,\ldots,k$ (in particular, $k=1$), so $\sigma(D)=\{0\}$, and D is nilpotent.

(b) Since A is diagonalizable, there is a nonsingular matrix $S\in M_n$ for which $A=SCS^{-1}$, where $C\in M_n$ is a diagonal matrix of the form $C=a_1I_1\oplus\cdots\oplus a_kI_k$, where the a_j are distinct and I_j are identity matrices.

Now $AD=DA$ implies that $S^{-1}DS$ is block diagonal, conformal to C. Set $F=S^{-1}BS$, then $AB-BA=D$ implies that $CF-FC=S^{-1}DS$. But the diagonal blocks of $CF-FC$ are all 0, so that $D=0$.

(c) Suppose $AB-BA=D$ where $AD=DA$ and A is nonderogatory. Write $A=PJP^{-1}$ where P is nonsingular and

$$J=J_1\oplus J_2\oplus\cdots\oplus J_k,$$

gives the Jordan blocks of A having distinct eigenvalues. Then $AD=DA$ gives $J(P^{-1}DP)=(P^{-1}DP)J$ or $J\tilde{D}=\tilde{D}J$ where $\tilde{D}=P^{-1}DP$.

Write $\tilde{D}=[D_{ij}]$ conformally with $J=\oplus_iJ_i$, then by Sylvester's Theorem, $D_{ij}=0$ if $i\neq j$ and $\tilde{D}=D_1\oplus D_2\oplus\cdots\oplus D_k$, where $D_iJ_i=J_iD_i$ (replacing D_{ii} by D_i), $1\leqslant i\leqslant k$.

Now $AB-BA=D$ gives $J\tilde{B}-\tilde{B}J=\tilde{D}$, where $\tilde{B}=P^{-1}BP$. Decompose $\tilde{B}$ conformally with J and $\tilde{D}$ as $\tilde{B}=[B_{ij}]$. Then $J_iB_{ij}-B_{ij}J_j=0$ if $i\neq j$ and $J_iB_{ii}-B_{ii}J_i=D_i$, so again, by Sylvester's Theorem, $B_{ij}=0$ if $i\neq j$. We therefore have

$$J_iB_{ii}-B_{ii}J_i=D_i \quad \text{where} \quad J_iD_i=D_iJ_i,\ 1\leqslant i\leqslant k.$$

We now prove a lemma:

LEMMA 1. *Let A,B and D belong to M_n where $AB-BA=D$. Let A be the Jordan block* $A=\begin{bmatrix}\lambda & 1 & 0 & 0 & \cdots & 0\\ 0 & \lambda & 1 & 0 & \cdots & 0\\ 0 & 0 & \lambda & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots\\ 0 & 0 & \cdots & \cdots & 0 & \lambda\end{bmatrix}$.

(i) *If $AD=DA$, then B is an upper triangular matrix whose eigenvalues are in arithmetic progression.*

(ii) *If $AD=DA$ and $BD=DB$, then $D=\begin{bmatrix}0 & E\\ 0 & 0\end{bmatrix}$, where E is upper triangular and $\operatorname{rank}(E)<n/2$.*

Proof. (i) From [3] (Theorem 3.2.4.2), if $AD = DA$, then D is of the form

$$D = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_n \\ 0 & d_1 & d_2 & \cdots & d_{n-1} \\ 0 & 0 & d_1 & \cdots & d_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & d_1 \end{bmatrix}.$$

But also from part (a) of this theorem, D is nilpotent, so $d_1 = 0$.

A calculation shows that if $B = [b_{ij}]$, then

$$AB - BA = \begin{bmatrix} b_{21} & b_{22} - b_{11} & b_{23} - b_{12} & b_{24} - b_{13} & \cdots & b_{2,n} - b_{1,n-1} \\ b_{31} & b_{32} - b_{21} & b_{33} - b_{22} & b_{34} - b_{23} & \cdots & b_{3,n} - b_{2,n-1} \\ b_{41} & b_{42} - b_{31} & b_{43} - b_{32} & b_{44} - b_{33} & \cdots & b_{4,n} - b_{3,n-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n,1} & b_{n,2} - b_{n-1,1} & b_{n,3} - b_{n-1,2} & \cdots & \cdots & b_{n,n} - b_{n-1,n-1} \\ 0 & -b_{n,1} & -b_{n,2} & \cdots & \cdots & -b_{n,n-1} \end{bmatrix}.$$

Equating this to D, it immediately follows that $b_{ij} = 0$ for $i > j$, so that B is an upper triangular matrix. This proves part (i) of the lemma, but we can see that B has a special form in the following way:

Set $b_{1,j} = \alpha_j$ for $j = 1, \ldots, n$. Then the equations

$$b_{i+1,i+1} - b_{i,i} = d_2, \quad i = 1, 2 \ldots, n-1,$$

imply that

$$b_{11} = \alpha_1, \;\; b_{22} = \alpha_1 + d_2, \;\; b_{33} = \alpha_1 + 2d_2, \ldots, b_{n,n} = \alpha_1 + (n-1)d_2,$$

and similarly

$$b_{12} = \alpha_2, \;\; b_{23} = \alpha_2 + d_3, \;\; b_{34} = \alpha_1 + 2d_3, \ldots, b_{n-1,n} = \alpha_2 + (n-2)d_3.$$

Continuing in this way we see that

$$B = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots\cdots & & \alpha_n \\ 0 & \alpha_1 + d_2 & \alpha_2 + d_3 & \alpha_3 + d_4 & \cdots\cdots & & \alpha_{n-1} + d_n \\ 0 & 0 & \alpha_1 + 2d_2 & \alpha_2 + 2d_3 & \cdots\cdots & & \alpha_{n-2} + 2d_{n-1} \\ 0 & 0 & 0 & \alpha_1 + 3d_2 & \cdots\cdots & & \alpha_{n-3} + 3d_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_1 + (n-1)d_2 \end{bmatrix}.$$

We see that the eigenvalues of B are in arithmetic progression.

(ii) Now we use the fact that $BD = DB$ and a calculation to get more detail on the structure of B and D. Let $1 < j \leqslant n$, then the $(1, j)$ entry of DB is

$$(\alpha_{j-1} + d_j)d_2 + (\alpha_{j-2} + 2d_{j-1})d_3 + (\alpha_{j-3} + 3d_{j-2})d_4$$
$$+ \cdots + (\alpha_2 + (j-2)d_3)d_{j-1} + (\alpha_1 + (j-1)d_2)d_j,$$

and the $(1,j)$ entry of BD is

$$\alpha_1 d_j + \alpha_2 d_{j-1} + \alpha_3 d_{j-2} + \cdots + \alpha_{j-1} d_2.$$

For the $(1,2)$ entry this gives $(\alpha_1 + d_2)d_2 = \alpha_1 d_2$, or $d_2 = 0$. The $(1,4)$ entry gives

$$(\alpha_3 + d_4)d_2 + (\alpha_2 + 2d_3)d_3 + (\alpha_1 + 3d_2)d_4 = \alpha_1 d_4 + \alpha_2 d_3 + \alpha_3 d_2,$$

or $d_3 = 0$. Inductively, suppose that we have shown $d_2 = 0, d_3 = 0, \ldots, d_{j-1} = 0$.

Consider the $(1, 2j-2)$ entry of BD and DB (when $2j - 2 \leqslant n$). Then we have

$$\begin{aligned}(\alpha_{2j-3} + d_{2j-2})d_2 + \cdots + (\alpha_j + (j-2)d_{j+1})d_{j-1} + (\alpha_{j-1} + (j-1)d_j)d_j \\ + \cdots + (\alpha_1 + (2j-3)d_2)d_{2j-2} \\ = \alpha_1 d_{2j-2} + \alpha_2 d_{2j-3} + \cdots + \alpha_{j-1} d_j + \cdots + \alpha_{2j-3} d_2.\end{aligned}$$

Since $d_2 = d_3 = \ldots = d_{j-1} = 0$, we have

$$\begin{aligned}\alpha_{j-1} d_j + (j-1)d_j^2 + \alpha_{j-2} d_{j+1} + \alpha_{j-3} d_{j+2} + \cdots + \alpha_1 d_{2j-2} \\ = \alpha_1 d_{2j-2} + \alpha_2 d_{2j-3} + \cdots + \alpha_{j-1} d_j,\end{aligned}$$

and this gives $d_j = 0$ for all j with $2j \leqslant n+2$. $\square$

Now apply Lemma 1(i) to J_i, B_{ii} and D_i to see that B_{ii} is upper triangular, $i = 1, \ldots, k$, so that both A and B can both be put into upper triangular form by the same matrix P, and (c) follows.

(d) If in addition $BD = DB$, then $\tilde{B}\tilde{D} = \tilde{D}\tilde{B}$ and $B_{ii}D_i = D_i B_{ii}$ for $i = 1, \ldots, k$. Applying Lemma 1(ii) we see that B_{ii} is of the form

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-2} + d_{n-1} & \alpha_{n-1} + d_n \\ 0 & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-3} + 2d_{n-2} & \alpha_{n-2} + 2d_{n-1} \\ 0 & 0 & 0 & \alpha_1 & \cdots & \cdots & \alpha_{n-3} + 3d_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_1 \end{bmatrix},$$

($d_j = 0$ for $j \leqslant n/2 + 1$) and D_i is of the form $\begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}$ where E is upper triangular. It follows that B has at most k distinct eigenvalues, $D^2 = 0$, and rank$(D) < n/2$.

(e) If all the eigenvalues of A have algebraic multiplicity no larger than 2, then the Jordan blocks of A are at most 2-by-2. Thus we can write $A = PJP^{-1}$ where

$$J = J_0 \oplus J_1 \oplus \cdots \oplus J_k,$$

where J_0 is a diagonal matrix all of whose eigenvalues occur with multiplicity at most 2, and $J_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$, $1 \leqslant i \leqslant k$, where all the λ_i's are distinct, and they are distinct from the diagonal entries of J_0, $1 \leqslant i \leqslant k$. Follow the steps of (c) to obtain matrices B_i and D_i as before, $0 \leqslant i \leqslant k$. Since J_0 is diagonal, (a) above implies that $D_0 = 0$, and Lemma 1(ii) implies that $D_i = 0$ for $1 \leqslant i \leqslant k$ (since $D_i \in M_2$), so we must have $D = 0$.

(f) Since $AD = -DA$,

$$A^2B - BA^2 = A(AB - BA) + (AB - BA)A = AD + DA = 0,$$

so $A^2B = BA^2$. Since A^2 is nonderogatory, B is a polynomial in A^2. It follows that $AB = BA$, so $D = 0$. □

The results of the following theorem (some of which now appear as exercises in 3.2 of [4]) are new, except for (d), which is a consequence of the Taussky and Zassenhaus result which says: if A is nonderogatory and $AX = XA^T$, then A is symmetric (see [9]). Theorem 2(d) does not require that D be a commutator.

THEOREM 2. *Let $A, B, D \in M_n$ with $AB - BA^T = D$.*

(a) *If $AD = DA^T$, then D is singular.*

(b) *If $AD = DA^T$ and A is diagonalizable, then $D = 0$.*

(c) *If $AD = DA^T$ and $DA = A^TD$, then D is nilpotent.*

(d) *If $AD = DA^T$ and A is nonderogatory, then D is symmetric (Taussky and Zassenhaus).*

(e) *If $AD = DA^T$ and A is nonderogatory, the geometric multiplicity of the eigenvalue 0 of D is greater than or equal to the number of distinct eigenvalues of A.*

Proof. (a) Choose R nonsingular with $A^T = RAR^{-1}$. Then

$$AB - B(RAR^{-1}) = D, \quad \text{so} \quad A(BR) - (BR)A = DR.$$

In addition, $AD = DA^T$ implies that $A(DR) = (DR)A$, so DR is the commutator of BR and A, and also A and DR commute, so Jacobson's lemma implies that DR is nilpotent, so D must be singular.

(b) As before we can write $A = SCS^{-1}$ for some invertible S and diagonal $C = a_1I_1 \oplus \cdots \oplus a_kI_k$, where the a_j are distinct. Now $AD = DA^T$ implies that $CS^{-1}D(S^T)^{-1} = S^{-1}D(S^T)^{-1}C$, so that again $S^{-1}D(S^T)^{-1}$ is block diagonal conformal to C. But

$$AB - BA^T = D \text{ implies } CS^{-1}B(S^T)^{-1} - S^{-1}B(S^T)^{-1}C = S^{-1}D(S^T)^{-1},$$

so again we must have $D = 0$.

(c) $AB - BA^T = D$, so $ABD - BA^TD = D^2$ or $A(BD) - (BD)A = D^2$ (since $A^TD = DA$). Write this as $A\tilde{B} - \tilde{B}A = \tilde{D}$ where $\tilde{B} = BD$, $\tilde{D} = D^2$.

Now $A\tilde{D} = AD^2 = DA^TD = D^2A = \tilde{D}A$, so that Theorem 1(a) applies to give $\tilde{D} = D^2$ nilpotent, and hence D is nilpotent.

(d) Let $S \in M_n$ be nonsingular and symmetric, and such that $A^T = SAS^{-1}$. Then $AD = DA^T = DSAS^{-1}$, so $A(DS) = (DS)A$.

Since A is nonderogatory, it follows that there is a polynomial p such that $DS = p(A)$. Then $SD^T = (DS)^T = p(A)^T = p(A^T) = p(SAS^{-1}) = Sp(A)S^{-1} = SD$, so $S(D^T - D) = 0$, or D is symmetric.

(e) Let $A = SJS^{-1}$ be the Jordan canonical form of A with S nonsingular and

$$J = J_{n_i}(\lambda_1) \oplus \cdots \oplus J_{n_d}(\lambda_d),$$

a direct sum of Jordan blocks with distinct eigenvalues. Let $\mathscr{D} = S^{-1}DS$ and $\mathscr{B} = S^{-1}BS^{-T}$; define $J_i = J_{n_i}(\lambda_i)$; and partition $\mathscr{D} = [\mathscr{D}_{ij}]_{i,j=1}^d$ and $\mathscr{B} = [\mathscr{B}_{ij}]_{i,j=1}^d$ conformally to J. Then

$$AD = DA^T \Rightarrow J\mathscr{D} = \mathscr{D}J^T \Rightarrow J_i\mathscr{D}_{ij} = \mathscr{D}_{ij}J_j^T,$$

so distinctness of eigenvalues and Sylvester's Theorem ensure that $\mathscr{D}_{ij} = 0$ if $i \neq j$.

Let $R_i \in M_{n_i}$ be nonsingular and such that $J_i^T = R_iJ_iR_i^{-1}$. Then

$$J_i\mathscr{D}_{ii} = \mathscr{D}_{ii}J_i^T \Rightarrow J_i\mathscr{D}_{ii} = \mathscr{D}_{ii}R_iJ_iR_i^{-1} \Rightarrow J_i(\mathscr{D}_{ii}R_i) = (\mathscr{D}_{ii}R_i)J_i.$$

Moreover, $D = AB - BA^T$ implies

$$\mathscr{D}_{ii} = J_i\mathscr{B}_{ii} - \mathscr{B}_{ii}J_i^T = J_i\mathscr{B}_{ii} - \mathscr{B}_{ii}R_iJ_iR_i^{-1},$$

or

$$\mathscr{D}_{ii}R_i = J_i(\mathscr{B}_{ii}R_i) - (\mathscr{B}_{ii}R_i)J_i.$$

Thus for each $i = 1, \ldots, d$, $\mathscr{D}_{ii}R_i$ is the commutator of $\mathscr{B}_{ii}R_i$ and J_i, and it commutes with J_i. Jacobson's Lemma ensures that each $\mathscr{D}_{ii}R_i$ is nilpotent, so each $\mathscr{D}_{ii}$ is singular. It follows that the null space of

$$\mathscr{D} = \mathscr{D}_{11} \oplus \cdots \oplus \mathscr{D}_{dd}$$

(and hence of D) has dimension at least d, that is, the geometric multiplicity of 0 as an eigenvalue of D is at least d. $\square$

3. Commutators and Quasi-Real Normal Matrices

We briefly survey a method for generalizing some results on quasi-real normal (QRN) matrices (see [1], [2]). Consider the commutator-type expression $AB + BA = D$, where $AD = DA^T$. Here is a non-trivial example, where A is a normal matrix (i.e., A is unitarily diagonalizable):

Let $I_{\lambda,\mu} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$, for $\lambda, \mu \in \mathbb{C}$. Set $A = I_{\lambda,\mu}$. The general form of D with $AD = DA^T$ is $D = \begin{bmatrix} e & f \\ f & -e \end{bmatrix}$. If we set $B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $e = 2\lambda a$ and $f = 2\lambda b$, we can check that the equation $AB+BA=D$ is satisfied. In this example, A is normal and both D^2 and B^2 are multiples of the 2-by-2 identity matrix, so their eigenvalues occur with multiplicity two and in fact the eigenvalues of B and D occur in $\pm$ pairs. We shall show that this is a fairly general situation.

The matrix A is actually an example of a *quasi-real normal* (QRN) matrix: $A \in M_n$ is QRN if (i) A is normal, (ii) $Ax=0$ implies $A\bar{x}=0$ and (iii) x is an eigenvector of A if and only if $\bar{x}$ is an eigenvector of A. The following was shown in [1]:

THEOREM 3. *A matrix $A \in M_n$ is QRN if and only if there is a unitary matrix of the form $U=[Y\ \bar{Y}\ Z]$, $Y \in M_{n,k}(\mathbb{C})$, $Z \in M_{n,n-2k}(\mathbb{R})$, and a diagonal matrix $\Lambda = L_1 \oplus L_2 \oplus L_3$ such that $A = U\Lambda U^*$, $L_1, L_2 \in M_k$ are nonsingular, and there are nonnegative integers d and r, positive integers $n_1,\dots,n_d,m_1,\dots,m_r$, and $2d+r$ distinct scalars $\lambda_1,\dots,\lambda_d,\mu_1,\dots,\mu_d,\nu_1,\dots,\nu_r$, such that $n_1+\cdots+n_d=k$, $m_1+\cdots+m_r=n-2k$, $L_1=\lambda_1 I_{n_1}\oplus\cdots\oplus\lambda_d I_{n_d}$, $L_2=\mu_1 I_{n_1}\oplus\cdots\oplus\mu_d I_{n_d}$, and $L_3=\nu_1 I_{m_1}\oplus\cdots\oplus\nu_r I_{m_r}$.*

Suppose that A,B and D are in M_n with

$$AB+BA=D \quad \text{and} \quad AD=DA^T,$$

where $A=U\Lambda U^*$ ($\Lambda = L_1\oplus L_2\oplus L_3$, $U=[Y\ \bar{Y}\ Z]$) is a QRN matrix. Then we can check that $A^T = U(L_2\oplus L_1\oplus L_3)U^*$ (see [1]), so that

$$AD=DA^T \Rightarrow (L_1\oplus L_2\oplus L_3)(U^*DU) = (U^*DU)(L_2\oplus L_1\oplus L_3).$$

Write $U^*DU=[D_{ij}]$ partitioned conformally with Λ. Then by Sylvester's Theorem (using $\sigma(L_i)\cap\sigma(L_j)=\emptyset$ for $i\neq j$) we have

$$U^*DU = \begin{bmatrix} 0 & D_{12} & 0 \\ D_{21} & 0 & 0 \\ 0 & 0 & D_{33} \end{bmatrix} = \begin{bmatrix} 0 & Y^*D\bar{Y} & 0 \\ Y^TDY & 0 & 0 \\ 0 & 0 & Z^TDZ \end{bmatrix},$$

since $U^*DU = \begin{bmatrix} Y^* \\ Y^T \\ Z^T \end{bmatrix} D[Y\ \bar{Y}\ Z]$.

The equation $AB+BA=D$ becomes $\Lambda(U^*BU)+(U^*BU)\Lambda = U^*DU$ or $\Lambda\tilde{B}+\tilde{B}\Lambda=\tilde{D}$ where $\tilde{B}=U^*BU$, $\tilde{D}=U^*DU$. Decompose $\tilde{B}$ conformally with Λ and assume that $\sigma(L_i)\cap\sigma(-L_i)=\emptyset$ and $\sigma(L_i)\cap\sigma(-L_3)=\emptyset$, $i=1,2$.

Equating the resulting matrices and again using Sylvester's Theorem gives $\tilde{B}$ (a form similar to that for $\tilde{D}$):

$$\tilde{B}=U^*BU = \begin{bmatrix} 0 & B_{12} & 0 \\ B_{21} & 0 & 0 \\ 0 & 0 & B_{33} \end{bmatrix} = \begin{bmatrix} 0 & Y^*B\bar{Y} & 0 \\ Y^TBY & 0 & 0 \\ 0 & 0 & Z^TBZ \end{bmatrix}.$$

We see, for example, that $\tilde{B}^2 = B_{12}B_{21}\oplus B_{21}B_{12}\oplus B_{33}^2$ (and similarly for $\tilde{D}^2$). Since the nonsingular Jordan structures of $B_{12}B_{21}$ and $B_{21}B_{12}$ are identical, and 0 is an eigenvalue of the same multiplicity for both matrices, the eigenvalues of $B_{12}B_{21}\oplus B_{21}B_{12}$

occur with even multiplicity (and similarly for $\tilde{D}^2$). There are various conditions on the spectrum of A which give rise to the above results. For example, if A is real and normal, $A = U\Lambda U^*$ where U is as above, $\Lambda = L \oplus \bar{L} \oplus R$, L can be chosen to consist of diagonal entries which lie in the open upper half plane, and the diagonal entries of R are real. In this case $L_1 = L, L_2 = \bar{L}$ and $L_3 = R$ where $\sigma(L_i) \cap \sigma(-L_i) = \emptyset$ and $\sigma(L_i) \cap \sigma(-L_3) = \emptyset$, $i = 1,2$ so the results above can be applied, leading to:

THEOREM 4. *Let $A \in M_n(\mathbb{R})$ be normal. If $B, D \in M_n$ satisfy the equations*

$$AB + BA = D \quad \text{and} \quad AD = DA^T,$$

then the subspace $H = \{x \in \mathbb{C}^n : Ax = A^T x\}$ and its orthogonal complement $H^\perp$ are both B and D invariant. In addition, for B and D considered as linear maps on the subspace $H^\perp$, the following holds:

(a) *The eigenvalues of B^2 and D^2 occur with even multiplicity.*

(b) *If B is real then D is real and the eigenvalues of both B and D occur in $\pm$ conjugate quadruplets with the same multiplicities.*

(c) *If $B = B^*$, then the eigenvalues of B are real and occur in $\pm$ pairs with the same multiplicities.*

Proof. (a) A real normal matrix is QRN, so the discussion prior to the theorem is applicable. Since the nonsingular Jordan structures of $B_{12}B_{21}$ and $B_{21}B_{12}$ are identical, and zero is an eigenvalue for both with the same multiplicity, the eigenvalues of $B_{12}B_{21} \oplus B_{21}B_{12}$ occur with even multiplicity. It therefore suffices to check that the subspace on which this matrix acts corresponds to the orthogonal complement of $H = \{x \in \mathbb{C}^n : Ax = A^T x\}$.

(b) If B is real, then $Y^* B \bar{Y} = \overline{Y^T B Y}$, so that $\tilde{B} = \begin{bmatrix} 0 & B_{12} \\ \bar{B}_{12} & 0 \end{bmatrix} \oplus B_{33}$. Now following the argument in [1], we see that the eigenvalues of $C = \begin{bmatrix} 0 & B_{12} \\ \bar{B}_{12} & 0 \end{bmatrix}$ occur in $\pm$ conjugate quadruplets. In fact it is shown in [1] that C is similar to a matrix of the form $-R \oplus R$, where R is real.

(c) In this case $C = \begin{bmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{bmatrix}$ is similar to a matrix of the form $-\Sigma \oplus \Sigma$, where Σ is a diagonal matrix whose diagonal entries are the singular values of B_{12} (see [1]). □

This theorem is true more generally when A is a QRN matrix with $\sigma(L_i) \cap \sigma(-L_i) = \emptyset$ and $\sigma(L_i) \cap \sigma(-L_3) = \emptyset$, $i = 1,2$. The methods outlined here will also work for certain other commutator-like expressions such as $AB + BA^T = D$ when $AD = DA$.

Acknowledgements. I thank Roger Horn for valuable suggestions resulting in improvements to this paper. In particular, I have included his proof of Theorem 2(e), which was an improvement on the original proof. Some of these results and other results about commutators are to appear in the second edition of [3] (see [4]). Theorem 2(b) was the result of discussions with Dennis Merino who I would also like to thank.

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(Received September 28, 2009)

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