Operators
and
Matrices
Volume 4, Number 2 (2010), 293–300

UNBOUNDED OPERATORS COMMUTING WITH THE COMMUTANT OF A RESTRICTED BACKWARD SHIFT

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(*Communicated by H. Berkovici*)

Abstract. It is shown that, in a proper coinvariant subspace of the shift operator on the Hardy space H^2, a densely defined operator that commutes with the commutant of the restricted backward shift is closable. A connection between this result and a case of the transitive algebra problem is discussed.

1. Introduction

The paper [4] characterizes the closed densely defined operators that commute with restricted backward shifts (of multiplicity 1). Here it will be proved that a densely defined operator that commutes with the commutant of a restricted backward shift is closable. The precise statement of the result and its proof are in Section 3, following a few preliminaries in Section 2.

The closability result was suggested by William Arveson, who was motivated by a link with the transitive algebra problem. These matters occupy Sections 4 and 5.

In the paper [2], H. Bercovici and coauthors study from a general viewpoint what they term the closability property for operator algebras. An algebra of operators on a Hilbert space is said to have this property if every densely defined operator in its commutant is closable. The closability results in [2] subsume the one proved here. In particular, it is proved in [2] that the commutant of any C_0 contraction has the closability property. The proof of the closability result given here is specific to the present context. It uses, in particular, the characterization from [4].

Notations

1. H^2 and H^∞ are the usual Hardy spaces for the unit disk $\mathbb{D}$. The functions in them will be identified with their boundary functions on $\partial\mathbb{D}$.

2. For λ in $\mathbb{D}$, k_λ denotes the kernel function in H^2 for the evaluation functional at $\lambda : k_\lambda(z) = 1/(1-\overline{\lambda}z)$.

3. S denotes the unilateral shift operator on H^2.

Mathematics subject classification (2010): 47B32, 47B35, 47D25.
Keywords and phrases: Restricted backward shift, closability, transitive algebra problem.

Paper OaM-04-16

4. Throughout, u will denote an inner function, assumed not to be a finite Blaschke product. (Any inner function that arises will be assumed to be normalized, i.e., having a positive initial nonvanishing Taylor coefficient at the origin.)

5. K_u^2 denotes the space $H^2 \ominus uH^2$ (the general proper infinite-dimensional invariant subspace of S^*).

6. S_u denotes the compression of S to K_u^2. The adjoint S_u^* is the restriction of S^* to K_u^2.

7. For u_1 and u_2 inner functions, their greatest common (normalized) inner divisor is denoted by $\text{g.c.i.d.}(u_1, u_2)$.

8. N denotes the Nevanlinna class, the family of functions φ in $\mathbb{D}$ writable as $\varphi = \psi/\chi$ with ψ and χ in H^∞ and $\chi \neq 0$. When such an expression for φ is written, it is to be understood that ψ and χ have no common nonconstant inner divisors. The Smirnov class N^+ consists of such ratios with χ an outer function.

9. For H a Hilbert space, $\mathscr{L}(H)$ denotes the algebra of bounded operators on H. The domain and graph of a possibly unbounded operator on H are denoted by $\mathscr{D}(H)$ and $\mathscr{G}(H)$, respectively. The operator W in $\mathscr{L}(H \oplus H)$ is defined by $W(x \oplus y) = y \oplus -x$. Note that if the operator A on H is densely defined, then $\mathscr{G}(A^*) = W\mathscr{G}(A)^\perp$.

2. Background on K_u^2

This section reviews the properties of the spaces K_u^2, and of the unbounded operators acting in them, needed for present purposes. Full details are in the papers [3] and [4].

The space K_u^2 carries a natural conjugation C, an antiunitary involution, defined by $(Cf)(z) = \bar{z}u(z)\overline{f(z)}$ $(z \in \partial\mathbb{D})$. When convenient, Cf will be denoted alternatively by $\widetilde{f}$.

The kernel function in K_u^2 for the evaluation functional at the point λ of $\mathbb{D}$ will be denoted by k_λ^u; it is given by $k_\lambda^u(z) = (1 - \overline{u(\lambda)}u(z))/(1 - \overline{\lambda}z)$. Its C-transform $\widetilde{k}_\lambda^u$ is given by $\widetilde{k}_\lambda^u(z) = (u(z) - u(\lambda))/(z - \lambda)$. The function $\widetilde{k}_0^u$ is a cyclic vector of S_u^*.

For ψ a function in H^∞, the compression to K_u^2 of the Toeplitz operator T_ψ will be denoted by A_ψ. The adjoint of A_ψ is the restriction of $T_{\overline{\psi}}$ to K_u^2 and will be denoted by $A_{\overline{\psi}}$; besides being the adjoint of A_ψ it is the C-transform of A_ψ: $A_{\overline{\psi}} = CA_\psi C$.

The local Smirnov class N_u^+ consists of all Nevanlinna functions $\varphi = \psi/\chi$ such that u and χ have no nonconstant common inner divisors. For each such φ, a closed, densely defined operator $A_{\overline{\varphi}}$ on K_u^2 is defined in [4]. This operator depends only on φ, not on its quotient representation, and if φ is in H^∞ it coincides with $A_{\overline{\varphi}}$ as defined earlier. The adjoint of $A_{\overline{\varphi}}$ is denoted by A_φ. The following properties hold.

LEMMA 1. *A_φ is the C-transform of $A_{\overline{\varphi}}$: $\mathscr{D}(A_\varphi) = C\mathscr{D}(A_{\overline{\varphi}})$ and $A_\varphi Cf = CA_{\overline{\varphi}}f$ for f in $\mathscr{D}(A_{\overline{\varphi}})$.*

LEMMA 2. *If $\varphi = \psi/\chi$ is in N_u^+, then $A_{\overline{\chi}}K_u^2$ is contained in $\mathscr{D}(A_{\overline{\varphi}})$, and $A_{\overline{\varphi}}A_{\overline{\chi}}h = A_{\overline{\psi}}h$ for h in K_u^2. Moreover $A_{\overline{\varphi}}$ is the closure of its restriction to $A_{\overline{\chi}}K_u^2$.*

LEMMA 3. *If φ_1 and φ_2 are in N_u^+, then $A_{\overline{\varphi_1}} = A_{\overline{\varphi_2}}$ if and only if u divides $\varphi_1 - \varphi_2$.*

LEMMA 4. *If w is in H^∞ and φ is in N_u^+, then $A_{\overline{w}}A_{\overline{\varphi}}f = A_{\overline{\varphi}}A_{\overline{w}}f = A_{\overline{\varphi w}}f$ for all f in $\mathscr{D}(A_{\overline{\varphi}})$.*

The main result from [4] is the following theorem.

THEOREM 1. *The closed densely defined operators on K_u^2 that commute with S_u^* are the operators $A_{\overline{\varphi}}$ with φ in N_u^+.*

Note that a closed densely defined operator on K_u^2 commuting with S_u^* commutes also with the closed unital operator algebra generated by S_u^*, in other words, with $A_{\overline{\psi}}$ for all ψ in H^∞, the operators comprising the commutant of S_u^*.

The proof of Theorem 1 in [4] is based on earlier work of Daniel Suárez [5].

3. Closability

THEOREM 2. *A densely defined operator on K_u^2 that commutes with $A_{\overline{\psi}}$ for all ψ in H^∞ is closable.*

Proof. Let A be the operator in question. By the commutativity assumption, its domain, $\mathscr{D}(A)$, is invariant under $A_{\overline{\psi}}$ for all ψ in H^∞. For f in $\mathscr{D}(A)$, we let $\mathscr{D}_f = \{A_{\overline{\psi}}f : \psi \in H^\infty\}$.

Step 1. We consider first the case where there is a function f in $\mathscr{D}(A)$ such that $\mathscr{D}_f$ is dense in K_u^2. It will be shown that A is then closable.

We let $A' = A \mid \mathscr{D}_f$, a densely defined operator that commutes with $A_{\overline{\psi}}$ for all ψ in H^∞. Let $\widetilde{A}'$ be the C-transform of A': $\mathscr{D}(\widetilde{A}') = C\mathscr{D}_f$, $\widetilde{A}' = CA'C$. The operators A' and $\widetilde{A}'$ are both densely defined, so their adjoints are well defined. Let $g = Af$. For ψ and χ in H^∞,

$$\begin{aligned}\langle A_{\overline{\psi}}f \oplus A_{\overline{\psi}}g, A_{\chi}\widetilde{g} \oplus -A_{\chi}\widetilde{f}\rangle &= \langle A_{\overline{\psi}}f, A_{\chi}\widetilde{g}\rangle - \langle A_{\overline{\psi}}g, A_{\chi}\widetilde{f}\rangle \\ &= \langle A_{\overline{\psi}\overline{\chi}}f, \widetilde{g}\rangle - \langle g, A_{\psi\chi}\widetilde{f}\rangle \\ &= \langle A_{\overline{\psi}\overline{\chi}}f, \widetilde{g}\rangle - \langle A_{\overline{\psi}\overline{\chi}}f, \widetilde{g}\rangle = 0.\end{aligned}$$

This shows that $\mathscr{G}(A')$ and $W\mathscr{G}(\widetilde{A}')$ are orthogonal, and so their closures are orthogonal. In particular, A' is closable; let $\overline{A}'$ denote its closure. By Theorem 1, there is a function φ in N_u^+ such that $\overline{A}' = A_{\overline{\varphi}}$.

We show that $\overline{A}'$ is the closure of A. If that is not true, then there is a function f' in $\mathscr{D}(A)$ that is not in $\mathscr{D}(\overline{A}')$. Let $\mathscr{G}'$ be the linear span of $\mathscr{G}(\overline{A}')$ and $f' \oplus Af'$. Then $\mathscr{G}'$ is closed and is the graph of an operator A'' commuting with S_u^*, its domain being the linear span of $\mathscr{D}(\overline{A}')$ and f'. By Theorem 1, there is a function φ' in N_u^+ such that $A'' = A_{\overline{\varphi'}}$. But since $A'' \mid \mathscr{D}(\overline{A}') = \overline{A}' = A_{\overline{\varphi}}$, it follows by Lemma 3 that u divides $\varphi - \varphi'$, and hence that $A_{\overline{\varphi'}} = A_{\overline{\varphi}}$, contrary to the supposition that f' is not in $\mathscr{D}(\overline{A}')$. Thus $\overline{A}' = A_\varphi$ is in fact the closure of A.

The result just established implies an extension of itself. Let f be any nonzero function in $\mathscr{D}(A)$. The S_u^*-invariant subspace generated by f then equals $K_{u_1}^2$ with u_1 an inner divisor of u. The operator $A \mid \mathscr{D}(A) \cap K_{u_1}^2$ commutes with $A_{\overline{\psi}} \mid K_{u_1}^2$ for all ψ in H^∞, and its domain contains $\{A_{\overline{\psi}} f : \psi \in H^\infty\}$, which dense in $K_{u_1}^2$. We can conclude that $A \mid \mathscr{D}(A) \cap K_{u_1}^2$ is closable.

Step 2. We consider next the case where there is a pair of functions f_1, f_2 in $\mathscr{D}(A)$ such that $\mathscr{D}_{f_1} + \mathscr{D}_{f_2}$ is dense in K_u^2 and $\overline{\mathscr{D}}_{f_1} \cap \overline{\mathscr{D}}_{f_2} = \{0\}$. It will be shown that A is then closable.

The closures $\overline{\mathscr{D}}_{f_1}, \overline{\mathscr{D}}_{f_2}$ are S_u^* invariant subspaces of K_u^2, so there are inner divisors u_1, u_2 of u such that $\overline{\mathscr{D}}_{f_1} = K_{u_1}^2$, $\overline{\mathscr{D}}_{f_2} = K_{u_2}^2$. In view of Step 1 we may as well assume u_1 and u_2 are proper divisors of u. The condition $\overline{\mathscr{D}}_{f_1} \cap \overline{\mathscr{D}}_{f_1} = \{0\}$, i.e., $K_{u_1}^2 \cap K_{u_2}^2 = \{0\}$, implies u_1 and u_2 are relatively prime as inner functions. The density of $\mathscr{D}_{f_1} + \mathscr{D}_{f_2}$ in K_u^2 implies u divides $u_1 u_2$. Hence $u = u_1 u_2$.

Let $f = f_1 + f_2$. It will be shown that $\mathscr{D}_f$ is dense in K_u^2. The desired conclusion will then follow by Step 1.

To prove $\mathscr{D}_f$ is dense in K_u^2, it will suffice to prove that if u_3 is a proper inner divisor of u then $A_{\overline{u}_3} f \neq 0$. Given such a u_3, it can be factored as $u_3 = u_1' u_2'$, where u_1', u_2' are inner divisors of u_1, u_2, respectively, at least one a proper divisor. Suppose u_1' is a proper divisor of u_1. Then u_3 divides $u_1' u_2$. Since $A_{\overline{u}_2} f_2 = 0$, we have

$$A_{\overline{u}_1' \overline{u}_2} f = A_{\overline{u}_1'} A_{\overline{u}_2} f_1 + A_{\overline{u}_1'} A_{\overline{u}_2} f_2 = A_{\overline{u}_2} A_{\overline{u}_1'} f_1.$$

Since u_1' properly divides u_1 and $\mathscr{D}_{f_1}$ is dense in $K_{u_1}^2$, we have $A_{\overline{u}_1'} f_1 \neq 0$. Since u_1 and u_2 are relatively prime, the operator $A_{\overline{u}_2}$ acts injectively on $K_{u_1}^2$. We can conclude that $A_{\overline{u}_2} A_{\overline{u}_1'} f_1 = A_{\overline{u}_1' \overline{u}_2} f \neq 0$. As u_3 divides $u_1' u_2$ it follows that also $A_{\overline{u}_3} f \neq 0$, as desired. If u_2' properly divides u_2, the same reasoning yields the same final result. We can conclude that $\mathscr{D}_f$ is dense in K_u^2, as desired.

Step 3. We strengthen the result in Step 2 by proving that A is closable if there are functions f_1, f_2 in $\mathscr{D}(A)$ such that $\mathscr{D}_{f_1} + \mathscr{D}_{f_2}$ is dense in K_u^2. As in Step 2, there are inner divisors u_1, u_2 of u such that $\overline{\mathscr{D}}_{f_1} = K_{u_1}^2$, $\overline{\mathscr{D}}_{f_2} = K_{u_2}^2$, and by Step 1 we can assume both are proper divisors of u. By Step 2 we can assume u_1 and u_2 are not relatively prime. Let $u_3 = \text{g.c.i.d.}(u_1, u_2)$. We then have factorizations $u_1 = u_1' u_3$, $u_2 = u_2' u_3$, $u = u_1' u_3 u_2'$, where the nonconstant inner functions u_1', u_3, u_2' are relatively prime in pairs.

Let $f_3 = A_{\overline{u}_3} f_2$, a function in $\mathscr{D}(A)$. We note that $\mathscr{D}_{f_3} = A_{\overline{u}_3} \mathscr{D}_{f_2}$. From the factorization $u_2 = u_3 u_2'$ we have the direct sum decomposition $K_{u_2}^2 = K_{u_3}^2 \oplus u_3 K_{u_2'}^2$, which together with the equality $\mathscr{D}_{f_3} = A_{\overline{u}_3} \mathscr{D}_{f_2}$ tells us that $\overline{\mathscr{D}}_{f_2} = K_{u_2'}^2$. Because $u = u_1 u_2'$, the space K_u^2 is spanned by $\mathscr{D}_{f_1}$ and $\mathscr{D}_{f_3}$. And because u_1 and u_2' are relatively prime, the intersection $\overline{\mathscr{D}_{f_1}} \cap \overline{\mathscr{D}_{f_3}}$ is trivial. The desired conclusion thus follows by Step 2. Also, the analysis in Step 2 shows that $\mathscr{D}_{f_1+f_3}$ is dense in K_u^2.

As in Step 1, the result just established implies an extension of itself: Let f_1 and f_2 be functions in $\mathscr{D}(A)$, and let $K_{u'}^2$ be the S_u^*-invariant subspace they generate. Then the operator $A \mid \mathscr{D}(A) \cap K_{u'}^2$ is closable. Moreover, there is a function f' in $\mathscr{D}(A) \cap K_{u'}^2$ such that $\mathscr{D}_{f'}$ is dense in $K_{u'}^2$.

Step 4. We suppose we are given a sequence $f_1, f_2, \dots$ of functions in $\mathscr{D}(A)$ such that $\overline{\mathscr{D}_{f_n}} = K_{u_n}^2$, where each inner function u_n is a proper divisor of u_{n+1}, and $\cup_1^\infty K_{u_n}^2$ is dense in K_u^2. Under these conditions, $u_n \to u$ pointwise in $\mathbb{D}$. We prove A is closable. Let $v_n = u/u_n$.

For each n, let C_n denote the conjugation in $K_{u_n}^2$. For f in $K_{u_n}^2$ and $m \geqslant n$, a simple argument shows that $C_m f = \overline{v}_m C f$, which we rewrite as $C_m f = A_{\overline{v}_m} C f$. As $m \to \infty$ we have $v_m \to 1$ boundedly pointwise in $\mathbb{D}$, implying that the operators A_{v_n} converge strongly to the identity. We can conclude that $C_m f \to C f$ weakly as $m \to \infty$. Here we assumed f is in $K_{u_n}^2$, but n was kept fixed, so the conclusion holds for all f in $\cup_{n=1}^\infty K_{u_n}^2$.

For each n let $A_n = A \mid \mathscr{D}(A) \cap K_{u_n}^2$, and let

$$A_\infty = A \mid \cup_{n=1}^\infty \mathscr{D}(A) \cap K_{u_n}^2.$$

Let f and f' be functions in $\mathscr{D}(A_\infty)$, and let $g = Af$, $g' = Af'$. Then $f \oplus g$ and $f' \oplus g'$ are in $\mathscr{G}(A_m)$ for m sufficiently large. We know from Step 1 that A_m is closable. By Theorem 1 and Lemma 1, the adjoint A_m^* is the C_m-transform of A_m. Therefore, for m large, $f \oplus g$ is orthogonal to $C_m g' \oplus -C_m f'$. Letting $m \to \infty$, we conclude that $f \oplus g$ and $Cg' \oplus -Cf'$ are orthogonal. So, letting A'_∞ denote the C-transform of A_∞, we have shown that $\mathscr{G}(A_\infty)$ and $W\mathscr{G}(A'_\infty)$ are orthogonal. Reasoning as in Step 1, we can conclude that A_∞ is closable, and then that A is closable (with $\overline{A_\infty} = \overline{A}$).

Step 5. The proof of the theorem will now be completed. We note that, if $\varphi = \psi/\chi$ is a function in N_u^+, then there is a function f in $\mathscr{D}(A_{\overline{\varphi}})$ such that $\mathscr{D}_f$ is dense in K_u^2. In fact, by Lemma 2 and the S_u^*-cyclicity of $\widetilde{k}_0^u$, the function $f = A_{\overline{\chi}} \widetilde{k}_0^u$ has this property.

We define inductively a transfinite sequence (f_α) of nonzero functions in $\mathscr{D}(A)$ indexed by a section of the countable ordinal numbers. For each α we let u_α denote the normalized inner function such that $\overline{\mathscr{D}_{f_\alpha}} = K_{u_\alpha}^2$. Our inductive procedure guarantees that u_β is a proper divisor of u_α for $\beta < \alpha$.

Initial Step. For f_1 we take any nonzero function in $\mathscr{D}(A)$.

Inductive Step. Suppose f_β has been defined for all $\beta < \alpha$.

(i) If α is not a limit ordinal and $\mathscr{D}_{f_{\alpha-1}}$ is dense in K_u^2, we terminate the sequence at the term $f_{\alpha-1}$.

(ii) If α is not a limit ordinal and $\mathscr{D}_{f_{\alpha-1}}$ is not dense in K_u^2, then because $\mathscr{D}(A)$ is dense in K_u^2, there is a function g in $\mathscr{D}(A)\setminus K_{u_{\alpha-1}}^2$. Let u_α be the inner function such that $K_{u_{\alpha-1}}^2$ is the closure of the linear span of $\mathscr{D}_{f_{\alpha-1}}$ and $\mathscr{D}_g$. By Step 3, the operator $A \mid \mathscr{D}(A)\cap K_{u_\alpha}^2$ is closable, and there is a function in $\mathscr{D}(A)\cap K_{u_\alpha}^2$, which we define to be f_α, such $\mathscr{D}_{f_\alpha}$ is dense in $K_{u_\alpha}^2$.

(iii) If α is a limit ordinal and $\cup_{\beta<\alpha}\mathscr{D}_{f_\beta}$ is dense in K_u^2, we terminate the sequence, i.e., we leave f_γ undefined for $\gamma \geqslant \alpha$.

(iv) If α is a limit ordinal and $\cup_{\beta<\alpha}\mathscr{D}_{f_\beta}$ is not dense in K_u^2, we take an increasing sequence $(\alpha_n)_1^\infty$ of nonlimit ordinals converging to α, and we let u_α be the inner function such that $K_{u_\alpha}^2$ is the closure of $\cup_1^\infty K_{u_{\alpha_n}}^2$. Step 4, applied to u_α in place of u, tells us that $A \mid \mathscr{D}(A)\cap K_{u_\alpha}^2$ is closable. By the remark at the beginning of Step 5, there is a function in the domain of the closure, which we define to be f_α, such that $\mathscr{D}_{f_\alpha}$ is dense in $K_{u_\alpha}^2$. The inductive step is now complete.

It is asserted that the sequence (f_α) terminates at a countable stage. In fact, the inner functions u_α are all divisors of u, and u_β is a proper divisor of u_α for $\beta < \alpha$. Pick a point z_0 in $\mathbb{D}$ such that $u(z_0) \neq 0$. Then the numbers $|u_\alpha(z_0)| - |u_{\alpha+1}(z_0)|$ are positive for all α such that $u_{\alpha+1}$ is defined, so their sum is bounded by 1, implying $u_{\alpha+1}$ is defined for only countably many α, as asserted.

Let $\overline{\alpha}$ be the least ordinal such that f_α is not defined. If $\overline{\alpha}$ is not a limit ordinal, then $\mathscr{D}_{f_{\overline{\alpha}-1}}$ is dense in $K_{u_1}^2$ and the closability of A follows by Step 1. If α is a limit ordinal, we take an increasing sequence $(\alpha_n)_1^\infty$ of nonlimit ordinals converging to $\overline{\alpha}$. Then $\cup K_{u_{\alpha_n}}^2$ must be dense in K_u^2, otherwise $u_{\overline{\alpha}}$ would be defined (see part (iv) of the induction). By Step 4, A is closable. □

4. Transitive Algebra Problem

An algebra $\mathscr{B}$ of operators on a Hilbert space H is called transitive if it has no invariant subspaces other than $\{0\}$ and H. The transitive algebra problem asks whether every transitive operator algebra on H is strongly dense in $\mathscr{L}(H)$, the algebra of all bounded operators on H. Although the problem has been around for 40+ years, and although experts by and large anticipate a negative answer, progress up to now has been rather scanty. The invariant subspace problem for Hilbert space operators is of course a special case.

In his paper [1] Arveson developed a general scheme for handling transitive operator algebras. Unbounded operators commuting with the algebra in question play the key role in the scheme. Arveson used his scheme to establish two results on transitive algebras: (i) A transitive operator algebra on H that contains a maximal abelian von Neumann algebra is strongly dense in $\mathscr{L}(H)$; (ii) A transitive operator algebra on the Hardy space H^2 that contains all analytic Toeplitz operators is strongly dense in $\mathscr{L}(H^2)$. Note that the analytic Toeplitz operators on H^2 form a maximal abelian subalgebra of $\mathscr{L}(H^2)$.

The commutant lifting theorem, applied to the compressed shift S_u on the space K_u^2, says that every (bounded) operator commuting with S_u is the compression of an analytic Toeplitz operator. Those compressions, then, form a maximal abelian subalgebra of $\mathscr{L}(K_u^2)$. When Arveson learned of Theorem 1 above (which, after conjugation, describes the closed densely defined operators commuting with S_u), he asked whether his scheme could be used to establish for the spaces K_u^2 the analogue of his result (ii) for H^2. Implementation of the scheme would involve two steps: (I) One needs to prove that a densely defined operator on K_u^2 that commutes with A_ψ for all ψ in H^∞ is closable; (II) Given a transitive operator algebra $\mathscr{B}$ on K_u^2 that contains A_ψ for all ψ in H^∞, one needs to prove that the only closed densely defined operators that commute with $\mathscr{B}$ are the scalar multiples of the identity.

Step (I) is accomplished in [2] and in Section 3 above. Step (II) has yet to be accomplished. Some minor initial progress is reported in the next section.

5. Arveson's Question – Simple Reductions

As above, we assume the inner function u is not a finite Blaschke product. Let φ be a nonconstant function in N_u^+, and let $\mathscr{B}_\varphi$ denote the algebra of all bounded operators on K_u^2 that commute with A_φ. After what has already been proven in [2] and above, Arveson's question boils down to the question whether $\mathscr{B}_\varphi$ is intransitive. A couple of reductions come easily.

PROPOSITION 1. *If φ and u have a nonconstant common inner divisor, then $\mathscr{B}_\varphi$ is intransitive.*

Proof. Let u_0 be a common inner divisor of φ and u, assumed nonconstant, and let $u_1 = u/u_0$. Then u divides φu_1, so $A_{\varphi u_1} = 0$ by Lemma 3. Since A_φ and A_{u_1} commute, we have the inclusion $A_{u_1}\mathscr{D}(A_\varphi) \subset \mathscr{D}(A_\varphi)$, and by (the conjugated version of) Lemma 4 $A_\varphi A_{u_1} \mid \mathscr{D}(A_\varphi) = A_{\varphi u_1} \mid \mathscr{D}(A_\varphi) = 0$. Since $\mathscr{D}(A_\varphi)$ is dense in K_u^2 and u_1 is a proper divisor of u, the image $A_{u_1}\mathscr{D}(A_\varphi)$ is nontrivial. We can conclude that A_φ has a nontrivial kernel. That kernel is shared by every bounded operator that commutes with A_φ, in other words, by every operator in $\mathscr{B}_\varphi$, implying that $\mathscr{B}_\varphi$ is intransitive. □

PROPOSITION 2. *If u has a zero in $\mathscr{D}$ then $\mathscr{B}_\varphi$ is intransitive.*

Proof. Let $u = \psi/\chi$, and assume u vanishes at the point λ of $\mathbb{D}$. Then the kernel function k_λ^u in K_u^2 for the evaluation functional at λ equals k_λ, the kernel function in H^2 for the evaluation functional at λ. By Lemma 2 the function $A_{\overline{\chi}} k_\lambda = \overline{\chi(\lambda)} k_\lambda$ belongs to $\mathscr{D}(A_{\overline{\varphi}})$, and hence k_λ is in $\mathscr{D}(A_{\overline{\varphi}})$. Lemma 2 also tells us that $A_{\overline{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. Applying the conjugation C, we conclude that the function $\widetilde{k}_\lambda^u$ is in $\mathscr{D}(A_\varphi)$, with $A_\varphi \widetilde{k}_\lambda^u = \varphi(\lambda)\widetilde{k}_\lambda^u$. The operator $A_\varphi - \varphi(\lambda)I$ thus has a nontrivial kernel. That kernel is invariant under all bounded operators commuting with A_φ, implying the intransitivity of $\mathscr{B}_\varphi$. □

The preceding two propositions reduce Arveson's question to the case in which u is a singular inner function, and, for the function φ in N_u^+, no nonconstant inner divisor of $\varphi - \lambda$, for any complex λ, is a proper divisor of u. To prove the corresponding algebra $\mathscr{B}_\varphi$ is intransitive one must show that it leaves invariant a nontrivial proper invariant subspace of the compressed shift S_u. The invariant subspaces of S_u are the subspaces $K_u^2 \cap u_0 H^2$ with u_0 an inner divisor of u; the subspace is proper if u_0 is not constant and nontrivial if u_0 is a proper divisor of u. Note that $K_u^2 \cap u_0 H^2 = u_0 K_{u/u_0}^2$, as one sees from the direct sum decomposition $K_u^2 = K_{u_0}^2 \oplus u_0 K_{u/u_0}^2$.

While the operator A_φ can be unbounded, even the case where it is bounded is nontrivial, or so it seems. Arveson's question awaits further study.

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(Received October 21, 2009)

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