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ON THE SPECTRUM OF TOEPLITZ OPERATORS WITH QUASI–HOMOGENEOUS SYMBOLS

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Abstract. In this paper, we fully describe the spectrum of a Toeplitz operator with quasi-homogeneous symbol and give formulas for the calculation of the spectral radius in the case where it is maximal. Finally, Theorem 4 gives equivalent conditions on the quasi-homogeneous function F to have $\|T_F\| = \|F\|_\infty$. As a corollary we obtain some necessary and sufficient conditions for such Toeplitz operators to verify the equality $\sigma(T_F) = F(\mathbb{D})$.

1. Introduction

Let dA denote the normalized Lebesgue area measure on the unit disc $\mathbb{D}$. The Bergman space L_a^2 is the Hilbert space consisting of analytic functions which are contained in $L^2(\mathbb{D}, dA)$. We recall some basic facts about L_a^2. The scalar product of two functions in $L^2(\mathbb{D}, dA)$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

The sequence $(e_n)_{n\in\mathbb{N}}$ where $e_n = \sqrt{n+1}z^n$, is an orthonormal basis of L_a^2 and L_a^2 is an Hilbert space with a reproducing kernel $K_z(w) = \frac{1}{(1-w\bar{z})^2}$. So, denoting $P^{L_a^2}$ the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2, for each $f \in L^2(\mathbb{D}, dA)$, for all $z \in \mathbb{D}$, we have

$$P^{L_a^2}(f)(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}\, dA(w).$$

For $F \in L^\infty(\mathbb{D}, dA)$, we define the Toeplitz operator with symbol F, $T_F : L_a^2 \longrightarrow L_a^2$ by the equation

$$T_F(g)(z) = P^{L_a^2}(Fg)(z) = \int_{\mathbb{D}} F(w)g(w)\overline{K_z(w)}\, dA(w).$$

We are particularly interested in a certain class of symbols: the bounded quasi-homogeneous functions defined and studied in [6] and [12]. We recall the definition in the bounded case:

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DEFINITION 1. A function $F \in L^\infty(\mathbb{D}, \mathrm{dA})$ is said to be m-quasi-homogeneous if there exists $f \in L^\infty(0,1)$ and $m \in \mathbb{Z}^*$ such that

$$\forall z \in \mathbb{D}, F(z) = f(|z|)e^{im\,\mathrm{Arg}(z)} \tag{1}$$

In this case we write $F \sim (f,m)$ and f is the radial part of F. If (1) holds for $m = 0$, F is said to be radial. Let us remark that $\|F\|_\infty = \|f\|_\infty$.

For $F \in L^2(\mathbb{D}, dA)$, $F(\mathbb{D})$ denotes the essential range of F defined by

$$F(\mathbb{D}) = \{z \in \mathbb{C} : \forall \varepsilon > 0, dA(F^{-1}(D(z,\varepsilon))) > 0\}$$

where $D(z,\varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$.

Finally, for an operator T, we denote $\sigma(T)$ the spectrum of T, $\sigma_e(T)$ its essential spectrum and $\|T\|_e = \inf_{K \in \mathscr{K}} \|T - K\|$, the essential norm of T.

In the following, we first show that the spectrum of T_F is a closed disc for any quasi-homogeneous symbol F. Then we give conditions for the spectral radius of T_F to be maximal, this means it is equal to $\|F\|_\infty$. These conditions depend on the Berezin transform, the mean value of the radial part of F near the boundary of $\mathbb{D}$, and other quantities which are related to the compacity (see [1], [5], [13]).

Moreover, while solving our question we obtain equivalent conditions for T_F to verify $\|T_F\| = \|F\|_\infty$. Remark that on the Hardy space, if $F \in L^\infty(\mathbb{T})$ then we have $\|T_F\|_e = \|T_F\| = \|F\|_\infty$ (see [8]) . This implies that the only compact Toeplitz operators on the Hardy space are the null ones. On the Bergman space, the same equality is true considering [14] bounded harmonic functions over $\mathbb{D}$. The double equality $\|T_F\|_e = \|T_F\| = \|F\|_\infty$ does not hold for all Toeplitz operators with quasi-homogeneous or radial symbols. In fact, if F is a bounded quasi-homogeneous function then $\|T_F\|_e = \|F\|_\infty$ if and only if $\|T_F\| = \|F\|_\infty$ and so we see that there is no compact Toeplitz operator with "maximal" norm.

In the final section, we answer the question: for which quasi-homogeneous symbol F, is the equality $\sigma(T_F) = F(\mathbb{D})$ true? This question is quite natural because there is an obvious link between the range of F and the spectrum of T_F. Indeed, on the Hardy space, $\mathbb{H}^2$, if F is a continuous bounded function on $\mathbb{T}$ then $\sigma(T_F) = \widehat{F(\mathbb{T})}$ the region bounded by the closed curve $F(\mathbb{T})$ (see [3]) and the essential spectrum is just $F(\mathbb{T})$ [11]. On the Bergman space, similar results have been obtained by G.McDonald and C.Sundberg [7]. They show that if F is a bounded harmonic function continuous on $\overline{\mathbb{D}}$ then $\sigma_e(T_F) = F^\#(\mathbb{T})$, where $F^\#$ denotes the extension of F to $\mathbb{T}$. Thus, if, in addition, F is real valued then $\sigma(T_F) = \sigma_e(T_F) = F(\mathbb{D})$. Finally, if F is a bounded analytic function then T_F is just the associated multiplication operator, so it is easy to prove that $\sigma(T_F) = F(\mathbb{D})$.

2. The spectrum of T_F

In this part, we show that the spectrum of T_F in the quasi-homogeneous case is always a disc. Before this, we recall a definition:

DEFINITION 2. Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence and E be a separable Hilbert space. An operator T on E is said to be a weighted shift on E with weight $(a_n)_{n\in\mathbb{N}}$ if and only if there exists a basis $(v_n)_{n\in\mathbb{N}}$ of E such that

$$Tv_n = a_n v_{n+1}.$$

Let us describe T_F in terms of weighted shifts over subspaces of L^2_a:

PROPOSITION 1. *Let F be a bounded m-quasi-homogeneous symbol.*

(1) *If $m \geqslant 1$ then T_F is the direct sum of weighted shifts.*

(2) *If $m \leqslant -1$ then T_F^* is the direct sum of weighted shifts.*

Proof. Let F be a bounded m-quasi-homogeneous function with $F \sim (f,m)$. An easy calculation of the scalar product shows that: $\langle T_F e_n, e_k\rangle = 0$ if $k \neq n+m$ and

$$T_F e_n = c_n(F) e_{n+m} \tag{2}$$

where

$$c_n(F) = 2\sqrt{n+1}\sqrt{n+m+1}\int_0^1 f(r) r^{2n+m+1} \boldsymbol{d}r$$

and $c_n(F) = 0$ if $n+m<0$. Now:

▶ if $m=1$, it is clear that T_F is a weighted shift with weight $(c_n(F))_{n\in\mathbb{N}}$.

▶ if $m>1$, for any integer $j \in \{0,\dots,m-1\}$, we denote

$$H_j = \overline{\mathrm{Vect}(e_j, e_{j+m}, \dots, e_{j+nm}, \dots)}$$

and $T_{F,j}$ the restriction of T_F to H_j. Then, we have

$$T_{F,j} e_{j+nm} = c_{j+nm}(F) e_{j+nm+m} = c_{j+nm}(F) e_{j+(n+1)m}.$$

Thus $T_{F,j}$ is a weighted shift on H_j with weight $(c_{j+nm}(F))_{n\in\mathbb{N}}$. Moreover, it is clear that
$L^2_a = H_0 \oplus \dots \oplus H_{m-1}$, where the H_j's are orthogonal. And $T_{F,j} H_j \subset H_j$, thus

$$T_F = T_{F,0} \oplus \dots \oplus T_{F,j} \oplus \dots \oplus T_{F,m-1}.$$

▶ if $m \leqslant -1$, it is easy to see that $T_F^* = T_{\bar{F}}$ and since $\bar{F}(z) = f(|z|)e^{-\mathrm{im}\mathrm{Arg}(z)}$, $\bar{F} \sim (\bar{f}, -m)$, so the previous case allows us to conclude. □

The quantity $\int_0^1 f(r) r^{n-1} \boldsymbol{d}r$ is called the n-th order Mellin coefficient of f. To find $\sigma(T_F)$, we will use the following result, obtained by A. Shields, which characterizes the spectrum of weighted shifts.

THEOREM 1. [9] *Let H be a separable Hilbert space and S a weighted shift with weight $(d_n)_{n\in\mathbb{N}}$ over H. Then $\sigma(S)$ is the closed disc with center 0 and radius $\rho(S)$ where $\rho(S)$ is the spectral radius of S. Moreover, we have*

$$\rho(S) = \lim_{p\to\infty}\left(\sup_{n\in\mathbb{N}}|d_n\ldots d_{n+k}\ldots d_{n+p}|\right)^{\frac{1}{p+1}}.$$

Finally, we have the following proposition:

PROPOSITION 2. *Let F be a bounded m-quasi-homogeneous function, then we have*

$$\sigma(T_F) = \rho(F)\overline{\mathbb{D}}, \tag{3}$$

where $\rho(F) = \max_{0\leqslant j\leqslant |m|-1}\rho_j(F)$ and

$$\rho_j(F) = \lim_{p\to\infty}(\sup_{n\in\mathbb{N}}(|c_{j+n|m|}(F)\ldots c_{j+(n+k)|m|}(F)\ldots c_{j+(n+p)|m|}(F)|))^{1/(p+1)}.$$

Proof. We consider the case $m \geqslant 1$. Let F be a bounded quasi-homogeneous function with $F\sim(f,m)$. Then, by Proposition 1, we have the decomposition $T_F = T_{F,0}\oplus\ldots\oplus T_{F,j}\oplus\ldots\oplus T_{F,m-1}$ where the $T_{F,j}$ are weighted shifts with weight $(c_{j+nm}(F))_{n\in\mathbb{N}}$. The sum is direct so we can write

$$\sigma(T_F) = \bigcup_{0\leqslant j\leqslant m-1}\sigma(T_{F,j}).$$

Using this and the previous theorem we have

$$\sigma(T_{F,j}) = \rho(T_{F,j})\overline{\mathbb{D}}, 0\leqslant j\leqslant m-1$$

thus, denoting $\rho_j(F) := \rho(T_{F,j})$, we have

$$\sigma(T_F) = \max_{0\leqslant j\leqslant m-1}\rho_j(F)\overline{\mathbb{D}}.$$

If $m\leqslant -1$, then $T_F^* = T_{\overline{F}}$ so $\sigma(T_F) = \overline{\sigma(T_F^*)} = \overline{\sigma(T_{\overline{F}})}$. But, $\overline{F}$ is $(-m)$-quasi-homogeneous and by the reasoning above, we have $\sigma(T_{\overline{F}}) = \rho\overline{\mathbb{D}}$, and we can conclude. □

From the discussion above, we know that $\sigma(T_F) = \sigma(T_{\overline{F}})$. Moreover F is m-quasi-homogeneous if and only if $\overline{F}$ is ($-m$)-quasi-homogeneous. Thus, since we are interested in $\sigma(T_F)$, in the following, we give results only for F a m-quasi-homogeneous function where $m\geqslant 1$ and let the reader deduce the corresponding results for $m\leqslant -1$.

On the Hardy space the spectrum and the essential spectrum of a Toeplitz operator with bounded symbol are always connected (see [3] and [11]). On the other hand, on the Bergman space McDonald and Sundberg show in [7] that if φ is harmonic on $\mathbb{D}$ and real or piecewise continuous on the boundary then the essential spectrum of T_φ is connected. Grudsky and Vasilevski have proved in [4] that this is true for any Toeplitz operator with radial symbol. The previous proposition shows that the spectrum of a Toeplitz operator with quasi-homogeneous symbol is always connected. Notice that in [10], Sundberg and Zheng gave an example of a harmonic symbol φ such that T_φ has disconnected spectrum and essential spectrum.

3. Calculation of $\rho(F)$

In this section, we simplify the expression of $\rho(F)$ wich depends on $(c_n(F))_{n\in\mathbb{N}}$ and we give a simpler characterization depending on the limit points of the sequence

$$C_n(f) = (n+1)\int_0^1 f(r)r^n \mathrm{d}r.$$

Now, by equation (2) we have,

$$\forall n \in \mathbb{N},\, c_{j+nm}(F) = 2\sqrt{j+nm+1}\sqrt{j+nm+m+1}\int_0^1 f(r)r^{2(j+nm)+m+1}\,dr.$$

One can verify that $\forall m \in \mathbb{N}$

$$C_{2n+m+1}(f) \sim c_n(F) \text{ as } n \to \infty. \tag{4}$$

we prove a simple lemma.

LEMMA 1. *Let $f \in L^\infty(0,1)$, $m \in \mathbb{N}^*$ and $F \sim (f,m)$. Then*

$$\lim_{n\to\infty} C_n(f) - C_{n+1}(f) = 0 \tag{5}$$

and

$$\lim_{n\to\infty} c_n(F) - c_{n+1}(F) = 0.$$

Proof. The proof of (5) can be found in [4]. Now, equation (4) and the boundedness of $(C_n(f))_{n\in\mathbb{N}}$ imply that

$$\lim_{n\to\infty} c_n(F) - c_{n+1}(F) = \lim_{n\to\infty} C_{2n+m+1}(f) - C_{2n+m+3}(f),$$

so, by equation (5), we have

$$\lim_{n\to\infty} c_n(F) - c_{n+1}(F) = 0. \qquad \square$$

Finally, let us state the main theorem of this section.

THEOREM 2. *Let $f \in L^\infty(0,1)$, m a non negative integer with $F \sim (f,m)$, and $l \geqslant 0$. The following assertions are equivalent:*

(1) $\max_j \rho_j(F) = l$;

(2) $\limsup_{n\to\infty} |c_n(F)| = l$;

(3) $\limsup_{n\to\infty} |C_n(f)| = l$.

Proof. We use the following general result: Let $l \geqslant 0$, if $(X_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} X_{n+1} - X_n = 0$, then for all $p,q \in \mathbb{N}^* \times \mathbb{N}$, $(X_n)_{n\in\mathbb{N}}$ and $(X_{pn+q})_{n\in\mathbb{N}}$ have the same limit points and in particular we have

$$\limsup_{n\to\infty} X_n = l \Longleftrightarrow \limsup_{n\to\infty} X_{pn+q} = l.(*)$$

Now we can prove the theorem.

► $(2) \Longleftrightarrow (3)$. By the result above, we have that:

$$\limsup_{n\to\infty} |C_n(f)| = l \Longleftrightarrow \limsup_{n\to\infty} |C_{2nm+m+1}(f)| = l,$$

so, by equations $(*)$ and (4),

$$\limsup_{n\to\infty} |C_n(f)| = l \Longleftrightarrow \limsup_{n\to\infty} |C_{2n+m+1}(f)| = l \Longleftrightarrow \limsup_{n\to\infty} |c_n(f)| = l.$$

► $(1) \Rightarrow (2)$. Suppose that $\limsup_n |c_n(F)| < l$, then for all $0 \leqslant j \leqslant m-1$,

$$\limsup_{n\to\infty} \left|c_{j+nm}(F)\right| < l.$$

By equation $(*)$ there exists $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $0 \leqslant j \leqslant m-1$, we have that

$$n > n_0 \Longrightarrow |c_{j+nm}(F)| < l - \varepsilon_0.$$

Thus $\forall n \in \mathbb{N}$, $p > n_0$ imply

$$|c_{j+nm}(F)\dots c_{j+(n+p)m}(F)|)^{\frac{1}{p+1}} \leqslant \|F\|_\infty^{\frac{n_0+1}{p+1}} (l-\varepsilon_0)^{\frac{p-n_0}{p+1}},$$

so that

$$\lim_{p\to\infty} \left(\sup_{n\in\mathbb{N}} |c_{j+nm}(F)\dots c_{j+(n+k)m}(F)\dots c_{j+(n+p)m}(F)| \right)^{\frac{1}{p+1}} \leqslant l - \varepsilon_0.$$

And so, for all $0 \leqslant j \leqslant m-1$, $\rho_j(F) \leqslant l - \varepsilon_0$.

By the same reasoning, if we suppose $\limsup_n |c_n(F)| > l$, we obtain that $\rho_j(F) > l$. This contradicts (1).

► $(2) \Rightarrow (1)$. Now suppose assertion (2) true. Then by $(*)$

$$\limsup_{n\to\infty} |c_{nm}(F)| = l.$$

So, let $(\gamma_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of integers such that

$$\lim_{n\to\infty} |c_{\gamma_n m}(F)| = l. \tag{6}$$

Let $\varepsilon > 0$, then by Lemma 1, we know that for all $p \in \mathbb{N}$, there exists $M'_p > 0$ such that

$$\left(n \geqslant M'_p \text{ and } 0 \leqslant k \leqslant p\right) \Longrightarrow |c_{\gamma_n m}(F) - c_{\gamma_n m+km}(F)| < \varepsilon.$$

Combining the previous equation and equation (6), it is clear that for all $p \in \mathbb{N}^*$ and $0 \leqslant k \leqslant p$, there exists $M_p \geqslant 0$ such that

$$|c_{\gamma_{M_p} m+km}(F)| \geqslant |c_{\gamma_{M_p} m}(F)| - |c_{\gamma_{M_p} m}(F) - c_{\gamma_{M_p} m+km}(F)| \geqslant l - 2\varepsilon.$$

Then

$$\sup_{n\in\mathbb{N}} \left|c_{nm}(F)\dots c_{nm+pm}(F)\right|^{1/(p+1)} \geqslant \left|c_{\gamma_{M_p} m}(F)\dots c_{\gamma_{M_p} m+pm}(F)\right|^{\frac{1}{p+1}} \geqslant l - 2\varepsilon,$$

and so for all $\varepsilon > 0$, $\rho_0(F) \geqslant l - 2\varepsilon$.

Moreover, there exists n_0 such that $n > n_0 \Longrightarrow |c_{nm}(F)| \leqslant l + \varepsilon$. Thus for $p > n_0$ we have

$$\sup_{n\in\mathbb{N}} \left|c_{nm}(F)\dots c_{(n+p)m}(F)\right|^{1/(p+1)} \leqslant \|F\|_\infty^{\frac{n_0+1}{p+1}} (l+\varepsilon)^{\frac{p-n_0}{p+1}}.$$

This implies $\rho_0(F) = l$. $\square$

Let F be a bounded quasi-homogeneous function with $F \sim (f,m)$. Then T_F is compact if and only if $\lim_{n\to\infty} c_n(F) = 0$ and T_f is compact if and only if $\lim_{n\to\infty} C_n(f) = 0$. Taking $l = 0$ in Theorem 2, we have that T_F is compact if and only if T_f is compact.

The following corollary is an immediate consequence of Proposition 2 and Theorem 2.

COROLLARY 1. *Let F be a bounded m-quasi-homogeneous function and $F \sim (f,m)$ then*

$$\sigma(T_F) = \limsup_{n\to\infty} |C_n(f)|\overline{\mathbb{D}}.$$

Now we give some more effective ways for calculating the spectral radius in the case where it is equal to $\|F\|_\infty$. We are looking for conditions equivalent to equation $\limsup_{n\to\infty} |C_n(f)| = \|f\|_\infty$.

4. Equivalent conditions to $\rho(T_F) = \|F\|_\infty$

In this section, we first give some simple conditions which imply $\rho(F) = \|F\|_\infty$ (which we have already shown to be equivalent to $\limsup_{n\to\infty} |C_n(f)| = \|f\|_\infty$). First, we establish an important inequality concerning some conditions which can be linked to the compacity problem.

DEFINITION 3. We denote by k_z the normalized reproducing kernel on L^2_a, so $k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$. Let $F \in L^\infty(\mathbb{D}, \mathrm{dA})$, the Berezin transform of F denoted $\tilde{F}$ is defined by:

$$\forall z \in \mathbb{D}, \tilde{F}(z) = \langle F k_z, k_z\rangle = \int_{\mathbb{D}} F(w) \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} \mathrm{dA}(w).$$

For $f \in L^\infty(0,1)$, we define the Berezin transform of f to be the Berezin transform of $F \sim (f,0)$ thus $\forall z \in \mathbb{D}$, $\tilde{f}(z) = \int_{\mathbb{D}} f(|w|)\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\mathrm{dA}(w)$.

The Berezin transform can be used to characterize compact Toeplitz operators. In [5], Korenblum and Zhu gave some equivalent conditions for a Toeplitz operator with radial symbol to be compact.

THEOREM 3. [5] *Let $f \in L^\infty(0,1)$ the following three assertions are equivalent:*

(1) $\lim_{n\to\infty}(n+1)\int_0^1 f(t)t^n \mathrm{d}t = 0$;

(2) $\lim_{\varepsilon\to 1}\frac{1}{1-\varepsilon}\int_\varepsilon^1 f(t)dt = 0$;

(3) $\lim_{z\to\partial\mathbb{D}} \tilde{f}(z) = 0$.

Let $F \in L^\infty(\mathbb{D}, \mathrm{d}A)$. In [1], S. Axler and D. Zheng give the following condition on the Berezin transform: $\lim_{|z|\to 1}\tilde{F}(z) = 0 \iff T_F$ is compact .In the radial case, $F \sim (f,0)$ with $f \in L^1([0,1], \mathrm{d}A)$, S. Grudsky and N. Vasilevski study in [4] the case where F is a radial L^1 function.

It is easy to obtain the same type of result concerning the radial part of the function in the quasi-homogeneous case.

Now, we give some properties concerning the Berezin transform of quasi-homogeneous functions. The following lemma is proved by Ž. Čučković in [2].

LEMMA 2. *Let $F \in L^\infty(\mathbb{D})$ be a bounded m-quasi-homogeneous function such that $F \sim (f,m)$, if $z = Re^{i\theta}$ then*

$$\tilde{F}(z) = 2(1-R^2)^2 R^{|m|} e^{im\theta} \sum_{n=0}^{\infty} \frac{n(n+|m|)}{2n+|m|+1} C_{2n+|m|}(f) R^{2(n-1)}.$$

Notice that this lemma tells us that the Berezin transform of a m-quasi-homogeneous function is another m-quasi-homogeneous function. But even if $F_0 \sim (f,0)$ and $F_m \sim (f,m)$ we see that $\tilde{F_0}$ and $\tilde{F_m}$ do not have the same radial part. Despite this fact, we show that $\tilde{F_0}$ and $\tilde{F_m}$ have the same "values" near the boundary of the disc $\mathbb{D}$.

LEMMA 3. *Let F be a bounded quasi-homogeneous function, $F \sim (f,m)$. We have*

$$\lim_{z\to\partial\mathbb{D}} |\tilde{F}(z)| - |\tilde{f}(z)| = 0.$$

In particular, we have

$$\limsup_{r\to 1} |\tilde{f}(r)| = \limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)|.$$

Proof. We will show that

$$\lim_{R\to 1}\left| |\tilde{F}(R)| - |\tilde{f}(R)| R^{|m|} \right| = 0.$$

It is equivalent to show that

$$\lim_{z\to\partial\mathbb{D}}\left||\tilde{F}(z)|-|\tilde{f}(z)|\times|z|^{|m|}\right|=0,$$

which easily implies the desired result.

By Lemma 2, if $z=R$, we have

$$\left||\tilde{F}(R)|-|\tilde{f}(R)|R^{|m|}\right|\leqslant 2(1-R^2)^2\sum_{n=0}^{\infty}\left|\frac{n(n+|m|)}{2n+|m|+1}C_{2n+|m|}(f)-\frac{n^2}{2n+1}C_{2n}(f)\right|R^{2n-2}\,.$$

Now, let $\varepsilon>0$. Since the sequence $(C_n(f))_{n\in\mathbb{N}}$ is uniformly bounded, $\frac{n(n+|m|)}{2n+|m|+1}$ is equivalent to $\frac{n^2}{2n+1}$ and $\lim_{n\to\infty}C_n(f)-C_{n+1}(f)=0$ ([4]), there exists $M_\varepsilon>0$, such that

$$n>M_\varepsilon\Longrightarrow\left|\frac{n(n+|m|)}{2n+|m|+1}C_{2n+|m|}(f)-\frac{n^2}{2n+1}C_{2n}(f)\right|\leqslant\frac{n^2}{2n+1}\varepsilon\,.$$

But this means that

$$\left|\tilde{F}(R)-|\tilde{f}(R)|R^{|m|}\right|\leqslant 2(1-R^2)^2\sum_{n=0}^{M_\varepsilon}\frac{n(n+|m|)}{2n+1}2\|f\|_\infty R^{2n-2}+(1-R^2)^2\sum_{n=M_\varepsilon}^{\infty}\varepsilon nR^{2n-2}.$$

And taking the limit as R tends to 1, we see that the first term tends to 0. The second one is smaller than the sum from indice 0, thus

$$\limsup_{R\to1}\left|\tilde{F}(R)-|\tilde{f}(R)|R^{|m|}\right|\leqslant\lim_{R\to1}(1-R^2)^2\sum_{n=0}^{\infty}\varepsilon nR^{2n-2}=\varepsilon.$$

Finally, for all $\varepsilon>0$, we have

$$\lim_{R\to1}\left||\tilde{F}(R)|-|\tilde{f}(R)|R^{|m|}\right|\leqslant\varepsilon$$

and the lemma is proved. $\square$

4.1. Sufficient conditions: some simple cases

Using Theorem 3, one can give some conditions which guarantee $\rho(F)=\|F\|_\infty$.

PROPOSITION 3. *Let F be a bounded m-quasi-homogeneous function and $F\sim(f,m)$. If any of the following conditions is true ;*

(1) $\lim_{t\to1}f(t)=L$ *with* $L\in\|F\|_\infty\mathbb{T}$,

(2) $\lim_{\varepsilon\to1}\frac{1}{1-\varepsilon}\int_\varepsilon^1 f(t)dt=L$ *with* $L\in\|F\|_\infty\mathbb{T}$,

(3) $\lim_{r\to1^-}\tilde{f}(r)=L$ *with* $L\in\|F\|_\infty\mathbb{T}$,

then F verifies $\limsup|C_n(f)|=\|F\|_\infty$.

Proof. If $\lim_{t\to 1} f(t)$ exists then it is easy to show that

$$\lim_{n\to\infty} C_n(f) = \lim_{n\to\infty}(n+1)\int_0^1 f(t)t^n \mathrm{d}t = \lim_{t\to 1} f(t).$$

Since $\forall n \in \mathbb{N}$, $C_n(f-L) = C_n(f) - L$, condition 1 implies the conclusion. Now, using $\widetilde{f-L} = \tilde{f} - L$ and applying Theorem 3, it is clear that if either (2) or (3) is true, then

$$\lim_{n\to\infty} |C_n(f)| = \|F\|_\infty. \qquad \square$$

The previous proposition deals with the case where $(C_n(f))_{n\in\mathbb{N}}$ has a limit. And applying Proposition 3 condition 1, we have:

EXAMPLE.

(1) If $F(z) = |z|^k e^{\mathrm{imArg}(z)}$ where $k \in \mathbb{N}^*$ and $m \in \mathbb{Z}^*$ then $\sigma(T_F) = F(\mathbb{D})$.

(2) Let $F(z) = f(|z|)e^{\mathrm{iArg}(z)}$ where $f(r) = \begin{cases} 2r-1 \text{ if } r \geqslant \frac{1}{2} \\ g(r) \text{ if } 0 \leqslant r \leqslant \frac{1}{2} \end{cases}$ and g is a function from $[0,1]$ to $[0,1]$. Then F is quasi-homogeneous and $\sigma(T_F) = F(\mathbb{D})$.

(3) Let F defined by $\forall z \in \mathbb{D}$, $F(z) = e^{\mathrm{imArg}(z)} \sin(1-|z|)^\alpha/(1-|z|)^\beta$ and $\alpha \geqslant \beta$. Then $\sigma(T_F) = F(\mathbb{D})$ if and only if $\alpha = \beta$.
It is clear that if $\alpha = \beta$ then $\lim_{r\to 1} \sin(1-|z|)^\alpha/(1-|z|)^\beta = 1 = \|F\|_\infty$.
If $\alpha > \beta$ then $\lim_{r\to 1} \sin(1-|z|)^\alpha/(1-|z|)^\beta = 0$ and $\sigma(T_F) = \{0\}$.

4.2. Equivalent conditions

In this section, we prove the equivalence result which follows.

THEOREM 4. *Let F be a bounded quasi-homogeneous function, $f \in L^\infty(0,1)$ and $m \in \mathbb{Z}^*$ such that $F \sim (f,m)$. The following conditions are equivalent*

a) $\|T_F\| = \|F\|_\infty$;

b) $\limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)| = \|F\|_\infty$;

c) $\limsup_{z\to\partial\mathbb{D}} \|T_F k_z\|_2 = \|F\|_\infty$;

d) $\|T_F\|_e = \|F\|_\infty$;

e) $\limsup_{n\to\infty} |C_n(f)| = \|F\|_\infty$;

f) $\limsup_{t\to 1} \left|\frac{1}{1-t}\int_t^1 f(r)dr\right| = \|f\|_\infty$.

Considering the equivalence between assertions $a)$, $b)$, $c)$ and $d)$, it is natural to ask if we have the same equivalence for other $F \in L^\infty(\mathbb{D})$. In particular, it would imply that in these cases a Toeplitz operator with maximal norm cannot be compact and assertion $b)$ would be equivalent to the condition that $\|T_F\| = \|F\|_\infty$.

To prove this theorem, we need some tools. First, we give an important inequality:

PROPOSITION 4. *Let $F \sim (f,m)$ a bounded quasi-homogeneous symbol, we have*

(1) $\|T_F\|_e = \limsup_{n\to\infty} |c_n(F)|$.

(2) $\limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)| \leqslant \limsup_{z\to\partial\mathbb{D}} \|T_F k_z\|_2 \leqslant \|T_F\|_e \leqslant \limsup_{\varepsilon\to 1} \left|\frac{1}{1-\varepsilon}\int_\varepsilon^1 f(r)dr\right| \leqslant \|F\|_\infty$.

Proof. (1) If we denote by K_n the compression of T_F to $\mathrm{Span}(1,z,\dots,z^n)$, then K_n is compact. Since $T_F e_n = c_n(F)e_{n+m}$ $\forall n \in \mathbb{N}$, we have

$$\|T_F - K_n\| = \sup_{k\geqslant n} |c_k(F)|.$$

This implies that

$$\|T_F\|_e \leqslant \limsup_{k\to\infty} |c_k(F)|.$$

Next, we consider $(\gamma_n)_{n\in\mathbb{N}}$ an strictly increasing sequence of integers such that $\lim_{n\to\infty} |c_{\gamma_n}(F)| = \limsup_{n\to\infty} |c_n(F)|$. Since (e_{γ_n}) converges weakly to 0, we have, for any compact operator K, $\lim_{n\to\infty} \|Ke_{\gamma_n}\| = 0$ thus

$$\lim_{n\to\infty} \|(T_F - K)e_{\gamma_n}\| = \lim_{n\to\infty} \|T_F e_{\gamma_n}\| = \lim_{n\to\infty} |c_{\gamma_n}(F)|,$$

and so

$$\limsup_{n\to\infty} |c_n(F)| \leqslant \|T_F - K\|.$$

Thus we have $\limsup_n |c_n(F)| \leqslant \|T_F\|_e$, completing the proof of the equality.

(2) Now, we prove the inequalities from left to right. First, we have

$$\forall z \in \mathbb{D}, |\tilde{F}(z)| = |\langle Fk_z, k_z\rangle| = |\langle T_F k_z, k_z\rangle| \leqslant \|T_F k_z\|_2.$$

This establishes the first inequality. Now, since k_z weakly converges to 0 as $z \to \partial\mathbb{D}$, we have that

$$\limsup_{z\to\partial\mathbb{D}} \|T_F k_z\|_2 \leqslant \inf_{K\in\mathscr{K}} \|T_F - K\| = \|T_F\|_e.$$

Finally, Theorem 2 implies

$$\limsup_{n\to\infty} |c_n(F)| = \limsup_{n\to\infty} |C_n(f)|,$$

and in the Theorem 3.3 of [4], S. Grudsky and N. Vasilevski show that

$$|C_n(f)| \leqslant k_n + \mathrm{const}\, n^2 \exp(-n^{1/3}) \quad \forall n \in \mathbb{N}$$

where $k_n = \sup_{1-n^{-2/3}<u<1} \frac{1}{1-u}\int_u^1 f(r)\mathrm{d}r$. Thus we see that

$$\limsup_{n\to\infty} |C_n(f)| \leqslant \limsup_{\varepsilon\to 1} \frac{1}{1-\varepsilon}\left|\int_\varepsilon^1 f(r)\mathrm{d}r\right|,$$

and the third relation is established. The last inequality is obvious since $\|F\|_\infty = \|f\|_\infty$. $\square$

As a consequence of this proposition, we have that $b) \Rightarrow c) \Rightarrow d) \Rightarrow e) \Rightarrow f)$ in Theorem 4. Next, we give some lemmas we will need to prove $f) \Rightarrow b)$.

4.2.1. Geometrical point of view.

REMARK. Let $r \in \mathbb{R}^+$ and let $(X_n)_{n\in\mathbb{N}}$ be a complex sequence. If $\forall n \in \mathbb{N}$, $|X_n| \leqslant r$, the following assertions are equivalent:

a)
$$\limsup_{n\to\infty} |X_n| = r;$$

b) there exists $L \in r\mathbb{T}$ such that
$$\liminf_{n\to\infty} |X_n - L| = 0.$$

This remark gives us an equivalent formulation for condition f) of Theorem 4 as
$$\exists L \in \|f\|_\infty \mathbb{T}, \liminf_{t\to 1} \left| \frac{1}{1-t} \int_t^1 (f(r) - L) dr \right| = 0,$$
and condition b) as
$$\exists L \in \|f\|_\infty \mathbb{T}, \liminf_{t\to 1} \left| \tilde{f}(r) - L \right| = 0.$$

NOTATION 1. For $A \subset \mathbb{C}$, we denote by $\mathscr{E}\mathscr{C}(A)$ the convex hull of A.

In the following, μ denotes the Lebesgue measure on $\mathbb{R}$.

LEMMA 4. (First geometric lemma) *Let E be a real measurable set with $\mu(E) > 0$ and $\varphi : E \to \mathbb{R}^+$ μ-integrable on E such that $\int_E \varphi d\mu > 0$ and $f \in L^\infty(E)$ then*
$$\frac{1}{\int_E \varphi d\mu} \int_E \varphi(\omega) f(\omega) d\mu(\omega) \in \overline{\mathscr{E}\mathscr{C}(f(E))},$$
the closure of $\mathscr{E}\mathscr{C} f(E)$.

Proof. This lemma is clear if φ is a simple function because in that case $\int_E \varphi d\mu$ is a barycenter. By density, we conclude. □

LEMMA 5. (Second geometric lemma) *Let $\rho > 0$, K a compact subset of $\overline{\rho\mathbb{D}}$ and $L \in \rho\mathbb{T} \backslash K$. If $(a_n)_{n\in\mathbb{N}}$, $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ are complex sequences such that*
$$a_n \in [0,1], M_n \in K, N_n \in \overline{\rho\mathbb{D}}, n \in \mathbb{N}. \qquad (**)$$

Then
$$\liminf_{n\to\infty} a_n > 0 \Longrightarrow \liminf_{n\to\infty} |a_n M_n + (1-a_n) N_n - L| > 0.$$

Proof. Let $L \in \rho\mathbb{T}$. We denote $\varphi_L : \mathbb{C}^2 \times]0,1] \to \mathbb{R}$ the application defined as follow

$$\varphi_L(M,N,a) = \frac{|aM+(1-a)N-L|}{a}.$$

Let $\delta > 0$. Since L is an extreme point of $\overline{\rho\mathbb{D}}$, for all $(M,N,a) \in K \times \overline{\rho\mathbb{D}} \times [\delta,1]$, it is clear that

$$aM+(1-a)N-L \neq 0$$

Since φ_L is a strictly positive continuous function on the compact set $K \times \overline{\rho\mathbb{D}} \times [\delta,1]$, it attains its minimum $\beta > 0$. So we have

$$|aM+(1-a)N-L| \geqslant \beta a \geqslant \beta\delta.$$

Notice that β does not depend on a, M or N. Now let $(a_n)_{n\in\mathbb{N}}$, (M_n) et (N_n) be sequences satisfying $(**)$.

Suppose $\liminf a_n > 0$, then denoting $\delta := \frac{1}{2}\liminf_{n\to\infty} a_n$, there exists $J > 0$ such that

$$n \geqslant J \Longrightarrow a_n \in [\delta,1].$$

With the same reasoning, if $n \geqslant J$ and $(M_n,N_n,a_n) \in K \times \overline{\rho\mathbb{D}} \times [\delta,1]$, there exists $\beta > 0$ such that

$$n \geqslant J \Longrightarrow |a_nM_n+(1-a_n)N_n-L| \geqslant \beta\delta.$$

It is now clear that

$$\liminf_{n\to\infty} |a_nM_n+(1-a_n)N_n-L| \geqslant \beta\delta > 0. \qquad \square$$

In the following, we will apply this lemma with sequences $(a_n)_{n\in\mathbb{N}}$ of the form

$$a_n = \mu(\{\omega \in [1-\frac{1}{n},1], |f(\omega)-L| > \varepsilon\}.$$

In order to express condition f) of Theorem 4 in terms of the Berezin transform, we introduce some more notation and give a lemma.

NOTATION 2. Let $f : (0,1) \to \mathbb{C}$, $L \in \mathbb{C}$, $\varepsilon > 0$ and $s \in \mathbb{R}^{+*}$: we denote

(1) $E^-_{L,s,\varepsilon} = \{\omega \in [1-\frac{1}{s},1], |f(\omega)-L| > \varepsilon\}$.

(2) $E_{L,s,\varepsilon} = \{\omega \in [1-\frac{1}{s},1], |f(\omega)-L| \leqslant \varepsilon\}$.

If L is fixed, we simply denote $E^-_{L,s,\varepsilon} = E^-_{s,\varepsilon}$ et $E_{L,s,\varepsilon} = E_{s,\varepsilon}$. The sets E and E^- obviously depend on f, but since f is always fixed, we will not use it as an index.

LEMMA 6. *Let $f \in L^\infty(0,1)$, $L \in \|f\|_\infty\mathbb{T}$, the following conditions are equivalent:*

a) *For all $\varepsilon > 0$, $\liminf_{s\to\infty} s\mu(E^-_{L,s,\varepsilon}) = 0$;*

b) $\liminf_{s\to\infty}\left|s\int_{1-1/s}^{1}(f(r)-L)dr\right|=0$.

Proof. $a)\Rightarrow b)$. We fix $\varepsilon>0$ and considering the given $f\in L^\infty(0,1)$ and $L\in\|f\|_\infty\mathbb{T}$ we define $E^-_{L,s,\varepsilon}$ and $E_{L,s,\varepsilon}$ as above. Now let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty}\gamma_n=+\infty$ such that

$$\liminf_{s\to\infty} s\mu(E^-_{L,s,\varepsilon})=\lim_{n\to\infty}\gamma_n\mu(E^-_{L,\gamma_n,\varepsilon}).$$

Then we have

$$\begin{aligned}\gamma_n\left|\int_{1-1/\gamma_n}^{1}(f(r)-L)dr\right| &\leqslant \gamma_n\int_{1-1/\gamma_n}^{1}|f(r)-L|dr\\ &\leqslant \gamma_n\mu(E_{\gamma_n,\varepsilon})\varepsilon+2\|f\|_\infty\gamma_n\mu(E^-_{\gamma_n,\varepsilon}).\end{aligned}$$

But by assumption $a)$, $\lim_{n\to\infty}\gamma_n\mu(E^-_{\gamma_n,\varepsilon})=0$, and it is clear that $\gamma_n\mu(E_{\gamma_n,\varepsilon})\leqslant 1$, thus

$$0\leqslant\limsup_{n\to\infty}\gamma_n\left|\int_{1-1/\gamma_n}^{1}(f(r)-L)dr\right|\leqslant\varepsilon.$$

This is true for all $\varepsilon>0$, then

$$\lim_{n\to\infty}\gamma_n\left|\int_{1-1/\gamma_n}^{1}(f(r)-L)dr\right|=0.$$

Now, for the converse $b)\Rightarrow a)$. Suppose $a)$ is false, then there exists $\varepsilon_0>0$ and $(\gamma_n)_n$ as in the previous case such that

$$\lim_{n\to\infty}\gamma_n\mu(E^-_{\gamma_n,\varepsilon_0})=\liminf_{s\to\infty}s\mu(E^-_{s,\varepsilon_0})>0.$$

We prepare ourselves to use our second geometric lemma.

First, let $\rho=\|f\|_\infty$ and $K=\mathscr{E}\mathscr{C}(\overline{\rho\mathbb{D}\backslash D(L,\varepsilon_0)})$. Then K is compact as the convex hull of a compact set, and $L\in\rho\mathbb{T}\backslash K$. Now, for all integers n, we denote

$$N_n=\frac{1}{\mu(E_{\gamma_n,\varepsilon_0})}\int_{E_{\gamma_n,\varepsilon}}f(r)dr,$$

and

$$M_n=\frac{1}{\mu(E^-_{\gamma_n,\varepsilon_0})}\int_{E^-_{\gamma_n,\varepsilon}}f(r)dr.$$

Then, by the first geometric lemma, $N_n\in\mathscr{E}\mathscr{C}(f(]0,1[)\subset\overline{\rho\mathbb{D}}$ and $M_n\in K$. Finally, denoting

$$a_n=\gamma_n\mu(E^-_{\gamma_n,\varepsilon_0}),$$

we see that for all $n\in\mathbb{N}$, $a_n\in[0,1]$ et $\gamma_n\mu(E_{\gamma_n,\varepsilon_0})=1-a_n$. Thus

$$\gamma_n\int_{1-1/\gamma_n}^{1}(f(r)-L)dr=a_nM_n+(1-a_n)N_n-L.$$

And by assumption

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \gamma_n \mu(E^-_{\gamma_n,\varepsilon_0}) = \liminf_{s\to\infty} s\mu(E^-_{s,\varepsilon_0}) > 0.$$

So, applying the second geometric lemma, we get

$$\liminf_{n\to\infty} |\gamma_n \int_{1-1/\gamma_n}^{1} f(r)dr - L| > 0 .$$

Thus $b)$ is false. $\square$

4.2.2. Proof of Theorem 4

First we show that $a) \Leftrightarrow d)$. Since $|c_n(F)| \leqslant \frac{2\sqrt{n+1}\sqrt{n+m+1}}{2n+m+2}\|F\|_\infty < \|F\|_\infty$, $\sup_n |c_n(F)| = \|F\|_\infty$ is equivalent to $\limsup_{n\to\infty} |c_n(F)| = \|F\|_\infty$. Thus, using the equality from assertion 1 of Proposition 4 $\|T_F\| = \sup_n |c_n(F)| = \|F\|_\infty \Longleftrightarrow \|T_F\|_e = \|F\|_\infty$.

Now using assertion 2 of Proposition 4, we see that the proof of Theorem 4 will be complete if we show that $f) \Rightarrow b)$. So we suppose $f)$. By Remark 4.2.1, we can find $L \in \|f\|_\infty \mathbb{T}$ such that

$$\liminf_{r\to 1} \left| \frac{1}{1-r} \int_r^1 (f(t) - L)\,\mathrm{dt} \right| = 0.$$

By Lemma 6, this implies

$$\forall \varepsilon > 0,\ \liminf_{s\to\infty} s.\mu(E^-_{L,s,\varepsilon}) = 0. \tag{7}$$

Now, in order to prove $b)$, it is enough to find, for each $\varepsilon > 0$, a sequence $(R_n^\varepsilon)_{n\in\mathbb{N}} \subset [0,1)$ such that $\lim_{n\to\infty} R_n^\varepsilon = 1$ and

$$\liminf_{n\to\infty} \left| \tilde{F}(R_n^\varepsilon) - L \right| < \varepsilon.$$

Using Lemma 3, it suffices to show the above inequality with $\tilde{F}$ replaced by $\tilde{f}$.

Since for any $R \in [0,1)$, we have

$$\begin{aligned}\left|\tilde{f}(R) - L\right| &\leqslant \int_{\mathbb{D}} |f(|w|) - L| \frac{(1-R^2)^2}{|1-Rw|^4} \mathrm{d}A(w) \\ &= (1-R^2)^2 \int_0^1 \rho |f(\rho) - L| \left(\frac{1}{\pi} \int_0^{2\pi} \frac{1}{|1-R\rho e^{i\theta}|^4} d\theta \right) d\rho.\end{aligned}$$

Evaluating the integral by taking $t = \tan\frac{\theta}{2}$, we see that

$$\left|\tilde{f}(R) - L\right| \leqslant (1-R^2)^2 \int_0^1 \rho |f(\rho) - L| \frac{2(1+R^2\rho^2)}{(1-R^2\rho^2)^3} d\rho.$$

Now, to simplify the above inequality as much as possible and transform our integral into a function that we know how to calculate, we use the obvious inequalities: $(1-R^2)\leqslant 4(1-R)^2$, $(1+R^2\rho^2)\leqslant 2$ and $(1-R^2\rho^2)^3\geqslant(1-R\rho)^3$. Thus we obtain

$$|\tilde{f}(R)-L|\leqslant 16(1-R)^2\int_0^1\frac{|f(\rho)-L|}{(1-R\rho)^3}d\rho. \tag{8}$$

Now, we fix $\varepsilon>0$. By equation (7), we can find $(\gamma_n)_{n\in\mathbb{N}}$ a sequence such that $\lim\gamma_n=+\infty$ and

$$\lim_{n\to\infty}\gamma_n\mu(E^-_{L,\gamma_n,\varepsilon})=0. \tag{9}$$

Then, if we denote $R_n=1-\sqrt{\frac{\mu(E^-_{\gamma_n,\varepsilon})}{\gamma_n}}$, we have for any $n\in\mathbb{N}$, $R_n\in[0,1)$ and $(R_n)_{\in\mathbb{N}}$ converges to 1. Now, we use the inequality (8) with $R=R_n$ and split the integral into the following parts,

$$A_{1,n}:=(1-R_n)^2\int_0^{1-\frac{1}{\gamma_n}}\frac{|f(\rho)-L|}{(1-R_n\rho)^3}d\rho,$$

$$A_{2,n}:=(1-R_n)^2\int_{E^-_{\gamma_n,\varepsilon}}\frac{|f(\rho)-L|}{(1-R_n\rho)^3}d\rho,$$

$$A_{3,n}:=(1-R_n)^2\int_{E_{\gamma_n,\varepsilon}}\frac{|f(\rho)-L|}{(1-R_n\rho)^3}d\rho.$$

Considering $A_{1,n}$, we have

$$A_{1,n}\leqslant(1-R_n)^2\int_0^{1-\frac{1}{\gamma_n}}\frac{2\|f\|_\infty}{(1-R_n\rho)^3}d\rho$$

and evaluating the integral, we get

$$A_{1,n}\leqslant\frac{\|f\|_\infty}{R_n}\left(\frac{1-R_n}{1-R_n(1-\frac{1}{\gamma_n})}\right)^2.$$

We write

$$\frac{1-R_n}{1-R_n(1-\frac{1}{\gamma_n})}=\frac{1}{1+\frac{R_n}{\gamma_n(1-R_n)}}.$$

and using equation (9), we have $\frac{R_n}{\gamma_n(1-R_n)}=\frac{1}{\sqrt{\gamma_n\mu(E^-_{\gamma_n,\varepsilon})}}-\frac{1}{\gamma_n}\longrightarrow\infty$. So

$$\lim_{n\to\infty}A_{1,n}=0.$$

Considering $A_{2,n}$, we have

$$\begin{aligned}A_{2,n}&\leqslant(1-R_n)^2\int_{E^-_{\gamma_n,\varepsilon}}\frac{2\|f\|_\infty}{(1-R_n\rho)^3}d\rho\\&\leqslant\frac{1}{(1-R_n)}2\|f\|_\infty\mu(E^-_{\gamma_n,\varepsilon})\\&\leqslant 2\|f\|_\infty\sqrt{\gamma_n\mu(E^-_{\gamma_n,\varepsilon})}\end{aligned}$$

and once again equation (9) gives us that

$$\lim_{n\to\infty} A_{2,n} = 0.$$

Finally,

$$\begin{aligned} A_{3,n} &\leqslant (1-R_n)^2 \int_{E_{\gamma_n,\varepsilon}} \frac{\varepsilon}{(1-R_n\rho)^3} d\rho \\ &\leqslant (1-R_n)^2 \int_0^1 \frac{\varepsilon}{(1-R_n\rho)^3} d\rho \\ &\leqslant \frac{\varepsilon(2-R_n)}{2} \leqslant \varepsilon. \end{aligned}$$

Thus

$$\liminf_{n\to\infty} \left|\tilde{f}(R_n) - L\right| \leqslant 16\varepsilon.$$

As a conclusion, for all $\varepsilon > 0$, we have

$$\liminf_{r\to 1} |\tilde{f}(r) - L| \leqslant 16\varepsilon,$$

so $\liminf_{r\to 1} \left|\tilde{f}(r) - L\right| = 0$ and the theorem is proved. □

Considering Proposition 4, we have

$$0 \leqslant \limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)| \leqslant \limsup_{z\to\partial\mathbb{D}} \|T_F k_z\|_2 \leqslant \|T_F\|_e \leqslant \|T_F\| \leqslant \|F\|_\infty$$

Theorem 2.2 of [1] together with Theorem 3 imply that if one of the quantities above equals 0 then so do the others, and Theorem 4 implies the same result with 0 replaced by $\|F\|_\infty$. Thus it is natural to ask the following question: let $F \sim (f,m)$ be a bounded quasi-homogeneous symbol, is the following equivalence true:

$$\|T_F\|_e = \limsup_{n\to\infty} |c_n(F)| = \limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)| = \limsup_{s\to 1^-} \left|\frac{1}{1-s}\int_s^1 f(r)\mathrm{d}r\right|.$$

The answer is no as can be shown using example 4 of [4].

5. The case $\sigma(T_F) = F(\mathbb{D})$

Now we are ready to easily answer the question: for F a bounded quasi-homogeneous symbol, under which assumptions is the equality $\sigma(T_F) = F(\mathbb{D})$ true?

THEOREM 5. *Let F be a bounded m-quasi-homogeneous function. Then $\sigma(T_F) = F(\mathbb{D})$ if and only if*

(*i*) $0 \in F(\mathbb{D})$;

(ii) $F(\mathbb{D})$ *is connected;*

(iii) *One of the following equivalent conditions is satisfied*

a) $\|T_F\| = \|F\|_\infty$;

b) $\limsup_{z\to\partial\mathbb{D}} |\tilde{F}(z)| = \|F\|_\infty$;

c) $\limsup_{z\to\partial\mathbb{D}} \|T_F k_z\|_2 = \|F\|_\infty$;

d) $\|T_F\|_e = \|F\|_\infty$;

e) $\limsup_{n\to\infty} |(n+1)\int_0^1 f(t)t^n \mathrm{d}t| = \|F\|_\infty$;

f) $\limsup_{t\to 1} \left|\frac{1}{1-t}\int_t^1 f(r)dr\right| = \|f\|_\infty$.

Proof. The equivalence *iii* comes from Theorem 4.

Now, if $\sigma(T_F) = F(\mathbb{D})$ then $F(\mathbb{D}) = \max_{0\leqslant j\leqslant m-1} \rho_j(F)\overline{\mathbb{D}}$, thus the assertions (i) and (ii) are true. Since $\max_{0\leqslant j\leqslant m-1} \rho_j(F) \leqslant \|F\|_\infty$ and $F(\mathbb{D})$ contains a complex number with module $\|F\|_\infty$, then $\|F\|_\infty \leqslant \max_{0\leqslant j\leqslant m-1} \rho_j(F)$ and (iii) is true using Proposition 2 and Theorem 4.

For the converse, let us suppose (i), (ii) and (iii). The equality $F(z) = f(|z|)e^{\mathrm{im}\mathrm{Arg}(z)}$ implies that $F(\mathbb{D})$ is rotation invariant so (i) and (ii) imply that $F(\mathbb{D}) = \|F\|_\infty\overline{\mathbb{D}}$. Assertion (iii), $e)$ means that $\max_j \rho_j(F) = \|F\|_\infty$, and so, by Proposition 2, $\sigma(T_F)$ also equals $\|F\|_\infty\overline{\mathbb{D}}$. $\square$

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