Operators
and
Matrices
Volume 4, Number 3 (2010), 385–403

CONVERSES OF JENSEN'S OPERATOR INEQUALITY

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(*Communicated by C.-K. Li*)

Abstract. We give a generalization of converses of Jensen's operator inequality for fields of positive linear mappings $(\phi_t)_{t\in T}$ such that $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. We consider different types of converse inequalities.

1. Introduction

Let f be an operator convex function defined on an interval I. Ch.Davis [2] proved a Schwarz inequality

$$f(\phi(x)) \leqslant \phi(f(x)),$$

where $\phi: \mathscr{A} \to B$ is a unital completely positive linear map from a C^*-algebra $\mathscr{A}$ to linear operators on a Hilbert space K, and x is a self-adjoint element in $\mathscr{A}$ with spectrum in I. Subsequently M.D.Choi [1] noted that it is enough to assume that ϕ is unital and positive. In fact, the restriction of ϕ to the commutative C^*-algebra generated by x and the identity operator $\mathbf{1}$ is automatically completely positive by a theorem of Stinespring [13].

B. Mond and J. Pečarić [11] proved the inequality

$$f\left(\sum_{i=1}^{n} \omega_i \phi_i(x_i)\right) \leqslant \sum_{i=1}^{n} \omega_i \phi_i(f(x_i)) \tag{1}$$

for an operator convex function f defined on an interval I, where $(\phi_1,\dots,\phi_n)$ is an n-tuple of unital positive linear maps $\phi_i: B(H) \to B(K)$, $(x_1,\dots,x_n)$ is an n-tuple of self-adjoint operators with spectra in I and $(\omega_1,\dots,\omega_n)$ is an n-tuple of non-negative real numbers with sum one.

Also, without the assumption of operator convexity, B. Mond and J. Pečarić [10, 12] showed the following extension of the converses of Jensen's inequality:

$$F\left[\sum_{i=1}^{n} \omega_i \phi_i(f(x_i)), f\left(\sum_{i=1}^{n} \omega_i \phi_i(x_i)\right)\right] \leqslant \max_{m\leqslant z\leqslant M} F\left[\alpha_f z+\beta_f, f(z)\right]\mathbf{1}, \tag{2}$$

Mathematics subject classification (2010): 47A63, 47A64.

Keywords and phrases: Converses of Jensen's operator inequality, convex function, Mond-Pečarić method, power functions, power means.

Paper OaM-04-20

for a convex function f defined on $[m,M]$, a real valued function $F(u,v)$ which is operator monotone in its first variable, where $(\phi_1,\ldots,\phi_n)$ is an n-tuple of unital positive linear maps $\phi_i : B(H) \to B(K)$, $(x_1,\ldots,x_n)$ is an n-tuple of self-adjoint operators with spectra in $[m,M]$ and $(\omega_1,\ldots,\omega_n)$ is an n-tuple of non-negative real numbers with sum one. Here we use the standard notation for a real valued continuous function $f : [m,M] \to \mathbb{R}$

$$\alpha_f := \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad \beta_f := \frac{Mf(m)-mf(M)}{M-m}.$$

J. Mićić, Y. Seo, S.-E. Takahasi and M. Tominaga [9] generalized (2) for a convex function f and any continuous function g on $[m,M]$.

Recently F. Hansen, J. Pečarić and I. Perić in [7] gave a general formulation of Jensen's operator inequality for unital field of positive linear mappings and its converses. They proved a generalization of (1) and (2) given in next two theorems. They say that a field $(\phi_t)_{t\in T}$ of mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ is unital if it is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = \mathbf{1}$, where $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras of operators on a Hilbert spaces H and K, respectively.

THEOREM A. *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I, and let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras on a Hilbert spaces H and K respectively. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leqslant \int_T \phi_t(f(x_t))\,d\mu(t) \tag{3}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in I.

THEOREM B. *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a unital field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$. Let $f,g : [m,M] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $f([m,M]) \subset U$, $g([m,M]) \subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex in the interval $[m,M]$, then*

$$F\left[\int_T \phi_t(f(x_t))\,d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \leqslant \sup_{m\leqslant z\leqslant M} F\left[\alpha_f z + \beta_f, g(z)\right]\mathbf{1}. \tag{4}$$

In the dual case (when f is concave) the opposite inequality holds in (4) with inf *instead of* sup.

Furthermore, J. I. Fujii, M. Nakamura, J. Pečarić and Y. Seo [4] observed the reverse inequality of Kadison's Schwarz inequality, without the assumption of the normalization of map Φ given in next lemma.

LEMMA C. *Let Φ be a positive linear map on $B(H)$ such that $\Phi(\mathbf{1}) = k\mathbf{1}$ for some positive scalar k. If A is a positive operator on H such that $0 < m\mathbf{1} \leqslant A \leqslant M\mathbf{1}$ for some scalars $m < M$, then for each $\lambda > 0$*

$$\Phi(A) \leqslant \lambda\Phi\left(A^{-1}\right)^{-1} + C(m,M,\lambda,k)\mathbf{1},$$

where

$$C(m,M,\lambda,k) = \begin{cases} k(m+M) - 2\sqrt{\lambda mM} & \text{if } m \leqslant \sqrt{\lambda mM}/k \leqslant M, \\ (k-\lambda/k)M & \text{if } \sqrt{\lambda mM}/k \leqslant m, \\ (k-\lambda/k)m & \text{if } M \leqslant \sqrt{\lambda mM}/k. \end{cases}$$

In this paper, using the idea given in Lemma C, we consider a generalization of Theorem A and Theorem B in the case when a field $(\phi_t)_{t\in T}$ of mappings $\phi_t : \mathscr{A} \to \mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. We consider some applications given in [6, 7, 8] under the new formulation.

2. Main results

Let T be a locally compact Hausdorff space, and let $\mathscr{A}$ be a C^*-algebra of operators on a Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathscr{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. If in addition μ is a bounded Radon measure on T and the function $t \to \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t\,d\mu(t)$, which is the unique element in the multiplier algebra

$$M(\mathscr{A}) = \{a \in B(H) \mid \forall x \in \mathscr{A} : ax + xa \in \mathscr{A}\}$$

such that

$$\varphi\left(\int_T x_t\,d\mu(t)\right) = \int_T \varphi(x_t)\,d\mu(t)$$

for every linear functional φ in the norm dual $\mathscr{A}^*$, cf. [5].

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another C^*-algebra $\mathscr{B}$ of operators on a Hilbert space K. We say that such a field is continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathscr{A}$.

THEOREM 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras on H and K respectively. Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in $\mathscr{A}$ with spectra in an interval I defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Then the inequality*

$$f\left(\frac{1}{k}\int_T \phi_t(x_t)\,d\mu(t)\right) \leqslant \frac{1}{k}\int_T \phi_t(f(x_t))\,d\mu(t) \tag{5}$$

holds for each operator convex function $f : I \to \mathbb{R}$ defined on I. In the dual case (when f is operator concave) the opposite inequality holds in (5).

Proof. This theorem follows from Theorem A, since $(\frac{1}{k}\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\frac{1}{k}\phi_t : \mathscr{A} \to \mathscr{B}$. $\square$

In the present context we may obtain results of the Li-Mathias type cf. [6, Chapter 3].

THEOREM 2.2. *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Let $f : [m,M] \to \mathbb{R}$, $g : [km,kM] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m,M]) \subset U$, $g([km,kM]) \subset V$ and F is bounded. If F is operator monotone in the first variable, then*

$$\begin{aligned}\inf_{km\leqslant z\leqslant kM} F\left[k\cdot h_1\left(\frac{1}{k}z\right),g(z)\right]\mathbf{1} &\leqslant F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \\ &\leqslant \sup_{km\leqslant z\leqslant kM} F\left[k\cdot h_2\left(\frac{1}{k}z\right),g(z)\right]\mathbf{1}\end{aligned} \tag{6}$$

holds for every operator convex function h_1 on $[m,M]$ such that $h_1 \leqslant f$ and for every operator concave function h_2 on $[m,M]$ such that $h_2 \geqslant f$.

Proof. We only prove RHS of (6). Let h_2 be operator concave function on $[m,M]$ such that $f(z) \leqslant h_2(z)$ for every $z \in [m,M]$. By using the functional calculus, it follows that $f(x_t) \leqslant h_2(x_t)$ for every $t \in T$. Applying the positive linear maps ϕ_t and integrating, we obtain

$$\int_T \phi_t\left(f(x_t)\right)d\mu(t) \leqslant \int_T \phi_t\left(h_2(x_t)\right)d\mu(t).$$

Furthermore, by using Theorem 2.1, we have

$$\frac{1}{k}\int_T \phi_t\left(h_2(x_t)\right)\,d\mu(t) \leqslant h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\,d\mu(t)\right)$$

and hence $\displaystyle\int_T \phi_t\left(f(x_t)\right)d\mu(t) \leqslant k\cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\,d\mu(t)\right)$. Since $m\,\phi_t(\mathbf{1}) \leqslant \phi_t(x_t) \leqslant M\,\phi_t(\mathbf{1})$, it follows that $km\mathbf{1} \leqslant \int_T \phi_t(x_t)d\mu(t) \leqslant kM\mathbf{1}$. Using operator monotonicity of $F(\cdot,v)$, we obtain

$$\begin{aligned}&F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \\ &\qquad\leqslant F\left[k\cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\,d\mu(t)\right), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \\ &\qquad\leqslant \sup_{km\leqslant z\leqslant kM} F\left[k\cdot h_2\left(\frac{1}{k}z\right),g(z)\right]\mathbf{1}. \qquad \square\end{aligned}$$

Applying RHS of (6) for a convex function f (or LHS of (6) for a concave function f) we obtain the following generalization of Theorem B.

THEOREM 2.3. *Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as in Theorem 2.2. Let $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex in the interval $[m,M]$, then*

$$F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \leqslant \sup_{km\leqslant z\leqslant kM} F\left[\alpha_f z+\beta_f k, g(z)\right]\mathbf{1}. \tag{7}$$

In the dual case (when f is concave) the opposite inequality holds in (7) with inf *instead of* sup.

Proof. We only prove the convex case. For convex f the inequality $f(z)\leqslant\alpha_f z+\beta_f$ holds for every $z\in[m,M]$. Thus, by putting $h_2(z)=\alpha_f z+\beta_f$ in RHS of (6) we obtain (7). □

Numerous applications of the previous theorem can be given (see [6]). Applying Theorem 2.3 for the function $F(u,v)=u-\lambda v$, we obtain the following generalization of [6, Theorem 2.4].

COROLLARY 2.4. *Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as in Theorem 2.2. If $f:[m,M]\to\mathbb{R}$ is convex in the interval $[m,M]$ and $g:[km,kM]\to\mathbb{R}$, then for any $\lambda\in\mathbb{R}$*

$$\int_T \phi_t\left(f(x_t)\right)d\mu(t)\leqslant \lambda\, g\left(\int_T \phi_t(x_t)d\mu(t)\right)+C\mathbf{1}, \tag{8}$$

where

$$C=\sup_{km\leqslant z\leqslant kM}\left\{\alpha_f z+\beta_f k-\lambda g(z)\right\}.$$

If furthermore λg is strictly convex differentiable, then the constant $C\equiv C(m,M,f,g,k,\lambda)$ can be written more precisely as

$$C=\alpha_f z_0+\beta_f k-\lambda g(z_0),$$

where

$$z_0=\begin{cases} g'^{-1}(\alpha_f/\lambda) & \text{for } \lambda g'(km)\leqslant\alpha_f\leqslant\lambda g'(kM),\\ km & \text{for } \lambda g'(km)\geqslant\alpha_f,\\ kM & \text{for } \lambda g'(kM)\leqslant\alpha_f.\end{cases}$$

In the dual case (when f is concave and λg is strictly concave differentiable) the opposite inequality holds in (8) with min *instead of* max *with the opposite condition while determining z_0.*

REMARK 2.5. We assume that $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ are as in Theorem 2.3. If $f:[m,M]\to\mathbb{R}$ is convex and $\lambda g:[km,kM]\to\mathbb{R}$ is strictly concave differentiable, then the constant $C\equiv C(m,M,f,g,k,\lambda)$ in (8) can be written more precisely as

$$C=\begin{cases}\alpha_f kM+\beta_f k-\lambda g(kM) & \text{for } \ \alpha_f-\lambda\alpha_{g,k}\geqslant 0,\\ \alpha_f km+\beta_f k-\lambda g(km) & \text{for } \ \alpha_f-\lambda\alpha_{g,k}\leqslant 0,\end{cases}$$

where

$$\alpha_{g,k}=\frac{g(kM)-g(km)}{kM-km}.$$

Setting $\phi_t(A_t)=\langle A_t\xi_t,\xi_t\rangle$ for $\xi_t\in H$ and $t\in T$ in Corollary 2.4 and Remark 2.5 give a generalization of all results from [6, Section 2.4]. For example, we obtain the following two corollaries.

COROLLARY 2.6. *Let $(A_t)_{t\in T}$ be a continuous field of positive operators on a Hilbert space H defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. We assume the spectra are in $[m,M]$ for some $0<m<M$. Let furthermore $(\xi_t)_{t\in T}$ be a continuous field of vectors in H such that $\int_T\|\xi_t\|^2d\mu(t)=k$ for some scalar $k>0$. Then for any real λ,q,p*

$$\int_T\langle A_t^p\xi_t,\xi_t\rangle d\mu(t)-\lambda\left(\int_T\langle A_t\xi_t,\xi_t\rangle d\mu(t)\right)^q\leqslant C, \tag{9}$$

where the constant $C\equiv C(\lambda,m,M,p,q,k)$ is

$$C=\begin{cases}(q-1)\lambda\left(\frac{\alpha_p}{\lambda q}\right)^{q/(q-1)}+\beta_p k & \text{for } \ \lambda q m^{q-1}\leqslant\frac{\alpha_p}{k^{q-1}}\leqslant\lambda q M^{q-1},\\ kM^p-\lambda(kM)^q & \text{for } \ \frac{\alpha_p}{k^{q-1}}\geqslant\lambda q M^{q-1},\\ km^p-\lambda(km)^q & \text{for } \ \frac{\alpha_p}{k^{q-1}}\leqslant\lambda q m^{q-1},\end{cases} \tag{10}$$

in the case $\lambda q(q-1)>0$ and $p\in\mathbb{R}\setminus(0,1)$
or

$$C=\begin{cases}kM^p-\lambda(kM)^q & \text{for } \ \alpha_p-\lambda k^{q-1}\alpha_q\geqslant 0,\\ km^p-\lambda(km)^q & \text{for } \ \alpha_p-\lambda k^{q-1}\alpha_q\leqslant 0,\end{cases} \tag{11}$$

in the case $\lambda q(q-1)<0$ and $p\in\mathbb{R}\setminus(0,1)$.

In the dual case: $\lambda q(q-1)<0$ and $p\in(0,1)$ the opposite inequality holds in (9) with the opposite condition while determining the constant C in (10). But in the dual case: $\lambda q(q-1)>0$ and $p\in(0,1)$ the opposite inequality holds in (9) with the opposite condition while determining the constant C in (11).

Constants α_p and β_p in terms above are the constants α_f and β_f associated with the function $f(z)=z^p$.

COROLLARY 2.7. *Let* $(A_t)_{t\in T}$ *and* $(\xi_t)_{t\in T}$ *be as in Corollary* 2.6. *Then for any real number* $r \neq 0$ *we have*

$$\int_T \langle \exp(rA_t)\,\xi_t,\xi_t\rangle d\mu(t) - \exp\left(r\int_T \langle A_t\xi_t,\xi_t\rangle d\mu(t)\right) \leqslant C_1, \tag{12}$$

$$\int_T \langle \exp(rA_t)\,\xi_t,\xi_t\rangle d\mu(t) \leqslant C_2\,\exp\left(r\int_T \langle A_t\xi_t,\xi_t\rangle d\mu(t)\right), \tag{13}$$

where the constant $C_1 \equiv C_1(r,m,M,k)$

$$C_1 = \begin{cases} \frac{\alpha}{r}\ln\left(\frac{\alpha}{r\mathrm{e}}\right) + k\beta & \textit{for}\;\; re^{rkm} \leqslant \alpha \leqslant re^{rkM}, \\ kM\alpha + k\beta - \mathrm{e}^{rkM} & \textit{for}\;\; re^{rkM} \leqslant \alpha, \\ km\alpha + k\beta - \mathrm{e}^{rkm} & \textit{for}\;\; re^{rkm} \geqslant \alpha \end{cases}$$

and the constant $C_2 \equiv C_2(r,m,M,k)$

$$C_2 = \begin{cases} \frac{\alpha}{re}e^{kr\beta/\alpha} & \textit{for}\;\; kre^{rm} \leqslant \alpha \leqslant kre^{rM}, \\ k\mathrm{e}^{(1-k)rm} & \textit{for}\;\; kre^{rm} \geqslant \alpha, \\ k\mathrm{e}^{(1-k)rM} & \textit{for}\;\; kre^{rM} \leqslant \alpha. \end{cases}$$

Constants α *and* β *in terms above are the constants* α_f *and* β_f *associated with the function* $f(z) = \mathrm{e}^{rz}$.

Proof. We set $f(z) \equiv g(z) = \mathrm{e}^{rz}$ and $\phi_t(A_t) = \langle A_t\xi_t,\xi_t\rangle, t \in T$, in Corollary 2.4. Then the problem is reduced to determine $\max_{km\leqslant z\leqslant kM} h(z)$ where $h(z) = \alpha z + k\beta - \mathrm{e}^{rz}$ in the inequality (12) and $h(z) = (\alpha z + k\beta)/\mathrm{e}^{rz}$ in the inequality (13). Applying the differential calculus we get C_1 and C_2. We omit the details. □

Applying the inequality $f(x) \leqslant \frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M)$ (for a convex function f on $[m,M]$) to positive operators $(A_t)_{t\in T}$ and using $0 < A_t \leqslant \|A_t\|\mathbf{1}$, we obtain the following theorem, which is a generalization of results from [7, 3].

THEOREM 2.8. *Let* f *be a convex function on* $[0,\infty)$ *and let* $\|\cdot\|$ *be a normalized unitarily invariant norm on* $B(H)$ *for some finite dimensional Hilbert space* H. *Let* $(\phi_t)_{t\in T}$ *be a field of positive linear maps* $\phi_t : B(H) \to B(K)$, *where* K *is a Hilbert space, defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ. *If the field* $t \to \phi_t(\mathbf{1})$ *is integrable with* $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ *for some positive scalar* k, *then for every continuous field of positive operators* $(A_t)_{t\in T}$ *we have*

$$\int_T \phi_t(f(A_t))\,d\mu(t) \leqslant kf(0)\mathbf{1} + \int_T \frac{f(\|A_t\|) - f(0)}{\|A_t\|}\phi_t(A_t)\,d\mu(t). \tag{14}$$

Especially, for $f(0) \leqslant 0$, *the inequality*

$$\int_T \phi_t(f(A_t))\,d\mu(t) \leqslant \int_T \frac{f(\|A_t\|)}{\|A_t\|}\phi_t(A_t)\,d\mu(t). \tag{15}$$

is valid.

Proof. This theorem follows from [7, Theorem 3.5] when we replace ϕ_t by $\frac{1}{k}\phi_t$, $t \in T$. $\square$

In the present context and by using subdifferentials we can give an estimation from below in the sense of Theorem 2.3. The following theorem is a generalization of [7, Theorem 3.8]. It follows from Theorem 2.2 applying LHS of (6) for a convex function f (or RHS of (6) for a concave function f).

THEOREM 2.9. *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Furthermore, let $f : [m,M] \to \mathbb{R}$, $g : [km,kM] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m,M]) \subset U$, $g([km,kM]) \subset V$, F is bounded and $f(y)+l(y)(t-y) \in U$ for every $y,t \in [m,M]$ where l is the subdifferential of f. If F is operator monotone in the first variable and f is convex on $[m,M]$, then*

$$F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t), g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \geqslant \inf_{km\leqslant z\leqslant kM} F\left[f(y)k+l(y)(z-yk), g(z)\right]\mathbf{1} \tag{16}$$

holds for every $y \in [m,M]$. In the dual case (when f is concave) the opposite inequality holds in (16) with sup *instead of* inf.

Proof. We only prove the convex case. Since f is convex we have $f(z) \geqslant f(y) + l(y)(z-y)$ for every $z,y \in [m,M]$. Thus, by putting $h_1(z) = f(y)+l(y)(z-y)$ in LHS of (6) we obtain (16). $\square$

Though $f(z) = \ln z$ is operator concave, the Schwarz inequality $\phi(f(x)) \leqslant f(\phi(x))$ does not hold in the case of non-unital ϕ. However, as applications of Corollary 2.4 and Theorem 2.9, we obtain the following corollary, which is a generalization of [6, Corollary 2.34].

COROLLARY 2.10. *Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as in Theorem 2.9 for $0 < m < M$. Then*

$$C_1\mathbf{1} \leqslant \int_T \phi_t\left(\ln(x_t)\right)d\mu(t) - \ln\left(\int_T \phi_t(x_t)d\mu(t)\right) \leqslant C_2\mathbf{1}, \tag{17}$$

where the constant $C_1 \equiv C_1(m,M,k)$

$$C_1 = \begin{cases} k\beta + \ln\left(\mathrm{e}/L(m,M)\right) & \text{for } km \leqslant L(m,M) \leqslant kM, \\ \ln\left(M^{k-1}/k\right) & \text{for } kM \leqslant L(m,M), \\ \ln\left(m^{k-1}/k\right) & \text{for } km \geqslant L(m,M), \end{cases}$$

the constant $C_2 \equiv C_2(m,M,k)$

$$C_2 = \begin{cases} \ln\left(\frac{L(m,M)^k k^{k-1}}{e^k m}\right) + \frac{m}{L(m,M)} & \textit{for} \;\; m \leqslant kL(m,M) \leqslant M \\ \ln\left(M^{k-1}/k\right) & \textit{for} \;\; kL(m,M) \geqslant M, \\ \ln\left(m^{k-1}/k\right) & \textit{for} \;\; kL(m,M) \leqslant m, \end{cases}$$

and the logarithmic mean $L(m,M)$ *is defined by* $L(m,M) = \frac{M-m}{\ln M - \ln m}$ *for* $M \neq m$ *and* $L(m,M) = m$ *for* $M = m$, β *is the constant* β_f *associated with the function* $f(z) = \ln z$.

Proof. We set $f(z) \equiv g(z) = \ln z$ in Corollary 2.4. Then we obtain the lower bound C_1 when we determine $\min\limits_{km \leqslant z \leqslant kM} (\alpha z + k\beta - \ln z)$.

Next, we shall obtain the upper bound C_2. We set $F(u,v) = u - v$ and $f(z) \equiv g(z) = \ln z$ in Theorem 2.9. We obtain

$$\begin{aligned} &\int_T \phi_t\left(\ln(x_t)\right) d\mu(t) - \ln\left(\int_T \phi_t(x_t) d\mu(t)\right) \\ &\qquad \leqslant \max\left\{\ln\left(\frac{y^k}{e^k km}\right) + \frac{km}{y}, \ln\left(\frac{y^k}{e^k kM}\right) + \frac{kM}{y}\right\}\mathbf{1} \end{aligned}$$

for every $y \in [m,M]$, since $h(z) = k\ln y + \frac{1}{y}(z - ky) - \ln z$ is a convex function and it implies that

$$\max_{km \leqslant z \leqslant kM} h(z) = \max\{h(km), h(kM)\}.$$

Now, if $m \leqslant kL(m,M) \leqslant M$, then we choose $y = kL(m,M)$. In this case we have $h(km) = h(kM)$. But, if $m \geqslant kL(m,M)$, then it follows $0 < k \leqslant 1$, which implies that $\max\{h(km), h(kM)\} = h(km)$ for every $y \in [m,M]$. In this case we choose $y = m$, since $h(y) = \ln\left(\frac{y^k}{e^k km}\right) + \frac{km}{y}$ is an increasing function in $[m,M]$. If $M \leqslant kL(m,M)$, then the proof is similar to above. □

By using subdifferentials, we also give generalizations of some results from [7, 3].

THEOREM 2.11. *Let* $(x_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with spectra in* $[m,M]$ *defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ, *and let* $(\phi_t)_{t\in T}$ *be a field of positive linear maps* $\phi_t : \mathscr{A} \to \mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^**-algebra* $\mathscr{B}$. *If the field* $t \to \phi_t(\mathbf{1})$ *is integrable with* $\int_T \phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ *for some positive scalar* k *and* $f : [m,M] \to \mathbb{R}$ *is a convex function then*

$$\begin{aligned} &f(y)k\mathbf{1} + l(y)\left(\int_T \phi_t(x_t) d\mu(t) - yk\mathbf{1}\right) \\ &\qquad \leqslant \int_T \phi_t\left(f(x_t)\right) d\mu(t) \\ &\qquad \leqslant f(x)k\mathbf{1} - x\int_T \phi_t\left(l(x_t)\right) d\mu(t) + \int_T \phi_t\left(l(x_t)x_t\right) d\mu(t) \end{aligned} \tag{18}$$

for every $x,y \in [m,M]$, where l is the subdifferential of f. In the dual case (f is concave) the opposite inequality holds.

Proof. We obtain this theorem by replacing ϕ_t by $\frac{1}{k}\phi_t$ in [7, Theorem 3.7]. For the sake of completeness we give the direct proof. Since f is convex in $[m,M]$, then for each $y \in [m,M]$ the inequality $f(x) \geqslant f(y)+l(y)(x-y)$ holds for every $x \in [m,M]$. By using the functional calculus in the variable x and applying the positive linear maps ϕ_t and integrating, we obtain LHS of (18). Next, since f is convex, then for each $x \in [m,M]$ the inequality $f(y) \leqslant f(x)-l(y)(x-y)$ holds for every $y \in [m,M]$. By using the functional calculus in the variable y, we obtain that $f(x_t) \leqslant f(x)\mathbf{1} - xl(x_t) + l(x_t)x_t$ holds for every $x \in [m,M]$ and $t \in T$. Applying the positive linear maps ϕ_t and integrating, we obtain RHS of (18). □

THEOREM 2.12. *Let $(x_t)_{t\in T}$ be a bounded continuous field of positive elements in a unital C^*-algebra $\mathscr{A}$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t\colon \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$ acting on a finite dimensional Hilbert space K, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Let $\|\cdot\|$ be unitarily invariant norm on $B(K)$ and let $f\colon [0,\infty) \to \mathbb{R}$ be an increasing function.*

1. *If $\|\mathbf{1}\| = 1$ and f is convex with $f(0) \leqslant 0$ then*

$$f\left(\frac{\|\int_T \phi_t(x_t)\,d\mu(t)\|}{k}\right) \leqslant \frac{\|\int_T \phi_t(f(x_t))\,d\mu(t)\|}{k}. \tag{19}$$

2. *If $\int_T \phi_t(x_t)\,d\mu(t) \leqslant \|\int_T \phi_t(x_t)\,d\mu(t)\|\mathbf{1}$ and f is concave then*

$$\frac{1}{k}\int_T \phi_t(f(x_t))\,d\mu(t) \leqslant f\left(\frac{\|\int_T \phi_t(x_t)\,d\mu(t)\|}{k}\right)\mathbf{1}. \tag{20}$$

Proof. We replace ϕ_t by $\frac{1}{k}\phi_t$ for $t \in T$ in [7, Theorem 3.9]. □

3. Ratio type inequalities

In this section, we consider the order among the following power functions of operators:

$$I_r(\mathbf{x},\phi) := \left(\int_T \phi_t\left(x_t^r\right)d\mu(t)\right)^{1/r} \qquad \text{if} \quad r \in \mathbf{R}\backslash\{0\}, \tag{21}$$

at these conditions: $(x_t)_{t\in T}$ is a bounded continuous field of positive operators in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ for some scalars $0 < m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and $(\phi_t)_{t\in T}$ is a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k.

In order to prove the ratio type order among power functions (21), we need some previous results given in the following two lemmas.

LEMMA 3.1. *Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ for some scalars $0<m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t:\mathscr{A}\to\mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t\to\phi_t(\mathbf{1})$ is integrable with $\int_T\phi_t(\mathbf{1})\,d\mu(t)=k\mathbf{1}$ for some positive scalar k.*

If $0<p\leqslant 1$, then

$$\int_T \phi_t\left(x_t^p\right)\,d\mu(t)\leqslant k^{1-p}\left(\int_T\phi_t(x_t)d\mu(t)\right)^p. \tag{22}$$

If $-1\leqslant p<0$ or $1\leqslant p\leqslant 2$, then the opposite inequality holds in (22).

Proof. We obtain this lemma by applying Theorem 2.1 for the function $f(z)=z^p$ and using the proposition that it is an operator concave function for $0<p\leqslant 1$ and an operator convex one for $-1\leqslant p<0$ and $1\leqslant p\leqslant 2$. □

The following lemma is a generalization of [8, Lemma 2].

LEMMA 3.2. *Assume that the conditions of Lemma* 3.1 *hold.*
If $0<p\leqslant 1$, then

$$k^{1-p}K(m,M,p)\left(\int_T\phi_t(x_t)d\mu(t)\right)^p\leqslant\int_T\phi_t\left(x_t^p\right)\,d\mu(t)\leqslant k^{1-p}\left(\int_T\phi_t(x_t)d\mu(t)\right)^p, \tag{23}$$

if $-1\leqslant p<0$ or $1\leqslant p\leqslant 2$, then

$$k^{1-p}\left(\int_T\phi_t(x_t)d\mu(t)\right)^p\leqslant\int_T\phi_t\left(x_t^p\right)\,d\mu(t)\leqslant k^{1-p}K(m,M,p)\left(\int_T\phi_t(x_t)d\mu(t)\right)^p, \tag{24}$$

if $p<-1$ or $p>2$, then

$$\begin{aligned}&k^{1-p}K(m,M,p)^{-1}\left(\int_T\phi_t(x_t)d\mu(t)\right)^p\\&\qquad\leqslant\int_T\phi_t\left(x_t^p\right)\,d\mu(t)\leqslant k^{1-p}K(m,M,p)\left(\int_T\phi_t(x_t)d\mu(t)\right)^p,\end{aligned} \tag{25}$$

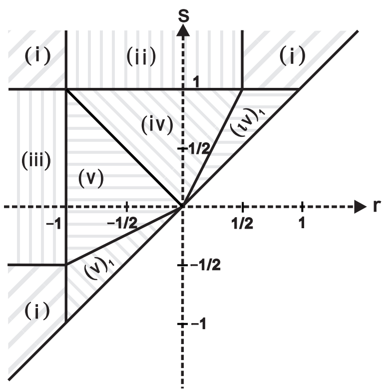
where a generalized Kantorovich constant $K(m,M,p)$ [6, §2.7] *is defined as*

$$K(m,M,p):=\frac{mM^p-Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\,\frac{M^p-m^p}{mM^p-Mm^p}\right)^p\quad\text{for all } p\in\mathbb{R}. \tag{$*$}$$

Proof. We obtain this lemma by applying Corollary 2.4 for the function $f(z) \equiv g(z) = z^p$ and choosing λ such that $C = 0$. $\square$

In the following theorem we give the ratio type order among power functions.

THEOREM 3.3. *Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ for some scalars $0 < m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar k. Let regions* (i) – (v)$_1$ *be as in Figure 1.*



(i)	r ≤ s, s ∉ (-1,1), r ∉ (-1,1) **or** 1/2 ≤ r ≤ 1 ≤ s **or** r ≤ -1 ≤ s ≤ -1/2,
(ii)	s ≥ 1, -1 < r < 1/2, r ≠ 0,
(iii)	r ≤ -1, -1/2 < s < 1, s ≠ 0,
(iv)	-s ≤ r < s/2, r ≠ 0, 0 < s ≤ 1,
(iv)$_1$	r ≤ s ≤ 2r, 0 < s ≤ 1,
(v)	r/2 < s ≤ -r, s ≠ 0, -1 ≤ r < 0,
(v)$_1$	2s ≤ r ≤ s, -1 ≤ r < 0.

Figure 1: *Regions in the (r,s)-plain*

If (r,s) in (i)*, then*

$$k^{\frac{s-r}{rs}}\, \Delta(h,r,s)^{-1}\, I_s(\mathbf{x},\phi) \leqslant I_r(\mathbf{x},\phi) \leqslant k^{\frac{s-r}{rs}}\, I_s(\mathbf{x},\phi),$$

if (r,s) in (ii) *or* (iii)*, then*

$$k^{\frac{s-r}{rs}}\, \Delta(h,r,s)^{-1}\, I_s(\mathbf{x},\phi) \leqslant I_r(\mathbf{x},\phi) \leqslant k^{\frac{s-r}{rs}}\, \Delta(h,r,s)\, I_s(\mathbf{x},\phi),$$

if (r,s) in (iv)*, then*

$$\begin{aligned} &k^{\frac{s-r}{rs}}\, \Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\, I_s(\mathbf{x},\phi) \leqslant I_r(\mathbf{x},\phi) \\ &\leqslant k^{\frac{s-r}{rs}}\, \min\{\Delta(h,r,1), \Delta(h,s,1)\Delta(h,r,s)\}\, I_s(\mathbf{x},\phi), \end{aligned}$$

if (r,s) in (v) *or* (iv)$_1$ *or* (v)$_1$*, then*

$$k^{\frac{s-r}{rs}}\, \Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\, I_s(\mathbf{x},\phi) \leqslant I_r(\mathbf{x},\phi) \leqslant k^{\frac{s-r}{rs}}\, \Delta(h,s,1)\, I_s(\mathbf{x},\phi),$$

where a generalized Specht ratio $\Delta(h,r,s)$ [6, § 2.7] *is defined as*

$$\Delta(h,r,s) = \left\{\frac{r(h^s - h^r)}{(s-r)(h^r-1)}\right\}^{1/s} \left\{\frac{s(h^r - h^s)}{(r-s)(h^s-1)}\right\}^{-1/r}, \qquad h = \frac{M}{m}. \tag{26}$$

Proof. This theorem follows from Lemma 3.2 by putting $p = s/r$ or $p = r/s$ and then using function order of positive operators cf. [6, Chapter 8]. We use the same technique as in the proof of [8, Theorem 11]. $\square$

4. Difference type inequalities

In order to prove the difference type order among power functions (21), we need some previous results given in the following lemma. It is a generalization of [8, Lemma 3].

LEMMA 4.1. *Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ for some scalars $0<m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k.*

If $0<p\leqslant 1$, then

$$\alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant k^{1-p}\left(\int_T \phi_t(x_t)d\mu(t)\right)^p, \tag{27}$$

if $-1\leqslant p<0$ or $1\leqslant p\leqslant 2$, then

$$k^{1-p}\left(\int_T \phi_t(x_t)d\mu(t)\right)^p \leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant \alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1}, \tag{28}$$

if $p<-1$ or $p>2$, then

$$py^{p-1}\int_T \phi_t(x_t)d\mu(t) + k(1-p)y^p\mathbf{1} \leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant \alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \tag{29}$$

for every $y\in[m,M]$. Constants α_p and β_p are the constants α_f and β_f associated with the function $f(z)=z^p$.

Proof. RHS of (27) and LHS of (28) are proven in Lemma 3.1. LHS of (27) and RHS of (28) and (29) follow from Corollary 2.4 for $f(z)=z^p$, $g(z)=z$ and $\lambda=\alpha_p$. LHS of (29) follows from LHS of (18) in Theorem 2.11 putting $f(y)=y^p$ and $l(y)=py^{p-1}$. $\square$

REMARK 4.2. Setting $y=(\alpha_p/p)^{1/(p-1)}\in[m,M]$ the inequality (29) gives

$$\begin{aligned}\alpha_p \int_T \phi_t(x_t)d\mu(t) &+ k(1-p)\left(\alpha_p/p\right)^{p/(p-1)}\mathbf{1}\\ &\leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant \alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1}\end{aligned} \tag{30}$$

for $p<-1$ or $p>2$.

Furthermore, setting $y = m$ or $y = M$ gives

$$\begin{aligned} & pm^{p-1}\int_T \phi_t(x_t)d\mu(t) + k(1-p)m^p\mathbf{1} \\ & \qquad \leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant \alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \end{aligned} \tag{31}$$

or

$$\begin{aligned} & pM^{p-1}\int_T \phi_t(x_t)d\mu(t) + k(1-p)M^p\mathbf{1} \\ & \qquad \leqslant \int_T \phi_t(x_t^p)d\mu(t) \leqslant \alpha_p \int_T \phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1}. \end{aligned} \tag{32}$$

We remark that the operator in LHS of (31) is positive for $p > 2$, since

$$\begin{aligned} 0 < km^p\mathbf{1} & \leqslant pm^{p-1}\int_T \phi_t(x_t)d\mu(t) + k(1-p)m^p\mathbf{1} \\ & \leqslant k(pm^{p-1}M + (1-p)m^p)\mathbf{1} < kM^p\mathbf{1} \end{aligned} \tag{33}$$

and the operator in LHS of (32) is positive for $p < -1$, since

$$\begin{aligned} 0 < kM^p\mathbf{1} & \leqslant pM^{p-1}\int_T \phi_t(x_t)d\mu(t) + k(1-p)M^p\mathbf{1} \\ & \leqslant k(pM^{p-1}m + (1-p)M^p)\mathbf{1} < km^p\mathbf{1}. \end{aligned} \tag{34}$$

(We have the inequality $pm^{p-1}M + (1-p)m^p < M^p$ in RHS of (33) and $pM^{p-1}m + (1-p)M^p < m^p$ in RHS of (34) by using Bernoulli's inequality.)

In the following theorem we give the difference type order among power functions.

THEOREM 4.3. *Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a unital C^*-algebra $\mathscr{A}$ with spectra in $[m,M]$ for some scalars $0 < m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Let regions* (i)$_1$ – (v)$_1$ *be as in Figure 2.*

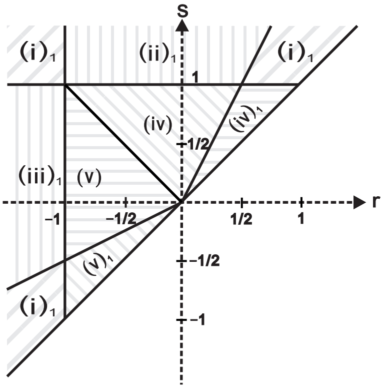
Then

$$C_2\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \leqslant C_1\mathbf{1}, \tag{35}$$

where constants $C_1 \equiv C_1(m,M,s,r,k)$ and $C_2 \equiv C_2(m,M,s,r,k)$ are

$$C_1 = \begin{cases} \widetilde{\Delta}_k, & \textit{for } (r,s) \textit{ in } \mathrm{(i)}_1 \textit{ or } \mathrm{(ii)}_1 \textit{ or } \mathrm{(iii)}_1; \\ \widetilde{\Delta}_k + \min\{C_k(s), C_k(r)\}, & \textit{for } (r,s) \textit{ in } \mathrm{(iv)} \textit{ or } \mathrm{(v)} \textit{ or } \mathrm{(iv)}_1 \textit{ or } \mathrm{(v)}_1; \end{cases}$$

$$C_2 = \begin{cases} (k^{1/s} - k^{1/r})m, & \textit{for } (r,s) \textit{ in } \mathrm{(i)}_1; \\ \widetilde{D}_k, & \textit{for } (r,s) \textit{ in } \mathrm{(ii)}_1; \\ \overline{D}_k, & \textit{for } (r,s) \textit{ in } \mathrm{(iii)}_1; \\ \max\left\{\widetilde{D}_k - C_k(s), \left(k^{1/s} - k^{1/r}\right) m - C_k(r)\right\}, & \textit{for } (r,s) \textit{ in } \mathrm{(iv)}; \\ \max\left\{\overline{D}_k - C_k(r), \left(k^{1/s} - k^{1/r}\right) m - C_k(s)\right\}, & \textit{for } (r,s) \textit{ in } \mathrm{(v)}; \\ (k^{1/s} - k^{1/r})m - \min\{C_k(r), C_k(s)\}, & \textit{for } (r,s) \textit{ in } \mathrm{(iv)}_1 \textit{ or } \mathrm{(v)}_1. \end{cases}$$



(i)$_1$	$r \leq -1,\ s \geq 1$ **or** $s/2 \leq r \leq s,\ s \geq 1$ **or** $r \leq s \leq r/2,\ r \leq -1$,
(ii)$_1$	$s \geq 1,\ -1 < r < s/2,\ r \neq 0$,
(iii)$_1$	$r \leq -1,\ r/2 < s < 1,\ s \neq 0$,
(iv)	$-s \leq r < s/2,\ r \neq 0,\ 0 < s \leq 1$,
(iv)$_1$	$r \leq s \leq 2r,\ 0 < s \leq 1$,
(v)	$r/2 < s \leq -r,\ s \neq 0,\ -1 \leq r < 0$,
(v)$_1$	$2s \leq r \leq s,\ -1 \leq r < 0$.

Figure 2: *Regions in the* (r,s)*-plain*

A constant $\widetilde{\Delta}_k \equiv \widetilde{\Delta}_k(m,M,r,s)$ *is*

$$\widetilde{\Delta}_k = \max_{\theta\in[0,1]} \left\{ k^{1/s}[\theta M^s + (1-\theta)m^s]^{1/s} - k^{1/r}[\theta M^r + (1-\theta)m^r]^{1/r} \right\},$$

a constant $\widetilde{D}_k \equiv \widetilde{D}_k(m,M,r,s)$ *is*

$$\widetilde{D}_k = \min\left\{ \left(k^{\frac{1}{s}} - k^{\frac{1}{r}}\right) m, k^{\frac{1}{s}} m \left(s\frac{M^r - m^r}{rm^r} + 1 \right)^{\frac{1}{s}} - k^{\frac{1}{r}} M \right\},$$

$\overline{D}_k \equiv \overline{D}_k(m,M,r,s) = -\widetilde{D}_k(M,m,s,r)$ *and a constant* $C_k(p) \equiv C_k(m,M,p)$ *is*

$$C_k(p) = k^{1/p} \cdot C(m^p, M^p, 1/p) \qquad \textit{for } p \neq 0,$$

where a constant $C(n,N,p)$ *is defined by*

$$C(n,N,p) = (p-1)\left(\frac{1}{p}\frac{N^p - n^p}{N-n}\right)^{p/(p-1)} + \frac{Nn^p - nN^p}{N-n} \qquad \textit{for all } p \in \mathbb{R} \tag{36}$$

(this is type of a generalized Kantorovich constant for difference, see [6, §2.7, Lemma 2.59]*).*

Proof. By the same technique as in the proof of [8, Theorem 7], we have this theorem. However, we give a proof for the sake of completeness. By Lemma 4.1 by putting $p = s/r$ or $p = r/s$ and then using the Löwner-Heinz inequality and the function order of positive operators, cf. [6, Chapter 8]:

$$\begin{aligned} &A \geqslant B > 0 \text{ and } \mathsf{Sp}(B) \subseteq [m,M] \text{ imply } A^p + C(m,M,p)\mathbf{1} \geqslant B^p > 0 \text{ for all } p > 1, \\ &A \geqslant B > 0 \text{ and } \mathsf{Sp}(A) \subseteq [m,M] \text{ imply } B^p + C(m,M,p)\mathbf{1} \geqslant A^p > 0 \text{ for all } p < -1, \end{aligned}$$

we have the following inequalities.

(a) If $r \leqslant s \leqslant -1$ or $1 \leqslant s \leqslant -r$ or $0 < r \leqslant s \leqslant 2r,\ s \geqslant 1$, then

$$\left(k^{1/s} - k^{1/r}\right) m\mathbf{1} \leqslant \left(k^{\frac{r-s}{rs}} - 1\right) I_r(\mathbf{x},\phi) \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \tag{37}$$
$$\leqslant \left(\widetilde{\alpha} \int_T \phi_t(x_t^r) d\mu(t) + k\widetilde{\beta}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) \leqslant \widetilde{\Delta}_k \mathbf{1}.$$

(b) If $0 < -r < s,\ s \geqslant 1$ or $0 < 2r < s,\ s \geqslant 1$, then

$$m\left(\tfrac{s}{r} m^{-r} \int_T \phi_t(x_t^r) d\mu(t) + k\tfrac{r-s}{r}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \tag{38}$$
$$\leqslant \left(\widetilde{\alpha} \int_T \phi_t(x_t^r) d\mu(t) + k\widetilde{\beta}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) \leqslant \widetilde{\Delta}_k \mathbf{1}.$$

(c) If $r \leqslant s,\ -1 \leqslant s < 0$ or $s \leqslant -r,\ 0 < s \leqslant 1$ or $0 < r \leqslant s \leqslant 2r,\ s \leqslant 1$, then

$$\left(\left(k^{1/s} - k^{1/r}\right) m - C_k(s)\right)\mathbf{1} \leqslant \left(k^{\frac{r-s}{rs}} - 1\right) I_r(\mathbf{x},\phi) - C_k(s)\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \tag{39}$$
$$\leqslant \left(\widetilde{\alpha} \int_T \phi_t(x_t^r) d\mu(t) + k\widetilde{\beta}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) + C_k(s)\mathbf{1} \leqslant \left(\widetilde{\Delta}_k + C_k(s)\right)\mathbf{1}.$$

(d) If $0 < -r < s \leqslant 1$ or $0 < 2r < s \leqslant 1$, then

$$m\left(\tfrac{s}{r} m^{-r} \int_T \phi_t(x_t^r) d\mu(t) + k\tfrac{r-s}{r}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) - C_k(s)\mathbf{1} \tag{40}$$
$$\leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi)$$
$$\leqslant \left(\widetilde{\alpha} \int_T \phi_t(x_t^r) d\mu(t) + k\widetilde{\beta}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) + C_k(s)\mathbf{1} \leqslant \left(\widetilde{\Delta}_k + C_k(s)\right)\mathbf{1}.$$

Moreover, we can obtain the following inequalities:

(a_1) If $1 \leqslant r \leqslant s$ or $-s \leqslant r \leqslant -1$ or $2s \leqslant r \leqslant s < 0,\ r \leqslant -1$, then

$$\widetilde{\Delta}_k \mathbf{1} \geqslant I_s(\mathbf{x},\phi) - \left(\bar{\alpha} \int_T \phi_t(x_t^s) d\mu(t) + k\bar{\beta}\mathbf{1}\right)^{1/r} \geqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \tag{41}$$
$$\geqslant \left(1 - k^{\frac{s-r}{rs}}\right) I_s(\mathbf{x},\phi) \geqslant \left(k^{1/s} - k^{1/r}\right) m\mathbf{1}.$$

(b_1) If $r < -s < 0,\ r \leqslant -1$ or $r < 2s < 0,\ r \leqslant -1$, then

$$\widetilde{\Delta}_k \mathbf{1} \geqslant I_s(\mathbf{x},\phi) - \left(\bar{\alpha} \int_T \phi_t(x_t^s) d\mu(t) + k\bar{\beta}\mathbf{1}\right)^{1/r} \geqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \tag{42}$$
$$\geqslant I_s(\mathbf{x},\phi) - M\left(\tfrac{r}{s} M^{-s} \int_T \phi_t(x_t^s) d\mu(t) + k\tfrac{s-r}{s}\mathbf{1}\right)^{1/r}.$$

(c_1) If $r \leqslant s,\ 0 < r \leqslant 1$ or $-s \leqslant r,\ -1 \leqslant r < 0$ or $2s \leqslant r \leqslant s < 0,\ r \geqslant -1$, then

$$(\widetilde{\Delta}_k + C_k(r))\mathbf{1} \geqslant I_s(\mathbf{x},\phi) - \left(\bar{\alpha} \int_T \phi_t(x_t^s) d\mu(t) + k\bar{\beta}\mathbf{1}\right)^{1/r} + C_k(r)\mathbf{1} \tag{43}$$
$$\geqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \geqslant \left(1 - k^{\frac{s-r}{rs}}\right) I_s(\mathbf{x},\phi) - C_k(r)\mathbf{1} \geqslant \left(\left(k^{1/s} - k^{1/r}\right) m - C_k(r)\right)\mathbf{1}.$$

(d_1) If $-1 \leqslant r < -s < 0$ or $-1 \leqslant r < 2s < 0$, then

$$(\widetilde{\Delta}_k + C_k(r))\mathbf{1} \geqslant I_s(\mathbf{x},\phi) - \left(\bar{\alpha}\int_T \phi_t(x_t^s)d\mu(t) + k\bar{\beta}\mathbf{1}\right)^{1/r} + C_k(r)\mathbf{1} \tag{44}$$
$$\geqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \geqslant I_s(\mathbf{x},\phi) - M\left(\tfrac{r}{s}M^{-s}\int_T \phi_t(x_t^s)d\mu(t) + k\tfrac{s-r}{s}\mathbf{1}\right)^{1/r} - C_k(r)\mathbf{1},$$

where we denote

$$\widetilde{\alpha} = \tfrac{M^s - m^s}{M^r - m^r}, \quad \widetilde{\beta} = \tfrac{M^r m^s - M^s m^r}{M^r - m^r}, \quad \bar{\alpha} = \tfrac{M^r - m^r}{M^s - m^s}, \quad \bar{\beta} = \tfrac{M^s m^r - M^r m^s}{M^s - m^s},$$
$$C(km^s, kM^s, 1/s) \;=\; k^{1/s}\, C(m^s, M^s, 1/s) = C_k(s),$$
$$\widetilde{\Delta}_k = \max_{z\in\bar{T}_1}\left\{k^{1/s}\left(\widetilde{\alpha}z + \widetilde{\beta}\right)^{1/s} - k^{1/r}z^{1/r}\right\} = \max_{z\in\bar{T}_2}\left\{k^{1/s}z^{1/s} - k^{1/r}\left(\bar{\alpha}z + \bar{\beta}\right)^{1/r}\right\},$$

and $\bar{T}_1$ and $\bar{T}_2$ denote the closed intervals joining m^r to M^r and m^s to M^s, respectively.

We will determine lower bounds in LHS of (b) and (d), in RHS of (b_1) and (d_1). On LHS of inequalities (38) and (40) we can apply the following inequality

$$m\left(\tfrac{s}{r}m^{-r}\int_T \phi_t(x_t^r)d\mu(t) + k\tfrac{r-s}{r}\mathbf{1}\right)^{1/s} - I_r(\mathbf{x},\phi) \tag{45}$$
$$\geqslant \min_{z\in\bar{T}_1}\left\{k^{1/s}m\left(\tfrac{s}{r}m^{-r}z + 1 - \tfrac{s}{r}\right)^{1/s} - k^{1/r}z^{1/r}\right\}\mathbf{1} = \widetilde{D}_k\mathbf{1}.$$

Using substitution $z = rm^r\left(x - \frac{1}{s}\right)$, finding the minimum of

$$h(z) = k^{1/s}m\left(\frac{s}{r}m^{-r}z + \frac{r-s}{r}\right)^{1/s} - k^{1/r}z^{1/r} \text{ on } \bar{T}_1$$

is equivalent to finding the minimum of $h_1(x) = k^{1/s}m\left(s(x-\frac{1}{r})\right)^{1/s} - k^{1/r}m\left(r(x-\frac{1}{s})\right)^{1/r}$ on $\bar{T} = [\frac{1}{s}+\frac{1}{r}, \frac{1}{s}+\frac{1}{r}\frac{M^r}{m^r}]$, where $r < s,\ s > 0$. The minimum value of the function h_1 on $\bar{T}$ is achieved at one end point of this interval. Really, functions h_1 and h_1' are continuous on $\bar{T}$. If there is a stationary point x_0 of h_1 in $\left(\frac{1}{s}+\frac{1}{r}, \frac{1}{s}+\frac{1}{r}\frac{M^r}{m^r}\right)$ then $h_1(x_0)$ is the maximum value, since $h_1''(x_0) = k^{\frac{1}{s}}m\left(s(x_0-\frac{1}{r})\right)^{1/s-2}\left(r(x_0-\frac{1}{s})\right)^{-1}(r-s)(x_0 + 1 - \frac{r+s}{rs}) < 0$. It follows that

$$\min_{z\in\bar{T}_1} h(z) = \min_{x\in\bar{T}} h_1(x) = \min\left\{h_1\left(\frac{1}{s}+\frac{1}{r}\right), h_1\left(\frac{1}{s}+\frac{1}{r}\frac{M^r}{m^r}\right)\right\} = \widetilde{D}_k.$$

So in the case (b) we obtain:

$$\widetilde{D}_k\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \leqslant \widetilde{\Delta}_k\mathbf{1} \tag{46}$$

and in the case (d) we obtain:

$$\left(\widetilde{D}_k - C_k(s)\right)\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \leqslant \left(\widetilde{\Delta}_k + C_k(s)\right)\mathbf{1}. \tag{47}$$

Similarly, for the RHS of (42) we obtain

$$
\begin{aligned}
& I_s(\mathbf{x},\phi) - M\left(\frac{r}{s}M^{-s}\int_T \phi_t(x_t^r)\,d\mu(t) + k\frac{s-r}{s}\mathbf{1}\right)^{1/r} \\
\geqslant & \min_{z\in \bar{T}_2}\left\{k^{1/s}z^{1/s} - k^{1/r}M\left(\frac{r}{s}M^{-s}z + 1 - \frac{r}{s}\right)^{1/r}\right\}\mathbf{1} \\
= & \min\left\{k^{1/s}m - k^{1/r}M\left(\frac{r}{s}\frac{m^s}{M^s} + 1 - \frac{r}{s}\right)^{1/r}, \left(k^{1/s} - k^{1/r}\right)M\right\}\mathbf{1} \\
= & \overline{D}_k\mathbf{1}.
\end{aligned}
$$

So in the case (b_1) we obtain:

$$
\overline{D}_k\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \leqslant \widetilde{\Delta}_k\mathbf{1} \tag{48}
$$

and in the case (d_1) we obtain:

$$
\left(\overline{D}_k - C_k(r)\right)\mathbf{1} \leqslant I_s(\mathbf{x},\phi) - I_r(\mathbf{x},\phi) \leqslant \left(\widetilde{\Delta}_k + C_k(r)\right)\mathbf{1}. \tag{49}
$$

Finally, we can obtain desired bounds C_1 and C_2 in (35), taking into account that (37) holds in the region (i)$_1$, (46) holds in (ii)$_1$, (48) holds in (iii)$_1$, (47) and (43) hold in (iv), (39) and (49) hold in (v), (39) and (43) hold in (iv)$_1$ and (v)$_1$. $\square$

REMARK 4.4. If we replace $(\phi_t)_{t\in T}$ by $(\frac{1}{k}\phi_t)_{t\in T}$ in Theorem 3.3 and Theorem 4.3 then we can obtain the order among operator means $M_r(\mathbf{x},\phi) := \left(\int_T \frac{1}{k}\,\phi_t\,(x_t^r)\,d\mu(t)\right)^{1/r}$, $r\in\mathbf{R}\backslash\{0\}$. The order among these means in the discrete case $T=\{1,\ldots,n\}$ is given in [8, Theorem 11] and [8, Theorem 7].

Note that in this case, for difference type inequalities we have $\widetilde{D}_k = \widetilde{D}_1 = m\left(s\frac{M^r-m^r}{rm^r}+1\right)^{\frac{1}{s}} - M$ and we can choose better bounds using that $C(m^r,M^r,1/r) \geqslant C(m^s,M^s,1/s)$ for $r\leqslant s$ and $M>m>0$ (see [8, Lemma 8]).

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(Received July 20, 2009)

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