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FURUTA INEQUALITY AND q–HYPONORMAL OPERATORS

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Abstract. This paper is to consider Furuta type inequalities and q-hyponormal operators. It is shown that the complete form and original form of Furuta inequality are equivalent to each other. Forward, we prove that the complete form and original form of Furuta inequality are equivalent to the order relations among Aluthge transforms on q-hyponormal operators. Lastly, a simplified and short proof of the order structure on powers of q-hyponormal operators is shown.

1. Introduction

A capital letter (such as T) means a bounded linear operator on a Hilbert space. $T \geqslant 0$ and $T > 0$ mean a positive operator and an invertible positive operator respectively. As a celebrated development of the classical Loewner-Heinz inequality ($A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for any $1 \geqslant p \geqslant 0$), Furuta [8] provided a kind of order preserving operator inequality.

THEOREM 1.1. (Furuta inequality, [8]) *Let $r \geqslant 0$, $p > 0$ and $A \geqslant B \geqslant 0$, then*

$$\left(B^{r/2}A^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geqslant \left(B^{r/2}B^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}},$$
$$\left(A^{r/2}A^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}}.$$

Tanahashi [21] proved the optimality of the outer exponent $\min\{1,p\}+r$, see [9] for related topics of Furuta inequality. In order to establish the order structure on powers of operators [29], the complete form of Furuta inequality was shown in [31].

THEOREM 1.2. (Complete form, [31]) *Let $q > 0$, $r \geqslant 0$, $p > p_0 > 0$ and $s(q) = \min\left\{p, 2p_0 + \min\{q,r\}\right\}$. Then $A \geqslant 0$ and $B \geqslant 0$ such that $A^q \geqslant B^q$ ensure*

$$(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{s(q)+r}{p_0+r}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{s(q)+r}{p+r}}.$$

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We call the theorem above the complete form of Furuta inequality because the case $p_0 = q = 1$ of it implies the essential part ($p > 1$) of Furuta inequality for $\frac{q+r}{s(1)+r} \in (0,1]$ by the Loewner-Heinz inequality. For convenience, we call Furuta inequality (Theorem 1.1) the original form of Furuta inequality. An equivalent relation between the complete form and original form of Furuta inequality will be proved in section 2 (Theorem 2.5).

It is known that there are many applications of Furuta type inequalities to the theory of q-hyponormal operators where $q > 0$. An operator T is called a q-hyponormal operator (introduced by [3, 6, 7]) if $(T^*T)^q \geqslant (TT^*)^q$, where T^* is the adjoint operator of T. If $q = 1$, T is called a hyponormal operator ([12]) and if $q = 1/2$, T is called a semi-hyponormal operator ([23]). It is clear that every q_2-hyponormal operator is q_1-hyponormal for $0 < q_1 \leqslant q_2$ by Loewner-Heinz inequality.

Hyponormal operators have been studied by many authors. See [13, 20, 24] for related topics and basic properties of hyponormal operators. In order to discuss q-hyponormal operators, Aluthge [1] introduced the so-called Aluthge transform $T(1/2,1/2) = |T|^{1/2}U|T|^{1/2}$ where U and $|T|$ are polar factors of T, and researched the Aluthge transform on q-hyponormal operators by using Furuta inequality. Aluthge's result [1, Theorem 1-2] on Aluthge transform $T(1/2,1/2)$ was extended to the (generalized) Aluthge transform $T(p,r) = |T|^p U|T|^r$ where $p > 0$ and $r > 0$.

THEOREM 1.3. ([1, 14, 28]) *Let $p > 0$, $r > 0$ and $q > 0$. If T is a q-hyponormal operator and $\gamma(q,p,r) = \min\{q+p, q+r, p+r\}$, then*

$$\big((T(p,r))^*T(p,r)\big)^{\frac{\gamma(q,p,r)}{p+r}} \geqslant \big(T(p,r)(T(p,r))^*\big)^{\frac{\gamma(q,p,r)}{p+r}}. \tag{1.1}$$

Moreover, the outer exponent $\gamma(q,p,r)$ in the Theorem above is optimal in the following sense.

THEOREM 1.4. ([26]) *For each $p > 0$, $r > 0$, $q > 0$ and $\alpha > 1$, there exists a q-hyponormal operator T such that*

$$\big((T(p,r))^*T(p,r)\big)^{\frac{\gamma(q,p,r)}{p+r}\alpha} \ngeqslant \big(T(p,r)(T(p,r))^*)^*\big)^{\frac{\gamma(q,p,r)}{p+r}\alpha}.$$

The Aluthge transform is an effective tool in operator theory and has received much attention in recent years, see [4, 5, 11, 18, 19, 22].

[31, Theorem 3.3] implies that the order relations among Aluthge transforms on q-hyponormal operators are determined by the complete form of Furuta inequality. In section 3, the converse will be proved, that is, the order relations among Aluthge transforms also deduces Furuta inequality via shift operator. These imply that Furuta inequality and the order relations among Aluthge transforms are equivalent to each other.

It is well known that if T is a hyponormal operator, T^2 is not hyponormal in general [13, Problem 209]. [2] proved that, for a q-hyponormal operator T where $0 < q \leqslant 1$, T^n is $\frac{q}{n}$-hyponormal. This result is a breakthrough in the study of [13, Problem 209]. Soon, some researchers obtained more precise results [10, 15, 25].

Inspired by the problem of powers of hyponormal operators, the order structure on powers of operators was introduced in [29]. The order structure on powers of operators consists of same-side structure and different-side structure. The same-side structure means the order relations between $T^{*^{n+m}}T^{n+m}$ and $T^{*^n}T^n$ (or $T^nT^{*^n}$ and $T^{n+m}T^{*^{n+m}}$), and the different-side structure means the order relations between $T^{*^m}T^m$ and $T^nT^{*^n}$ where m, n are positive integers. Therefore the original problem of powers of hyponormal operators belongs to different-side structure on powers of hyponormal operators. In section 4, a simplified, short and self contained proof (only one page) of the structure on powers of q-hyponormal operators is given.

The order structure among Aluthge transforms on operators can be defined in a manner similar to the order structure on powers of operators, that is, the same-side structure means the order relations between $(T(p,r))^*T(p,r)$ and $(T(p_0,r))^*T(p_0,r)$ (or $T(p,r)(T(p,r))^*$ and $T(p,r_0)(T(p,r_0))^*$), and the different-side structure means the relations between $(T(p,r))^*T(p,r)$ and $T(p,r)(T(p,r))^*$ where $p > p_0 > 0$, $r > r_0 > 0$.

In this paper, section 2 is devoted to the equivalent relations between the complete form and original form of Furuta inequality. In section 3, the equivalent relations between the Furuta type inequalities and the order structure among Aluthge transforms on q-hyponormal operators are obtained. At the end, we give a simplified proof of the structure on powers of q-hyponormal operators.

2. Furuta type inequalities

Recall that $s(q) = \min\{p, 2p_0 + \min\{q,r\}\}$. In order to prove Theorem 2.5, Theorem 2.1 is proved in advance.

THEOREM 2.1. *Let $r > 0$, $0 < p_0 < p$, $A \geqslant 0$ and $B \geqslant 0$.*

(1) *If* $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$, *then for each r, p_0 and p, the following inequalities are equivalent to each other:*

$$\left(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}} \geqslant \left(B^{\frac{p_0}{2}}B^rB^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}}, \tag{2.1}$$

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{p+r}{p_0+r}} \geqslant (A^{r/2}B^pA^{r/2})^{\frac{p+r}{p+r}}. \tag{2.2}$$

In particular, (2.1) implies (2.2) without the condition $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$.

(2) *If* $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$, *then for each r, p_0 and p, the following inequalities are equivalent to each other:*

$$\left(A^{\frac{p_0}{2}}B^rA^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}} \leqslant \left(A^{\frac{p_0}{2}}A^rA^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}}, \tag{2.3}$$

$$(B^{r/2}A^{p_0}B^{r/2})^{\frac{p+r}{p_0+r}} \leqslant (B^{r/2}A^pB^{r/2})^{\frac{p+r}{p+r}}. \tag{2.4}$$

In particular, (2.3) implies (2.4) without the condition $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$.

Theorem 2.1 says that the order relations between $B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}}$ and $B^{\frac{p_0}{2}}B^rB^{\frac{p_0}{2}}$ is equivalent to the order relations between $A^{r/2}B^{p_0}A^{r/2}$ and $A^{r/2}B^pA^{r/2}$ under some kernel conditions.

Theorem 2.1 can be regarded as a parallel result to the following result which implies that class $A(p,r)$ coincides with class $wA(p,r)$ [16, Theorem 3]. For $A \geqslant 0$, A^0 means the projection $P_{(\ker A)^\perp}$.

THEOREM 2.2. ([16, 17]) *Let $r>0$, $0 \leqslant p_0 < p$, $A \geqslant 0$ and $B \geqslant 0$.*

(1) *If $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$, then for each r, p_0 and p, the following inequalities are equivalent to each other:*

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}} \geqslant \left(B^{\frac{p}{2}}B^rB^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}}, \tag{2.5}$$

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{p_0+r}{p_0+r}} \geqslant (A^{r/2}B^{p}A^{r/2})^{\frac{p_0+r}{p+r}}. \tag{2.6}$$

In particular, (2.5) implies (2.6) without the condition $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$.

(2) *For each r, p_0 and p, the following inequalities are equivalent to each other:*

$$\left(A^{\frac{p}{2}}B^rA^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}} \leqslant \left(A^{\frac{p}{2}}A^rA^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}}, \tag{2.7}$$

$$(B^{r/2}A^{p_0}B^{r/2})^{\frac{p_0+r}{p_0+r}} \leqslant (B^{r/2}A^{p}B^{r/2})^{\frac{p_0+r}{p+r}}. \tag{2.8}$$

Note that the outer exponent $p+r$ in (2.2) and (2.4) is greater than the outer exponent p_0+r in (2.6) and (2.8), and the outer exponent $p+r$ is optimal in Theorem 2.5 ([31, Theorem 1.4]).

LEMMA 2.3. ([16]) *Let X be an operator, S and T self-adjoint operators such that*

$$X^*SX \geqslant X^*TX.$$

If $\ker X^* \subseteq \ker S$ and $\ker X^* \subseteq \ker T$, then $S \geqslant T$.

Proof of Theorem 2.1. $(2.1) \Rightarrow (2.2)$. By [9, p. 129], (2.2) is equivalent to

$$A^{\frac{r}{2}}B^{\frac{p_0}{2}}(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}})^{\frac{p-p_0}{p_0+r}}B^{\frac{p_0}{2}}A^{\frac{r}{2}} \geqslant A^{\frac{r}{2}}B^{\frac{p_0}{2}}B^{p-p_0}B^{\frac{p_0}{2}}A^{\frac{r}{2}}. \tag{2.9}$$

By (2.1), (2.9) holds.

$(2.2) \Rightarrow (2.1)$. (2.9) follows by (2.2). By Lemma 2.3 and $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$, it is enough to prove $\ker(AB^{\frac{p_0}{2}}) = \ker(A^{\frac{r}{2}}B^{\frac{p_0}{2}})$. In fact,

$$x \in \ker(AB^{\frac{p_0}{2}}) \Leftrightarrow B^{\frac{p_0}{2}}x \in \ker A \Leftrightarrow B^{\frac{p_0}{2}}x \in \ker(A^{\frac{r}{2}}) \Leftrightarrow x \in \ker(A^{\frac{r}{2}}B^{\frac{p_0}{2}}).$$

So that $\ker(A^{\frac{r}{2}}B^{\frac{p_0}{2}}) = \ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$.

$(2.3) \Rightarrow (2.4)$. By [9, p. 129], (2.4) is equivalent to

$$B^{\frac{r}{2}}A^{\frac{p_0}{2}}(A^{\frac{p_0}{2}}B^rA^{\frac{p_0}{2}})^{\frac{p-p_0}{p_0+r}}A^{\frac{p_0}{2}}B^{\frac{r}{2}} \leqslant B^{\frac{r}{2}}A^{\frac{p_0}{2}}A^{p-p_0}A^{\frac{p_0}{2}}B^{\frac{r}{2}}. \tag{2.10}$$

By (2.3), (2.10) holds.

(2.4) $\Rightarrow$ (2.3). (2.10) follows by (2.4). By Lemma 2.3 and $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$, it is sufficient to prove $\ker(BA^{\frac{p_0}{2}}) = \ker(B^{\frac{r}{2}}A^{\frac{p_0}{2}})$. In fact,

$$x \in \ker(BA^{\frac{p_0}{2}}) \Leftrightarrow A^{\frac{p_0}{2}}x \in \ker B \Leftrightarrow A^{\frac{p_0}{2}}x \in \ker(B^{\frac{r}{2}}) \Leftrightarrow x \in \ker(B^{\frac{r}{2}}A^{\frac{p_0}{2}}).$$

So that $\ker(B^{\frac{r}{2}}A^{\frac{p_0}{2}}) = \ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$. Hence the proof is complete. $\square$

We remark that the proofs of Theorem 2.1 and 2.2 are different: the proof of Theorem 2.1 depends on Lemma 2.3 and proof of Theorem 2.2 depends on the following result.

LEMMA 2.4. ([17]) *Let $\alpha \in (0,1]$, $A \geqslant 0$ and $B \geqslant 0$, then*

$$\lim_{\varepsilon \to +0} A^{1/2}B^{1/2}\big((B^{1/2}AB^{1/2})^{\alpha} + \varepsilon I\big)^{-1}B^{1/2}A^{1/2} = (A^{1/2}BA^{1/2})^{1-\alpha}.$$

Note that the assertion that (2.2) ensures (2.1) is not true without the condition $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$ (cf. [17, Remark 1]): Let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $a > 1$, then $\ker(AB^{\frac{p_0}{2}}) \nsubseteq \ker B$ and

$$\big(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}}\big)^{\frac{p-p_0}{r+p_0}} = \begin{pmatrix} a^{\frac{r(p-p_0)}{p_0+r}} & 0 \\ 0 & 0 \end{pmatrix} \ngeqslant \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \big(B^{\frac{p_0}{2}}B^rB^{\frac{p_0}{2}}\big)^{\frac{p-p_0}{r+p_0}},$$

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{p+r}{p_0+r}} = \begin{pmatrix} a^{\frac{r(p+r)}{p_0+r}} & 0 \\ 0 & 0 \end{pmatrix} \geqslant \begin{pmatrix} a^r & 0 \\ 0 & 0 \end{pmatrix} = (A^{r/2}B^pA^{r/2})^{\frac{p+r}{p+r}}.$$

Though (2.8) implies (2.7) without the condition $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$, it is unknown if (2.4) ensures (2.3) is true without $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$.

THEOREM 2.5. *Let $r > 0$, $0 < p_0 < p$, $A \geqslant B \geqslant 0$. If $p \leqslant s(1)$, then for each r, p_0 and p, the following inequalities hold and they are equivalent to each other:*

$$\big(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}}\big)^{\frac{p-p_0}{r+p_0}} \geqslant \big(B^{\frac{p_0}{2}}B^rB^{\frac{p_0}{2}}\big)^{\frac{p-p_0}{r+p_0}},$$

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{p+r}{p_0+r}} \geqslant (A^{r/2}B^pA^{r/2})^{\frac{p+r}{p+r}}.$$

Proof. The two inequalities in the theorem follow by the original form and complete form of Furuta inequality respectively. Note that $A \geqslant B \geqslant 0$ implies $\ker A \subseteq \ker B$, by Theorem 2.1, the proof is complete if $\ker A \subseteq \ker B$ ensures $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$. In fact,

$$x \in \ker(AB^{\frac{p_0}{2}}) \Leftrightarrow B^{\frac{p_0}{2}}x \in \ker A \Rightarrow B^{\frac{p_0}{2}}x \in \ker B \Leftrightarrow x \in \ker(B^{\frac{p_0}{2}+1}) = \ker B. \qquad \square$$

Theorem 2.5 says that the original form and complete form of Furuta inequality are equivalent to each other.

A fact should be pointed out that Theorem 2.5 can be proved without using Theorem 2.1 if $A \geqslant B \geqslant 0$, A and B are invertible.

3. Structure among Aluthge transforms implies Furuta inequality

THEOREM 3.1. *Let $p > p_0 > 0$, $r > r_0 > 0$, $q > 0$, $s(q) = \min\{p, 2p_0 + \min\{q,r\}\}$ and $\widetilde{s}(q) = \min\{r, 2r_0 + \min\{q,p\}\}$. For each p, p_0, r, r_0 and q, the following assertions* (3.1)–(3.4) *imply* (3.5)–(3.8) *respectively.*

(1) *If T is a q-hyponormal operator, then*

$$\big((T(p,r))^*T(p,r)\big)^{\frac{s(q)+r}{p+r}} \geqslant \big((T(p_0,r))^*T(p_0,r)\big)^{\frac{s(q)+r}{p_0+r}}, \tag{3.1}$$

$$\big((T(p,r))^*T(p,r)\big)^{\frac{\min\{p,q\}+r}{p+r}} \geqslant (T^*T)^{\min\{p,q\}+r}, \tag{3.2}$$

$$\big(T(p,r)(T(p,r))^*\big)^{\frac{\widetilde{s}(q)+p}{r+p}} \leqslant \big(T(p,r_0)(T(p,r_0))^*\big)^{\frac{\widetilde{s}(q)+p}{r_0+p}}, \tag{3.3}$$

$$\big(T(p,r)(T(p,r))^*\big)^{\frac{\min\{r,q\}+p}{r+p}} \leqslant (T^*T)^{\min\{r,q\}+p}. \tag{3.4}$$

(2) *If $A \geqslant 0$, $B \geqslant 0$ such that $A^q \geqslant B^q$, then*

$$(B^{r/2}A^pB^{r/2})^{\frac{s(q)+r}{p+r}} \geqslant (B^{r/2}A^{p_0}B^{r/2})^{\frac{s(q)+r}{p_0+r}}, \tag{3.5}$$

$$(B^{r/2}A^pB^{r/2})^{\frac{\min\{p,q\}+r}{p+r}} \geqslant B^{\min\{p,q\}+r}, \tag{3.6}$$

$$(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\widetilde{s}(q)+p}{r+p}} \leqslant (A^{\frac{p}{2}}B^{r_0}A^{\frac{p}{2}})^{\frac{\widetilde{s}(q)+p}{r_0+p}}, \tag{3.7}$$

$$(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\min\{r,q\}+p}{r+p}} \leqslant A^{\min\{r,q\}+p}. \tag{3.8}$$

Theorem 3.1 says that Furuta inequality (Theorem 1.1–1.2) is equivalent to the same-side structure among Aluthge transforms.

(1.1) follows by (3.2) and (3.4) easily for $\gamma(q,p,r) = \min\{\min\{p,q\}+r, \min\{r,q\} + p\}$. Theorem 3.1 (1) is a direct result of the complete form of Furuta inequality ([31, Theorem 3.3]).

In order to prove the result, the following lemma will be needed.

LEMMA 3.2. (Shift operator, [26, 27, 29, 30]) *For $A \geqslant 0$ and $B \geqslant 0$ on a Hilbert space $\mathscr{H}$, define operators U and D on $\bigoplus_{k=-\infty}^{\infty}\mathscr{H}_k$ where $\mathscr{H}_k \cong \mathscr{H}$ as follows:*

$$U = \begin{pmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & 1 & (0) & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}, \quad D = \begin{pmatrix} \ddots & & & & \\ & B^{\frac{1}{2}} & & & \\ & & (A^{\frac{1}{2}}) & & \\ & & & A^{\frac{1}{2}} & \\ & & & & \ddots \end{pmatrix}$$

where $(\cdot)$ shows the place of the $(0,0)$ matrix element, and $T = UD$. Then the following assertions hold for each $q > 0$, $p > p_0 > 0$, $r > r_0 > 0$ and $\beta > 0$:

(1) *T is q-hyponormal if and only if $A^q \geqslant B^q$.*

(2) $(|T^*|^r|T|^{2p}|T^*|^r)^\beta \geqslant |T^*|^{2(p+r)\beta}$ *if and only if* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^\beta \geqslant B^{(p+r)\beta}$.

(3) $|T|^{2(p+r)\beta} \geqslant (|T|^p|T^*|^{2r}|T|^p)^\beta$ *if and only if* $A^{(p+r)\beta} \geqslant (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^\beta$.

(4) $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{\beta}{p+r}} \geqslant (|T^*|^r|T|^{2p_0}|T^*|^r)^{\frac{\beta}{p_0+r}}$ *if and only if*

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\beta}{p+r}} \geqslant (B^{\frac{r}{2}}A^{p_0}B^{\frac{r}{2}})^{\frac{\beta}{p_0+r}}.$$

(5) $(|T|^p|T^*|^{2r_0}|T|^p)^{\frac{\beta}{p+r_0}} \geqslant (|T|^p|T^*|^{2r}|T|^p)^{\frac{\beta}{p+r}}$ *if and only if*

$$(A^{\frac{p}{2}}B^{r_0}A^{\frac{p}{2}})^{\frac{\beta}{p+r_0}} \geqslant (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\beta}{p+r}}.$$

(6) $(T^{*^{n+m}}T^{n+m})^{\frac{\beta}{n+m}} \geqslant (T^{*^n}T^n)^{\frac{\beta}{n}}$ *if and only if*

$$\left(B^{l/2}A^{n+m-l}B^{l/2}\right)^{\frac{\beta}{n+m}} \geqslant \left(B^{l/2}A^{n-l}B^{l/2}\right)^{\frac{\beta}{n}} \text{ where } l = 1,\dots,n-1,$$
$$\left(B^{l/2}A^{n+m-l}B^{l/2}\right)^{\frac{\beta}{n+m}} \geqslant B^\beta \text{ where } l = n,\dots,n+m-1.$$

(7) $(T^nT^{n^*})^{\frac{\beta}{n}} \geqslant (T^{n+m}T^{n+m^*})^{\frac{\beta}{n+m}}$ *if and only if*

$$\left(A^{j/2}B^{n-j}A^{j/2}\right)^{\frac{\beta}{n}} \geqslant \left(A^{j/2}B^{n+m-j}A^{j/2}\right)^{\frac{\beta}{n+m}} \text{ where } j = 1,2,\dots,n-1,$$
$$A^\beta \geqslant \left(A^{j/2}B^{n+m-j}A^{j/2}\right)^{\frac{\beta}{n+m}} \text{ where } j = n,n+1,\dots,n+m-1.$$

(8) $(T^{*^m}T^m)^{\frac{\beta}{m}} \geqslant (T^nT^{*^n})^{\frac{\beta}{n}}$ *if and only if*

$$A^\beta \geqslant B^\beta \text{ and}$$
$$(B^{\frac{l}{2}}A^{m-l}B^{\frac{l}{2}})^{\frac{\beta}{m}} \geqslant B^\beta \text{ where } l = 1,2,...,m,$$
$$A^\beta \geqslant (A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}})^{\frac{\beta}{n}} \text{ where } j = 1,2,...,n.$$

(9) $\left((T(p,r))^*T(p,r)\right)^{\frac{\beta}{p+r}} \geqslant \left(T(p,r)(T(p,r))^*\right)^{\frac{\beta}{p+r}}$ *if and only if*

$$\left(B^{r/2}A^pB^{r/2}\right)^{\frac{\beta}{p+r}} \geqslant B^\beta \text{ and}$$
$$(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\beta}{r+p}} \leqslant A^\beta.$$

Proof of Theorem 3.1. It is enough to prove that (3.1) implies (3.5) because the others can be proved similarly.

If $A \geqslant 0$, $B \geqslant 0$ such that $A^q \geqslant B^q$, define T as in Lemma 3.2, then T is q-hyponormal by (1) of Lemma 3.2. Thus the inequality (3.1) holds. By the property of the polar decomposition of T^*, (3.1) is equivalent to

$$\left(|T^*|^r|T|^{2p}|T^*|^r\right)^{\frac{s(q)+r}{p+r}} \geqslant \left(|T^*|^r|T|^{2p_0}|T^*|^r\right)^{\frac{s(q)+r}{p_0+r}}.$$

Therefore, the inequality (3.5) follows by (4) of Lemma 3.2. $\square$

Similarly to the proof of Theorem 3.1, the following result can be proved by (9) of Lemma 3.2.

THEOREM 3.3. *Let $p>0$, $r>0$, $q>0$ and $\gamma(q,p,r)=\min\{q+p,q+r,p+r\}$. For each p, r and q, the following assertion (1) and (2) are equivalent to each other:*

(1) *If T is a q-hyponormal operator, then*

$$\left((T(p,r))^*T(p,r)\right)^{\frac{\gamma(q,p,r)}{p+r}} \geqslant \left(T(p,r)(T(p,r))^*\right)^{\frac{\gamma(q,p,r)}{p+r}}. \tag{3.9}$$

(2) *If $A \geqslant 0$, $B \geqslant 0$ such that $A^q \geqslant B^q$, then*

$$\begin{aligned}(B^{r/2}A^pB^{r/2})^{\frac{\gamma(q,p,r)}{p+r}} &\geqslant B^{\gamma(q,p,r)},\\ (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\gamma(q,p,r)}{r+p}} &\leqslant A^{\gamma(q,p,r)}.\end{aligned} \tag{3.10}$$

By the viewpoint of this section and Lemma 3.2, the problems below on Furuta inequality and Aluthge transform are equivalent.

PROBLEM 3.4. *When $q>r>0$ and $2p_0+r=2p_0+\min\{q,r\}<p$, is the outer exponent $s(q)+r=2p_0+2r$ in (3.1) of Theorem 3.1 optimal in the sense of Theorem 1.3?*

PROBLEM 3.5 ([31]). *When $q>r>0$ and $2p_0+r=2p_0+\min\{q,r\}<p$, is the outer exponent $s(q)+r=2p_0+2r$ in (3.5) of Theorem 3.1 optimal?*

4. Simplified proof

THEOREM 4.1. (Order structure, [15, 27, 29]) *Let m, n be positive integers, $q>0$ and $\gamma(q,m,n)=\min\{q,m,n+1\}$. If T is q-hyponormal, then the following assertions hold:*

(1) *Same-side structure,*

$$\left({T^*}^{m+1}T^{m+1}\right)^{\frac{\min\{q,m\}+1}{m+1}} \geqslant \left(T^*T\right)^{\min\{q,m\}+1}, \tag{4.1}$$

$$\left({T^*}^{m+n+1}T^{m+n+1}\right)^{\frac{\gamma(q,m,n)+n+1}{m+n+1}} \geqslant \left({T^*}^{n+1}T^{n+1}\right)^{\frac{\gamma(q,m,n)+n+1}{n+1}}, \tag{4.2}$$

$$\left(T^{m+1}{T^*}^{m+1}\right)^{\frac{\min\{q,m\}+1}{m+1}} \leqslant \left(TT^*\right)^{\min\{q,m\}+1}, \tag{4.3}$$

$$\left(T^{m+n+1}{T^*}^{m+n+1}\right)^{\frac{\gamma(q,m,n)+n+1}{m+n+1}} \leqslant \left(T^{n+1}{T^*}^{n+1}\right)^{\frac{\gamma(q,m,n)+n+1}{n+1}}. \tag{4.4}$$

(2) *Different-side structure,*

$$\left(T^{*^{m+1}}T^{m+1}\right)^{\frac{\min\{q,m+1\}}{m+1}} \geqslant (TT^*)^{\min\{q,m+1\}}, \tag{4.5}$$

$$(T^*T)^{\min\{q,n+1\}} \geqslant \left(T^{n+1}T^{*^{n+1}}\right)^{\frac{\min\{q,n+1\}}{n+1}}, \tag{4.6}$$

$$\left(T^{*^{m+1}}T^{m+1}\right)^{\frac{\min\{q,m+1,n+1\}}{m+1}} \geqslant \left(T^{n+1}T^{*^{n+1}}\right)^{\frac{\min\{q,m+1,n+1\}}{n+1}}. \tag{4.7}$$

[29] showed Theorem 4.1 except (4.1) and (4.3). Ito [15] obtained (4.1) and (4.3). The case $q \leqslant 2$ of (4.2) and (4.4) are the main results of [27].

In order to give a short and simplified proof, a variant on original form of Furuta inequality is prepared.

THEOREM 4.2. (Variant on original form) *Let $q>0$, $r \geqslant 0$, $p>0$, $A \geqslant 0$ and $B \geqslant 0$. Then $A^q \geqslant B^q$ ensures*

$$\left(B^{r/2}A^pB^{r/2}\right)^{\frac{\min\{q,p\}+r}{p+r}} \geqslant \left(B^{r/2}B^pB^{r/2}\right)^{\frac{\min\{q,p\}+r}{p+r}},$$
$$\left(A^{r/2}A^pA^{r/2}\right)^{\frac{\min\{q,p\}+r}{p+r}} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{\frac{\min\{q,p\}+r}{p+r}}.$$

Proof of Theorem 4.1. The assertions (4.1), (4.3), (4.5)–(4.7), (4.2) and (4.4) will be proved in turn.

Step 1. To prove (4.1) by induction and the original form of Furuta inequality (Theorem 4.2): When $m=1$, (4.1) becomes

$$\left(T^{*^{1+1}}T^{1+1}\right)^{\frac{\min\{q,1\}+1}{1+1}} \geqslant \left(T^*T\right)^{\min\{q,1\}+1}.$$

By the property of the polar decomposition of T^*, the inequality above is equivalent to

$$\left(|T^*||T|^2|T^*|\right)^{\frac{\min\{q,1\}+1}{1+1}} \geqslant |T^*|^{2(\min\{q,1\}+1)}. \tag{4.8}$$

(4.8) follows by the q-hyponormality of T and the original form of Furuta inequality (Theorem 4.2). Assume that (4.1) holds for $m=k(\geqslant 1)$, this together with the q-hyponormality of T and $\min\{\min\{q,k\}+1,q\} = \min\{q,k+1\}$ deduce that

$$\left(T^{*^{k+1}}T^{k+1}\right)^{\frac{\min\{q,k+1\}}{k+1}} \geqslant \left(TT^*\right)^{\min\{q,k+1\}}.$$

Then, by applying the original form of Furuta inequality to $|T^{k+1}|^{\frac{2}{k+1}}$ and $|T^*|^2$,

$$\left(|T^*||T^{k+1}|^2|T^*|\right)^{\frac{\min\{q,k+1\}+1}{k+1+1}} \geqslant |T^*|^{2(\min\{q,k+1\}+1)}. \tag{4.9}$$

By the property of the polar decomposition of T^*, (4.9) is equivalent to

$$\left(T^{*^{k+1+1}}T^{k+1+1}\right)^{\frac{\min\{q,k+1\}+1}{k+1+1}} \geqslant \left(T^*T\right)^{\min\{q,k+1\}+1}.$$

Therefore (4.1) holds.

Step 2. To prove (4.3): The proof of (4.3) is similar to that of (4.1).

Step 3. To prove (4.5)–(4.7): (4.5) is a direct result of (4.1) and the q-hyponormality of T because of $\min\{\min\{q,m\}+1,q\} = \min\{q,m+1\}$. The proof of (4.6) is similar to (4.5). (4.7) follows by (4.5) and (4.6).

Step 4. To prove (4.2): If $m=1$, then $\min\{\min\{q,n+1\},1\} = \gamma(q,m,n)$. Thus, by (4.6) and applying the original form of Furuta inequality to $|T|^2$ and $|T^{*^{n+1}}|^{\frac{2}{n+1}}$,

$$\left(|T^{*^{n+1}}||T|^2|T^{*^{n+1}}|\right)^{\frac{\gamma(q,m,n)+n+1}{1+n+1}} \geqslant |T^{*^{n+1}}|^{2(\gamma(q,m,n)+n+1)}. \tag{4.10}$$

In the case of $m \geqslant 2$, by (4.7) and applying the original form of Furuta inequality to $|T^m|^{\frac{2}{m}}$ and $|T^{*^{n+1}}|^{\frac{2}{n+1}}$,

$$\left(|T^{*^{n+1}}||T^m|^2|T^{*^{n+1}}|\right)^{\frac{\gamma(q,m,n)+n+1}{m+n+1}} \geqslant |T^{*^{n+1}}|^{2(\gamma(q,m,n)+n+1)}. \tag{4.11}$$

So (4.2) holds by (4.10)–(4.11) and the property of the polar decomposition of $T^{*^{n+1}}$.

Step 5. To prove (4.4): The proof of (4.4) is similar to that of (4.2). □

Step 1 of Proof of Theorem 4.1 is a short proof of Ito's result [15, Theorem 1]. The outer exponents of the inequalities in Theorem 4.1 are optimal except the cases $n+1 < \min\{q,m\}$ of (4.2) and (4.4) by [27, Theorem 3.1] and [29, Theorem 3.2].

PROBLEM 4.3 ([29]). *When $n+1 < \min\{q,m\}$, is the outer exponent $\gamma(q,m,n)+n+1 = 2(n+1)$ in (4.2) and (4.4) optimal?*

By Lemma 3.2 (6)–(7), Problem 4.3 is a special case of Problem 3.5.

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