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CHARACTERIZATION OF ξ–LIE MULTIPLICATIVE ISOMORPHISMS

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Abstract. Let $\mathscr{A}$ and $\mathscr{A}'$ be two algebras over a field $\mathbb{F}$ and $\xi \in \mathbb{F}$ a scalar. A map $\Phi : \mathscr{A} \to \mathscr{A}'$ is called a ξ-Lie multiplicative isomorphism if Φ is bijective and satisfies $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ for all $A, B \in \mathscr{A}$. The additivity of ξ-Lie multiplicative isomorphisms on prime algebras is discussed. A characterization of ξ-Lie multiplicative isomorphisms between matrix algebras over a field of characteristic not 2 and a characterization of ξ-Lie multiplicative isomorphisms between infinite dimensional Banach space standard operator algebras are obtained.

1. Introduction

Commutation relations between self-adjoint operators in a complex Hilbert space H play an important role in the interpretation of quantum mechanical observable and the analysis of their spectra. Accordingly, such relations have been extensively studied in the mathematical literature (see [12]). An interesting related aspect concerns the commutativity up to a factor for pairs of elements. Let A and B be two elements in an algebra. If A and B satisfy the algebraic relation $AB = \xi BA$ for some nonzero scalar ξ, we say that A and B are commutative up to a factor ξ. More recently, the commutativity up to a factor for pairs of operators has been studied in the context of quantum groups, and their matrix realizations give examples of operator pairs commuting up to a factor (see, for example, [3], [7], [8] and [14]). The concept of commutativity is closely related to the concept of Lie products, i.e., $[A,B] = AB - BA$. It is clear that two elements are commutative if and only if their Lie product is zero. Motivated by product and Lie product, for each scalar ξ, we can define ξ-Lie product of A and B by $[A,B]_\xi = AB - \xi BA$. It is clear that ξ-Lie product is the usual product if $\xi = 0$; the Lie product if $\xi = 1$; the Jordan product if $\xi = -1$. It is also obvious that A commutes with B up to a factor ξ if and only if their ξ-Lie product is zero.

Let $\mathscr{A}$ and $\mathscr{A}'$ be two algebras over a field $\mathbb{F}$. Recall that a map $\Phi : \mathscr{A} \to \mathscr{A}'$ is called a multiplicative map if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathscr{A}$; is called a Lie multiplicative map if $\Phi([A,B]) = [\Phi(A), \Phi(B)]$; is called a Jordan multiplicative map

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if $\Phi(AB+BA)=\Phi(A)\Phi(B)+\Phi(B)\Phi(A)$. Here we may introduce a concept of ξ-Lie multiplicative maps which unifies the above three kinds of maps. A map $\Phi:\mathscr{A}\to\mathscr{A}'$ is called a ξ-Lie multiplicative map if $\Phi([A,B]_\xi)=[\Phi(A),\Phi(B)]_\xi$ for all $A,B\in\mathscr{A}$. In addition, a map $\Phi:\mathscr{A}\to\mathscr{A}'$ is called a ξ-Lie multiplicative isomorphism if Φ is bijective and ξ-Lie multiplicative; is called a ξ-Lie ring isomorphism if Φ is bijective, additive and ξ-Lie multiplicative. A linear (conjugate linear) ξ-Lie ring isomorphism between two algebras is called a ξ-Lie isomorphism (conjugate ξ-Lie isomorphism).

The question when a multiplicative map is additive is studied by many mathematicians. It is a well known result due to Matindale [10] that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Recently Matindale's result has been generalized in several directions, such as multiplicative maps and Jordan multiplicative maps between standard operator algebras or nest algebras (see [1], [9] and the references therein). However, it was proved in [2] that, if $\mathscr{R}$, $\mathscr{R}'$ be prime rings with $\mathscr{R}$ being unital and containing a nontrivial idempotent, and if $\Phi:\mathscr{R}\to\mathscr{R}'$ is a Lie multiplicative bijective map, then $\Phi(T+S)=\Phi(T)+\Phi(S)+Z'_{T,S}$ for all $T,S\in\mathscr{R}$, where $Z'_{T,S}$ is an element in the center $\mathscr{Z}'$ of $\mathscr{R}'$ depending on T and S. This result reveals that the Lie multiplicativity of a map does not imply its additivity anymore. Then, an interesting question is, for $\xi\neq 1$, whether every ξ-Lie multiplicative bijection between algebras is necessarily additive? The purpose of this paper is to answer this question and to show that every ξ-Lie multiplicative bijective map with $\xi\neq 1$ from an algebra with some properties (weaker than primeness) onto another algebra must be additive. Then, based on this result, we give a complete characterization of ξ-Lie multiplicative isomorphisms on matrix algebras over a field of characteristic not 2 and ξ-Lie multiplicative isomorphisms on infinite dimensional Banach space standard operator algebras.

This paper is organized as follows. In Section 2, we discuss the additivity of ξ-Lie multiplicative isomorphisms. Let $\mathscr{A}$ and $\mathscr{A}'$ be two algebras over a field $\mathbb{F}$. Assume that $\mathscr{A}$ is unital with unit I and contains a nontrivial idempotent P. Let ξ be a scalar and $\Phi:\mathscr{A}\to\mathscr{A}'$ be a ξ-Lie multiplicative bijective map. We prove that, if $\xi\neq 1$ and if $\mathscr{A}$ satisfies $P_iAP_j=0$ whenever $P_iAP_j\mathscr{A}P_l=0$ or $P_l\mathscr{A}P_iAP_j=0$ $(P_1=P, P_2=I-P_1, 1\leqslant i,j,l\leqslant 2)$ (particularly, this is the case when $\mathscr{A}$ is prime), then Φ is additive and thus a ξ-Lie ring isomorphism; if $\xi=1$ and if $\mathscr{A}$ is prime, then Φ is almost additive (Theorem 2.1). In Section 3, we discuss the question of characterizing the unital ξ-Lie multiplicative isomorphisms on matrix algebras and infinite dimensional Banach space standard operator algebras. For the matrix algebra case, let $M_n(\mathbb{F})$ be the algebra of all $n\times n$ matrices $(n>1)$ over a field $\mathbb{F}$ of characteristic not 2. We first give a characterization of additive maps on $M_n(\mathbb{F})$ preserving zero ξ-Lie products in both directions with $\xi\neq 0,1$ (Theorem 3.1). Then this result, together with Theorem 2.1, is applied to show that, for a ξ-Lie multiplicative isomorphism $\Phi:M_n(\mathbb{F})\to M_n(\mathbb{F})$, there exists a field automorphism $\tau:\mathbb{F}\to\mathbb{F}$ and a nonsingular matrix $T\in M_n(\mathbb{F})$ such that (1) if $\xi=1$ and $\mathbb{F}$ is also of characteristic not 3, then either $\Phi(A)=TA_\tau T^{-1}+h(A)I$ for all $A\in M_n(\mathbb{F})$ or $\Phi(A)=-T(A_\tau)^{\mathrm{tr}}T^{-1}+h(A)I$ for all $A\in M_n(\mathbb{F})$, where $h:M_n(\mathbb{F})\to\mathbb{F}$ is a functional satisfying $h([A,B])=0$ for all A,B; (2) if $\xi=-1$, then either $\Phi(A)=TA_\tau T^{-1}$ for all A or $\Phi(A)=T(A_\tau)^{\mathrm{tr}}T^{-1}$ for all A; (3) if $\xi\neq\pm 1$, then $\Phi(A)=TA_\tau T^{-1}$ for all A and τ satisfies $\tau(\xi)=\xi$ (Theorem 3.2).

Here A^{tr} denotes the transpose of A and A_τ denotes the matrix obtained from A by applying τ to all the entries of A. For the infinite dimensional case, by use of Theorem 2.1 and a characterization of additive maps preserving zero ξ-Lie products in [6], we characterize all ξ-Lie multiplicative isomorphisms on Banach space standard operator algebras (Theorem 3.4).

2. The additivity of ξ-Lie multiplicative maps

In this section, we discuss the additivity of ξ-Lie multiplicative bijective maps between general algebras. The following is our main result.

THEOREM 2.1. *Let $\mathscr{A}$, $\mathscr{A}'$ be two algebras over a field $\mathbb{F}$. Assume that $\mathscr{A}$ is unital with unit I and contains a nontrivial idempotent P. Denote $P_1 = P$, $P_2 = I - P_1$ and let ξ be a scalar. Assume that $\Phi : \mathscr{A} \to \mathscr{A}'$ is a ξ-Lie multiplicative bijective map, that is,*

$$\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A) \quad \forall A,B \in \mathscr{A}. \tag{2.1}$$

(1) If $\xi = 1$ and $\mathscr{A}$ is prime, then $\Phi(A+B) = \Phi(A) + \Phi(B) + Z_{A,B}$ for all $A,B \in \mathscr{A}$, where $Z_{A,B}$ is an element in the center $\mathscr{Z}$ of $\mathscr{A}'$ depending on A and B.

(2) If $\xi \neq 1$ and $\mathscr{A}$ satisfies the condition that $P_iAP_j = 0$ whenever $P_iAP_j\mathscr{A}P_l = 0$ or $P_l\mathscr{A}P_iAP_j = 0$ $(1 \leqslant i,j,l \leqslant 2)$, then Φ is additive and thus a ξ-Lie ring isomorphism.

Proof. The main technique we will use in the following arguments will be called a *standard argument*: Suppose that $S,A,B \in \mathscr{A}$ are such that $\Phi(S) = \Phi(A) + \Phi(B)$. Multiplying this equation by $\Phi(T)$ $(T \in \mathscr{A})$ from the left and the right, respectively, we get $\Phi(T)\Phi(S) = \Phi(T)\Phi(A) + \Phi(T)\Phi(B)$ and $\Phi(S)\Phi(T) = \Phi(A)\Phi(T) + \Phi(B)\Phi(T)$. Then

$$\Phi(T)\Phi(S) - \xi\Phi(S)\Phi(T) = \Phi(T)\Phi(A) - \xi\Phi(A)\Phi(T) + \Phi(T)\Phi(B) - \xi\Phi(B)\Phi(T).$$

It follows that

$$\Phi(TS - \xi ST) = \Phi(TA - \xi AT) + \Phi(TB - \xi BT).$$

Moreover, if we have

$$\Phi(TA - \xi AT) + \Phi(TB - \xi BT) = \Phi(TA - \xi AT + TB - \xi BT),$$

then by the injectivity of Φ, we get

$$TS - \xi ST = TA - \xi AT + TB - \xi BT.$$

We'll prove the theorem by considering three cases.

Case 1. $\xi = 1$.

By [2, Main Theorem] and its proof, it is easily seen that the condition $\mathscr{A}'$ is prime is superfluous. Hence the statement (1) of Theorem 2.1 is true.

Case 2. $\xi = 0$.

By using a similar argument to that of [10], one can show that Φ is additive.

Case 3. $\xi \neq 0, 1$.

We'll divide the proof into checking several claims.

Claim 1. $\Phi(0) = 0$.

Since Φ is surjective, we can find an element $A \in \mathscr{A}$ such that $\Phi(A) = 0$. Therefore $\Phi(0) = \Phi(0A - \xi A0) = \Phi(0)\Phi(A) - \xi\Phi(A)\Phi(0) = 0$.

In the sequel, we set $\mathscr{A}_{ij} = P_i\mathscr{A}P_j$, $i, j = 1, 2$, and thus $\mathscr{A} = \mathscr{A}_{11}\dot{+}\mathscr{A}_{12}\dot{+}\mathscr{A}_{21}\dot{+}\mathscr{A}_{22}$. For an element $S_{ij} \in \mathscr{A}$, we always mean that $S_{ij} \in \mathscr{A}_{ij}$.

Claim 2. *For every A_{ii} and A_{ij}, we have $\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij})$, $1 \leqslant i \neq j \leqslant 2$.*

Since Φ is surjective, we may find an element $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{2.2}$$

For $T_{jj} \in \mathscr{A}_{jj}$, applying a standard argument to Eq.(2.2), we obtain

$$\Phi(ST_{jj} - \xi T_{jj}S) = \Phi(A_{ii}T_{jj} - \xi T_{jj}A_{ii}) + \Phi(A_{ij}T_{jj} - \xi T_{jj}A_{ij}) = \Phi(A_{ij}T_{jj}).$$

Therefore $ST_{jj} - \xi T_{jj}S = A_{ij}T_{jj} \in \mathscr{A}_{ij}$ for every $T_{jj} \in \mathscr{A}_{jj}$, which implies that $S_{ji} = S_{jj} = 0$. Hence the equation $ST_{jj} - \xi T_{jj}S = A_{ij}T_{jj}$ becomes $S_{ij}T_{jj} = A_{ij}T_{jj}$. So $S_{ij} = A_{ij}$.

For $T_{ij} \in \mathscr{A}_{ij}$, applying a standard argument to Eq.(2.2) again, we have

$$\Phi(T_{ij}S - \xi ST_{ij}) = \Phi(T_{ij}A_{ii} - \xi A_{ii}T_{ij}) + \Phi(T_{ij}A_{ij} - \xi A_{ij}T_{ij}) = \Phi(-\xi A_{ii}T_{ij}).$$

Therefore $T_{ij}S - \xi ST_{ij} = -\xi A_{ii}T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. Note that $S_{ji} = S_{jj} = 0$ and $\xi \neq 0$. We have $S_{ii}T_{ij} = A_{ii}T_{ij}$ since ξ is invertible. Hence $S_{ii} = A_{ii}$. Consequently, $S = A_{ii} + A_{ij}$.

Similarly, one can check that

Claim 3. *For every A_{ii} and A_{ji}, we have that $\Phi(A_{ii} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ji})$, $1 \leqslant i \neq j \leqslant 2$.*

Claim 4. Φ *is additive on* $\mathscr{A}_{12}$.

For any $A_{12}, B_{12} \in \mathscr{A}_{12}$, since

$$\begin{aligned} A_{12} + B_{12} &= (P_1 + B_{12})(A_{12} + P_2) \\ &= (P_1 + B_{12})(A_{12} + P_2) - \xi(A_{12} + P_2)(P_1 + B_{12}), \end{aligned}$$

using Eq.(2.1), Claim 2-3, we get

$$\begin{aligned} \Phi(A_{12} + B_{12}) &= \Phi(P_1 + B_{12})\Phi(A_{12} + P_2) - \xi\Phi(A_{12} + P_2)\Phi(P_1 + B_{12}) \\ &= (\Phi(P_1) + \Phi(B_{12}))(\Phi(A_{12}) + \Phi(P_2)) \\ &\quad -\xi(\Phi(A_{12}) + \Phi(P_2))(\Phi(P_1) + \Phi(B_{12})) \\ &= (\Phi(P_1)\Phi(A_{12}) - \xi\Phi(A_{12})\Phi(P_1)) + (\Phi(P_1)\Phi(P_2) - \xi\Phi(P_2)\Phi(P_1)) \\ &\quad +(\Phi(B_{12})\Phi(A_{12}) - \xi\Phi(A_{12})\Phi(B_{12})) \\ &\quad +(\Phi(B_{12})\Phi(P_2) - \xi\Phi(P_2)\Phi(B_{12})) \\ &= \Phi(P_1A_{12} - \xi A_{12}P_1) + \Phi(P_1P_2 - \xi P_2P_1) \\ &\quad +\Phi(B_{12}A_{12} - \xi A_{12}B_{12}) + \Phi(B_{12}P_2 - \xi P_2B_{12}) \\ &= \Phi(A_{12}) + \Phi(B_{12}). \end{aligned}$$

Claim 5. Φ *is additive on* $\mathscr{A}_{21}$.

Let $A_{21}, B_{21} \in \mathscr{A}_{21}$. Note that

$$\begin{aligned} A_{21} + B_{21} &= (A_{21} + P_2)(P_1 + B_{21}) \\ &= (A_{21} + P_2)(P_1 + B_{21}) - \xi(P_1 + B_{21})(A_{21} + P_2). \end{aligned}$$

Now we can complete the proof by using a computation similar to that in the proof of Claim 4.

Claim 6. Φ *is additive on* $\mathscr{A}_{ii}$, $i = 1, 2$.

Let $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(B_{ii}). \tag{2.3}$$

Let $j \neq i$. For $T_{jj} \in \mathscr{A}_{jj}$, applying a standard argument to Eq.(2.3), we get $T_{jj}S - \xi S T_{jj} = 0$. It follows that $S_{ij} = S_{ji} = S_{jj} = 0$.

Now it remains to prove that $S_{ii} = A_{ii} + B_{ii}$. For $T_{ij} \in \mathscr{A}_{ij}$, applying a standard argument to Eq.(2.3) again, we get $\Phi(T_{ij}S - \xi S T_{ij}) = \Phi(-\xi A_{ii} T_{ij}) + \Phi(-\xi B_{ii} T_{ij})$. Hence by the injectivity of Φ and Claim 4-5, we have $T_{ij}S - \xi S T_{ij} = -\xi(A_{ii} + B_{ii})T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. Since $S_{ij} = S_{ji} = S_{jj} = 0$, it follows that $S_{ii}T_{ij} = (A_{ii} + B_{ii})T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. Thus we get $S_{ii} = A_{ii} + B_{ii}$.

Claim 7. Φ *is additive on* $P_1\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12}$.

Let $A_{11}, B_{11} \in \mathscr{A}_{11}$, $A_{12}, B_{12} \in \mathscr{A}_{12}$. Then by Claim 2, Claim 4 and Claim 6, we see that

$$\begin{aligned} \Phi((A_{11} + A_{12}) + (B_{11} + B_{12})) &= \Phi((A_{11} + B_{11}) + (A_{12} + B_{12})) \\ &= \Phi(A_{11} + B_{11}) + \Phi(A_{12} + B_{12}) \\ &= \Phi(A_{11}) + \Phi(B_{11}) + \Phi(A_{12}) + \Phi(B_{12}) \\ &= (\Phi(A_{11}) + \Phi(A_{12})) + (\Phi(B_{11}) + \Phi(B_{12})) \\ &= \Phi(A_{11} + A_{12}) + \Phi(B_{11} + B_{12}). \end{aligned}$$

Claim 8. $\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22})$.

Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{22}). \tag{2.4}$$

Then applying a standard argument to Eq.(2.4), we have

$$P_1 S - \xi S P_1 = P_1 A_{11} - \xi A_{11} P_1 = (1 - \xi)A_{11}.$$

By a simple computation, we get that $S_{12} = S_{21} = 0$ and $S_{11} = A_{11}$.

Next we prove that $S_{22} = A_{22}$. For $T_{22} \in \mathscr{A}_{22}$, applying a standard argument to Eq.(2.4) again, we get $T_{22}S - \xi S T_{22} = T_{22}A_{22} - \xi A_{22} T_{22}$. Since we have shown that $S_{12} = S_{21} = 0$, the above equation reduces to $T_{22}(S_{22} - A_{22}) - \xi(S_{22} - A_{22})T_{22} = 0$ for every $T_{22} \in \mathscr{A}_{22}$. Particularly, taking $T_{22} = P_2$ we get $S_{22} = A_{22}$. Consequently, $S = A_{11} + A_{22}$.

Claim 9. $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.

Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}). \tag{2.5}$$

For $T_{12} \in \mathscr{A}_{12}$, applying a standard argument to Eq.(2.5), we get

$$\begin{aligned}\Phi(T_{12}S - \xi ST_{12}) &= \Phi(T_{12}A_{12} - \xi A_{12}T_{12}) + \Phi(T_{12}A_{21} - \xi A_{21}T_{12}) \\ &= \Phi(T_{12}A_{21} - \xi A_{21}T_{12}).\end{aligned}$$

By the injectivity of $\Phi,$ we have that

$$T_{12}S - \xi ST_{12} = T_{12}A_{21} - \xi A_{21}T_{12} \tag{2.6}$$

for every $T_{12} \in \mathscr{A}_{12}$. Multiplying this equation by P_1 from the right, we get $T_{12}S_{21} = T_{12}A_{21}$ for every $T_{12} \in \mathscr{A}_{12}$. It follows that $S_{21} = A_{21}$.

An argument similar to what has led to the equation $S_{21} = A_{21}$ proves that $S_{12} = A_{12}$ also holds. The following we will prove that $S_{11} = S_{22} = 0$.

Applying a standard argument to Eq.(2.5) again, we have

$$\Phi(P_2S - \xi SP_2) = \Phi(-\xi A_{12}) + \Phi(A_{21})$$

and

$$\Phi(SP_1 - \xi P_1S) = \Phi(-\xi A_{12}) + \Phi(A_{21}).$$

Therefore $\Phi(P_2S - \xi SP_2) = \Phi(SP_1 - \xi P_1S)$, which implies that $P_2S - \xi SP_2 = SP_1 - \xi P_1S$. A simple computation gets $S_{11} = S_{22} = 0$. Consequently, $S = A_{12} + A_{21}$.

Claim 10. $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ be such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}). \tag{2.7}$$

Then by Claim 2 and Claim 3, we have

$$\Phi(S) = \Phi(A_{11} + A_{12}) + \Phi(A_{21}) \tag{2.8}$$

and

$$\Phi(S) = \Phi(A_{11} + A_{21}) + \Phi(A_{12}). \tag{2.9}$$

For $T_{21} \in \mathscr{A}_{21}$, applying a standard argument to Eq.(2.8), we have

$$\Phi(T_{21}S - \xi ST_{21}) = \Phi(T_{21}A_{11} + T_{21}A_{12} - \xi A_{12}T_{21}).$$

Hence by the injectivity of Φ, we obtain

$$T_{21}S - \xi ST_{21} = T_{21}A_{11} + T_{21}A_{12} - \xi A_{12}T_{21}. \tag{2.10}$$

for every $T_{21} \in \mathscr{A}_{21}$. Multiplying this equation by P_2 from the right, we get that $T_{21}S_{12} = T_{21}A_{12}$ holds for every $T_{21} \in \mathscr{A}_{21}$. So $S_{12} = A_{12}$. Similarly, for $T_{12} \in \mathscr{A}_{12}$, applying a standard argument to Eq.(2.9), we can get $S_{21} = A_{21}$.

For $T_{22} \in \mathscr{A}_{22}$, applying a standard argument to Eq.(2.8), and using Claim 9, we have

$$\begin{aligned}\Phi(T_{22}S - \xi ST_{22}) &= \Phi(T_{22}(A_{11} + A_{12}) - \xi(A_{11} + A_{12})T_{22}) \\ &\quad + \Phi(T_{22}A_{21} - \xi A_{21}T_{22}) \\ &= \Phi(-\xi A_{12}T_{22}) + \Phi(T_{22}A_{21}) \\ &= \Phi(-\xi A_{12}T_{22} + T_{22}A_{21}),\end{aligned}$$

that is, $T_{22}S-\xi ST_{22}=-\xi A_{12}T_{22}+T_{22}A_{21}$. Since $S_{12}=A_{12}$ and $S_{21}=A_{21}$, it follows that $S_{22}=0$. Multiplying Eq.(2.10) by P_2 from the left, we get that $T_{21}(S_{11}-A_{11})=\xi S_{22}T_{21}=0$ holds for every $T_{21}\in\mathscr{A}_{21}$. Hence $S_{11}=A_{11}$. Consequently, $S=A_{11}+A_{12}+A_{21}$.

Claim 11. $\Phi(A_{11}+A_{12}+A_{21}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})$.

Let $S=S_{11}+S_{12}+S_{21}+S_{22}\in\mathscr{A}$ be such that

$$\Phi(S)=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}). \tag{2.11}$$

Then we have

$$\begin{aligned}\Phi(P_1S-\xi SP_1)&=\Phi(P_1)\Phi(S)-\xi\Phi(S)\Phi(P_1)\\&=\Phi(P_1)(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))\\&\quad-\xi(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))\Phi(P_1)\\&=\Phi((1-\xi)A_{11})+\Phi(A_{12})+\Phi(-\xi A_{21})\\&=\Phi((1-\xi)A_{11}+A_{12}-\xi A_{21}).\end{aligned}$$

It follows that $P_1S-\xi SP_1=(1-\xi)A_{11}+A_{12}-\xi A_{21}$. By a simple computation, we get $S_{11}=A_{11},\ S_{12}=A_{12},\ S_{21}=A_{21}$.

For $T_{12}\in\mathscr{A}_{12}$, applying a standard argument to Eq.(2.11), we have

$$\Phi(T_{12}S-\xi ST_{12})=\Phi(-\xi A_{11}T_{12})+\Phi(T_{12}A_{21}-\xi A_{21}T_{12})+\Phi(T_{12}A_{22}).$$

Furthermore, applying a standard argument to the above equation, and using Claim 2 and Claim 4, we have

$$\begin{aligned}&\Phi(P_1(T_{12}S-\xi ST_{12})-\xi(T_{12}S-\xi ST_{12})P_1)\\&=\Phi(-\xi A_{11}T_{12})+\Phi(T_{12}A_{21}-\xi T_{12}A_{21})+\Phi(T_{12}A_{22})\\&=\Phi(-\xi A_{11}T_{12}+T_{12}A_{21}-\xi T_{12}A_{21}+T_{12}A_{22}).\end{aligned}$$

Thus we get

$$T_{12}S_{21}+T_{12}S_{22}-\xi S_{11}T_{12}-\xi T_{12}S_{21}=-\xi A_{11}T_{12}+T_{12}A_{21}-\xi T_{12}A_{21}+T_{12}A_{22}.$$

Since we have shown that $S_{11}=A_{11},\ S_{12}=A_{12},\ S_{21}=A_{21}$, it follows that $T_{12}S_{22}=T_{12}A_{22}$ for every $T_{12}\in\mathscr{A}_{12}$ and hence $S_{22}=A_{22}$. Consequently, $S=A_{11}+A_{12}+A_{21}+A_{22}$.

Claim 12. *For any $A,B\in\mathscr{A}$, we have* $\Phi(A+B)=\Phi(A)+\Phi(B)$.

Let $A=A_{11}+A_{12}+A_{21}+A_{22}$ and $B=B_{11}+B_{12}+B_{21}+B_{22}$ be in $\mathscr{A}$. Then using Claim 2-11, we have

$$\begin{aligned}\Phi(A+B)&=\Phi((A_{11}+B_{11})+(A_{12}+B_{12})+(A_{21}+B_{21})+(A_{22}+B_{22}))\\&=\Phi(A_{11}+B_{11})+\Phi(A_{12}+B_{12})+\Phi(A_{21}+B_{21})+\Phi(A_{22}+B_{22})\\&=\Phi(A_{11})+\Phi(B_{11})+\Phi(A_{12})+\Phi(B_{12})\\&\quad+\Phi(A_{21})+\Phi(B_{21})+\Phi(A_{22})+\Phi(B_{22})\\&=\Phi(A_{11}+A_{12}+A_{21}+A_{22})+\Phi(B_{11}+B_{12}+B_{21}+B_{22})\\&=\Phi(A)+\Phi(B).\end{aligned}$$

That is, Φ is additive. Combining Case (2) and (3), the proof of the statement (2) is completed.

The proof is finished. $\square$

3. A characterization of ξ-Lie multiplicative isomorphisms

In this section, we apply Theorem 2.1 to give a complete classification of unital ξ-Lie multiplicative isomorphisms on matrix algebras and infinite dimensional Banach space standard operator algebras.

For the matrix algebra case, we first give a characterization of additive maps on $M_n(\mathbb{F})$ which preserve zero ξ-Lie products in both directions with $\xi \neq 0,1$ (i.e. $AB = \xi BA$ if and only if $\Phi(A)\Phi(B) = \xi\Phi(B)\Phi(A)$).

THEOREM 3.1. *Let $M_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices $(n > 1)$ over a field $\mathbb{F}$ of characteristic not 2. Suppose that $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a unital additive surjection and ξ is a scalar with $\xi \neq 0,1$. Then Φ preserves zero ξ-Lie product in both directions if and only if one of the followings holds:*

(1) *There exists a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ satisfying $\tau(\xi) = \xi$, and there exists a nonsingular matrix $T \in M_n(\mathbb{F})$ such that $\Phi(A) = TA_\tau T^{-1}$ for all $A \in M_n(\mathbb{F})$.*

(2) *There exists a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ satisfying $\tau(\xi) = \xi^{-1}$, and there exists a nonsingular matrix $T \in M_n(\mathbb{F})$ such that $\Phi(A) = T(A_\tau)^{\mathrm{tr}} T^{-1}$ for all $A \in M_n(\mathbb{F})$.*

Here A^{tr} denotes the transpose of A and A_τ denotes the matrix obtained from A by applying τ to all the entries of A.

Proof. For $A \in M_n(\mathbb{F})$, if $\mathrm{rank}(A) = 1$, then it is obvious that there exist $x, f \in \mathbb{F}^n$, and a scalar k such that $A = xf^t$ and $A^2 = kA$. We will use symbol $x \otimes f$ to stand for xf^t.

Clearly, we need only to prove the necessity. Assume that Φ preserves zero ξ-Lie product in both directions, that is, $AB - \xi BA = 0$ if and only if $\Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A) = 0$ for all $A, B \in M_n(\mathbb{F})$. We finish the proof by checking several claims.

Claim 1. $\Phi(0) = 0$ and Φ is injective.

Let $\Phi(A) = 0$. Then for any $B \in M_n(\mathbb{F})$, we have $\Phi(A)\Phi(B) = \xi\Phi(B)\Phi(A)$, and hence $AB = \xi BA$ for every $B \in M_n(\mathbb{F})$. In particular, $A = \xi A$, thus $A = 0$ since $\xi \neq 1$.

Claim 2. Φ preserves rank one idempotents in both directions.

We first prove that Φ preserves idempotents. Let $P \in M_n(\mathbb{F})$ be an idempotent. Then $P(I-P) = \xi(I-P)P$ implies that $\Phi(P)(I-\Phi(P)) = \xi(I-\Phi(P))\Phi(P)$. Since $\xi \neq 1$, it follows that $\Phi(P) = \Phi(P)^2$. That is, $\Phi(P)$ is an idempotent.

Now suppose that an idempotent P is of rank one while $\Phi(P)$ is not of rank one. Then $\Phi(P)$ can be written as a sum of an idempotent and a rank one idempotent in $M_n(\mathbb{F})$. Since Φ^{-1} satisfies the same hypotheses as Φ does, what we have just proved shows that the rank one idempotent P can be also written as a sum of two nonzero idempotents, a contradiction. So Φ preserves rank one idempotents. Applying the same discussion to Φ^{-1}, we obtain that Φ^{-1} preserves idempotents and rank one idempotents. Hence Φ preserves rank one idempotents in both directions.

Claim 3. Φ preserves rank one matrices in both directions.

Let P be a rank one idempotent. Then for every nonzero scalar λ, we have $(\lambda P)(I-P) = \xi(I-P)(\lambda P)$, so $\Phi(\lambda P)(I-\Phi(P)) = \xi(I-\Phi(P))\Phi(\lambda P)$, that is,

$$(1-\xi)\Phi(\lambda P)=\Phi(\lambda P)\Phi(P)-\xi\Phi(P)\Phi(\lambda P). \tag{3.1}$$

Since $\Phi(P)$ is a rank one matrix and $\xi \neq 0$, multiplying $\Phi(P)$ in (3.1) from the left and the right, respectively, we get

$$\Phi(\lambda P)\Phi(P)=\Phi(P)\Phi(\lambda P)\Phi(P)=\Phi(P)\Phi(\lambda P). \tag{3.2}$$

Combining (3.1), (3.2) with $\xi \neq 1$, we have

$$\Phi(\lambda P)=\Phi(\lambda P)\Phi(P)=\Phi(P)\Phi(\lambda P)=\Phi(P)\Phi(\lambda P)\Phi(P),$$

which implies that $\Phi(\lambda P)$ is of rank one, and there exists $f_P(\lambda)\in\mathbb{F}$ such that $\Phi(\lambda P)=f_P(\lambda)\Phi(P)$.

Next we prove that Φ preserves rank one nilpotent matrices. Let $N=x\otimes f$ be a rank one nilpotent matrix. Take $f_1\in\mathbb{F}^n$ such that $\langle x,f_1\rangle=1$ and let $f_2=f_1-f$. Then $P_i=x\otimes f_i$ $(i=1,2)$ are rank one idempotents and $N=P_1-P_2=x\otimes f_1-x\otimes f_2$. Let $\Phi(P_i)=y_i\otimes g_i$, $y_i,g_i\in\mathbb{F}^n$. By Claim 2, $\langle y_i,g_i\rangle=1$. Note that $\mathbb{F}$ is of characteristic not 2 and $P=\frac{1}{2}(P_1+P_2)$ is a rank one idempotent. So $\Phi(P)=\frac{1}{2}(y_1\otimes g_1+y_2\otimes g_2)$ is a rank one idempotent, which implies that either y_1,y_2 are linear dependent or g_1,g_2 are linear dependent. Without loss of generality, assume $y_1=y_2$. Thus $\Phi(N)=y_1\otimes g_1-y_1\otimes g_2$ is a rank one nilpotent matrix.

So far we have proved that Φ preserves rank one matrices. A same discussion is applied to Φ^{-1}, we get that Φ^{-1} preserves rank one matrices. Hence Φ preserves rank one matrices in both directions.

Claim 4. There exists a field automorphism $\tau:\mathbb{F}\to\mathbb{F}$ and a nonsingular matrix $T\in M_n(\mathbb{F})$ such that either

(a) $\Phi(A)=TA_\tau T^{-1}$ for all $A\in M_n(\mathbb{F})$, or

(b) $\Phi(A)=T(A_\tau)^{\mathrm{tr}}T^{-1}$ for all $A\in M_n(\mathbb{F})$, where A_τ denotes the matrix obtained from A by applying τ to all the entries of A.

Since Φ is additive and preserves rank one matrices in both directions, it is well known that such maps have the form (a) or (b) (for eg., ref. [4, 13]).

Claim 5. The statements (1)-(2) in the theorem hold true.

If Φ takes the form $\Phi(A)=TA_\tau T^{-1}$ for all $A\in M_n(\mathbb{F})$, then, for any $A,B\in M_n(\mathbb{F})$ with $AB=\xi BA\neq 0$, we have $A_\tau B_\tau=\xi B_\tau A_\tau$. That is, $(AB)_\tau=\xi(BA)_\tau$. This implies that $\tau(\xi)=\xi$. So the statement (1) holds true.

If Φ take the form $\Phi(A)=T(A_\tau)^{\mathrm{tr}}T^{-1}$ for all $A\in M_n(\mathbb{F})$, then, for $A,B\in M_n(\mathbb{F})$ with $AB=\xi BA\neq 0$, we have $(A_\tau)^{\mathrm{tr}}(B_\tau)^{\mathrm{tr}}=\xi(B_\tau)^{\mathrm{tr}}(A_\tau)^{\mathrm{tr}}$, that is, $(BA)_\tau=\xi(AB)_\tau$. So $(BA)_\tau=\xi(\xi BA)_\tau=\xi\tau(\xi)(BA)_\tau$, and hence $\xi\tau(\xi)=1$, i.e., $\tau(\xi)=\xi^{-1}$. So (2) holds true, completing the proof. $\square$

The next result gives a characterization of ξ-Lie multiplicative isomorphisms on $M_n(\mathbb{F})$.

THEOREM 3.2. *Let $M_n(\mathbb{F})$ be the algebra of all $n\times n$ matrices $(n>1)$ over a field $\mathbb{F}$ of characteristic not 2 and ξ be a scalar. Then a unital bijective map Φ :*

$M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is ξ-Lie multiplicative (i.e., $\Phi(AB-\xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ for all $A,B \in M_n(\mathbb{F})$) if and only if the following statements hold:

(1) $\xi = 1$ *and $\mathbb{F}$ is of characteristic not 3. There exists a functional $h : M_n(\mathbb{F}) \to \mathbb{F}$ with $h([A,B]) = 0$ for all A,B, a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$, and a nonsingular matrix $T \in M_n(\mathbb{F})$ such that either $\Phi(A) = TA_\tau T^{-1} + h(A)I$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = -T(A_\tau)^{\mathrm{tr}}T^{-1} + h(A)I$ for all $A \in M_n(\mathbb{F})$.*

(2) $\xi = -1$. *There exists a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and a nonsingular matrix $T \in M_n(\mathbb{F})$ such that either $\Phi(A) = TA_\tau T^{-1}$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = T(A_\tau)^{\mathrm{tr}}T^{-1}$ for all $A \in M_n(\mathbb{F})$.*

(3) $\xi \neq \pm 1$. *There exists a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ satisfying $\tau(\xi) = \xi$ and a nonsingular matrix $T \in M_n(\mathbb{F})$ such that $\Phi(A) = TA_\tau T^{-1}$ for all $A \in M_n(\mathbb{F})$.*

Proof. The "if" part is clear. Let us check the "only if" part.

Assume that Φ is a ξ-Lie multiplicative isomorphism.

For the case of $\xi = 1$, we need a result in [2, Corollary] which states that every Lie multiplicative isomorphism ϕ from a prime ring $\mathscr{R}$ of characteristic neither 2 nor 3 onto a prime ring $\mathscr{R}'$ is of the form $\phi = \psi + h$, where ψ is a ring isomorphism or a negative ring anti-isomorphism and h is a map from $\mathscr{R}$ into the extended centroid of $\mathscr{R}'$ vanishing all commutators. Applying this result to $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$, it is obvious that the statement (1) holds true.

If $\xi = -1$, the statement (2) holds by Theorem 2.1 and Theorem 3.1.

If $\xi = 0$, then, by Theorem 2.1, Φ is a ring isomorphism and thus has the form stated in (3) as $\tau(0) = 0$.

If $\xi \neq 0, \pm 1$, then, by Theorem 2.1 and Theorem 3.1, there exist a nonsingular matrix T and a field automorphism τ such that $\tau(\xi) = \xi$ and $\Phi(A) = TA_\tau T^{-1}$ for all A or $\tau(\xi) = \xi^{-1}$ and $\Phi(A) = T(A_\tau)^{\mathrm{tr}}T^{-1}$ for all A. We claim that the last form never occur. Assume, on the contrary, that the last one occurs, then, the condition $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ implies that $-\xi^{-1} = 1$ and hence $\xi = -1$, a contradiction. So (3) holds true, completing the proof. $\square$

Since the only field automorphism of the real field $\mathbb{R}$ is the identity, the following corollary is immediate.

COROLLARY 3.3. *Let $M_n(\mathbb{R})$ be the algebra of all real $n \times n$ matrices $(n > 1)$ and ξ be a real number. Then a unital bijective map $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is ξ-Lie multiplicative if and only if the following statements hold:*

(1) $\xi = 1$. *There exists a functional $h : M_n(\mathbb{R}) \to \mathbb{R}$ with $h([A,B]) = 0$ for all A,B and a nonsingular matrix $T \in M_n(\mathbb{R})$ such that either $\Phi(A) = TAT^{-1} + h(A)I$ for all $A \in M_n(\mathbb{R})$ or $\Phi(A) = -TA^{\mathrm{tr}}T^{-1} + h(A)I$ for all $A \in M_n(\mathbb{R})$.*

(2) $\xi = -1$. *Then there exists a nonsingular matrix $T \in M_n(\mathbb{R})$ such that either $\Phi(A) = TAT^{-1}$ for all $A \in M_n(\mathbb{R})$ or $\Phi(A) = TA^{\mathrm{tr}}T^{-1}$ for all $A \in M_n(\mathbb{R})$, where A^{tr} denotes the transpose of A.*

(3) $\xi \neq \pm 1$. *There exists a nonsingular matrix $T \in M_n(\mathbb{R})$ such that $\Phi(A) = TAT^{-1}$ for all $A \in M_n(\mathbb{R})$.*

Recall that a subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$, the algebra of all bounded linear operators on a Banach space X, is called a standard operator algebra if it contains all finite rank operators and the identity operator I.

For the infinite dimensional case, we have the following result.

THEOREM 3.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on infinite dimensional Banach spaces X and Y over the real or complex field $\mathbb{F}$, respectively. Assume that $\Phi:\mathscr{A}\to\mathscr{B}$ is a unital bijection and ξ is a scalar. Then Φ is ξ-Lie multiplicative if and only if one of the followings holds:*

(1) $\xi=1$. *There exists a functional $h:\mathscr{A}\to\mathbb{F}$ with $h([A,B])=0$ for all A,B, and either there exists an invertible bounded linear or conjugate linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}+h(A)I$ for all $A\in\mathscr{A}$ or there exists an invertible bounded linear or conjugate linear operator $T:X^*\to Y$ such that $\Phi(A)=-TA^*T^{-1}+h(A)I$ for all $A\in\mathscr{A}$. In the last case, X and Y are reflexive.*

(2) $\xi=-1$. *Either there exists an invertible bounded linear or conjugate linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$ or there exists an invertible bounded linear or conjugate linear operator $T:X^*\to Y$ such that $\Phi(A)=TA^*T^{-1}$ for all $A\in\mathscr{A}$. In the last case, X and Y are reflexive.*

(3) $\xi\in\mathbb{R}\setminus\{\pm1\}$. *There exists an invertible bounded linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$ if $\mathbb{F}=\mathbb{R}$; there exists an invertible bounded linear or conjugate linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$ if $\mathbb{F}=\mathbb{C}$.*

(4) $\xi\in\mathbb{C}\setminus\mathbb{R}$. *There exists an invertible bounded linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$.*

We remark that, by Theorem 3.4, a ξ-Lie multiplicative isomorphism on infinite dimensional Banach space standard operator algebra is automatically continuous. It follows from Theorem 3.2 that every continuous unital ξ-Lie multiplicative isomorphism on $M_n(\mathbb{C})$ has the same form stated in Theorem 3.4.

To prove Theorem 3.4, we need a result from [6], which gives a characterization of the unital additive surjective maps between standard operator algebras on infinite dimensional Banach spaces that preserve zero ξ-Lie products in both directions with $\xi\neq0,\pm1$. To the convenience for readers, we list this result in [6] as a lemma here.

LEMMA 3.5. ([6, Theorem 2.1]) *Let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on real or complex infinite dimensional Banach spaces X and Y, respectively. Assume that $\Phi:\mathscr{A}\to\mathscr{B}$ is a unital additive surjection. Then Φ preserves commutativity up to a factor ξ in both directions with $\xi\neq0,\pm1$ if and only if one of the followings holds:*

(1) *If $\xi\in\mathbb{R}$, then there exists an invertible bounded linear or conjugate linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$.*

(2) *If $\xi\in\mathbb{C}\setminus\mathbb{R}$ and $|\xi|\neq1$, then there exists an invertible bounded linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$.*

(3) *If $|\xi|=1$, then either there exists an invertible bounded linear operator $T:X\to Y$ such that $\Phi(A)=TAT^{-1}$ for all $A\in\mathscr{A}$ or there exists an invertible bounded conjugate linear operator $T:X^*\to Y$ such that $\Phi(A)=TA^*T^{-1}$ for all $A\in\mathscr{A}$. In the last case, X and Y are reflexive.*

Now we give the proof of Theorem 3.4.

Proof. The "if" part is clear. Let us check the "only if" part. Assume that Φ is a ξ-Lie multiplicative isomorphism.

For the case of $\xi = 1$, by [11], it is easily seen that the statement (1) holds true.

If $\xi = -1$, the statement (2) holds by Theorem 2.1 and a result in [15], which gives a characterization of unital additive surjective maps preserving Jordan zero products in both directions between standard operator algebras on infinite dimensional Banach spaces.

If $\xi = 0$, then, by Theorem 2.1, Φ is a ring isomorphism and thus has the form stated in (3) (see [5]).

If $\xi \neq 0, \pm 1$, then, by Theorem 2.1, Φ is additive, and thus preserves zero ξ-Lie products in both directions. So, Lemma 3.5 is applied. The remain is to check that the form $\Phi(A) = TA^*T^{-1}$ for all A in (3) of Lemma 3.5 can not occur. If, on the contrary, $\Phi(A) = TA^*T^{-1}$ for all A, then $T(AB - \xi BA)^*T^{-1} = \Phi([A,B]_\xi) = [\Phi(A), \Phi(B)]_\xi = T(BA - \overline{\xi}AB)^*T^{-1}$. It follows that $AB - \xi BA = BA - \overline{\xi}AB$ holds for all $A, B \in \mathscr{A}$, which is impossible since $\xi \neq -1$. The proof is complete. $\square$

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