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A CONDITIONAL EXPECTATION TYPE OPERATOR ON L^p SPACES

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Abstract. In this paper we discuss some of the basic operator-theoretic characterizations for conditional expectation type operator $T = EM_u$ on L^p spaces.

1. Introduction and Preliminaries

Let $L(X,\Sigma,\mu)$ be a σ-finite measure space. For any complete σ-finite subalgebra $\mathscr{A} \subseteq \Sigma$ with $1 \leqslant p \leqslant \infty$, the L^p-space $L^p(X,\mathscr{A},\mu|\mathscr{A})$ is abbreviated by $L^p(\mathscr{A})$, and its norm is denoted by $\|.\|_p$. We understand $L^p(\mathscr{A})$ as a Banach subspace of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

For any non-negative Σ-measurable function f as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathscr{A}$-measurable function $E(f)$ such that

$$\int_A Ef d\mu = \int_A f d\mu, \qquad \text{for all } A \in \mathscr{A}.$$

Hence we obtain an operator E from $L^p(\Sigma)$ onto $L^p(\mathscr{A})$ which is called conditional expectation operator associated with the σ-algebra $\mathscr{A}$. This operator will play a major role in our work, and we list here some of its useful properties:

- If g is $\mathscr{A}$-measurable then $E(fg) = E(f)g$.
- $|E(f)|^p \leqslant E(|f|^p)$.
- $\|E(f)\|_p \leqslant \|f\|_p$.
- If $f \geqslant 0$ then $E(f) \geqslant 0$; if $f > 0$ then $E(f) > 0$.

Let f be a real-valued measurable function. Consider the set $B_f = \{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$. The function f is said to be conditionable with respect to $\mathscr{A}$, if $\mu(B_f) = 0$. If f is complex-valued, then f is conditionable if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. We denote the linear space of all conditionable Σ-measurable functions on X by $L^0(\Sigma)$. It is known that $|E(f)|^2 = E(|f|^2)$ if and only if $f \in L^0(\mathscr{A})$. For more details on the properties of E see [5], [6] and [9].

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Recall that an $\mathscr{A}$-atom of the measure μ is an element $A \in \mathscr{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called non-atomic. It is well-known fact that every σ-finite measure space $(X, \mathscr{A}, \mu_{|_{\mathscr{A}}})$ can be partitioned uniquely as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathscr{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [12]). Note that since $\mathscr{A}$ is σ- finite, it follows that $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$.

Combination of conditional expectation operator E and multiplication operator M_u appears more often in the service of the study of other operators such as multiplication operators, weighted composition operators and Lambert operators (see [8] and [7]). These operators are closely related to averaging operators on order ideals in Banach lattices and to operators called conditional expectation-type operators introduced in [1]. In this paper, we investigate some of the basic operator-theoretic questions for the conditional type operator $T = EM_u$ between L^p spaces. For a beautiful exposition of the study of weighted conditional expectation operators on L^p-spaces, see [6] and the references therein.

2. The operator $\mathbf{T} = \mathbf{EM_u}$

Let $1 \leqslant p \leqslant \infty$. We shall always take $u \in L^0(\Sigma)$ for which $uf \in L^0(\Sigma)$ for all $f \in L^p(\Sigma)$. In other words, the operator $T = EM_u$ is defined on all $L^p(\Sigma)$. A straightforward calculation shows that for $1 \leqslant p < \infty$, the adjoint operator $T^* : L^q(\mathscr{A}) \to L^q(\Sigma)$ is given by $T^* f = \bar{u} f$, where $\frac{1}{p} + \frac{1}{q} = 1$ (note that we can consider $T^* : L^q(\Sigma) \to L^q(\Sigma)$ as $T^* = M_{\bar{u}} E$). Let $1 \leqslant q < \infty$. It is proved by Alan Lambert in [8] that T^* is a bounded operator if and only if $E(|u|^q) \in L^\infty(\mathscr{A})$. In this case $\|T^*\| = \|E(|u|^q)\|_\infty^{1/q}$. In the case $q = \infty$, we claim that T^* is bounded if and only if $u \in L^\infty(\Sigma)$ and its norm is given by $\|T^*\| = \|u\|_\infty$. Indeed, if $u \in L^\infty(\Sigma)$ and $f \in L^\infty(\mathscr{A})$, we have

$$\begin{aligned} \|\bar{u} f\|_{L^\infty(\mathscr{A})} &= \sup_{A\in\mathscr{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |\bar{u} f| d\mu \\ &\leqslant \|u\|_\infty \sup_{A\in\mathscr{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |f| d\mu = \|u\|_\infty \|f\|_{L^\infty(\mathscr{A})}. \end{aligned}$$

It follows that $T^*(L^\infty(\mathscr{A})) \subseteq L^\infty(\mathscr{A}) \subseteq L^\infty(\Sigma)$, and $\|T^*\| \leqslant \|u\|_\infty$. On the other hand, if T^* is bounded, then

$$\|u\|_\infty = \|\bar{u}\chi_X\|_\infty = \|T^*\chi_X\|_\infty \leqslant \|T^*\| < \infty.$$

These observations establish the following proposition.

PROPOSITION 2.1. *(a) $T = EM_u$ defines a bounded linear operator from $L^1(\Sigma)$ into $L^1(\mathscr{A})$ if and only if $u \in L^\infty(\Sigma)$. In this case $\|T\| = \|u\|_\infty$.*

(b) Let $1 < p < \infty$. T defines a bounded operator from $L^p(\Sigma)$ into $L^p(\mathscr{A})$ if and only if $E(|u|^q) \in L^\infty(\mathscr{A})$, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case $\|T\| = \|E(|u|^q)\|_\infty^{1/q}$.

In the following theorem we investigate a necessary and sufficient condition for T to be compact.

THEOREM 2.2. *Let $1<p<\infty$. Suppose $(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ can be partitioned as $X=(\bigcup_{n\in\mathbb{N}}A_n)\cup B$. Then the bounded linear operator $T=EM_u$ from $L^p(\Sigma)$ into $L^p(\mathscr{A})$ is compact if and only if $u(B)=0$ ($u(x)=0$ for μ-almost all $x\in B$) and for any $\varepsilon>0$, the set $\{n\in\mathbb{N}:\mu(A_n\cap D_\varepsilon(u))>0\}$ is finite, where $D_\varepsilon(u)=\{x\in X:E(|u|)(x)\geqslant\varepsilon\}$.*

Proof. Suppose T is a compact operator. First we show that $u(B)=0$. Suppose the contrary i.e., $\mu\{x\in B:u(x)\neq 0\})>0$. Then there is $\delta>0$ and $B_0\in\mathscr{A}\cap B$ such that $0<\mu(B_0\cap D_\delta(u))<\infty$. Since $J_0:=B_0\cap D_\delta(u)\in\mathscr{A}\cap B_0$ has no atoms, hence we can choose a sequence $\{B_n\}_{n\in\mathbb{N}}\subseteq\mathscr{A}\cap B_0$, such that $J_{n+1}\subseteq J_n\subseteq J_0$, $0<\mu(J_{n+1})=\frac{\mu(J_n)}{2}$, where $J_n:=B_n\cap D_\delta(u)$. Note that for all $n\in\mathbb{N}$, J_n is $\mathscr{A}$-measurable. Put

$$f_n=\frac{\bar{u}|u|^{\frac{q-p}{p}}\chi_{J_n}}{\{\|E(|u|^q)\|_\infty\mu(J_n)\}^{\frac{1}{p}}},\quad n\in\mathbb{N}.$$

Boundedness of T implies that $E(|u|^q)\in L^\infty(\mathscr{A})$ and hence $\|f_n\|_p\leqslant 1$. Now, for any $m,n\in\mathbb{N}$ with $m>n$ we have

$$\|Tf_n-Tf_m\|_p^p=\int_X|E(u(f_n-f_m))|^p d\mu$$

$$=\int_X\frac{[E(|u|^{\frac{q}{p}+1})]^p}{\|E(|u|^q)\|_\infty}\left|\frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{p}}}-\frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{p}}}\right|^p d\mu\geqslant\frac{\delta^{(\frac{q}{p}+1)p}}{\|E(|u|^q)\|_\infty}\int_{J_n\setminus J_m}\frac{d\mu}{\mu(J_n)}$$

$$=\frac{\delta^{q+p}}{\|E(|u|^q)\|_\infty}\frac{\mu(J_n\setminus J_m)}{\mu(J_n)}=\frac{\delta^{q+p}}{\|E(|u|^q)\|_\infty}\left(1-\frac{\mu(J_m}{\mu(J_n)}\right)>\frac{\delta^{q+p}}{2\|E(|u|^q)\|_\infty},$$

which shows that the sequence $\{Tf_n\}_{n\in\mathbb{N}}$ does not contain a convergent subsequence. But this is a contradiction.

Now, we show that for any $\varepsilon>0$ the set $\{n\in\mathbb{N}:\mu(A_n\cap D_\varepsilon(u))>0\}$ is finite. By the way of contradiction, for some $\varepsilon>0$, there is a subsequence $\{A_k\}_{k\in\mathbb{N}}$ of disjoint atoms in $\mathscr{A}$ such that $\mu(A_k\cap D_\varepsilon(u))>0$, for all $k\in\mathbb{N}$. Put $G_k=A_k\cap D_\varepsilon(u)$. Hence, we obtain a sequence of pairwise disjoint sets $\{G_k\}_{k\in\mathbb{N}}$ such that for every $k\in\mathbb{N}$, $G_k\in\mathscr{A}$ and $0<\mu(G_k)=\mu(A_k)<\infty$. For any $k\in\mathbb{N}$, take $f_n=\bar{u}|u|^{\frac{q-p}{p}}\chi_{G_n}/(\|E(|u|^q)\|_\infty\mu(G_n))^{1/p}$. Then $\|f_n\|_p\leqslant 1$. Since for each $n\neq m$, $G_n\cap G_m=\emptyset$, it follows that

$$\|Tf_n-Tf_m\|_p^p\geqslant\int_X\frac{(E(|u|))^{q+p}\chi_{G_n}}{\|E(|u|^q)\|_\infty\mu(G_n)}d\mu+\int_X\frac{(E(|u|))^{q+p}\chi_{G_m}}{\|E(|u|^q)\|_\infty\mu(G_m)}d\mu\geqslant\frac{2\varepsilon^{q+p}}{|E(|u|^q)\|_\infty},$$

which contradicts the compactness of T.

Conversely, suppose that $u(B)=0$ and for an arbitrary $\varepsilon>0$, there exist at most finite $\mathscr{A}$-atoms $\{A_\varepsilon^k\}_{k=1}^n\subseteq\{A_n\}_{n\in\mathbb{N}}$ such that $\mu(A_\varepsilon^k\cap D_\varepsilon(u))>0$. Put $B_\varepsilon=\cup_{k=1}^n A_\varepsilon^k$. Then $E(|u|)<\varepsilon$ on $X\setminus B_\varepsilon$ and hence $|u|<\varepsilon$ on $X\setminus(B_\varepsilon\cup B)$. Set $v=\chi_{B_\varepsilon}u$ and $T_1=EM_v$. It is easy to see that $u=v=0$ on B and $u=v$ on B_ε. Now, since

$B_\varepsilon \cup B \in \mathscr{A}$, then foe each $f \in L^p(\Sigma)$ we have that

$$\begin{aligned}\|(T-T_1)f\|_p^p &= \int_X |E(u-v)f|^p d\mu = \int_{X\setminus(B_\varepsilon\cup B)} |E(uf)|^p d\mu \\ &\leqslant \int_{X\setminus(B_\varepsilon\cup B)} E(|uf|^p)d\mu = \int_{X\setminus(B_\varepsilon\cup B)} |uf|d\mu \leqslant \varepsilon^p \int_X |f|^p d\mu = \varepsilon^p \|f\|_p^p.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}T_1 f &= E(\chi_{B_\varepsilon} uf) = E(\sum_{k=1}^n \chi_{A_\varepsilon^k} uf) = \sum_{k=1}^n E(\chi_{A_\varepsilon^k} uf) \\ &= \sum_{k=1}^n E(uf)(A_\varepsilon^k)\chi_{A_\varepsilon^k} = \sum_{k=1}^n (Tf)(A_\varepsilon^k)\chi_{A_\varepsilon^k}.\end{aligned}$$

Therefore, T_1 has finite rank and hence T is compact. □

REMARK 2.3. Under the same assumptions as in Theorem 2.2, if we take $f_n = \bar{u}\chi_{J_n}/(\|u\|_\infty \mu(J_n))$, then by the same method used in the proof of Theorem 2.2, $T = EM_u$ from $L^1(\Sigma)$ into $L^1(\mathscr{A})$ is compact if and only if $u(B) = 0$ and for any $\varepsilon > 0$, the set $\{x \in X : E(|u|)(x) \geqslant \varepsilon\}$ consists of finitely many atoms.

In the following theorem we show that if $T = EM_u$ is weakly compact on $L^1(\Sigma)$, then it is compact. Recall that the operator $T : L^1(\Sigma) \to L^1(\Sigma)$ is said to be weakly compact if it maps bounded subsets of $L^1(\Sigma)$ into weakly sequentially compact subsets of $L^1(\Sigma)$. We begin with the following lemma, which can be deduced from Theorem IV.8.9, and its Corollaries 8.10, 8.11 in [4].

LEMMA 2.4. *Let H be a weakly sequentially compact set in $L^1(\Sigma)$. Then for each decreasing sequence $\{E_n\}$ in Σ such that* $\lim_{n\to\infty} \mu(E_n) = 0$ *or $\cap_{n=1}^\infty E_n = \emptyset$, the sequence of integrals $\{\int_{E_n} |h| d\mu\}$ converges to zero uniformly for h in H.*

THEOREM 2.5. *Suppose (X, Σ, μ) can be partitioned as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$. Then the bounded operator $T = EM_u$ is a weakly compact operator on $L^1(\Sigma)$ if and only if it is compact.*

Proof. It suffices to show the " only if " part. To prove the theorem, we use the method which inspired by Takagi [10]. Let T be a weakly compact operator on $L^1(\Sigma)$. We first show that $u(B) = 0$. To obtain a contradiction, we may assume that for some $\delta > 0$ and $B_0 \subseteq B$, $0 < \mu(B_0 \cap D_\delta(u)) < \infty$. By the same argument in the proof of Theorem 2.2, as B_0 is non-atomic, we can find a decreasing sequence $\{B_n\} \subseteq B_0 \cap \Sigma$ with $0 < \mu(B_n) < \frac{1}{n}$ and $0 < \mu(B_n \cap D_\delta(u)) < \infty$. Let U be the closed unit ball of $L^1(\Sigma)$. Since $T(U)$ is weakly sequentially compact, we can apply Lemma 2.4, with $H = T(U)$ and $E_n = B_n$. Choose $\varepsilon = \delta^2/\|u\|_\infty$. Then there exists an $n_o \in \mathbb{N}$ such that

$$\int_{B_{n_o}} |Tf| d\mu < \frac{\delta^2}{\|u\|_\infty}, \quad f \in U. \tag{2.1}$$

On the other hand if we take $f_{n_o} = \bar{u}\chi_{J_{n_o}}/(\|u\|_\infty \mu(J_{n_o}))$, we have

$$\begin{aligned}\int_{B_{n_o}} |Tf| d\mu &= \int_{B_{n_o}} \left|E\left(\frac{u\bar{u}\chi_{J_{n_o}}}{\|u\|_\infty \mu(J_{n_o})}\right)\right| d\mu \\ &= \int_{B_{n_o}} E\left(\frac{|u|^2\chi_{J_{n_o}}}{\|u\|_\infty \mu(J_{n_o})}\right) d\mu = \frac{1}{\|u\|_\infty \mu(J_{n_o})} \int_{B_{n_o}} |u|^2 \chi_{J_{n_o}} d\mu \\ &= \frac{1}{\|u\|_\infty \mu(J_{n_o})} \int_{J_{n_o}} |u|^2 d\mu \geqslant \frac{\delta^2}{\|u\|_\infty}.\end{aligned}$$

Since $f_{n_o} \in U$, this contradicts (2.1). According to the Theorem 2.2, it remains to show that for any $\varepsilon > 0$, the set $\mathrm{A} := \{n \in \mathbb{N} : \mu(A_n \cap D_\varepsilon(u)) > 0\}$ is finite. To this end, without loss of generality, we can assume that $\mathrm{A} = \mathbb{N}$ for some $\varepsilon > 0$. Put $\mathrm{K}_n = \{A_k : k \geqslant n\}$. It follows that $\cap_{n=1}^\infty K_n = \emptyset$. Applying Lemma 2.4 once more, there exists an $\mathrm{N} \in \mathbb{N}$ such that

$$\int_{\mathrm{K_N}} |Tf| d\mu < \frac{\varepsilon^2}{\|u\|_\infty}, \quad f \in U.$$

Now, for any n with $n \geqslant \mathrm{N}$, let $g_n = \bar{u}\chi_{A_n}/(\|u\|_\infty \mu(A_n))$. Then we have

$$\int_{\mathrm{K}_N} |Tg_n| d\mu = \int_{\mathrm{K}_N} E\left(\frac{|u|^2\chi_{A_n}}{\|u\|_\infty \mu(A_n)}\right) d\mu = \frac{1}{\|u\|_\infty \mu(A_n)} \int_{A_n} |u|^2 d\mu \geqslant \frac{\varepsilon^2}{\|u\|_\infty}.$$

Since $g_n \in U$, this contradicts (2.1). This completes the proof of the theorem. □

COROLLARY 2.6. *Let $1 \leqslant p < \infty$ and $E(|u|) > 0$ a.e. on X. If the bounded operator $T = EM_u : L^p(\Sigma) \to L^p(\mathscr{A})$ is (weakly) compact, then $\mathscr{A}$ is purely atomic.*

Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces. The set of all bounded linear operators from $\mathscr{K}$ into $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{K},\mathscr{H})$. If $\mathscr{H} = \mathscr{K}$, $\mathscr{B}(\mathscr{H},\mathscr{H})$ will be written by $\mathscr{B}(\mathscr{H})$. For $A \in \mathscr{B}(\mathscr{K},\mathscr{H})$, the range and the null-space of A are denoted by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively. If $A \in \mathscr{B}(\mathscr{H})$, the spectrum of A is denoted by $\mathrm{Sp}(A)$.

Now, we consider matrix form of $T = EM_u$. Notice that $L^2(\Sigma)$ is the direct sum of the $\mathscr{R}(E) = L^2(\mathscr{A})$ with $\mathscr{N}(E) = \{f - Ef : f \in L^2(\Sigma)\}$. With respect to the direct sum decomposition, $L^2(\Sigma) = L^2(\mathscr{A}) \oplus \mathscr{N}(E)$, the matrix form of T is

$$T = \begin{bmatrix} ETE & ET(I-E) \\ (I-E)TE & (I-E)T(I-E) \end{bmatrix} = \begin{bmatrix} M_{Eu} & EM_u \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

In this sequel, we investigate closedness of range and spectrum of T on $L^2(\Sigma)$. We begin with the following lemma, which can be deduced from Theorem 2.3 in [2] and Example 7 in [3].

LEMMA 2.7. *Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces. Suppose that $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{K})$ and $C \in \mathscr{B}(\mathscr{K},\mathscr{H})$.*

(i) If A and B are normal operators, then $Sp\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = Sp(A) \cup Sp(B)$.

(ii) If $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed, then the range $\mathscr{R}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right)$ is closed if and only if at least one of $\dim\mathscr{N}(A^*)$ *or* $\dim\mathscr{N}(B)$ *is finite.*

THEOREM 2.8. *Suppose that the operator $T = EM_u : L^2(\Sigma) \to L^2(\mathscr{A})$ is bounded. Then*

(i) $Sp(T) \cup \{0\} = ess\ range\ \{E(u)\} \cup \{0\}$.

(ii) Let $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$. Then T has closed range if and only if $|E(u)| > 0$ a.e. on X except at most on finitely many atoms.

Proof. (i) If $\mathscr{A} \neq \Sigma$, then $\mathscr{R}(T) \subseteq L^2(\mathscr{A}) \subset L^2(\Sigma)$. Therefore T is not surjective and so $0 \in \mathrm{Sp}(T)$. On the other hand, by Lemma 2.7 (i), since $\mathrm{Sp}(M_{Eu}) =$ ess range $\{E(u)\}$, the result holds.

(ii) It is known that the multiplication operator M_{Eu} has closed range if and only if $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$. Now, by Lemma 2.7 (i) and (2.2) we have:

$$\mathscr{R}(T) \text{ is closed} \iff \mathscr{R}\left(\begin{bmatrix} M_{Eu} & EM_u \\ 0 & 0 \end{bmatrix}\right) \text{ is closed} \iff \dim\mathscr{N}(M_{\overline{Eu}}) < \infty$$
$$\iff |E(u)| > 0 \text{ a.e. on } X \text{ except at most on finitely many atoms.}$$

It is well known that every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathscr{N}(U) = \mathscr{N}(T)$, then this decomposition is called the polar decomposition.

Now, by the operator matrices method we obtain the polar decomposition of $T = EM_u$. Direct computations show that

$$T^*T = \begin{bmatrix} M_{|E(u)|^2} & EM_{u\overline{Eu}} \\ M_{\bar{u}Eu} & M_{\bar{u}}EM_u \end{bmatrix} \text{ and } |T| = \begin{bmatrix} M_{\frac{|E(u)|^2}{\sqrt{E(|u|^2)}}} & EM_{\frac{u\overline{Eu}}{\sqrt{E(|u|^2)}}} \\ M_{\frac{\bar{u}E(u)-|E(u)|^2}{\sqrt{E(|u|^2)}}} & M_{\frac{\bar{u}-\overline{Eu}}{\sqrt{E(|u|^2)}}}EM_u \end{bmatrix}.$$

Then for each $f \in L^2(\Sigma)$ we have that

$$|T|\begin{bmatrix} Ef & f-Ef \end{bmatrix} = \begin{bmatrix} M_{\frac{|E(u)|^2}{\sqrt{E(|u|^2)}}} & EM_{\frac{u\overline{Eu}}{\sqrt{E(|u|^2)}}} \\ M_{\frac{\bar{u}E(u)-|E(u)|^2}{\sqrt{E(|u|^2)}}} & M_{\frac{\bar{u}-\overline{Eu}}{\sqrt{E(|u|^2)}}}EM_u \end{bmatrix}\begin{bmatrix} Ef \\ f-Ef \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\overline{E(u)}E(uf)}{\sqrt{E(|u|^2)}} & \frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}} - \frac{\overline{E(u)}E(uf)}{\sqrt{E(|u|^2)}} \end{bmatrix}.$$

Notice that, since for each conditionable function u, $E(|u|) = 0$ implies that $E(u) = 0 = u$, we used the notational convention of $\frac{u}{\sqrt{E(|u|^2)}}$ for $\frac{u}{\sqrt{E(|u|^2)}}\chi_{\sigma(u)}$.

Now, since the mapping $f \mapsto \begin{bmatrix} Ef & f-Ef \end{bmatrix}$ is an isometric isomorphism from $L^2(\Sigma)$ onto $L^2(\mathscr{A}) \oplus \mathscr{N}(E)$, then we get that $|T|(f) = \frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}}$. Hence for any $f \in L^2(\Sigma)$, $E(uf) = U(\frac{\bar{u}E(uf)}{\sqrt{E(|u|^2)}})$. It is easy to check that $U(f) = \frac{E(uf)}{\sqrt{E(|u|^2)}}$ and U is a partial isometry (see [6]). These calculations establish the following proposition. □

PROPOSITION 2.9. *The polar decomposition of* $T = EM_u$ *on* $L^2(\Sigma)$ *is* $U|T|$, *where* $U = M_{1/\sqrt{E(|u|^2)}}T$ *and* $|T| = M_{\bar{u}/\sqrt{E(|u|^2)}}T$.

Let $p \in (0,\infty)$. Recall that an operator A on a Hilbert space $\mathscr{H}$ is p-hyponormal if $(A^*A)^p \geqslant (AA^*)^p$; A is ∞-hyponormal if A is p-hyponormal for all p; and A is p-quasihyponormal if $A^*(A^*A)^pA \geqslant A^*(AA^*)^pA$. For all unit vectors $x \in \mathscr{H}$, if $\||A|^pU|A|^px\| \geqslant \||A|^px\|^2$, then A is called a p-paranormal operator. By using the property of real quadratic forms (see [11]), A is p-paranormal if and only if

$$|A|^pU^*|A|^{2p}U|A|^p - 2k|A|^{2p} + k^2 \geqslant 0, \quad \text{for all } k \geqslant 0. \tag{2.3}$$

The following lemma is significant amount of consideration for the next computations.

LEMMA 2.10. *Let* $f \in L^2(\Sigma)$ *and* $Af := \bar{u}E(uf)$. *Then for all* $p \in (0,\infty)$

$$A^pf = \bar{u}[E(|u|^2)]^{p-1}E(uf).$$

Proof. Suppose $f \in L^2(\Sigma)$, then by induction we obtain

$$A^{\frac{1}{n}}f = \bar{u}[E(|u|^2)]^{\frac{1-n}{n}}E(uf), \quad n \in \mathbb{N}.$$

Now the reiteration of powers of operator $A^{\frac{1}{n}}$, yields

$$A^{\frac{m}{n}}f = \bar{u}[E(|u|^2)]^{\frac{(1-n)m}{n}}[E(|u|^2)]^{m-1}E(uf), \quad m,n \in \mathbb{N}.$$

Finally, by using of the functional calculus the desired formula is proved. □

LEMMA 2.11. *Let* $T = EM_u$ *be a bounded operator on* $L^2(\Sigma)$. *Then* T *is* ∞-*hyponormal if and only id* $u \in L^\infty(\mathscr{A})$.

Proof. By Lemma 2.10, it is easy to verify that $(T^*T)^p = M_{\bar{u}[E(|u|^2)]^{p-1}}T$ and $(TT^*)^p = M_{[E(|u|^2)]^p}$, for all $0 < p < \infty$. Then we get that $(T^*T)^p \geqslant (TT^*)^p$ if and only if

$$M_{[E(|u|^2)]^{p-1}}(M_{\bar{u}}T - M_{E(|u|^2)}) \geqslant 0 \Longleftrightarrow M_{\bar{u}}T - M_{E(|u|^2)} \geqslant 0,$$

where we have used the fact that $T_1T_2 \geqslant 0$ if $T_1 \geqslant 0$, $T_2 \geqslant 0$ and $T_1T_2 = T_2T_1$ for all $T_i \in \mathscr{B}(\mathscr{H})$. Thus for any $0 < f \in L^2(\mathscr{A})$ we have

$$\begin{aligned}0 \leqslant (M_{\bar{u}}Tf - M_{E(|u|^2)}f, f) &= \int_X (\bar{u}E(uf) - E(|u|^2)f)\bar{f}d\mu \\ &= \int_X (\bar{u}E(u) - E(|u|^2))|f|^2d\mu = \int_X (|E(u)|^2 - E(|u|^2))|f|^2d\mu.\end{aligned}$$

Since $f > 0$, this gives $|E(u)|^2 \geqslant E(|u|^2)$. On the other hand we always have $|E(u)|^2 \leqslant E(|u|^2)$. Hence $u \in L^\infty(\mathscr{A})$. Notice that if $u \in L^\infty(\mathscr{A})$, then it is easy to see that $(T^*T)^p \geqslant (TT^*)^p$. □

THEOREM 2.12. *Let $T = EM_u$ be a bounded operator on $L^2(\Sigma)$. Then the following are equivalent:*

(i) *T is ∞-hyponormal.*

(ii) *T is p-hyponormal.*

(iii) *T is p-quasihyponormal.*

(iv) *T is p-paranormal.*

(v) $u \in L^\infty(\mathscr{A})$.

Proof. By Lemma 2.11, we complete the proof by showing $(iii) \Leftrightarrow (v)$ and $(iv) \Leftrightarrow (v)$ below.

$(iii) \Leftrightarrow (v)$ By Lemma 2.10, it is easy to verify that $T^*(TT^*)^pT = M_{\bar{u}[E(|u|^2)]^p}T$ and $T^*(T^*T)^pT = M_{\bar{u}|E(u)|^2[E(|u|^2)]^{p-1}}T$. Therefore, $T^*(T^*T)^p \geqslant T^*(TT^*)^pT$ if and only if $M_{[E(|u|^2)]^{p-1}}(M_{\bar{u}|E(u)|^2-\bar{u}E(|u|^2)}T) \geqslant 0$. Therefore, for any $0 < f \in L^2(\mathscr{A})$ we have

$$0 \leqslant \int_X (\bar{u}|E(u)|^2 - \bar{u}E(|u|^2))E(u)|f|^2 d\mu = \int_X (|E(u)|^4 - |E(u)|^2 E(|u|^2))|f|^2 d\mu.$$

It follows that $|E(u)|^2 \geqslant E(|u|^2)$ and hence $|E(u)|^2 = E(|u|^2)$. Thus $u \in L^\infty(\mathscr{A})$. Conversely, if $u \in L^\infty(\mathscr{A})$, then

$$T^*(T^*T)^pT = T^*(TT^*)^pT = M_{\bar{u}|u|^{2p}}T,$$

which proves the desired implication.

We now prove $(iv) \Leftrightarrow (v)$. Since $|T|(f) = \frac{\bar{u}}{\sqrt[4]{E(|u|^2)}}E(\frac{uf}{\sqrt[4]{E(|u|^2)}})$, by Lemma 2.10 we get that

$$|T|^p(f) = \bar{u}[E(|u|^2)]^{\frac{p-2}{2}}E(uf), \quad f \in L^2(\Sigma).$$

Also since $U^*(f) = \frac{\bar{u}}{\sqrt{E(|u|^2)}}E(f)$, by a direct computation, we have

$$|T|^pU^*|T|^{2p}U|T|^pf = \bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2E(uf), \quad f \in L^2(\Sigma).$$

By condition (2.3), T is p-paranormal if and only if

$$k^2 - 2kM_{\bar{u}[E(|u|^2)]^{p-1}}T + M_{\bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2}T \geqslant 0, \quad \text{for all } k \geqslant 0$$

$$\Longleftrightarrow M_{\bar{u}[E(|u|^2)]^{2p-2}|E(u)|^2}T \geqslant (M_{\bar{u}[E(|u|^2)]^{p-1}}T)^2 = M_{\bar{u}[E(|u|^2)]^{2p-2}|E(|u|^2)}T.$$

Therefore, for any $0 < f \in L^2(\mathscr{A})$ we have

$$\int_X |E(u)|^2(E(|u|^2)^{2p-2}\left(|E(u)|^2 - E(|u|^2)\right)|f|^2 d\mu \geqslant 0.$$

It follows that $|E(u)|^2 \geqslant E(|u|^2)$ and hence $u \in L^\infty(\mathscr{A})$. Conversely, if $u \in L^\infty(\mathscr{A})$, it is easy to check that condition (2.3) holds for all $k \geqslant 0$. Hence the proof is complete. $\square$

EXAMPLE 2.13. Let $X = [-1,1]$, $d\mu = dx$, Σ the Lebesgue sets, and $\mathscr{A}$ the σ-subalgebra generated by the symmetric sets about the origin. Now any real valued function on X can be written uniquely as a sum of an even function and an odd function, one simply uses the functions $f_e(x) = (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Put $0 < a \leqslant 1$. Then for each $f \in L^2(\Sigma)$ we have $\int_{-a}^{a} E(f)(x)dx = \int_{-a}^{a} f_e(x)dx$ and consequently, $Ef = f_e$. This example is due to Alan Lambert [8]. Now, if u is an even and continuous function on X, then $T = EM_u$ is ∞-hyponormal and hence is p-paranormal. Note that if $u(x) = 1 + x$, then T is not p-paranormal.

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