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MULTIPLIERS OF MULTIDIMENSIONAL FOURIER ALGEBRAS

I. G. TODOROV AND L. TUROWSKA

Abstract. Let G be a locally compact σ-compact group. Motivated by an earlier notion for discrete groups due to Effros and Ruan, we introduce the multidimensional Fourier algebra $A^n(G)$ of G. We characterise the completely bounded multidimensional multipliers associated with $A^n(G)$ in several equivalent ways. In particular, we establish a completely isometric embedding of the space of all n-dimensional completely bounded multipliers into the space of all Schur multipliers on G^{n+1} with respect to the (left) Haar measure. We show that in the case G is amenable the space of completely bounded multidimensional multipliers coincides with the multidimensional Fourier-Stieltjes algebra of G introduced by Ylinen. We extend some well-known results for abelian groups to the multidimensional setting.

1. Introduction

A classical result in Harmonic Analysis asserts that a bounded function defined on a locally compact abelian group G is a multiplier of the Fourier algebra $A(G)$ of G precisely when it is the Fourier transform of a regular Borel measure on the character group $\hat{G}$ of G. After the seminal work of P. Eymard [10], Harmonic Analysis on general locally compact groups has been closely related to the theory of C*- and von Neumann algebras. More recent work of E. Effros, M. Neufang, Zh.-J. Ruan, V. Runde, N. Spronk and others shows that Operator Space Theory plays a significant role in the subject. The operator space structure of $A(G)$ has thus become an indispensable tool in non-commutative Harmonic Analysis. J. de Cannière and U. Haagerup [4] defined the set $M^{cb}A(G)$ of completely bounded multipliers of $A(G)$, and M. Bozejko and G. Fendler [3] provided a characterisation of $M^{cb}A(G)$ which, combined with a classical result of A. Grothendieck [13] and a result of V. Peller [17] shows that $M^{cb}A(G)$ can be isometrically identified with the space of all Schur multipliers of Toeplitz type. An alternative proof of this result was given by P. Jolissaint [14]. N. Spronk [21] showed that this identification is in fact a complete isometry. We refer the reader to Sections 5 and 6 of G. Pisier's monograph [18] for an account of Schur multipliers.

Building on an earlier work on bimeasures on locally compact groups [11], [12], K. Ylinen [22] defined a multivariable version $B^n(G)$ of the Fourier-Stieltjes algebra of a locally compact group. A multivariable version of the Fourier algebra of a discrete group was introduced by E. Effros and Zh.-J. Ruan in [7], and its completely bounded

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multipliers were characterised in terms of a multilinear matrix version of classical Schur multipliers, introduced in the same paper.

In [15], multidimensional Schur multipliers associated with measure spaces were introduced and identified with a natural extended Haagerup tensor product [9] up to an isometry. In the present paper, we show that this identification is a complete isometry. We define the n-dimensional Fourier algebra $A^n(G)$ of an arbitrary locally compact group and show that it is a closed ideal of $B^n(G)$. We characterise the set $M_n^{cb}A(G)$ of completely bounded multipliers associated with $A^n(G)$ in several equivalent ways (Proposition 5.4, Theorem 5.5, Theorem 5.7). In particular, we show that there exists a completely isometric inclusion of $M_n^{cb}A(G)$ into the space of all $n+1$-dimensional Schur multipliers on G with respect to the (left) Haar measure. Its image is a space of multidimensional Schur multipliers of Toeplitz type. Our results imply that if G is amenable then $B^n(G)$ can be completely isometrically identified with $M_n^{cb}A(G)$. In the case G is abelian, we show that $B^n(G)$ can be identified with more general classes of multipliers on G arising from partitions of the variables (Theorem 6.4). In particular, every multiplier of $A^n(G)$ is in this case automatically completely bounded. We obtain a multidimensional version of the classical result that if $\varphi \in \ell^\infty(\mathbb{Z})$ then the function $\tilde{\varphi} \in \ell^\infty(\mathbb{Z}\times\mathbb{Z})$ given by $\tilde{\varphi}(x,y) = \varphi(x-y)$ is a Schur multiplier if and only if φ is the Fourier transform of a regular Borel measure on the unit circle.

2. Preliminaries

We begin by recalling some basic notions and results from P. Eymard's work [10]. If H and K are Hilbert spaces we let $\mathscr{B}(H,K)$ be the space of all bounded linear operators from H into K. We write $\mathscr{B}(H) = \mathscr{B}(H,H)$. Throughout the paper, G will denote a locally compact σ-compact group with a left Haar measure m and a neutral element e. As usual, $L^p(G)$, $p = 1,2$, will denote the space of all complex valued Borel functions f on G such that $|f|^p$ is integrable with respect to m. Integration against m with respect to the variable x will be denoted by dx. The space $L^1(G)$ is an involutive Banach algebra; its enveloping C*-algebra is the *group C*-algebra* $C^*(G)$ of G. We denote by $W^*(G)$ the enveloping von Neumann algebra of $C^*(G)$ and let $\omega : G \to W^*(G)$ be the canonical homomorphism of G into $W^*(G)$. Let λ be the left regular representation of $L^1(G)$ on the Hilbert space $L^2(G)$; the closure of its image in the operator norm is the *reduced C*-algebra* $C_r^*(G)$ of G, and its closure in the weak operator topology is the *group von Neumann algebra* $\mathrm{VN}(G)$ of G. We use the symbol λ to also denote the left regular representation of G on $L^2(G)$.

Let $B(G) = C^*(G)^*$ be the *Fourier-Stieltjes algebra* of G; if $f \in B(G)$ then f can be identified with a function (denoted in the same way and) given by $f(x) = \langle f, \omega(x)\rangle$. Any such f has the form $f(x) = (\pi(x)\xi, \eta)$ for some unitary representation $\pi : G \to \mathscr{B}(H)$ and vectors $\xi, \eta \in H$, and the space $B(G)$ is a Banach algebra with respect to the pointwise product. By $A(G)$ we denote as usual the *Fourier algebra* of G, that is, the ideal of $B(G)$ of all functions f of the form $f(x) = (\lambda_x\xi, \eta)$ where $\xi, \eta \in L^2(G)$. Then $A(G)$ can be canonically identified with the predual of $\mathrm{VN}(G)$: if $f(x) = (\lambda_x\xi, \eta)$, $x \in G$, then $\langle f, T\rangle = (T\xi, \eta)$, $T \in \mathrm{VN}(G)$.

We next recall some notions and facts from Operator Space Theory. We refer the reader to [1], [8], [16] and [19] for further details. An *operator space* is a closed subspace $\mathscr{E}$ of $\mathscr{B}(H,K)$ for some Hilbert spaces H and K. If $n,m \in \mathbb{N}$, we will denote by $M_{n,m}(\mathscr{E})$ the space of all n by m matrices with entries in $\mathscr{E}$ and let $M_n(\mathscr{E}) = M_{n,n}(\mathscr{E})$. Note that $M_{n,m}(\mathscr{E})$ can be identified in a natural way with a subspace of $\mathscr{B}(H^m,K^n)$ and hence carries a natural operator norm. If $n=\infty$ or $m=\infty$, we will denote by $M_{n,m}(\mathscr{E})$ the space of all (singly or doubly infinite) matrices with entries in $\mathscr{E}$ which represent a bounded linear operator between the corresponding amplifications of the Hilbert spaces and set $M_\infty(\mathscr{E}) = M_{\infty,\infty}(\mathscr{E})$. We also write $M_{n,m} = M_{n,m}(\mathbb{C})$ and $M_\infty = M_{\infty,\infty}(\mathbb{C})$. If $\mathscr{E}$ and $\mathscr{F}$ are operator spaces, a linear map $\Phi : \mathscr{E} \to \mathscr{F}$ is called *completely bounded* if the map $\Phi^{(k)} : M_k(\mathscr{E}) \to M_k(\mathscr{F})$, given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$, is bounded for each $k \in \mathbb{N}$ and $\|\Phi\|_{cb} \stackrel{def}{=} \sup_k \|\Phi^{(k)}\| < \infty$. The map Φ is called a *complete isometry* if $\Phi^{(k)}$ is an isometry for each $k \in \mathbb{N}$, and a *complete contraction* if $\|\Phi\|_{cb} \leqslant 1$.

If $\mathscr{E}$ (resp. $\mathscr{F}$) is a linear space and $\|\cdot\|_k$ is a norm on $M_k(\mathscr{E})$ (resp. $M_k(\mathscr{F})$), $k \in \mathbb{N}$, then one may speak of completely bounded, completely contractive and completely isometric mappings from $\mathscr{E}$ into $\mathscr{F}$ as described above. Ruan's celebrated abstract characterisation of operator spaces identifies a set of axioms on the family $(\|\cdot\|_k)_{k=1}^\infty$ of norms in order that $\mathscr{E}$ be completely isometric to an operator space; see [8] for a description of these axioms and applications. An *operator space structure* on a linear space $\mathscr{E}$ is a family $(\|\cdot\|_k)_{k=1}^\infty$, where $\|\cdot\|_k$ is a norm on $M_k(\mathscr{E})$, with respect to which $\mathscr{E}$ is completely isometric to an operator space.

Let $\mathscr{E},\mathscr{E}_1,\dots,\mathscr{E}_n$ be operator spaces, $\Phi : \mathscr{E}_1 \times \dots \times \mathscr{E}_n \to \mathscr{E}$ be a multilinear map and

$$\Phi^{(k)} : M_k(\mathscr{E}_1) \times M_k(\mathscr{E}_2) \times \dots \times M_k(\mathscr{E}_n) \to M_k(\mathscr{E})$$

be the multilinear map given by

$$\Phi^{(k)}(a^1,\dots,a^n)_{p,q} = \sum_{p_2,\dots,p_n} \Phi(a^1_{p,p_2}, a^2_{p_2,p_3},\dots,a^n_{p_n,q}), \tag{1}$$

where $a^i = (a^i_{p,q}) \in M_k(\mathscr{E}_i)$, $1 \leqslant p,q \leqslant k$. The map Φ is called completely bounded if there exists $C>0$ such that for all $k \in \mathbb{N}$ and all elements $a^i \in M_k(\mathscr{E}_i)$, $i=1,\dots,n$, we have

$$\|\Phi^{(k)}(a^1,\dots,a^n)\| \leqslant C\|a^1\|\dots\|a^n\|.$$

If $\mathscr{E}$ and $\mathscr{E}_i$, $i=1,\dots,n$, are dual operator spaces we say that Φ is *normal* if it is weak* continuous in each variable. We denote by $CB^\sigma(\mathscr{E}_1 \times \dots \times \mathscr{E}_n, \mathscr{E})$ the set of all normal completely bounded multilinear maps from $\mathscr{E}_1 \times \dots \times \mathscr{E}_n$ into $\mathscr{E}$; this space can be equipped with an operator space structure in a canonical way (see [9]).

E. Christensen and A. Sinclair [6] gave a characterisation of completely bounded (resp. normal completely bounded) multilinear maps defined on the direct product of finitely many C*-algebras (resp. von Neumann algebras). We will need the following generalisation of Corollaries 5.7 and 5.9 of [6] whose proof is a straightforward generalisation of the proof of Corollary 5.9 of [6]. If $\mathscr{A}$ is a set we let $\mathscr{A}^n = \underbrace{\mathscr{A} \times \dots \times \mathscr{A}}_{n}$.

If $\mathcal{M}$ is a von Neumann algebra and $\mathcal{R}_j \subseteq \mathcal{M}$, $j = 1, \ldots, n-1$, are von Neumann subalgebras, we say that a mapping $\Phi : \mathcal{M}^n \to B(H)$ is $(\mathcal{R}_1, \ldots, \mathcal{R}_{n-1})$-modular if

$$\Phi(a_1 r_1, a_2 r_2, \ldots, a_n) = \Phi(a_1, r_1 a_2, \ldots, r_{n-1} a_n),$$

for all $a_1, \ldots, a_n \in \mathcal{M}$, $r_j \in \mathcal{R}_j$, $j = 1, \ldots, n-1$.

THEOREM 2.1. *Let $\mathcal{M} \subseteq \mathcal{B}(K)$ be a von Neumann algebra, $\mathcal{R}_j \subseteq \mathcal{M}$ be a von Neumann subalgebra, $j = 1, \ldots, n-1$, H be a Hilbert space and $\Phi : \mathcal{M}^n \to \mathcal{B}(H)$ be a multilinear map. The following are equivalent:*

(i) Φ is completely bounded, normal and $(\mathcal{R}_1, \ldots, \mathcal{R}_{n-1})$-modular;

(ii) there exists an index set J and operators $V_j \in M_J(\mathcal{R}'_j)$, $j = 1, \ldots, n-1$, $V_0 \in \mathcal{B}(K^J, H)$ and $V_n \in M_{1,J}(H, K^J)$ such that for all $a_1, \ldots, a_n \in \mathcal{M}$, we have

$$\Phi(a_1, \ldots, a_n) = V_0(a_1 \otimes 1_J) V_1 (a_2 \otimes 1_J) V_2 \ldots V_{n-1}(a_n \otimes 1_J) V_n.$$

Moreover, if (i) holds then $\|\Phi\|_{cb}$ equals the infimum of $\|V_0\| \ldots \|V_n\|$ over all representations of Φ as in (ii) and this infimum is attained.

Tensor products will play a substantial role in the paper. We denote by $V \odot W$ the algebraic tensor product of the vector spaces V and W. If $\mathcal{E}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{E}_2 \subseteq \mathcal{B}(H_2)$ are operator spaces and $u \in \mathcal{E}_1 \odot \mathcal{E}_2$, the *Haagerup norm* of u is given by

$$\|u\|_h = \inf \left\{ \left\| \sum_{j=1}^k a_j a_j^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^k b_j^* b_j \right\|^{\frac{1}{2}} : u = \sum_{j=1}^k a_j \otimes b_j \right\}.$$

The completion $\mathcal{E}_1 \otimes_h \mathcal{E}_2$ of $\mathcal{E}_1 \odot \mathcal{E}_2$ with respect to $\| \cdot \|_h$ is the *Haagerup tensor product* of $\mathcal{E}_1$ and $\mathcal{E}_2$. We refer the reader to [8] for its properties and to [9] for the definition and properties of the extended Haagerup tensor product $\mathcal{E}_1 \otimes_{eh} \mathcal{E}_2$ and the normal Haagerup tensor product $\mathcal{E}_1 \otimes_{\sigma h} \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. We recall the canonical identifications $(\mathcal{E}_1 \otimes_h \mathcal{E}_2)^* = \mathcal{E}_1^* \otimes_{eh} \mathcal{E}_2^*$ and $(\mathcal{E}_1 \otimes_{eh} \mathcal{E}_2)^* = \mathcal{E}_1^* \otimes_{\sigma h} \mathcal{E}_2^*$. If $\delta \in \mathcal{E}_1^*$ then the left slice map $L_\delta : \mathcal{E}_1 \otimes_{eh} \mathcal{E}_2 \to \mathcal{E}_2$ is the unique completely bounded map given on elementary tensors by $L_\delta(a \otimes b) = \delta(a) b$ [9]. Similarly, for $\delta \in \mathcal{E}_2^*$ one defines the right slice map $R_\delta : \mathcal{E}_1 \otimes_{eh} \mathcal{E}_2 \to \mathcal{E}_1$.

If $\mathcal{X}$ is a Banach space we denote by $b_1(\mathcal{X})$ the unit ball of $\mathcal{X}$. Banach space duality is denoted by $\langle \cdot, \cdot \rangle$. We denote by 1_H the identity operator on a Hilbert space H and, for a cardinal J, write $1_J = 1_{\ell^2(J)}$. The identity operator on $\ell^2(\mathbb{N})$ is often denoted simply by 1.

3. The operator space of Schur multipliers

In this section we recall the definition of multidimensional Schur multipliers associated with measure spaces and prove a completely isometric version of the characterisation result, Theorem 3.4, of [15].

Let (X_i,μ_i), $i=1,\dots,n$, be standard measure spaces and

$$\Gamma(X_1,\dots,X_n)=L^2(X_1\times X_2)\odot\dots\odot L^2(X_{n-1}\times X_n),$$

where the direct products are equipped with the corresponding product measures. We identify the elements of $\Gamma(X_1,\dots,X_n)$ with functions on

$$X_1\times X_2\times X_2\times\dots\times X_{n-1}\times X_{n-1}\times X_n$$

in the obvious fashion. We equip $\Gamma(X_1,\dots,X_n)$ with the Haagerup tensor norm $\|\cdot\|_{\mathrm{h}}$, where the L^2-spaces are given their opposite operator space structure (see [19]) arising from the identification $f\longleftrightarrow T_f$ of $L^2(X\times Y)$ with the class of Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$ where, for $f\in L^2(X\times Y)$, we let T_f be the (Hilbert-Schmidt) operator given by

$$(T_f\xi)(y)=\int_X f(x,y)\xi(x)dx,\qquad \xi\in L^2(X),\ y\in Y, \tag{2}$$

dx denoting integration with respect to the corresponding measure on X. If $f\in L^2(X\times Y)$ we let $\|f\|_{\mathrm{op}}$ be equal to the operator norm of T_f.

For each $\varphi\in L^\infty(X_1\times\dots\times X_n)$ let

$$S_\varphi:\Gamma(X_1,\dots,X_n)\to L^2(X_1\times X_n)$$

be the map sending $f_1\otimes\dots\otimes f_{n-1}\in\Gamma(X_1,\dots,X_n)$ to the function which maps (x_1,x_n) to

$$\int\varphi(x_1,\dots,x_n)f_1(x_1,x_2)f_2(x_2,x_3)\dots f_{n-1}(x_{n-1},x_n)dx_2\dots dx_{n-1}.$$

It was shown in Theorem 3.1 of [15] that S_φ is a bounded mapping when $\Gamma(X_1,\dots,X_n)$ is equipped with the projective norm where each of its terms is given the L^2-norm, and that $\|S_\varphi\|=\|\varphi\|_\infty$.

DEFINITION 3.1. A function $\varphi\in L^\infty(X_1\times\dots\times X_n)$ is called a Schur multiplier (relative to the measure spaces $(X_1,\mu_1),\dots(X_n,\mu_n)$) if there exists $C>0$ such that $\|S_\varphi(u)\|_{\mathrm{op}}\leqslant C\|u\|_{\mathrm{h}}$, for all $u\in\Gamma(X_1,\dots,X_n)$. The smallest constant C with this property is denoted by $\|\varphi\|_{\mathrm{m}}$.

Let $H_i=L^2(X_i)$, $i=1,\dots,n$, and $\varphi\in L^\infty(X_1\times\dots\times X_n)$ be a Schur multiplier. It was shown in Section 3 of [15] that φ induces a normal completely bounded multilinear map

$$\tilde S_\varphi:\mathscr{B}(H_{n-1},H_n)\times\dots\times\mathscr{B}(H_1,H_2)\to\mathscr{B}(H_1,H_n)$$

such that $\|\tilde S_\varphi\|_{cb}=\|\varphi\|_{\mathrm{m}}$ and $\tilde S_\varphi(T_{f_{n-1}},\dots,T_{f_1})=S_\varphi(f_1\otimes\dots\otimes f_{n-1})$, for all $f_i\in L^2(X_i\times X_{i+1})$, $i=1,\dots,n$. We denote by $\mathscr{S}=\mathscr{S}(X_1,\dots,X_n)$ the collection of all Schur multipliers in $L^\infty(X_1\times\dots\times X_n)$. It follows that $\mathscr{S}$ can be canonically embedded into $CB^\sigma(\mathscr{B}(H_{n-1},\ H_n)\times\dots\times\mathscr{B}(H_1,H_2),\mathscr{B}(H_1,H_n))$. Thus, $\mathscr{S}$ inherits an operator space structure from the latter space. More precisely, if $\varphi=(\varphi_{p,q})\in M_k(\mathscr{S})$ we have $\|\varphi\|_{\mathrm{m},k} \stackrel{def}{=} \|(\tilde{S}_{\varphi_{p,q}})\|_{\mathrm{cb}}$, where $\tilde{S}_\varphi = (\tilde{S}_{\varphi_{p,q}})$ is identified with a normal completely bounded multilinear map from $\mathscr{B}(H_{n-1},H_n) \times \ldots \times \mathscr{B}(H_1,H_2)$ into $M_k(\mathscr{B}(H_1,H_n))$. Note that a matrix $\varphi = (\varphi_{p,q}) \in M_k(\mathscr{S})$ can be viewed as a map $\varphi : X_1 \times \ldots \times X_n \to M_k$ by letting $\varphi(x_1,\ldots,x_n) = (\varphi_{p,q}(x_1,\ldots,x_n)) \in M_k$.

The following result is a matricial version of Theorem 3.4 of [15].

THEOREM 3.2. *Let $\varphi = (\varphi_{p,q}) \in M_k(\mathscr{S})$. The following are equivalent:*

(i) $\|\varphi\|_{\mathrm{m},k} < 1$;

(ii) there exist essentially bounded functions $a_1 : X_1 \to M_{\infty,k}$, $a_n : X_n \to M_{k,\infty}$ and $a_i : X_i \to M_\infty$, $i = 2,\ldots,n-1$, such that, for almost all $(x_1,\ldots,x_n) \in X_1 \times \ldots \times X_n$, we have

$$\varphi(x_1,\ldots,x_n) = a_n(x_n)a_{n-1}(x_{n-1})\ldots a_1(x_1) \textit{ and } \operatorname*{esssup}_{x_i\in X_i} \prod_{i=1}^{n} \|a_i(x_i)\| < 1.$$

Proof. (i)$\Rightarrow$(ii) Let $\mathscr{D}_i$ be the multiplication masa of $L^\infty(X_i)$. The proof of Theorem 3.4 of [15] implies that the mapping

$$\tilde{S}_\varphi \stackrel{def}{=} (\tilde{S}_{\varphi_{p,q}}) : \mathscr{B}(H_{n-1},H_n) \times \ldots \times \mathscr{B}(H_1,H_2) \to M_k(\mathscr{B}(H_1,H_n))$$

is normal, completely bounded, and $(\mathscr{D}_n,\ldots,\mathscr{D}_1)$-modular in the sense that

$$\tilde{S}_\varphi(A_nT_{n-1}A_{n-1},\ldots,T_1A_1)$$

$$= (A_n \otimes 1_k)\tilde{S}_\varphi(T_{n-1},A_{n-1}T_{n-2},\ldots,A_2T_1)(A_1 \otimes 1_k),$$

whenever $A_i \in \mathscr{D}_i$, $i = 1,\ldots,n$. A modification of Corollary 5.9 of [6] shows that there exist operators $V_1 : H_1^k \to H_1^\infty$, $V_i : H_i^\infty \to H_i^\infty$, $i = 2,\ldots,n-1$ and $V_n : H_n^\infty \to H_n^k$ such that the entries of V_i belong to $\mathscr{D}_i$, $\Pi_{i=1}^n \|V_i\| < 1$ and

$$\tilde{S}_\varphi(T_{n-1},\ldots,T_1) = V_n(T_{n-1} \otimes I)\ldots(T_1 \otimes I)V_1,$$

for all $T_i \in \mathscr{B}(H_i,H_{i+1})$, $i = 1,\ldots,n$. If $V_i = (A_{s,t}^i)_{s,t}$, where $A_{s,t}^i$ is the multiplication operator corresponding to $a_{s,t}^i \in L^\infty(X_i)$ let $a_i : X_i \to M_\infty$ be the function given by $a_i(x_i) = (a_{s,t}^i(x_i))_{s,t}$, $x_i \in X_i$, $i = 1,\ldots,n$. Define $a_1 : X_1 \to M_{\infty,k}$ and $a_n : X_n \to M_{k,\infty}$ similarly. Then $\operatorname{esssup}_{x_i\in X_i} \prod_{i=1}^n \|a_i(x_i)\| = \Pi_{i=1}^n \|V_i\| < 1$.

Let V_n^p (resp. V_1^q) be the pth row (resp. the qth column) of V_n (resp. V_1). Let $a_n^p : X_n \to M_{1,\infty}$ (resp. $a_1^q : X_1 \to M_{\infty,1}$) be the function corresponding to V_n^p (resp. V_1^q). We have that

$$\tilde{S}_{\varphi_{p,q}}(T_{n-1},\ldots,T_1) = V_n^p(T_{n-1} \otimes I)V_{n-1}\ldots V_2(T_1 \otimes I)V_1^q,$$

for all $T_i \in \mathscr{B}(H_i,H_{i+1})$, $i = 1,\ldots,n-1$. It follows from Theorem 3.4 of [15] that

$$\varphi_{p,q}(x_1,\ldots,x_n) = a_n^p(x_n)a_{n-1}(x_{n-1})\ldots a_2(x_2)a_1^q(x_1), \quad \text{a.e. } x_i \in X_i.$$

Since this holds for all $p,q = 1,\dots,k$, we have that

$$\varphi(x_1,\dots,x_n) = a_n(x_n)a_{n-1}(x_{n-1})\dots a_2(x_2)a_1(x_1)$$

for almost all $x_i \in X_i$, $i = 1,\dots,n$.

(ii)$\Rightarrow$(i) In the notation of (i) we have that

$$\varphi_{p,q}(x_1,\dots,x_n) = a_n^p(x_n)a_{n-1}(x_{n-1})\dots a_2(x_2)a_1^q(x_1),$$

for almost all $x_i \in X_i$, $i = 1,\dots,n$, which in turn implies that

$$\tilde{S}_{\varphi_{p,q}}(T_{n-1},\dots,T_1) = V_n^p(T_{n-1}\otimes I)V_{n-1}\dots V_2(T_1\otimes I)V_1^q,$$

and hence that

$$\tilde{S}_{\varphi}(T_{n-1},\dots,T_1) = V_n(T_{n-1}\otimes I)V_{n-1}\dots V_2(T_1\otimes I)V_1,$$

for all $T_i \in \mathscr{B}(H_i,H_{i+1})$, $i = 1,\dots,n-1$. It follows that $\|S_\varphi\| < 1$ and so $\|\varphi\|_{\mathrm{m},k} <$ 1. $\square$

REMARK 3.3. Theorem 3.2 amounts to the statement that the identification of the set of all n-dimensional Schur multipliers on $X_1 \times \dots \times X_n$ with the extended Haagerup tensor product $L^\infty(X_n)\otimes_{eh}\dots\otimes_{eh}L^\infty(X_1)$ discussed in the remark after Theorem 3.4 of [15] is completely isometric.

4. The multidimensional Fourier-Stieltjes algebra

In this section we recall the notion of the Fourier transform of a completely bounded multilinear map on the direct product of finitely many group C*-algebras studied in [22], which will provide the basis for our study of multidimensional multipliers. We discuss a description of the multidimensional Fourier-Stieltjes algebra in terms of tensor products and explain its relation to the one dimensional case as well as to the notion of a bimeasure studied in [11].

Let $n \in \mathbb{N}$. An *n-measure* on G is a completely bounded multilinear map $\Phi : C^*(G)^n \to \mathbb{C}$. We note that the term "bimeasure" was used in [11] to designate a bounded bilinear form on $C_0(G)\times C_0(H)$, where G and H are locally compact groups. We will show below that in the case $H = G$ is abelian, the notion of a bimeasure agrees with that of a 2-measure. In general, however, this notion is different from ours; we note that a multivariable generalisation of it was introduced and studied in [23].

We let $M^n(G)$ denote the space of all n-measures on G; by the universal property of the Haagerup tensor product, we have that

$$M^n(G) \equiv \Bigg(\underbrace{C^*(G)\otimes_h\dots\otimes_h C^*(G)}_{n}\Bigg)^*.$$

We equip $M^n(G)$ with the standard operator space structure of a dual operator space arising from the above identification. Suppose that $\Phi \in M^n(G)$. It is standard (see p. 156 of [22]) to extend Φ to a normal completely bounded map

$$\tilde{\Phi} : \underbrace{W^*(G) \otimes_{\sigma h} \ldots \otimes_{\sigma h} W^*(G)}_{n} \to \mathbb{C}.$$

Let

$$B^n(G) = \{f \in L^\infty(G^n) : \text{ there exists } \Phi \in M^n(G) \text{ such that}$$
$$f(x_1,\ldots,x_n) = \tilde{\Phi}(\omega(x_1),\ldots,\omega(x_n)),\ x_1,\ldots,x_n \in G\}. \tag{3}$$

Since $\{\omega(x) : x \in G\}$ generates $W^*(G)$ as a von Neumann algebra, we have that the element $\Phi \in M^n(G)$ associated with $f \in B^n(G)$ in (3) is unique. We call f the Fourier transform of Φ and write $f = \hat{\Phi}$. Thus, $B^n(G)$ is in one-to-one correspondence with $M^n(G)$; we equip it with the operator space structure arising from this correspondence. Thus, if $(f_{p,q}) \in M_k(B^n(G))$ and $\Phi_{p,q} \in M^n(G)$ is such that $\hat{\Phi}_{p,q} = f_{p,q}$, we have that $\|(f_{p,q})\|_{M_k(B^n(G))} = \|(\Phi_{p,q})\|_{M_k(M^n(G))}$. Since the map $x \to \omega(x)$ is weak* continuous, the space $B^n(G)$ consists of separately continuous functions. By Corollary 5.4 of [22], $B^n(G)$ is closed under the pointwise product. By [2],

$$B^n(G) \equiv \underbrace{B(G) \otimes_{eh} \ldots \otimes_{eh} B(G)}_{n} \tag{4}$$

up to a complete isometry. We note that if $f \in B^n(G)$ and $a_i \in L^1(G)$, $i = 1,\ldots,n$, then

$$\langle a_1 \otimes \ldots \otimes a_n, f\rangle = \int_{G^n} f(x_1,\ldots,x_n) a_1(x_1) \ldots a_n(x_n) dx_1 \ldots dx_n. \tag{5}$$

Indeed, (5) is obviously true if f is an elementary tensor, and by linearity, if f is in the algebraic tensor product of n copies of $B(G)$. If $f \in B^n(G)$ then there exists a bounded net $\{f_\nu\}_\nu$ in the algebraic tensor product of n copies of $B(G)$ which tends to f in the topology determined by the duality between $B^n(G)$ and $\underbrace{W^*(G) \odot \ldots \odot W^*(G)}_{n}$ [9]. But then

$$\begin{aligned} f_\nu(x_1,\ldots,x_n) &= \langle f_\nu, \omega(x_1) \otimes \ldots \otimes \omega(x_n)\rangle \\ &\to \langle f, \omega(x_1) \otimes \ldots \otimes \omega(x_n)\rangle = f(x_1,\ldots,x_n) \end{aligned}$$

for all $x_1,\ldots,x_n \in G$ and (5) follows from the Lebesgue Dominated Convergence Theorem.

The following fact proved in [22] will be of importance to us.

THEOREM 4.1. [22] *A function f belongs to $B^n(G)$ if and only if there exist a Hilbert space H, vectors $\xi, \eta \in H$ and strongly continuous unitary representations π_i of G on H, $i = 1,\ldots,n$, such that*

$$f(x_n,\ldots,x_1) = (\pi_n(x_n) \ldots \pi_1(x_1)\xi, \eta),\ \ x_1,\ldots,x_n \in G.$$

Moreover, the norm of f equals the infimum of the products $\|\xi\|\|\eta\|$ over all representations of f of the above form.

Theorem 4.1 has the following consequence.

COROLLARY 4.2. *The functions from $B^n(G)$ are jointly continuous.*

Proof. The statement follows from Theorem 4.1 and the fact that operator multiplication is strongly continuous on bounded subsets. □

We recall that an operator space which is also a Banach algebra is called a completely contractive Banach algebra if the product is completely contractive with respect to the operator projective tensor norm (see [8] for the definition of this tensor norm). It is known that $B(G)$ is a completely contractive Banach algebra; the next result shows that this remains true in the multivariable setting.

PROPOSITION 4.3. *The operator space $B^n(G)$ is a completely contractive Banach algebra.*

Proof. Let $(f_{p,q}) \in M_k(B^n(G))$ (resp. $(g_{s,t}) \in M_m(B^n(G))$) and $\Phi_{p,q}$ (resp. $\Psi_{s,t}$) be the n-measure such that $\hat{\Phi}_{p,q} = f_{p,q}$ (resp. $\hat{\Psi}_{s,t} = g_{s,t}$). Let $\Phi = (\Phi_{p,q})$ (resp. $\Psi = (\Psi_{s,t})$); then Φ (resp. Ψ) can be viewed as a completely bounded mapping from $C^*(G)^n$ into M_k (resp. M_m). Moreover, $\|(f_{p,q})\|_{M_k(B^n(G))} = \|\Phi\|_{cb}$ and $\|(g_{s,t})\|_{M_m(B^n(G))} = \|\Psi\|_{cb}$.

Since Φ is completely bounded, there exist representations $\pi_1,\dots,\pi_n$ of $C^*(G)$ on Hilbert spaces $H_1,\dots,H_n$, respectively, and operators $V_0,\dots,V_{n+1}$, where $V_{n+1} : H_n \to \mathbb{C}^k$, $V_0 : \mathbb{C}^k \to H_1$ and $V_i : H_i \to H_{i+1}$, $i = 1,\dots,n$, such that

$$\Phi(a_1,\dots,a_n) = V_{n+1}\pi_n(a_n)V_n \dots V_1\pi_1(a_1)V_0, \quad a_1,\dots,a_n \in C^*(G), \tag{6}$$

and $\|\Phi\|_{cb} = \|V_0\| \dots \|V_{n+1}\|$. We determine representations $\rho_1,\dots,\rho_n$ and operators $W_0,\dots,W_{n+1}$ associated with Ψ in a similar fashion.

Consider the mapping $\Omega : C^*(G)^n \to M_{km}$, where for $a_1,\dots,a_n \in C^*(G)$ we set $\Omega(a_1,\dots,a_n)$ to be equal to

$$(V_{n+1} \otimes W_{n+1})(\pi_n \otimes \rho_n(a_n))(V_n \otimes W_n) \dots (V_1 \otimes W_1)(\pi_1 \otimes \rho_1(a_1))(V_0 \otimes W_0).$$

Let $\tilde{\Phi}$ (resp. $\tilde{\Psi}$, $\tilde{\Omega}$) be the canonical extensions of Φ (resp. Ψ, Ω) to a normal completely bounded map from $W^*(G)^n$ into M_k (resp. M_m, M_{km}). We have that $\tilde{\Phi}$ is given as in (6) but with the extension $\tilde{\pi}_i$ of π_i to $W^*(G)$ in the place of π_i. Similar formulas hold for $\tilde{\Psi}$ and $\tilde{\Omega}$.

It now follows that

$$\tilde{\Omega}(\omega(x_1),\dots\omega(x_n)) = \tilde{\Phi}(\omega(x_1),\dots\omega(x_n)) \otimes \tilde{\Psi}(\omega(x_1),\dots\omega(x_n)),$$

for all $x_1,\dots,x_n \in G$. Thus,

$$\begin{aligned}\|(f_{p,q}g_{s,t})_{p,q,s,t}\|_{M_{km}(B^n(G))} &= \|\tilde{\Omega}\|_{cb} \leqslant \|V_0 \otimes W_0\| \dots \|V_{n+1} \otimes W_{n+1}\| \\ &= \|V_0\| \dots \|V_{n+1}\| \|W_0\| \dots \|W_{n+1}\| \\ &= \|(f_{p,q})_{p,q}\|_{M_k(B^n(G))} \|(g_{s,t})_{s,t}\|_{M_m(B^n(G))}.\end{aligned}$$

The proof is complete. $\square$

We note that Theorem 4.1 implies that $B^1(G)$ coincides with the Fourier-Stieltjes algebra $B(G)$ of the group G introduced by Eymard [10].

Suppose that G is abelian and $n = 2$. In this case $M^2(G)$ coincides with the set of all bimeasures on the character group $\hat{G}$ of G studied in [12], while $B^2(G)$ coincides with the set of their Fourier transforms. Indeed, let $\Phi \in M^2(G)$. Since G is abelian, $C^*(G)$ is canonically *-isomorphic to $C_0(\hat{G})$. Thus, Φ can be considered as a bounded bilinear form on $C_0(\hat{G}) \times C_0(\hat{G})$ (in other words, a *bimeasure* on $\hat{G}$ in the sense of [12]). On the other hand, for any locally compact Hausdorff space X there exists a canonical injection $\iota : \mathscr{L}^\infty(X) \to C_0(X)^{**}$ (where $\mathscr{L}^\infty(X)$ is the algebra of all bounded Borel functions on X) given by $\iota(f)(\mu) = \int_X f d\mu$, $\mu \in C_0(X)^*$. Let $\Phi_1 : \mathscr{L}^\infty(\hat{G}) \times \mathscr{L}^\infty(\hat{G}) \to \mathbb{C}$ be the extension of Φ described in Corollary 1.3 of [12]. If $x \in G$ let $\check{x}$ be the character of $\hat{G}$ corresponding to x^{-1}. It is straightforward to check that

$$\iota(\check{x}) = \omega(x). \tag{7}$$

We next observe that

$$\tilde{\Phi}(\iota(f), \iota(g)) = \Phi_1(f, g), \quad f, g \in \mathscr{L}^\infty(\hat{G}). \tag{8}$$

To this end, let μ_1 and μ_2 be probability measures associated with Φ through Grothendieck's inequality and let $\{f_\alpha\} \subseteq C_0(\hat{G})$ and $\{g_\alpha\} \subseteq C_0(\hat{G})$ be bounded nets such that $f_\alpha \to \iota(f)$ and $g_\alpha \to \iota(g)$ in the weak* topology of $W^*(G)$. Then $f_\alpha \to f$ in $L^2(\hat{G}, \mu_1)$ and $g_\alpha \to g$ in $L^2(\hat{G}, \mu_2)$. By the definition of $\Phi_1(f, g)$ (see [12]), we have that it is the limit of the net $\{\Phi(f_\alpha, g_\alpha)\}_\alpha$. Identity (8) now follows by approximation.

Now note that (7) and (8) imply

$$\Phi_1(\check{x}, \check{y}) = \tilde{\Phi}(\omega(x), \omega(y)), \quad x, y \in G.$$

It follows from Definition 1.10 of [12] that $B^2(G)$ coincides with the set of all Fourier transforms of bimeasures on $\hat{G}$.

5. Multipliers of $A^n(G)$: non-abelian groups

In this section, we introduce the multidimensional Fourier algebra $A^n(G)$ of a locally compact group G. For each partition $\mathscr{P}$ of the set $\{n, \dots, 1\}$ into k subsets, we define a completely isometric embedding of $A^k(G)$ into $A^n(G)$. Using these embeddings, we define the (completely bounded) multipliers of G relative to $\mathscr{P}$. We characterise the completely bounded multipliers corresponding to the partition with $k = 1$ in a number of ways, generalising results from [7] and [21].

Let

$$A^n(G) = \{f \in L^\infty(G^n) : \text{ there exists a normal c.b. multilinear map}$$

$$\Phi : \mathrm{VN}(G)^n \to \mathbb{C} \text{ such that } f(x_n, \dots, x_1) = \Phi(\lambda_{x_n}, \dots, \lambda_{x_1})\}.$$

Since $\{\lambda_x : x \in G\}$ generates $\mathrm{VN}(G)$ as a von Neumann algebra, the element Φ associated with $f \in A^n(G)$ in the above definition is unique. As before, we call f the Fourier transform of Φ and write $f = \hat{\Phi}$. Set $\mathrm{VN}(G)^{\otimes^n_{\sigma h}} = \underbrace{\mathrm{VN}(G) \otimes_{\sigma h} \dots \otimes_{\sigma h} \mathrm{VN}(G)}_{n}$. By [9], $A^n(G)$ can be identified with the predual of the operator space $\mathrm{VN}(G)^{\otimes^n_{\sigma h}}$ (see [9]). Hence, $A^n(G)$ possesses a canonical operator space structure; up to a complete isometry,

$$A^n(G) \equiv \underbrace{A(G) \otimes_{eh} \dots \otimes_{eh} A(G)}_{n}.$$

In particular, $\|f\|_{A^n(G)}$ is by definition equal to the completely bounded norm of its associated map Φ. Moreover, the elements $f \in A^n(G)$ have the form

$$f(x_n, \dots, x_1) = \langle \lambda_{x_n} \otimes \dots \otimes \lambda_{x_1}, f \rangle, \ \ x_n, \dots, x_1 \in G.$$

It follows from Corollary 5.7 of [6] that a function $f \in L^\infty(G^n)$ belongs to $A^n(G)$ if and only if there exists an index set J, operators $V_i \in \mathscr{B}(L^2(G)^J)$, $i = 1, \dots, n-1$ and vectors $\xi, \eta \in L^2(G)^J$ such that for all $x_n, \dots, x_1 \in G$ we have

$$f(x_n, \dots, x_1) = \big((\lambda_{x_n} \otimes 1_J) V_{n-1} (\lambda_{x_{n-1}} \otimes 1_J) V_{n-2} \dots (\lambda_{x_1} \otimes 1_J) \xi, \eta \big). \tag{9}$$

Moreover, $\|f\|_{A^n(G)}$ is equal to the infimum of $\|V_1\| \dots \|V_{n-1}\| \|\xi\| \|\eta\|$ over all representations of the form (9) and this infimum is attained.

A fundamental fact proved by Eymard [10] is that $A(G)$ is an ideal of $B(G)$. We now prove the multidimensional version of this result. In the case G is discrete, this was stated in [7] (p. 214).

THEOREM 5.1. *$A^n(G)$ is a closed ideal of $B^n(G)$.*

Proof. We only consider the case $n = 2$; the general case can be treated similarly. Let $f \in A^2(G)$. Then $f(x,y) = ((\lambda_x \otimes 1_J) V (\lambda_y \otimes 1_J) \xi, \eta)$ for some index set J, vectors $\xi, \eta \in L^2(G)^J$ and a bounded operator $V \in \mathscr{B}(L^2(G)^J)$. Letting π be the ampliation of multiplicity J of the left regular representation of $C^*(G)$ on $L^2(G)^J$ and $\Phi \in (C^*(G) \otimes_h C^*(G))^*$ be given by $\Phi(a,b) = (\pi(a) V \pi(b) \xi, \eta)$ we see that $f = \hat{\Phi}$ and hence $f \in B^2(G)$. Thus, $A^2(G) \subseteq B^2(G)$; from the injectivity of the extended Haagerup tensor product it is clear that $A^2(G)$ is closed.

Now let $f \in A^2(G)$ be given as in the first paragraph and $g \in B^2(G)$. By Theorem 4.1, $g(x,y) = (\pi(x)\rho(y)\xi', \eta')$ for some representations $\pi, \rho : G \to H$ and vectors $\xi', \eta' \in H$. Thus,

$$(fg)(x,y) = (((\lambda_x \otimes 1_J \otimes \pi(x)))(V \otimes 1_H)(\lambda_y \otimes 1_J \otimes \rho(y))(\xi \otimes \xi'), \eta \otimes \eta').$$

By [4, Lemma 2.1], there exist unitary operators U and W and index sets J' and J'' such that $U(\lambda_x \otimes 1_J \otimes \pi(x))U^* = \lambda_x \otimes 1_{J'}$ and $W(\lambda_y \otimes 1_J \otimes \rho(y))W^* = \lambda_y \otimes 1_{J''}$. It follows that

$$(fg)(x,y) = (((\lambda_x \otimes 1_{J'}) T (\lambda_y \otimes 1_{J''}) \xi_0, \eta_0),$$

where $T = U(V \otimes I_H)W^*$, $\xi_0 = W(\xi \otimes \xi')$ and $\eta_0 = U(\eta \otimes \eta')$. This clearly implies that $fg \in A^2(G)$. $\square$

Suppose that $1 \leqslant k \leqslant n$. By a block (k,n)-partition we mean a partition of the ordered set $\{n, n-1, \ldots, 1\}$ into k subsets of the form $\{\{n, \ldots, n_{k-1}\}, \ldots, \{n_1 - 1, \ldots, 1\}\}$ where $n \geqslant n_{k-1} > \ldots > n_1 > 1$. Suppose that $\mathscr{P}$ is the block (k,n)-partition associated with the sequence $n \geqslant n_{k-1} > \ldots > n_1 > 1$ as above. We define a mapping $\theta_{\mathscr{P}} : A^k(G) \to A^n(G)$ by letting $(\theta_{\mathscr{P}} f)(x_n, \ldots, x_1) = f(y_k, \ldots, y_1)$ where $y_i = x_{n_i - 1} \ldots x_{n_{i-1}}$, $i = 1, \ldots, k$, and we have set $n_0 = 1$, $n_k = n+1$. It follows from (9) that $\theta_{\mathscr{P}}$ maps $A^k(G)$ into $A^n(G)$. We let $\theta = \theta_{\mathscr{P}_0}$ where $\mathscr{P}_0$ is the $(1,n)$-partition; thus, θ maps $A(G)$ into $A^n(G)$.

If $\mathscr{A}$ and $\mathscr{B}$ are algebras and $\mathscr{P}$ is the (k,n)-partition associated with the sequence $n \geqslant n_{k-1} > \ldots > n_1 > 1$, we say that a map $\Phi : \mathscr{A}^n \to \mathscr{B}$ is $\mathscr{P}$-modular if

$$\Phi(a_n, \ldots, a_i a, a_{i-1}, \ldots, a_1) = \Phi(a_n, \ldots, a_i, a a_{i-1}, \ldots, a_1)$$

whenever $a, a_1, \ldots, a_n \in \mathscr{A}$ and $i \notin \{1, n_1, \ldots, n_{k-1}\}$.

PROPOSITION 5.2. *For each block (k,n)-partition $\mathscr{P}$, the map $\theta_{\mathscr{P}} : A^k(G) \to A^n(G)$ is a completely isometric homomorphism. Moreover,*

$$\operatorname{ran} \theta_{\mathscr{P}} = \{\hat{\Psi} : \Psi : \mathrm{VN}(G)^n \to \mathbb{C} \textit{ is } \mathscr{P}\textit{-modular}\}.$$

Proof. Suppose that $\mathscr{P}$ is associated with the sequence $n \geqslant n_{k-1} > \ldots > n_1 > 1$. It is obvious that $\theta_{\mathscr{P}}$ is linear and multiplicative. Suppose that $(f_{p,q}) \in M_r(A^k(G))$ and let $\Phi_{p,q} : \mathrm{VN}(G)^k \to \mathbb{C}$ be such that $\hat{\Phi}_{p,q} = f_{p,q}$. Set $\Phi = (\Phi_{p,q})$; then Φ can be viewed as a completely bounded multilinear mapping from $\mathrm{VN}(G)^k$ into M_r. There exist an index set J and operators $V_1, \ldots, V_{k-1} \in \mathscr{B}(L^2(G)^J)$, $V_0 : \mathbb{C}^r \to L^2(G)^J$ and $V_k : L^2(G)^J \to \mathbb{C}^r$ such that

$$\Phi(\lambda_{y_k}, \ldots, \lambda_{y_1}) = V_k(\lambda_{y_k} \otimes 1_J) V_{k-1} (\lambda_{y_{k-1}} \otimes 1_J) V_{k-2} \ldots V_1 (\lambda_{y_1} \otimes 1_J) V_0$$

and $\|\Phi\|_{cb} = \Pi_{i=0}^k \|V_i\|$. Let $\Psi_{p,q} : \mathrm{VN}(G)^n \to \mathbb{C}$ be such that $\hat{\Psi}_{p,q} = \theta_{\mathscr{P}}(f_{p,q})$, $1 \leqslant p, q \leqslant r$ and $\Psi = (\Psi_{p,q})$. Then

$$\Psi(\lambda_{x_n}, \ldots, \lambda_{x_1}) = V_k(\lambda_{x_n \ldots x_{n_{k-1}}} \otimes 1_J) V_{k-1} \ldots (\lambda_{x_{n_1 - 1} \ldots x_1} \otimes 1_J) V_0. \tag{10}$$

It follows that

$$\|(\theta_{\mathscr{P}}(f_{p,q}))\|_{M_r(A^n(G))} \leqslant \Pi_{i=0}^k \|V_i\| = \|(f_{p,q})\|_{M_r(A^k(G))},$$

Thus, $\theta_{\mathscr{P}}$ is completely contractive.

Suppose that for some $f \in A^k(G)$ we have $\theta_{\mathscr{P}}(f) = 0$. This implies that $f(x_n \ldots x_{n_{k-1}}, \ldots, x_{n_1 - 1} \ldots x_1) = 0$ for all $x_i \in G$, $i = 1, \ldots, n$. Setting $x_i = e$ whenever $i \notin \{1, n_1, \ldots, n_{k-1}\}$, we see that $f = 0$. Thus, $\theta_{\mathscr{P}}$ is injective.

Fix $f = (f_{p,q}) \in M_r(A^k(G))$. It is clear from (10) that the element $\Psi = (\Psi_{p,q})$ for which $\hat{\Psi}_{p,q} = \theta_{\mathscr{P}}(f_{p,q})$ is $\mathscr{P}$-modular. By Theorem 2.1,

$$\|\theta_{\mathscr{P}}^{(r)}(f)\|_{M_r(A^n(G))} = \inf \Pi_{i=0}^k \|V_i\|,$$

where the infimum is taken over all operators V_i for which $\Psi(\lambda_{x_n},\dots,\ \lambda_{x_1})$ equals the right hand side of (10), for all $x_1,\dots,x_n \in G$. Since θ is injective, if (10) is a representation for Ψ then

$$f(y_k,\dots,y_1) = V_k(\lambda_{y_k}\otimes 1_J)V_{k-1}(\lambda_{y_{k-1}}\otimes 1_J)V_{k-2}\dots(\lambda_{x_1}\otimes 1_J)V_0,$$

for all $y_1,\dots,y_k \in G$. It follows that $\|f\|_{M_r(A^k(G))} \leqslant \Pi_{i=0}^k \|V_i\|$ and so $\|f\|_{M_r(A^k(G))} \leqslant \|\theta_{\mathscr{P}}^{(r)}(f)\|_{M_r(A^n(G))}$. Thus, $\theta_{\mathscr{P}}$ is a complete isometry.

Let $\Psi : \mathrm{VN}(G)^n \to \mathbb{C}$ be $\mathscr{P}$-modular. It remains to show that $\hat{\Psi} \in \operatorname{ran}\theta_{\mathscr{P}}$. By Theorem 2.1, there exist an index set and operators $V_1,\dots,V_{k-1}$ and vectors ξ,η such that

$$\Psi(a_n,\dots,a_1) = \left((a_n\dots a_{n_k}\otimes 1_J)V_{k-1}\dots V_1(a_{n_1-1}\dots a_1\otimes 1_J)\xi,\eta\right),$$

$a_1,\dots,a_n \in \mathrm{VN}(G)$. Letting $f \in A^k(G)$ be the function

$$f(y_k,\dots,y_1) = \left((\lambda_{y_k}\otimes 1_J)V_{k-1}(\lambda_{y_{k-1}}\otimes 1_J)V_{k-2}\dots V_1(\lambda_{y_1}\otimes 1_J)\xi,\eta\right),$$

we see that $\theta_{\mathscr{P}}(f) = \hat{\Psi}$. $\square$

DEFINITION 5.3. Let $\mathscr{P}$ be a block (k,n)-partition. We call a function $\varphi \in L^\infty(G^n)$ a $\mathscr{P}$-multiplier of $A(G)$ if

$$f \in A^k(G) \ \Rightarrow\ \varphi\theta_{\mathscr{P}}(f) \in A^n(G).$$

We denote by $M_{\mathscr{P}}A(G)$ the collection of all $\mathscr{P}$-multipliers of $A(G)$.

If $\varphi \in M_{\mathscr{P}}A(G)$ and the map $f \to \varphi\theta_{\mathscr{P}}(f)$ from $A^k(G)$ into $A^n(G)$ is completely bounded we call φ a completely bounded (or c.b.) $\mathscr{P}$-multiplier of $A(G)$. We denote by $M^{cb}_{\mathscr{P}}A(G)$ the collection of all c.b. $\mathscr{P}$-multipliers of $A(G)$.

If $\mathscr{P}$ is the block $(1,n)$-partition we set $M_nA(G) = M_{\mathscr{P}}A(G)$ and $M_n^{cb}A(G) = M^{cb}_{\mathscr{P}}A(G)$.

REMARKS. (i) If $k = n = 1$ the above definition reduces to that of multipliers and completely bounded multipliers of $A(G)$.

(ii) An application of the Closed Graph Theorem shows that if $\varphi \in M_{\mathscr{P}}A(G)$ then the map $f \to \varphi\theta_{\mathscr{P}}(f)$ from $A^k(G)$ into $A^n(G)$ is bounded.

PROPOSITION 5.4. *Let $\mathscr{P}$ be the block (k,n)-partition associated with the sequence $n \geqslant n_{k-1} > \dots > n_1 > 1$. The following are equivalent:*

(i) $\varphi \in M^{cb}_{\mathscr{P}}A(G)$;

(ii) The map

$$(\lambda_{x_n},\dots,\lambda_{x_1}) \to \varphi(x_n,\dots,x_1)\lambda_{x_n\dots x_{n_k}}\otimes\lambda_{x_{n_k-1}\dots x_{n_{k-1}}}\otimes\dots\otimes\lambda_{x_{n_1-1}\dots x_1}$$

extends to a c.b. normal map $\Phi_\varphi : \mathrm{VN}(G)^n \to \mathrm{VN}(G)^{\otimes^k_{\sigma h}}$.

Proof. Suppose that the map $T_\varphi : A^k(G) \to A^n(G)$ given by $f \to \varphi\theta(f)$ is completely bounded. Then its adjoint

$$T_\varphi^* : \mathrm{VN}(G)^{\otimes^n_{\sigma h}} \to \mathrm{VN}(G)^{\otimes^k_{\sigma h}}$$

is completely bounded. For $x_1,\dots,x_n \in G$ set $y_k = x_n \dots x_{n_k}, \dots, y_1 = x_{n_1-1}\dots x_1$. If $f \in A(G)$ we have

$$\begin{aligned}
&\langle T_\varphi^*(\lambda_{x_n}\otimes\dots\otimes\lambda_{x_1}), f\rangle = \langle \lambda_{x_n}\otimes\dots\otimes\lambda_{x_1}, T_\varphi f\rangle\\
&= \langle \lambda_{x_n}\otimes\dots\otimes\lambda_{x_1}, \varphi\theta(f)\rangle = (\varphi\theta(f))(x_n,\dots,x_1)\\
&= \varphi(x_n,\dots,x_1)f(y_k,\dots,y_1) = \langle \varphi(x_n,\dots,x_1)\lambda_{y_k}\otimes\dots\otimes\lambda_{y_1}, f\rangle.
\end{aligned}$$

Thus, the map Φ_φ in (ii) can be taken to be T_φ^*. Conversely, if (ii) holds then the map Φ_φ in (ii) has a completely bounded predual T_φ and the chain of equalities above implies (i). □

The mapping $\varphi \to \Phi_\varphi$ from Proposition 5.4 is an embedding of $M^{cb}_{\mathscr{P}}A(G)$ into the space of all normal completely bounded maps from $\mathrm{VN}(G)^{\otimes^n_{\sigma h}}$ into $\mathrm{VN}(G)^{\otimes^k_{\sigma h}}$ and hence gives rise to an operator space structure on $M^{cb}_{\mathscr{P}}A(G)$. Namely, given a matrix

$$\varphi = (\varphi_{p,q}) \in M_m(M^{cb}_{\mathscr{P}}A(G))$$

we let $\|\varphi\|_{M_m(M^{cb}_{\mathscr{P}}A(G))} = \|\Phi_\varphi\|_{cb}$, where $\Phi_\varphi \stackrel{def}{=} (\Phi_{\varphi_{p,q}})$ is the corresponding mapping from $\mathrm{VN}(G)^{\otimes^n_{\sigma h}}$ into $M_m(\mathrm{VN}(G)^{\otimes^k_{\sigma h}})$.

In the next theorem, we relate the completely bounded $\mathscr{P}$-multipliers to multidimensional Schur multipliers in the case where $\mathscr{P}$ is the $(1,n)$-partition. It generalises Theorem 4.1 of [7], which concerns discrete groups, to arbitrary locally compact groups.

THEOREM 5.5. *Let $\varphi \in L^\infty(G^n)$ and $\mathscr{S}$ be the space of all $n+1$-dimensional Schur multipliers with respect to the left Haar measure on G. The following are equivalent:*

(i) $\varphi \in M^{cb}_n A(G)$;

(ii) The function $\tilde{\varphi} \in L^\infty(G^{n+1})$ given by

$$\tilde{\varphi}(x_1,\dots,x_{n+1}) = \varphi(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)$$

belongs to $\mathscr{S}$.

Moreover, if $k \in \mathbb{N}$ and $\varphi_{p,q} \in M^{cb}_n A(G)$, $1 \leqslant p,q \leqslant k$, then

$$\|(\varphi_{p,q})\|_{M_k(M^{cb}_n A(G))} = \|(\tilde{\varphi}_{p,q})\|_{M_k(\mathscr{S})}.$$

Proof. (i)⇒(ii) Let $\varphi = (\varphi_{p,q}) \in M_k(M^{cb}_n A(G))$ with $\|\varphi\|_{M_k(M^{\mathrm{cb}}_n A(G))} < 1$, $\Phi_{\varphi_{p,q}}$ be the completely bounded normal map from Proposition 5.4, and $\Phi_\varphi = (\Phi_{\varphi_{p,q}})$. By

[6], there exist operators $V_i \in \mathscr{B}(L^2(G)^\infty)$, $i=2,\dots,n$, $V_1 \in \mathscr{B}(L^2(G)^k, L^2(G)^\infty)$ and $V_{n+1} \in \mathscr{B}(L^2(G)^\infty, L^2(G)^k)$ such that $\Pi_{i=1}^{n+1}\|V_i\| < 1$ and

$$(\varphi_{p,q}(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)\lambda_{x_{n+1}^{-1}}\lambda_{x_1})_{p,q}$$
$$= V_{n+1}(\lambda_{x_{n+1}^{-1}}\lambda_{x_n}\otimes 1)V_n(\lambda_{x_n^{-1}}\lambda_{x_{n-1}}\otimes 1)V_{n-1}\dots(\lambda_{x_2{}^{-1}}\lambda_{x_1}\otimes 1)V_1, \tag{11}$$

where the ampliations are of infinite countable multiplicity. Let $a_1 : G\to\mathscr{B}(L^2(G)^k, L^2(G)^\infty)$ and $a_{n+1} : G \to \mathscr{B}(L^2(G)^\infty, L^2(G)^k)$ be given as follows:

$$a_1(x_1) = (\lambda_{x_1}\otimes 1)V_1(\lambda_{x_1^{-1}}\otimes 1_k), \ \ a_{n+1}(x_{n+1}) = (\lambda_{x_{n+1}}\otimes 1_k)V_{n+1}(\lambda_{x_{n+1}^{-1}}\otimes 1).$$

Let also $a_i : G \to \mathscr{B}(L^2(G)^\infty)$, $i=2,\dots,n$, be given by

$$a_i(x_i) = (\lambda_{x_i}\otimes 1)V_i(\lambda_{x_i^{-1}}\otimes 1), \quad x_i \in G.$$

It follows from (11) that, for all $x_1,\dots,x_{n+1}$, we have

$$\begin{aligned}&\varphi(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)\otimes 1_{L^2(G)}\\ &= (\varphi_{p,q}(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)1_{L^2(G)})_{p,q}\\ &= (\lambda_{x_{n+1}}\otimes 1_k)(\varphi_{p,q}(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1)\lambda_{x_{n+1}^{-1}}\lambda_{x_1})_{p,q}(\lambda_{x_1^{-1}}\otimes 1_k)\\ &= a_{n+1}(x_{n+1})a_n(x_n)\dots a_1(x_1).\end{aligned}$$

Let ξ be a unit vector in $L^2(G)$ and E be the projection onto the one dimensional subspace of $L^2(G)$ generated by ξ. The last identity implies that $\varphi(x_{n+1}^{-1}x_n,\dots,x_2^{-1}x_1) = (Ea_{n+1}(x_{n+1}))a_n(x_n)\dots a_2(x_2)(a_1(x_1)E)$, for all $x_i \in G$, $i = 1,\dots,n+1$. It follows from Theorem 3.2 that $\tilde{\varphi}_{p,q} \in \mathscr{S}$ and

$$\|(\tilde{\varphi}_{p,q})\|_{\mathrm{m},k} \leqslant \Pi_{i=1}^{n+1}\|V_i\| < 1.$$

(ii)⇒(i) Let $\varphi \in L^\infty(G^n)$ and suppose that $\tilde{\varphi}$ is a Schur multiplier with respect to the left Haar measure. By Theorem 3.4 of [15], the function $\psi \in L^\infty(G^{n+1})$ given by $\psi(y_1,\dots,y_{n+1}) = \tilde{\varphi}(y_1^{-1},\dots,y_{n+1}^{-1})$, $y_1,\dots,y_{n+1}\in G$, is also a Schur multiplier with respect to the left Haar measure. Set $y_i = x_i^{-1}x_{i+1}^{-1}\dots x_n^{-1}s$, $i=1,\dots,n$, and $y_{n+1}=s$. We have that

$$\psi(y_1,\dots,y_{n+1}) = \varphi(y_{n+1}y_n^{-1}, y_ny_{n-1}^{-1},\dots,y_2y_1^{-1}) = \varphi(x_n,x_{n-1},\dots,x_1).$$

By Theorem 3.4 of [15], there exist functions $a_i : G\to M_\infty$, $i=2,\dots,n$, $a_1 : G\to M_{\infty,1}$ and $a_{n+1} : G \to M_{1,\infty}$ such that

$$\psi(y_1,\dots,y_{n+1}) = a_{n+1}(y_{n+1})a_n(y_n)\dots a_1(y_1), \quad y_1,\dots,y_{n+1}\in G.$$

For each $i=2,\dots,n$, let $A_i \in \mathscr{B}(L^2(G)\otimes\ell^2)$ be the operator corresponding in a canonical way to a_i. Namely, A_i is given by $(A_i\tilde{\xi})(s) = a_i(s)\tilde{\xi}(s)$, $s\in G$, where we have identified $L^2(G)\otimes\ell^2$ with the space $L^2(G;\ell_2)$ of all square integrable ℓ^2-valued functions on G. Similarly, let $A_1 \in \mathscr{B}(L^2(G), L^2(G) \otimes \ell^2)$ and $A_{n+1} \in \mathscr{B}(L^2(G) \otimes \ell^2, L^2(G))$ be the operators corresponding to a_1 and a_{n+1}, respectively.

Let $f \in A(G)$. Then there exist $\xi, \eta \in L^2(G)$ such that

$$\theta(f)(x_n, \dots, x_1) = (\lambda_{x_n \dots x_1} \xi, \eta) = \int_G \xi(x_1^{-1} \dots x_n^{-1} s) \overline{\eta(s)} ds.$$

We have

$$\begin{aligned}
& (\varphi\theta(f))(x_n, \dots, x_1) \\
&= \varphi(x_n, \dots, x_1) f(x_n \dots x_1) \\
&= \int_G \varphi(x_n, \dots, x_1) \xi(x_1^{-1} \dots x_n^{-1} s) \overline{\eta(s)} ds \\
&= \int_G \psi(x_1^{-1} \dots x_n^{-1} s, \dots, x_n^{-1} s, s) \xi(x_1^{-1} \dots x_n^{-1} s) \overline{\eta(s)} ds \\
&= \int_G a_{n+1}(s) a_n(x_n^{-1} s) \dots a_1(x_1^{-1} \dots x_n^{-1} s) \xi(x_1^{-1} \dots x_n^{-1} s) \overline{\eta(s)} ds .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (A_{n+1}(\lambda_{x_n} \otimes 1) A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi, \eta) \\
&= \big((\lambda_{x_n} \otimes 1) A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi, A_{n+1}^* \eta\big) \\
&= \int_G \big(((\lambda_{x_n} \otimes 1) A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi)(s), (A_{n+1}^* \eta)(s)\big)_{\ell_2} ds \\
&= \int_G a_{n+1}(s)((\lambda_{x_n} \otimes 1) A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi)(s) \overline{\eta(s)} ds \\
&= \int_G a_{n+1}(s)(A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi)(x_n^{-1} s) \overline{\eta(s)} ds \\
&= \int_G a_{n+1}(s) a_n(x_n^{-1} s)(\lambda_{x_{n-1}} \otimes 1) \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi)(x_n^{-1} s) \overline{\eta(s)} ds \\
&= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= \int_G a_{n+1}(s) a_n(x_n^{-1} s) \dots a_1(x_1^{-1} \dots x_n^{-1} s) \xi(x_1^{-1} \dots x_n^{-1} s) \overline{\eta(s)} ds .
\end{aligned}$$

It follows that

$$(\varphi\theta(f))(x_n, \dots, x_1) = (A_{n+1}(\lambda_{x_n} \otimes 1) A_n \dots A_2(\lambda_{x_1} \otimes 1) A_1 \xi, \eta) \tag{12}$$

and hence $\varphi\theta(f) \in A^n(G)$. Thus, $\varphi \in M_n A(G)$ and, by Remark (ii) after Definition 5.3, the map $f \to \varphi\theta(f)$ is bounded. Equation (12) implies that if Φ_φ is its adjoint then

$$\Phi_\varphi(\lambda_{x_n} \otimes \dots \otimes \lambda_{x_1}) = A_{n+1}(\lambda_{x_n} \otimes 1) \dots (\lambda_{x_1} \otimes 1) A_1, \quad x_1, \dots, x_n \in G. \tag{13}$$

Thus, Φ_φ is completely bounded, and hence $\varphi \in M_n^{cb} A(G)$.

Now suppose that $\varphi = (\varphi_{p,q}) \in M_k(L^\infty(G^n))$ and that $\|(\tilde{\varphi}_{p,q})\|_{\mathrm{m},k} < 1$. Let $\psi_{p,q}$ be the map corresponding to $\varphi_{p,q}$ as specified in the case $k = 1$ above and $\psi =$

$(\psi_{p,q})$. Theorem 3.2 implies that $\|\psi\|_{\mathrm{m},k} = \|\tilde{\varphi}\|_{\mathrm{m},k} < 1$. Thus, in the notation of Theorem 3.2, $\|\tilde{S}_\psi\|_k < 1$, where $\tilde{S}_\psi = (\tilde{S}_{\psi_{p,q}})_{p,q}$ is the canonical normal completely bounded multilinear map from $\mathscr{B}(L^2(G)) \times \ldots \times \mathscr{B}(L^2(G))$ into $M_k(\mathscr{B}(L^2(G)))$. By Theorem 3.2, we can write $\psi(y_1,\ldots,y_{n+1}) = a_{n+1}(y_{n+1}) \ \ldots \ a_1(y_1)$, where $a_i : G \to M_\infty$, $i = 2,\ldots,n$, $a_1 : G \to M_{\infty,k}$ and $a_{n+1} : G \to M_{k,\infty}$ are functions such that $\mathrm{esssup}_{y_1,\ldots,y_{n+1}\in G} \Pi_{i=1}^{n+1} \|a_i(y_i)\| < 1$. As before, let $A_i \in \mathscr{B}(L^2(G)^\infty)$, $i = 2,\ldots,n$, $A_1 \in \mathscr{B}(L^2(G)^k, L^2(G)^\infty)$ and $A_{n+1} \in \mathscr{B}(L^2(G)^\infty, L^2(G)^k)$ be the operators corresponding to the a_i's in the canonical way. Let A_{n+1}^p (resp. A_1^q) be the pth row (resp. the qth column) of A_{n+1} (resp. A_1). By (13), $\Phi_{\varphi_{p,q}}(\lambda_{x_n} \otimes \ldots \otimes \lambda_{x_1}) = A_{n+1}^p(\lambda_{x_n} \otimes 1)A_n \ldots A_2(\lambda_{x_1} \otimes 1)A_1^q$, for all $x_1,\ldots,x_n \in G$. It follows that if $\Phi_\varphi = (\Phi_{\varphi_{p,q}})$ then (13) holds in the case under consideration as well. Since $\Pi_{i=1}^{n+1}\|A_i\| < 1$, we conclude that $\|\Phi_\varphi\|_{cb} < 1$ or, equivalently, $\|\varphi\|_{M_k(M_n^{cb}A(G))} < 1$. □

COROLLARY 5.6. *We have that $B^n(G) \subseteq M_n^{cb}A(G)$. Moreover, the inclusion map is a complete contraction.*

Proof. The inclusion follows from Theorem 4.1, Theorem 5.5 and Theorem 3.4 of [15].

Let $\varphi = (\varphi_{p,q}) \in M_k(B^n(G))$, $\|\varphi\|_{M_k(B^n(G))} < 1$ and $\Phi : C^*(G)^n \to M_k$ be the completely bounded mapping associated with φ. By Theorem 5.2 of [6], there exist Hilbert spaces $H_1,\ldots,H_n$, representations $\pi_i : C^*(G) \to \mathscr{B}(H_i)$ and operators $V_1 \in \mathscr{B}(H,\mathbb{C}^k)$, $V_{n+1} \in \mathscr{B}(\mathbb{C}^k,H)$ and $V_i \in \mathscr{B}(H)$, $i = 2,\ldots,n$, such that

$$\Phi(a_1,\ldots,a_n) = V_1\pi_1(a_1)V_2\ldots V_n\pi_n(a_n)V_{n+1}$$

and $\Pi_{i=1}^{n+1}\|V_i\| < 1$. Let $\tilde{\pi}_i : W^*(G) \to \mathscr{B}(H)$ be the canonical normal extension of π_i, $i = 1,\ldots,n$. Since the extension $\tilde{\Phi}$ of Φ to a normal completely bounded map from $W^*(G)^n$ into M_k is unique, we have that

$$\tilde{\Phi}(b_1,\ldots,b_n) = V_1\tilde{\pi}_1(b_1)V_2\ldots V_n\tilde{\pi}_n(b_n)V_{n+1}, \quad b_1,\ldots,b_n \in W^*(G).$$

Let $a_1(y_1) = \tilde{\pi}_n(\omega(y_1))V_{n+1}$, $a_2(y_2) = \tilde{\pi}_{n-1}(\omega(y_2))V_n\tilde{\pi}_n(\omega(y_2^{-1}))$, $\ldots$, $a_{n+1}\ (y_{n+1}) = V_1\tilde{\pi}_1(\omega(y_{n+1}^{-1}))$. Then

$$\begin{aligned}\tilde{\varphi}(y_1,\ldots,y_{n+1}) &= \tilde{\Phi}(\omega(y_{n+1}^{-1})\omega(y_n),\ldots,\omega(y_2^{-1})\omega(y_1)) \\ &= a_{n+1}(y_{n+1})\ldots a_1(y_1)\end{aligned}$$

and $\mathrm{esssup}_{y_1,\ldots,y_{n+1}\in G} \Pi_{i=1}^{n+1}\|a_i(y_i)\| < 1$. Theorems 3.2 and 5.5 imply that the norm of φ as an element of $M_k(M_n^{cb}A^n(G))$ is less than one. Thus, the inclusion $B^n(G) \subset M_n^{cb}A(G)$ is a complete contraction. □

We recall that $C_r^*(G)$ is the reduced C*-algebra of G. We write $C_r^*(G)^{\otimes_h^n}$ for $\underbrace{C_r^*(G) \otimes_h \ldots \otimes_h C_r^*(G)}_{n}$. Let $B_r(G) = C_r^*(G)^*$ and $B_r^n(G) = (C_r^*(G)^{\otimes_h^n})^*$. It is standard to identify the elements of $B_r(G)$ with functions from $B(G)$ in such a way that the duality between $B_r(G)$ and $C_r^*(G)$ is given by $\langle b,\lambda(f)\rangle = \int f(x)b(x)dx,\ f\in L^1(G)$. We equip $B_r(G)$ and $B_r^n(G)$ with the canonical operator space structure as dual operator spaces. Let M be the completely contractive mapping from $C_r^*(G)^{\otimes_h^n}$ to $C_r^*(G)$ which maps $\lambda(f_1)\otimes\ldots\otimes\lambda(f_n)$ (for $f_1,\ldots,f_n\in L^1(G)$) to $\lambda(f)$, where

$$f(x)=\int_{G^n} f_1(x_1)f_2(x_1^{-1}x_2)\ldots f_n(x_{n-1}^{-1}x)dx_1\ldots dx_{n-1}.$$

It is easy to check that the adjoint mapping M^* maps $f\in B_r(G)$ to $\theta(f)\in B_r^n(G)$ (here $\theta(f)(x_1,\ldots,x_n)=f(x_1\ldots x_n)$). We define $M_n^{cb}B_r(G)$ to be the space of all $\varphi\in L^\infty(G^n)$ such that the mapping $T_\varphi : f\mapsto \varphi\theta(f)$ is completely bounded as a map from $B_r(G)$ to $B_r^n(G)$. We note that this map is normal. In fact, if $f_1,\ldots,f_n\in L^1(G)$ then

$$\begin{aligned}
&\langle \varphi\theta(f),\lambda(f_1)\otimes\ldots\otimes\lambda(f_n)\rangle\\
&=\int_{G^n}\varphi(x_1,\ldots,x_n)f(x_1\ldots x_n)f_1(x_1)\ldots f_n(x_n)dx_1\ldots dx_n\\
&=\langle f,\lambda(g)\rangle,
\end{aligned}$$

where $g(x)$ equals

$$\int f_1(x_1)f_2(x_1^{-1}x_2)\ldots f_n(x_{n-1}^{-1}x)\varphi(x_1,x_1^{-1}x_2,\ldots,x_{n-1}^{-1}x)dx_1\ldots dx_{n-1};$$

it is easy to see that $g\in L^1(G)$. Therefore T_φ has a predual M_φ which is given by $\lambda(f_1)\otimes\ldots\otimes\lambda(f_n)\mapsto\lambda(g)$. If $\varphi\in M_n^{cb}B_r(G)$ then M_φ is completely bounded and $\|\varphi\|_{M_n^{cb}B_r(G)}=\|M_\varphi\|_{cb}$. From the definition of the operator space structure of $B_r(G)$, we have that if $(\varphi_{p,q})\in M_k(M_{cb}^nB_r(G))$ then $\|(\varphi_{p,q})\|=\|M_\varphi\|_{cb}$, where $M_\varphi=(M_{\varphi_{p,q}})$ is the corresponding mapping from $C_r^*(G)^{\otimes_h^n}$ to $M_k(C_r^*(G))$.

The following theorem supplements Theorem 5.5 and provides a multidimensional version of Proposition 4.1 of [21].

THEOREM 5.7. *Let $\varphi\in M_k(L^\infty(G^n))$. Then the following are equivalent*

(i) $\varphi\in b_1(M_k(M_n^{cb}A(G)))$;

(ii) the multilinear mapping $M_\varphi : (\lambda(f_1),\ldots,\lambda(f_n))\mapsto(\lambda(f_{ij}))$, where $f_1,\ \ldots,$ $f_n\ \in\ L^1(G)$ and $f_{ij}(x)$ equals

$$\int f_1(x_1)f_2(x_1^{-1}x_2)\ldots f_n(x_{n-1}^{-1}x)\varphi_{ij}(x_1,x_1^{-1}x_2,\ldots,x_{n-1}^{-1}x)dx_1\ldots dx_{n-1}$$

extends to a complete contraction from $C_r^*(G)^{\otimes_h^n}$ into $M_k(C_r^*(G))$;

(iii) $\varphi\in b_1(M_k(M_n^{cb}B_r(G)))$.

Proof. For the sake of technical simplicity we assume that $n=2$; the general case can be treated similarly.

(i)⇒(ii) Let $\varphi=(\varphi_{p,q})\in b_1(M_k(M_2^{cb}A(G)))$. By Proposition 5.4, there exist operators

$$V_0\in\mathscr{B}(L^2(G)^k,L^2(G)^\infty),\ V_1\in\mathscr{B}(L^2(G)^\infty)\text{ and }V_2\in\mathscr{B}(L^2(G)^\infty,L^2(G)^k)$$

such that $\|V_0\|\|V_1\|\|V_2\| \leqslant 1$ and

$$\varphi(x_2,x_1)\lambda_{x_2x_1} = V_2(\lambda_{x_2}\otimes 1)V_1(\lambda_{x_1}\otimes 1)V_0. \tag{14}$$

Let $f_1 = (f_1^{p,q}) \in M_{k,r}(C_r^*(G))$ and $f_2 = (f_2^{p,q}) \in M_{r,k}(C_r^*(G))$. We denote by $\lambda(f_1)\odot\lambda(f_2) \in M_k(C_r^*(G)\otimes_h C_r^*(G))$ a $k\times k$-matrix whose (p,q) entry equals $\sum_{s=1}^r \lambda(f_{p,s}^1)\otimes\lambda(f_{s,q}^2)$. If $f_{p,q}^l \in L^1(G)$, $l=1,2$, then

$$\begin{aligned}
&M_\varphi^{(k)}(\lambda(f_1)\odot\lambda(f_2))\\
&=\left(\sum_{s=1}^r \int f_{p,s}^1(x_1)f_{s,q}^2(x_1^{-1}x_2)\varphi(x_1,x_1^{-1}x_2)\lambda(x_2)dx_1dx_2\right)_{p,q}\\
&=\left(\sum_{s=1}^r \int f_{p,s}^1(x_1)f_{s,q}^2(x_2)\varphi(x_1,x_2)\lambda(x_1x_2)dx_1dx_2\right)_{p,q}\\
&=\left(\int \sum_{s=1}^r f_{p,s}^1(x_1)f_{s,q}^2(x_2)V_2(\lambda_{x_1}\otimes 1)V_1(\lambda_{x_2}\otimes 1)V_0dx_1dx_2\right)_{p,q}\\
&=\left(\sum_{s=1}^r V_2((\int f_{p,s}^1(x_1)\lambda_{x_1}dx_1)\otimes 1)V_1((\int f_{s,q}^2(x_2)\lambda_{x_2}dx_2)\otimes 1)V_0\right)_{p,q}\\
&=\left(\sum_{s=1}^r V_2(\lambda(f_{p,s}^1)\otimes 1)V_1(\lambda(f_{s,q}^2)\otimes 1)V_0\right)_{p,q}.
\end{aligned}$$

Therefore

$$\|M_\varphi^{(k)}(\lambda(f_1)\odot\lambda(f_2)\| \leqslant \|V_0\|\|V_1\|\|V_2\|\|\lambda(f_1)\|\|\lambda(f_2)\|$$

and hence $\|M_\varphi^{(k)}\| \leqslant 1$.

(ii)$\Leftrightarrow$(iii) Follows trivially from the definition of the operator structure of $M_n^{cb}B_r(G)$.

(iii)$\Rightarrow$(i) We only consider the case $k=1$. Let $\varphi \in M_n^{cb}B_r(G)$, $\|\varphi\| \leqslant 1$ and $\psi \in A(G)\cap C_c(G)$, where $C_c(G)$ is the space of compactly supported functions on G. We can find $g \in A(G)$ such that $g=1$ on the support of ψ so that $\psi g = \psi$. As $\theta(g) \in A^n(G)$ and $A^n(G)$ is an ideal in $B_r^n(G)$ we have $\varphi\theta(\psi) = \varphi\theta(\psi)\theta(g) \in A^n(G)$. Since the $A^n(G)$-norm and $B_r^n(G)$-norm coincide on $A^n(G)$ and $A(G)\cap C_c(G)$ is dense in $A(G)$ we obtain that φ is in $b_1(M_n(G))$. Similar arguments show that φ is a completely contractive multiplier. □

We next supply some corollaries of the previous results.

COROLLARY 5.8. *Let G be an amenable locally compact group. Then $B^n(G) = M_n^{cb}A(G)$ completely isometrically.*

Proof. If G is amenable then $B^n(G) = B_r^n(G)$ completely isometrically. Hence, by Theorem 5.7, $M_n^{cb}A(G) = M_n^{cb}B(G)$ completely isometrically. Since $B(G)$ contains the constant functions, it is easy to see that $M_n^{cb}B(G) = B^n(G)$ completely isometrically. □

COROLLARY 5.9. *Let $\mathscr{P}$ be the block (k,n)-partition associated with the sequence $n \geqslant n_k > \ldots > n_1 > 1$ such that each block contains at least two elements, and $\varepsilon_i = \pm 1$, $i = 1,\ldots,n$. Assume that G is amenable. Then the function $\psi : G^n \to \mathbb{C}$ given by*

$$\psi(s_n,\ldots,s_1) = \varphi(s_1^{\varepsilon_1}\ldots s_{n_1-1}^{\varepsilon_{n_1-1}},\ldots,s_{n_{k-1}}^{\varepsilon_{n_{k-1}}}\ldots s_n^{\varepsilon_n})$$

is a Schur multiplier with respect to the left Haar measure if and only if $\varphi \in B^k(G)$.

Proof. We prove the statement for $k = 2$ and a partition of the form $\mathscr{P} = \{\{n,\ldots,m\}, \{m-1,\ldots,1\}\}$; the other cases are similar. Assume ψ is a Schur multiplier. Then $\psi(s_n,\ldots,s_1) = a_1(s_1)\ldots a_n(s_n)$ for some (essentially bounded) functions $a_i : G \to M_\infty$, $i = 2,\ldots,n-1$, $a_n : G \to M_{\infty,1}$ and $a_1 : G \to M_{1,\infty}$. Therefore, the function

$$(s_1,s_2,s_3) \mapsto \varphi(s_3^{-1}s_2, s_2^{-1}s_1) = \psi(s_1^{\varepsilon_n}, s_2^{-\varepsilon_{n-1}}, e,\ldots,e, s_2^{\varepsilon_2}, s_3^{-\varepsilon_1})$$

is a Schur multiplier and hence by Theorem 5.5, $\varphi \in M_2^{cb}A(G) = B^2(G)$.

Let now $\varphi \in B^2(G)$. By Theorem 4.1, there exist representations π_1, π_2 of G on H and vectors ξ, η such that $\varphi(s_2,s_1) = (\pi_2(s_2)\pi_1(s_1)\xi,\eta)$, and

$$\psi(s_n,\ldots,s_1) = (\pi_2(s_1^{\varepsilon_1}\ldots s_{m-1}^{\varepsilon_{m-1}})\pi_1(s_m^{\varepsilon_m}\ldots s_n^{\varepsilon_n})\xi,\eta).$$

Theorem 3.4 of [15] now easily implies ψ is a Schur multiplier. □

REMARK 5.10. Since if G is abelian then $B(G) = \{\hat{\mu} : \mu \in M(\hat{G})\}$, Corollary 5.9 implies the following classical result: If G is a discrete abelian group and $\varphi \in l^\infty(G)$ then the function ψ given by $\psi(x,y) = \varphi(y^{-1}x)$ is a Schur multiplier if and only if $\varphi = \hat{\mu}$ for some measure $\mu \in M(\hat{G})$.

Here is a more general result:

COROLLARY 5.11. *Let G be a locally compact abelian group, $m_1,\ldots,m_n = \pm 1$, $\varphi \in L^\infty(G)$ and ψ be the function given by*

$$\psi(s_n,\ldots,s_1) = \varphi(s_1^{m_1}\ldots s_n^{m_n}), \quad s_1,\ldots,s_n \in G.$$

Then ψ is a Schur multiplier (with respect to the Haar measure) if and only if $\varphi = \hat{\mu}$ for some measure $\mu \in M(\hat{G})$. In this case, $\|\psi\|_{\mathrm{m}} = \|\mu\|$.

We close this section with a multidimensional version of [5, Theorem 1]. We use the notation from Proposition 5.4. Recall [10] that if $f \in A(G)$ and $T \in \mathrm{VN}(G)$ then $fT \in \mathrm{VN}(G)$ is the operator given by the duality relation $\langle g, fT\rangle = \langle fg, T\rangle$.

PROPOSITION 5.12. *Let $\Phi : \mathrm{VN}(G)^n \to \mathrm{VN}(G)$ be a normal completely bounded multilinear map. Then $\Phi = \Phi_\varphi$ for some $\varphi \in M_n^{cb}A(G)$ if and only if*

$$\Phi(\theta(f)(S_1 \otimes \ldots \otimes S_n)) = f\Phi(S_1 \otimes \ldots \otimes S_n), \tag{15}$$

for all $f \in A(G)$ and all $S_1, \ldots, S_n \in \mathrm{VN}(G)$.

Proof. Since Φ is a normal completely bounded map, $\Phi = \Psi^*$ for a completely bounded map from $A(G)$ to $A^n(G)$,

$$\langle \Phi(\theta(f)(S_1 \otimes \ldots \otimes S_n)), h \rangle = \langle S_1 \otimes \ldots \otimes S_n, \theta(f)\Psi(h) \rangle$$

and

$$\langle f\Phi(S_1 \otimes \ldots \otimes S_n), h \rangle = \langle S_1 \otimes \ldots \otimes S_n, \Psi(fh) \rangle$$

Thus, if Φ satisfies (15) then $\theta(f)\Psi(h) = \Psi(fh)$ for all f, $h \in A(G)$. Since $A(G)$ is commutative, $\theta(f)\Psi(h) = \theta(h)\Psi(f)$ and therefore $\Psi(h) = \varphi\theta(h)$ for some function φ on G^n. Since Ψ is completely bounded, $\varphi \in M_n^{cb}A(G)$. Moreover,

$$\langle \Phi(\lambda_{x_n} \otimes \ldots \otimes \lambda_{x_1}), h \rangle = \langle \lambda_{x_n} \otimes \ldots \otimes \lambda_{x_1}, \varphi\theta(h) \rangle =$$
$$\varphi(x_n, \ldots, x_1)h(x_n \ldots x_1) = \langle \varphi(x_n, \ldots, x_1)\lambda_{x_n \ldots x_1}, h \rangle,$$

that is, $\Phi = \Phi_\varphi$. $\square$

6. The abelian case

In this section we assume that G is abelian. We denote by $\hat{G}$ the character group of G. Let $C_0(G)$ be the algebra of continuous functions vanishing at infinity on G. The Haagerup tensor product $\underbrace{C_0(G) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} C_0(G)}_{n}$ will be denoted by $V_{\mathrm{h}}^n(G)$. The dual space of $V_{\mathrm{h}}^n(G)$ is the space of n-measures on $\hat{G}$. Let $C_b(G)$ be the C^*-algebra of continuous bounded functions on G and $\mathscr{V}^n(G) = \underbrace{C_b(G) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} C_b(G)}_{n}$.

Denote by $\hat{G}_d$ the group $\hat{G}$ equipped with the discrete topology and recall that the Bohr compactification $\bar{G}$ of G is the dual of $\hat{G}_d$. We note that there is a canonical inclusion of $V_{\mathrm{h}}^n(G)^*$ into $V_{\mathrm{h}}^n(\bar{G})^*$: for $\Phi \in V_{\mathrm{h}}^n(G)^*$ define $\bar{\Phi} \in V_{\mathrm{h}}^n(\bar{G})^*$ by

$$\bar{\Phi}(a_1 \otimes \ldots \otimes a_n) = \tilde{\Phi}(\iota(a_1|_G) \otimes \ldots \otimes \iota(a_n|_G)), \;\; a_1, \ldots, a_n \in C(\bar{G}),$$

where $\tilde{\Phi}$ is the extension of Φ to a normal completely bounded multilinear map from $(C_0(G)^{**})^{\otimes_{\sigma h}^n}$ to $\mathbb{C}$, and $\iota : C_b(G) \to C_0(G)^{**}$ is the canonical injection.

We claim that

$$\|\bar{\Phi}\|_{V_{\mathrm{h}}^n(\bar{G})^*} = \|\Phi\|_{V_{\mathrm{h}}^n(G)^*}. \tag{16}$$

If $a_k = (a^k_{i,j})$, $k = 1,\dots,n$, are n by n matrices let $a_1 \odot \dots \odot a_n$ be the n by n matrix whose (i,j)-entry is equal to

$$a^1_{i,i_1} \otimes a^2_{i_1,i_2} \otimes \dots \otimes a^n_{i_{n-1},j}.$$

To show (16), first note that if $a_1 \odot \dots \odot a_n \in V^n_{\mathrm{h}}(\bar{G})$ is a function of unit Haagerup norm then

$$|\bar{\Phi}(a_1 \odot \dots \odot a_n)| = |\tilde{\Phi}(\iota(a_1|_G) \odot \dots \odot \iota(a_n|_G))| \leqslant \|\Phi\|,$$

where for $a = (a_{ij}) \in M_{k,l}(C(\bar{G}))$ we denote by $a|_G$ the matrix $(a_{ij}|_G)$. Hence, $\|\bar{\Phi}\|_{V^n_{\mathrm{h}}(\bar{G})^*} \leqslant \|\Phi\|_{V^n_{\mathrm{h}}(G)^*}$. Conversely, let $\bar{a}$ denote the canonical extension of a function a from $C_0(G)$ to a function from $C(\bar{G})$ and $\bar{u} \in V^n_{\mathrm{h}}(\bar{G})$ denote the corresponding extension of an element $u \in V^n_{\mathrm{h}}(G)$. Thus, if $u = a_1 \odot \dots \odot a_n$ then $\bar{u} = \bar{a}_1 \odot \dots \odot \bar{a}_n$. It follows that $\|\bar{u}\|_{V^n_{\mathrm{h}}(\bar{G})} \leqslant \|u\|_{V^n_{\mathrm{h}}(G)}$ and hence

$$\begin{aligned}
\|\Phi\|_{V^n_{\mathrm{h}}(G)^*} &= \sup\{|\Phi(u)| : u \in V^n_{\mathrm{h}}(G), \|u\|_{\mathrm{h}} \leqslant 1\} \\
&= \sup\{|\bar{\Phi}(\bar{u})| : u \in V^n_{\mathrm{h}}(G), \|u\|_{\mathrm{h}} \leqslant 1\} \\
&\leqslant \sup\{|\bar{\Phi}(v)| : v \in V^n_{\mathrm{h}}(\bar{G}), \|v\|_{\mathrm{h}} \leqslant 1\} \\
&= \|\bar{\Phi}\|_{V^n_{\mathrm{h}}(\bar{G})^*}.
\end{aligned}$$

Thus (16) is established. We hence have a canonical isometric embedding of $M^n(\hat{G})$ into $M^n(\hat{G}_d)$, which gives rise to an isometric embedding of $B^n(\hat{G})$ into $B^n(\hat{G}_d)$. The next proposition generalises [12, Theorem 3.3] to the multidimensional case. We note that the proof we give is new in the case $n = 2$ as well.

PROPOSITION 6.1. *Let $f \in B^n(\hat{G}_d)$. Then $f \in B^n(\hat{G})$ if and only if f is continuous.*

Proof. It is clear that if $f \in B^n(\hat{G})$ then f is continuous. For the converse direction we use induction on n. If $n = 1$ the claim follows from a classical result of Eberlein [20, Theorem 1.9.1]. Suppose that $n > 1$ and fix a continuous function f from $B^n(\hat{G}_d)$. For an element $\gamma \in \hat{G}$ let $\delta_\gamma \in B(\hat{G}_d)^*$ be the evaluation functional, $\delta_\gamma(h) = h(\gamma)$, $h \in B(\hat{G})$. Using the identification (4), we let $L_{\delta_\gamma} : B^n(\hat{G}) \to B^{n-1}(\hat{G})$ be the corresponding slice map. We have that $L_{\delta_\gamma}(f) \in B^{n-1}(\hat{G}_d)$ and that $L_{\delta_\gamma}(f)$ is continuous. By the induction assumption, $L_{\delta_\gamma}(f) \in B^{n-1}(\hat{G})$. Since every element of $B(\hat{G}_d)^*$ can be approximated in the weak* topology by a bounded net consisting of linear combinations of the functionals δ_γ, $\gamma \in \hat{G}$, we conclude that $L_\delta(f) \in B^{n-1}(\hat{G})$ for every $\delta \in B(\hat{G}_d)^*$. An application of [21, Theorem 2.2] shows that $f \in B(\hat{G}_d) \otimes_{eh} B^{n-1}(\hat{G})$. Repeating the above argument with a right slice map in the place of L_δ shows that $f \in B^n(\hat{G})$. $\square$

The following lemma generalises a theorem of Eberlein [20, Theorem 1.9.1] to the multidimensional case.

LEMMA 6.2. *Let $\phi \in L^\infty(\hat{G}^n)$. The following are equivalent:*
(i) $\phi \in B^n(\hat{G})$;
(ii) ϕ is continuous and there exists a constant $C > 0$ such that

$$\left|\sum c_{i_1\dots i_n}\phi(\chi_{i_1},\dots,\chi_{i_n})\right| \leqslant C\left\|\sum c_{i_1,\dots,i_n}\chi_{i_1}\otimes\dots\otimes\chi_{i_n}\right\|_{\mathscr{V}^n(G)},$$

where $\chi_{i_k} \in \hat{G}$ and the summation is over a finite number of indices $(i_1,\dots,i_n)$.

Proof. For notational simplicity we assume $n = 2$.

(i)$\Rightarrow$(ii) Let $\phi \in B^2(\hat{G})$; by Corollary 4.2, ϕ is continuous and since $\omega(\chi_i) = \iota(\check{\chi}_i)$, where $\check{\chi}_i(x) = \overline{\chi_i(x)} = \chi_i(x^{-1})$ (see (7)), we have

$$\begin{aligned}\left|\sum c_{ij}\phi(\chi_i,\chi_j)\right| &= \left|\tilde{\Phi}(\sum c_{ij}\iota(\check{\chi}_i)\otimes\iota(\check{\chi}_j)\right|\\ &\leqslant \|\Phi\|\left\|\sum c_{ij}\iota(\check{\chi}_i)\otimes\iota(\check{\chi}_j)\right\|_{C_0(G)^{**}\otimes_{\mathrm{h}}C_0(G)^{**}}\\ &= \|\Phi\|\left\|\sum c_{ij}\chi_i\otimes\chi_j\right\|_{\mathscr{V}^2(G)}.\end{aligned}$$

The last equality follows from the injectivity of the Haagerup tensor product.

(ii)$\Rightarrow$(i) Assume first that G is compact. Then $\hat{G}$ is discrete. Let $T : C_0(G)\odot C_0(G) \to \mathbb{C}$ be the mapping given by $T(\sum c_{ij}\chi_i\otimes\chi_j) = \sum c_{ij}\phi(\chi_i,\chi_j)$. Then $|T(f)| \leqslant C\|f\|_{\mathscr{V}^2(G)} = C\|f\|_{V^2_{\mathrm{h}}(G)}$ for finite sums $f = \sum c_{ij}\chi_i\otimes\chi_j$ and therefore T can be extended to a bounded linear functional on $V^2_{\mathrm{h}}(G)$. Thus, there exists $u \in M^2(\hat{G})$ such that

$$\sum c_{ij}\phi(\chi_i,\chi_j) = \langle u, \sum c_{ij}\chi_i\otimes\chi_j\rangle.$$

In particular, $\phi(\chi_1,\chi_2) = \langle u,\chi_1\otimes\chi_2\rangle$, that is, $\phi = \hat{u}_1 \in B^2(\hat{G})$, where $\langle u_1,\chi_i\otimes\chi_j\rangle = \langle u,\check{\chi}_i\otimes\check{\chi}_j\rangle$.

If G is not compact let $\bar{G}$ be the Bohr compactification of G. Extending each $\chi \in \hat{G}$ to a character on $\bar{G}$ we define a linear functional T on the space of all functions f on $\bar{G}\times\bar{G}$ of the form $f(x,y) = \sum c_{ij}\chi_i(x)\chi_j(y)$, $x,y \in \bar{G}$, where $\chi_i,\chi_j \in \hat{G}$, by letting, for f as above, $T(f) = \sum c_{ij}\phi(\chi_i,\chi_j)$. Let $i \in \mathbb{N}$, $g_i = \sum_k c^i_k\chi_{k,i}$ and $h_i = \sum_j d^i_j\psi_{j,i}$ be trigonometric polynomials on $\bar{G}$, where $\chi_{k,i},\psi_{j,i} \in \hat{G}$. Then

$$\begin{aligned}\left|T\left(\sum_i g_i\otimes h_i\right)\right| &= \left|\sum_{i,k,j} c^i_k d^i_j\phi(\chi_{k,i},\psi_{j,i})\right| \leqslant C\left\|\sum_{i,k,j} c^i_k d^i_j\chi_{k,i}\otimes\psi_{j,i}\right\|_{\mathscr{V}^2(G)}\\ &= C\left\|\sum_i g_i\otimes h_i\right\|_{\mathscr{V}^2(G)} = C\left\|\sum_i g_i\otimes h_i\right\|_{V^2_{\mathrm{h}}(\bar{G})}.\end{aligned}$$

The last equality follows from the injectivity of the Haagerup tensor product and the fact that $C_b(G)$ is completely isometrically embedded in $C(\bar{G})$. Thus, T can be extended to a bounded linear functional on $V_{\mathrm{h}}^2(\bar{G})$ and hence $\phi(\chi_1,\chi_2) = \langle u, \chi_1 \otimes \chi_2 \rangle$ for $u \in M^2(\hat{\hat{G}}) = M^2(\hat{G}_d)$, and $\phi \in B^2(\hat{G}_d)$. Since ϕ is continuous, Proposition 6.1 implies that $\phi \in B^2(\hat{G})$. $\square$

The following lemma is a multidimensional version of [20, Theorem 3.8.1].

LEMMA 6.3. *Let $\varphi \in L^\infty(G^n)$. Assume $\varphi\theta(g) \in B^n(G)$ for every $g \in A(G)$. Then $\varphi \in B^n(G)$.*

Proof. We only consider the case $n = 2$; the general case can be treated in a similar way. Let $T : A(G) \to B^2(G)$ be the linear mapping defined by $T(g) = \varphi\theta(g)$. We show that T is continuous. If $g_n \to g$ in $A(G)$ and $\varphi\theta(g_n) \to \hat{u}$ in $B^2(G)$, where $u \in M^2(G)$, then

$$\hat{u}(h_1,h_2) = \lim_{n\to\infty} \varphi(h_1,h_2)g_n(h_1h_2) = \varphi(h_1,h_2)g(h_1h_2),$$

hence $\hat{u} = \varphi\theta(g)$. By the Closed Graph Theorem, T is continuous and $\|\varphi\theta(g)\|_{B^2(G)} \leqslant C\|g\|_{A(G)}$.

Given $h_1,\dots,h_n \in G$, $\varepsilon > 0$, there exists $f \in A(G)$, $\|f\|_{A(G)} \leqslant 1+\varepsilon$, such that $f(h_ih_j) = 1$, for all i,j. Let $u \in M^2(G)$ be such that $\hat{u} = \varphi\theta(f)$. Then

$$\begin{aligned} \left|\sum c_{ij}\varphi(h_i,h_j)\right| &= \left|\sum c_{ij}\varphi(h_i,h_j)f(h_ih_j)\right| = \left|\sum c_{ij}\hat{u}(h_i,h_j)\right| \\ &= \left|\tilde{u}\left(\sum c_{ij}\iota(\check{h}_i)\otimes\iota(\check{h}_j)\right)\right| \\ &\leqslant C(1+\varepsilon)\left\|\sum c_{ij}h_i\otimes h_j\right\|_{\mathscr{V}^2(\hat{G})}, \end{aligned}$$

where $\tilde{u}$ is the extention of u to a normal completely bounded linear map from $(C_0(G)^{**})^n$ to $\mathbb{C}$ and $\iota : C_b(G) \to C_0(G)^{**}$ is the canonical inclusion. Given open sets V_1, $V_2 \subset G$ with compact closures we can find $f \in A(G)$ such that $\theta(f)$ is constant on $V_1 \times V_2$. Therefore, φ is continuous on $V_1 \times V_2$, and hence φ is continuous on $G \times G$. By Lemma 6.2, $\varphi \in B^2(G)$. $\square$

In the next corollary, we denote by M_g the operator of multiplication by the function g.

THEOREM 6.4. *For every block (k,n)-partition $\mathscr{P}$, we have that $B^n(G) = M^{cb}_{\mathscr{P}}(G) = M_{\mathscr{P}}(G)$.*

Proof. Let $\mathscr{P}_1$ (resp. $\mathscr{P}_2$) be the block $(1,n)$- (resp. (n,n)-)partition. We have that $\theta_{\mathscr{P}_2}$ is the identity map. For any block (k,n)-partition $\mathscr{P}$ we have that

$$\operatorname{ran}\theta_{\mathscr{P}_1} \subseteq \operatorname{ran}\theta_{\mathscr{P}} \subseteq \operatorname{ran}\theta_{\mathscr{P}_2} = A^n(G).$$

Thus,

$$M_{\mathscr{P}_2}A(G) \subseteq M_{\mathscr{P}}A(G) \subseteq M_{\mathscr{P}_1}A(G),$$

and similarly for the completely bounded multipliers. By Theorem 5.1, $B^n(G) \subseteq M_{\mathscr{P}_2}A(G)$. By Lemma 6.3, $M_{\mathscr{P}_1}A(G) \subset B^n(G)$ and hence $B^n(G) = M_{\mathscr{P}}(G)$.

The fact that $B^n(G) = M^{cb}_{\mathscr{P}}A(G)$ follows in the same way, using Proposition 4.3 and the fact that for completely contractive Banach algebras A, one has $A \subseteq M_{cb}A$. □

COROLLARY 6.5. *Let $\Psi : A(G) \to A^n(G)$ be a bounded linear map such that $\Psi M_\chi = M_{\theta(\chi)}\Psi$ for any $\chi \in \hat{G}$. Then $\Psi(f) = \varphi\theta(f)$, $f \in A(G)$, for some $\varphi \in B^n(G)$.*

Proof. It follows from the proof of Theorem 5.12 that $\Psi(f) = \varphi\theta(f)$ for some bounded function φ on G. Thus $\varphi \in M_nA(G)$. The statement now follows from Theorem 6.4. □

REFERENCES

[1] D. P. BLECHER AND C. LE MERDY, *Operator algebras and their modules – an operator space approach*, Oxford University Press, 2004.

[2] D. P. BLECHER AND R. SMITH, *The dual of the Haagerup tensor product*, J. London Math. Soc. (2), **45** (1992), 126–144.

[3] M. BOZEJKO AND G. FENDLER, *Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Colloquium Math., **63** (1992), 311–313.

[4] J. DE CANNIÈRE AND U. HAAGERUP, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math., **107**, 2 (1985), 455–500.

[5] C. CECCHINI, *Operators on* $\mathrm{VN}(G)$ *commuting with* $A(G)$, Colloquium Math., **43** (1980), 137–142.

[6] E. CHRISTENSEN AND A. M. SINCLAIR, *Representations of completely bounded multilinear operators*, J. Funct. Anal., **72** (1987), 151–181.

[7] E. G. EFFROS AND ZH.-J. RUAN, *Multivariable multipliers for groups and their operator algebras*, Proceedings of Symposia in Pure Mathematics, **51** (1990), Part 1, 197–218.

[8] E. G. EFFROS AND ZH.-J. RUAN, *Operator Spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000. xvi+363 pp.

[9] E. G. EFFROS AND ZH.-J. RUAN, *Operator spaces tensor products and Hopf convolution algebras*, J. Operator Theory, **50** (2003), 131–156.

[10] P. EYMARD, *L'alg è bre de Fourier d'un groupe localement compact*, Bulletin de la S.M.F., **92** (1964), 181–236.

[11] J. E. GILBERT, T. ITO AND B.M. SCHREIBER, *Bimeasure algebras on locally compact groups*, J. Funct. Anal., **64** (1985), 134–162.

[12] C. GRAHAM AND B. M. SCHREIBER, *Bimeasure algebras on LCA groups*, Pacific J. Math., **115**, 1 (1984), 91–127.

[13] A. GROTHENDIECK, *Resume de la theorie metrique des produits tensoriels topologiques*, Boll. Soc. Mat. Sao-Paulo, **8** (1956), 1–79.

[14] P. JOLISSAINT, *A characterisation of completely bounded multipliers of Fourier algebras*, Colloquium Math., **63** (1992), 311–313.

[15] K. JUSCHENKO, I. G. TODOROV AND L. TUROWSKA, *Multidimensional operator multipliers*, Trans. Amer. Math. Soc., **361**, 9 (2009), 4683–4720.

[16] V. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge University Press, 2002.

[17] V. V. PELLER, *Hankel operators in the perturbation theory of unitary and selfadjoint operators*, Funktsional. Anal. i Prilozhen., **19**, 2 (1985), 37–51, 96.

[18] G. PISIER, *Similarity problems and completely bounded maps*, Second, expanded edition. Includes the solution to "The Halmos problem". Lecture Notes in Mathematics, 1618. Springer-Verlag, Berlin, 2001. viii+198 pp.

[19] G. PISIER, *Introduction to Operator Space Theory*, Cambridge University Press, 2003.

[20] W. RUDIN, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York-London 1962 ix+285 pp.

[21] N. SPRONK, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras*, Proc. London Math. Soc. (3), **89**, 1 (2004), 161–192.

[22] K. YLINEN, *Non-commuattive Fourier transforms of bounded bilinear forms and completely bounded multilinear operators*, J. Funct. Anal., **79** (1988), 144–165.

[23] G. ZHAO AND B.M. SCHREIBER, *Algebras of multilinear forms on groups*, Contemp. Math., **189** (1995), 497-511.

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I. G. Todorov
Department of Pure Mathematics
Queen's University Belfast
Belfast BT7 1NN, United Kingdom
e-mail: i.todorov@qub.ac.uk

L. Turowska
Department of Mathematical Sciences
Chalmers University of Technology
and the University of Gothenburg
Gothenburg SE-412 96, Sweden
e-mail: turowska@chalmers.se