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NORMALS, SUBNORMALS AND AN OPEN QUESTION

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To Zoltán Sebestyén
for his 65th birthday anniversary

Abstract. An acute look at basic facts concerning unbounded subnormal operators is taken here. These operators have the richest structure and are the most exciting among the whole family of beneficiaries of the normal ones. Therefore, the latter must necessarily be taken into account as the reference point for any exposition of subnormality. So as to make the presentation more appealing a kind of comparative survey of the bounded and unbounded case has been set forth.

This piece of writing serves rather as a practical guide to this largely impenetrable territory than an exhausting report.

We begin with bounded operators pointing out those well known properties of normal and subnormal operators, which in unbounded case become much more complex. Then we are going to show how the situation looks like for their unbounded counterparts. The distinguished example of the creation operator coming from the quantum harmonic oscillator crowns the theory. Finally we discuss an open question, one of those which seem to be pretty much intriguing and hopefully inspiring.

By an unbounded operator we mean a not necessarily bounded one, nevertheless it is always considered to be densely defined, always in a complex Hilbert space. If we want to emphasis an operator to be everywhere defined we say it is on, otherwise we say it is in. Unconventionally though suggestively, $\boldsymbol{B}(\mathcal{H})$ denotes all the bounded operators on $\mathcal{H}$. If A is an operator, then $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stands for its domain, kernel (null space) and range respectively; if A is closable, its closure is denoted by $\bar{A}$.

Let us mention some books where unbounded normal operators are treated: they are [4], [12, Chapter XV, Section 12], [56] and [59]. To bounded subnormal operators the book [11] is totally devoted.

Despite the ambitious plan the topics presented here are a rather selective. Also some of the arguments used in the proofs have to be extended. The material, though still developing, is sizeable enough to cover a large monograph; this is the project [41] already in progress.

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The haven of tranquility of bounded operators

Foremost topics of normality

Normal operators: around the definition. An operator $N \in \boldsymbol{B}(\mathcal{H})$ is said to be *normal* if it commutes with its Hilbert space adjoint, that is if

$$NN^* = N^*N. \tag{1}$$

This purely algebraic definition can be made spatial through a standard argument: N is normal if and only if

$$\|Nf\| = \|N^*f\|, \quad f \in \mathcal{H}. \tag{2}$$

Let us notify the following.

TRIVIALITY 1. *N is normal if and only if* (2) *holds for f's from a dense linear space*[1] *$\mathcal{D}$ only.*

Spectral representation. The most powerful tool for normal operators is its spectral representation. Though different people may have different understanding of it, everyone agrees that the most appealing is its spatial version below.

THEOREM 2. (Spectral Theorem) *An operator N is normal if and only if it is a spectral integral of the identity function on $\mathbb{C}$ with respect to a spectral measure E on $\mathbb{C}$. Such a spectral measure E is uniquely determined and its closed support coincides with the spectrum of N.*

From spectral representation to $\mathcal{L}^2$–model. What is sometimes meant by spectral theorem, tailored to the simplest possible situation and as such pretty often satisfactory in use, is the following.

COROLLARY 3. *If an operator N is normal and $*$–cyclic, then there is positive measure μ on $\operatorname{sp}(N)$ such that N is unitarily equivalent to the operator M_Z of multiplication by the independent variable on $\mathcal{L}^2(\mu)$.*

N is *$*$–cyclic* means[2] here that there is a vector $e \in \mathcal{H}$ (called a $*$–cyclic vector of N) such that the linear space

$$\{p(N^*,N)e\colon\ p \in \mathbb{C}[Z,\bar{Z}]\} \tag{3}$$

is dense in $\mathcal{H}$; this notion appears as one of the very sensitive when passing to unbounded operators.

The converse to Corollary 3 is trivial. We state it here because of the further role it is going to play.

[1] If we want to have a linear space closed we always make it clear.

[2] This definition makes sense for any operator as long as the involved monomials are kept to be ordered like $N^{*k}N^l$.

FACT 4. *Suppose* μ *is compactly supported positive measure*[3] *on* $\mathbb{C}$. *Then the operator* M_Z *of multiplication by the independent variable is normal and* $*$*-cyclic with the cyclic vector* $e = 1$.

Some supplementary information is in what follows.

COROLLARY 5. *Let* N *and* μ *be as in* Corollary 3. *Then*

$$\langle N^m e, N^n e\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \mu(\mathrm{d}z), \quad m,n = 0,1,\ldots$$

The spectrum. For the spectrum of a normal operator N we have

$$\mathrm{sp}(N) = \mathrm{sp}_{\mathrm{ap}}(N)$$

where the right hand side stands for the approximate point spectrum. As the adjoint N^* is normal as well the same refers to it; the apparent equality $\mathrm{sp}(N) = \overline{\mathrm{sp}(N^*)}$ is applicable here.

The finest points of subnormal operators

Here we would like to itemize the topics, which are well known in the theory of bounded subnormal operators (cf. [11]), and which we are going to juxtapose with those for unbounded operators.

Normal dilations and subnormality. Given $A \in \boldsymbol{B}(\mathcal{H})$, a normal operator $N \in \boldsymbol{B}(\mathcal{K})$, $\mathcal{K}$ contains isometrically $\mathcal{H}$, is said to be a (*power*) *dilation* of A if

$$A^n f = PN^n f, \quad f \in \mathcal{H}, \quad n = 0,1,\ldots \tag{4}$$

with P being the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$; if N is a dilation of A then so is N^* for A^*.

If for $S \in \boldsymbol{B}(\mathcal{H})$ there is N normal in $\mathcal{K}$ such that instead of (4) we have

$$Sf = Nf, \quad f \in \mathcal{H}, \tag{5}$$

then we say that S is *subnormal*. If S is subnormal and N is its normal extension then N^* is a normal dilation of S^*. In addition to this we have, cf. [57, §5]

PROPOSITION 6. *The following conditions are equivalent:*

(a) B *is an extension of* A;

(b) B *is a dilation of* A *and* B^*B *is a dilation of* A^*A;

(c) $B^{*i}B^j$ *is a dilation of* $A^{*i}A^j$ *for any* $i,j = 0,1,\ldots$.

[3] We call a measure positive if it takes non-negative values.

Another way of writing (5), both illustrative and precise, is

$$S \subset N; \tag{6}$$

use of $\subset$ suggests the graph connotation.

Halmos' positive definiteness and Bram's characterization of subnormality. It is an immediate consequence of normality of N in (5) that a subnormal operator $S \in \boldsymbol{B}(\mathcal{H})$ must necessarily satisfy a kind of *positive definiteness* condition introduced by Halmos in [18]:

$$\sum_{m,n} \langle S^m f_n, S^n f_m \rangle \geqslant 0, \quad \text{for any finite sequence } (f_k)_k \subset \mathcal{H}.$$

THEOREM 7. *$S \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if it satisfies the positive definiteness condition and*

$$\sum_{m,n} \langle S^{m+1} f_n, S^{n+1} f_m \rangle \leqslant C \sum_{m,n} \langle S^m f_n, S^n f_m \rangle, \quad \textit{for any finite sequence } (f_k)_k \subset \mathcal{H} \tag{7}$$

with some $C \geqslant 0$.

Bram's result[4] [6] says the boundedness condition (7) in Halmos' Theorem 7[5] is superfluous. It turns out that the boundedness condition (7) comes back in the unbounded case under some forms of growth conditions.

Another characterization is in [1]; it is interesting because it provides with a matricial construction of the extension space independent of S. In principal it does lead to minimal extensions, cf. Proposition 8.

Minimality and uniqueness of extensions. For S subnormal and its normal extension N let us take into consideration the following three situations.

(M_1) If $\mathcal{H} \subset \mathcal{K}_1 \subset \mathcal{K}$ and $N\restriction_{\mathcal{K}_1}$ turns out to be normal then either $\mathcal{K}_1 = \mathcal{H}$ or $\mathcal{K}_1 = \mathcal{K}$.

For E being the spectral measure of N and $\mathcal{D}$ a linear subspace of $\mathcal{H}$ set

$$\mathcal{S}_{\mathcal{D}} \stackrel{\text{def}}{=} \operatorname{clolin}\{E(\sigma)f\colon\ \sigma \text{ Borel subset of } \mathbb{C},\ f \in \mathcal{D}\} \tag{8}$$

$$\mathcal{C}_{\mathcal{D}} \stackrel{\text{def}}{=} \operatorname{lin}\{N^{*m}N^n f\colon\ f \in \mathcal{D},\ m,n = 0,1,\ldots\}. \tag{9}$$

(M_2) $\mathcal{S}_{\mathcal{H}}$ is $\mathcal{K}$.

(M_3) The closure of $\mathcal{C}_{\mathcal{H}}$ is $\mathcal{K}$.

The standard fact of the theory says the conditions (M_1), (M_2) and (M_3) are equivalent. If this happens we speak of *minimality* of N. Notice minimal normal extensions always exist, both (M_2) and (M_3) provide with an algorithm to determine them. Moreover,

PROPOSITION 8. *Two minimal normal extensions of a subnormal S are $\mathcal{H}$–equivalent, that is there is a unitary similarity between them which remains identity on the space $\mathcal{H}$.*

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[4] A short replacement for Bram's main argument concerning redundancy of (7) can be found in [42]. The argument from [42] is present in [58, p. 509]

[5] For another proof of Halmos' theorem look at [57]

The hazardous terrain of unbounded operators

All the operators from now on are densely defined; if A is such, $\mathcal{D}(A)$ always stands for its domain. The closure of A, if it is closable, is denoted by $\overline{A}$. If $\mathcal{D}$ is a linear subspace of $\mathcal{D}(A)$ then $A|_{\mathcal{D}}$ stands for the *restriction* of A to $\mathcal{D}$, if $\mathcal{D}$ is invariant for A we consider $A|_{\mathcal{D}}$ as an operator in the Hilbert space $\overline{\mathcal{D}}$.

Normal operators and their spectral representation again

The definition of normality in unbounded case is much the same, more precisely, a closed operator is said to be *normal* if (1) holds [6]. However, it turns out that a version of (2) is more easy-to-use: N is normal if and only if

$$\begin{gathered}\mathcal{D}(N)=\mathcal{D}(N^*),\\ \|Nf\|=\|N^*f\|,\quad f\in\mathcal{D}(N).\end{gathered}\tag{10}$$

Now closeness of N is implicit in (10).

The plain version of spectral theorem. As in the bounded case all the versions of spectral representation are available. The spectral theorem, Theorem 2, is true as stated due to the vast flexibility of the spectral integral. We are going to state it here with more particulars enhancing some of them which are pertinent to unbounded operators; of course, they are present in the bounded case as well.

THEOREM 9. (Spectral Theorem, the extras included) *An operator N is normal if and only if it is a spectral integral of the identity function on $\mathbb{C}$ with respect to a spectral measure E on $\mathbb{C}$, that is*

1° $\langle Nf,g\rangle=\int_{\mathbb{C}} z\langle E(\mathrm{d}z)f,g\rangle$ *for all $f\in\mathcal{D}(N)$ and $g\in\mathcal{H}$.*

Moreover, if this happens then

2° $\mathcal{D}(N)=\{f\in\mathcal{H}\colon\ \int_{\mathbb{C}}|z|^2\langle E(\mathrm{d}z)f,f\rangle<+\infty\}$;

3° *for every Borel measurable non-negative function ϕ on $\mathbb{C}$ and $f\in\mathcal{D}(N)$*

$$\int_{\mathbb{C}}\phi(x)\langle E(\mathrm{d}z)Nf,Nf\rangle=\int_{\mathbb{C}}\phi(x)\,|z|^2\langle E(\mathrm{d}z)f,f\rangle,$$

in particular,

$$\|Nf\|^2=\int_X|z|^2\langle E(\mathrm{d}z)f,f\rangle,\quad f\in\mathcal{D}(N)\tag{11}$$

and

$$E(\sigma)N\subset NE(\sigma)\ \textit{for all Borel sets}\ \sigma;\tag{12}$$

4° *the spectral measure E is uniquely determined and its closed support coincides with the spectrum of N.*

[6] It is always tacitly understood that the domain of a composition of two operators is the maximal possible one. One has to notice that the adventure with domains of unbounded operators already starts here.

This is more or less all what survives from surroundings of the spectral theorem when passing from the bounded case to the unbounded one.

Invariant and reducing subspaces. A closed subspace $\mathcal{L}$ of $\mathcal{H}$ is *invariant* for A if $A(\mathcal{L}\cap\mathcal{D}(A))\subset\mathcal{L}$; then the *restriction* $A\restriction_{\mathcal{L}} \overset{\text{def}}{=} A|_{\mathcal{L}\cap\mathcal{D}(A)}$ is a operator in $\mathcal{L}$ becomes clear. If A is a closable (closed) operator in $\mathcal{H}$ then so is $A\restriction_{\mathcal{L}}$; this is so because the notions are topological, with topology in the graph space. The closed subspace $\mathcal{L}$ is invariant for A if and only if $PAP=AP$, where P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$.

On the other hand, a linear subspace $\mathcal{D}\subset\mathcal{D}(A)$ is said to be *invariant* for an operator A in $\mathcal{H}$ if $A\mathcal{D}\subset\mathcal{D}$. If this happens and $\mathcal{D}$ is not dense in $\mathcal{H}$ we consider the *restriction* $A|_{\mathcal{D}}$ as a densely defined operator in $\overline{\mathcal{D}}$. However, if $\mathcal{D}$ is a dense in $\mathcal{H}$ then $A|_{\mathcal{D}}$ is still a densely defined operator in $\mathcal{H}$.

The above two concepts of invariance and restriction look much alike. If a linear subspace $\mathcal{D}$ is invariant for A then $A|_{\mathcal{D}}\subset A\restriction_{\overline{\mathcal{D}}}$ whereas $\overline{A|_{\mathcal{D}}}=\overline{A}\restriction_{\overline{\mathcal{D}}}$ provided A is closable[7]. This makes the difference more transparent.

A step further, a closed subspace $\mathcal{L}$ *reduces* an operator A if both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ are invariant for A as well as $P\mathcal{D}(A)\subset\mathcal{D}(A)$; all this is the same as to require $PA\subset AP$. The restriction $A\restriction_{\mathcal{L}}$ is called a *part* of A in $\mathcal{L}$.

$\mathcal{C}^\infty$–vectors. For an operator A set

$$\mathcal{D}^\infty(A)\overset{\text{def}}{=}\bigcap_{n=0}^{\infty}\mathcal{D}(A^n),$$

$$\mathcal{D}^\infty(A,A^*)\overset{\text{def}}{=}\bigcap_{\substack{A_1,\dots A_n\in\{A^*,A\}\\ \text{any finite choice}}}\mathcal{D}(A_1\cdots A_n).$$

It is customary to refer to vectors in any of these two classes as to $\mathcal{C}^\infty$–ones.

One has to notify that

$$\mathcal{D}^\infty(A^*,A)=\mathcal{D}^\infty(A,A^*)\subset\mathcal{D}^\infty(A^*)\cap\mathcal{D}^\infty(A).$$

If $f\in\mathcal{D}^\infty(A)$ then $p(A)f\in\mathcal{D}^\infty(A)$ for any $p\in\mathbb{C}[Z]$ as well, if $f\in\mathcal{D}^\infty(A^*,A)$ then $p(A^*,A)f\in\mathcal{D}^\infty(A^*,A)$ for any $p\in\mathbb{C}[Z,\overline{Z}]$; the latter regardless any commutativity property between A and A^*, cf. footnote 2.

A vector $f\in\mathcal{D}^\infty(A)$ may belong to one of the following classes: $\mathcal{B}(A)$ (*bounded*), $\mathcal{A}(A)$ (*analytic*) or $\mathcal{Q}(A)$ (*quasianalytic*). While the last two are rather pretty well known we give here the definition of *bounded vectors*, they are those f's in $\mathcal{D}^\infty(A)$ for which there are a,b such that $\|A^nf\|\leqslant ab^n$, $n=0,1,\dots$ It is clear that

$$\mathcal{B}(A)\subset\mathcal{A}(A)\subset\mathcal{Q}(A).$$

The first two are linear subspaces whereas the third is not[8].

[7] Identifying operators with their graphs we can write $A|_{\mathcal{D}}=A\cap(\mathcal{D}\times\mathcal{D})$ and $A\restriction_{\mathcal{D}}=A\cap(\overline{D}\times\overline{D})$. Hence the equality follows.

[8] There are two more notions: seminanalytic and Stieltjes vectors, they are rather less popular, cf. [41]

A core. This is an important invention for unbounded operators when a need not to consider them closed becomes strong. Let us call here that this appear more often than someone may imagine, take an operator with invariant domain, if it is closed, then in the vast majority of cases it turns out to be necessarily bounded, see [28]. If someone does deal with a closed operator and in spite of this wants to consider an invariant domain a core comes to rescue. Thus $\mathcal{D} \subset \mathcal{D}(A)$ is a *core* of a closable[9] operator A if $\overline{A|_{\mathcal{D}}} = \overline{A}$. Trivially, a domain $\mathcal{D}(A)$ is always a core of A and, on the other hand, a core must necessarily be dense. The essence of the notion of core is in offering additional 'domains' for an operator. On this occasion we recall a practical notion: a closable N is called *essentially normal* if $\overline{N}$ is normal.

A handy necessary and sufficient condition for $\mathcal{D}$ to be a core of A is the following implication to hold

$$\text{for } f \in \mathcal{D}(A) \text{ such that } \langle f,g\rangle + \langle Af,Ag\rangle = 0 \text{ for all } g \in \mathcal{D} \text{ implies } f = 0. \tag{13}$$

The observation which follows fits within the character of this section and makes intrinsic use of the notion of core.

PROPOSITION 10. *Bounded vectors of a normal operator form a core of it. Therefore, a normal operator decomposes as an orthogonal sum of a sequence of bounded normal operators.*

Proof. Due to 2° in Spectral Theorem 9 for any bounded set $\sigma \subset \mathbb{C}$ and $f \in$ the vector $E(\sigma)f$ is in $\mathcal{D}(N)$ and, by (12), $E(\sigma)NE(\sigma)f = NE(\sigma)f$. This means that the linear space $\mathcal{B}(\sigma) \stackrel{\text{def}}{=} \{E(\sigma)f\colon\ f \in \mathcal{H}\}$ is invariant for N. Because $\cup_\sigma \mathcal{B}(\sigma)$ is dense every N^n is normal as well. Therefore, by (11), for any $f \in \mathcal{H}$ and any bounded set σ $E(\sigma)f$ is a bounded vector. To check that they all together constitute a core proceed as follows. Due to (12), $E(\sigma)N^*Nf = N^*E(\sigma)Nf = N^*NE(\sigma)f$ and therefore condition (13) gives

$$E(\sigma)f + E(\sigma)N^*Nf = 0$$

Because σ is an arbitrary bounded Borel set we infer that $f + N^*Nf = 0$, hence $f = 0$.

Decomposing $\mathbb{C}$ as a disjoint sum of bounded Borel sets we get the orthogonal decomposition in question. More precisely, if $\{\sigma_n\}_n$ is such a partition of $\mathbb{C}$ then the subspaces $E(\sigma_n)\mathcal{H}$ are mutually orthogonal and reduce N; this is due to (12). Notice that because the parts $N\restriction_{E(\sigma_n)N}$ are bounded a graph argument guarantees the orthogonal sum of the parts is a closed operator. Now because bounded vectors form a core of N the final conclusion comes out. $\square$

COROLLARY 11. *Any $\mathcal{D} \in \{\mathcal{B}(N), \mathcal{A}(N), \operatorname{lin} \mathcal{Q}(N)\}$ is a core of a normal operator N.*

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[9] A core may be defined even for non-closable operators because in fact the graph topology is behind the notion.

Resemblance of normality: formal normality. The first and very serious surprise comes when one asks what happens now to Triviality 1. In the unbounded case one gets nothing but $\mathcal{D} \subset \mathcal{D}((N|_{\mathcal{D}})^*)$. If $\mathcal{D}$ is a core of N then (2) can be stated as

$$\begin{aligned} &\mathcal{D}(N) \subset \mathcal{D}(N^*), \\ &\|Nf\| = \|N^*f\|, \quad f \in \mathcal{D}(N). \end{aligned} \tag{14}$$

and nothing more. Therefore, we have to call those N's somehow. Because (14) and (2) look much alike, the name in use for operators satisfying (14) is: *formally normal*. Though there is a tiny difference in definitions of normality and formal normality, '$=$' is replaced by '$\subset$', the consequences are rather significant as we are going to realize later.

Notice that if N is formally normal then it must necessarily be closable. Moreover, its closure $\overline{N}$ is formally normal as

$$\mathcal{D}(\overline{N}) \subset \mathcal{D}(N^*). \tag{15}$$

Moreover, if N is formally normal and $\mathcal{D}(N)$ is a core of N^* then N is essentially normal.

PROPOSITION 12. *Suppose N is formally normal in $\mathcal{H}$. If N_1 is a normal operator N_1 in $\mathcal{H}$ such that $N_1 \subset N$ then N is normal too and $N_1 = N$.*

Proof. Because $N_1 \subset N$, $N^* \subset N_1^*$ and consequently

$$\mathcal{D}(N^*) \subset \mathcal{D}(N_1^*) = \mathcal{D}(N_1) \subset \mathcal{D}(N),$$

which makes N normal and equal to N_1. $\square$

The operator of multiplication by independent variable. Let μ be a positive measure on $\mathbb{C}$ of finite moments [10]. Denote by $\mathcal{P}(\mu)$ the polynomials in $\mathbb{C}[Z,\overline{Z}]$ regarded as members of $\mathcal{L}^2(\mu)$. Define the operator M_Z of multiplication by the independent variable in $\mathcal{L}^2(\mu)$ as

$$\begin{aligned} &\mathcal{D}(M_Z) \stackrel{\mathrm{def}}{=} \{f \in \mathcal{L}^2(\mu) \colon \int_{\mathbb{C}} |zf(z)|^2 \mu(\mathrm{d}z) < \infty\}, \\ &(M_Z f)(z) \stackrel{\mathrm{def}}{=} zf(z),\ z \in \operatorname{supp}\mu, \quad M_Z \colon f \to M_Z f. \end{aligned}$$

Notice that the characteristic (indicator) functions 1_σ of Borel subsets of $\mathbb{C}$ are in $\mathcal{D}(M_Z)$. Therefore M_Z is densely defined, and because $(M_Z)^* = M_{\overline{Z}}$ as well as $(M_{\overline{Z}})^* = M_Z$, the operator M_Z is closed and, consequently, it is normal.

Suppose $\mathcal{P}(\mu)$ is dense $\mathcal{L}^2(\mu)$. Then $M_Z|_{\mathcal{P}(\mu)}$ is a densely defined operator. Is it essentially normal? In general not because

[10] We say μ has finite moments if $\int_{\mathbb{C}} |z|^{2n} \mu(\mathrm{d}z) < \infty$ for all $n = 0, 1, \ldots$. This is what we are taking for granted in this paper once and for all.

FACT 13. *$M_Z|_{\mathcal{P}(\mu)}$ is essentially normal if and only if $\mathcal{P}(\mu)$ is a core of M_Z. This happens if and only if $\mathcal{P}(\mu)$ is dense in $\mathcal{L}^2((1+|Z|^2)\mu)$* [11].

The second conclusion of the above follows immediately from the fact that the space $\mathcal{L}^2((1+|Z|^2)\mu)$ bears the graph norm with respect to the operator M_Z.

Remark 14. The operator $M_Z|_{\mathcal{P}(\mu)}$ is formally normal in the closure $\overline{\mathcal{P}(\mu)}$ of $\mathcal{P}(\mu)$ in $\mathcal{L}^2(\mu)$–norm regardless $\mathcal{P}(\mu)$ is dense in $\mathcal{L}^2(\mu)$ or not and has a normal extension M_Z in $\mathcal{L}^2(\mu)$. In other words, $M_Z|_{\mathcal{P}(\mu)}$ is always formally normal, has a normal extension in $\mathcal{L}^2(\mu)$ though it may act within a <u>smaller</u> space $\overline{\mathcal{P}(\mu)}$.

Repairing $*$–cyclicity. The notion $*$–cyclicity, as defined in the greyish area around (3) for bounded operators, for unbounded ones requires (3) to hold for $f \in \mathcal{D}^\infty(N,N^*)$. The above considerations show that this definition is not satisfactory in the unbounded case for quite a number of reasons: neither Corollary 3 nor Fact 4 holds true in particular. Therefore, call now N *$*$-cyclic* [12] with a cyclic vector $e \in \mathcal{D}^\infty(N^*,N)$ if the set [13] (3) is a core of N. Under this modification both Corollary 3 and Fact 4 <u>revive</u>.

A word about spectral properties. An example of an ultradeterminate measure is the Gaussian one, that is $\mathrm{e}^{-|x|^2}\,\mathrm{d}x$. The polynomials in $\mathcal{P}(\mu)$ constitute a core of M_Z and all the oddities are left apart. However, here $\mathrm{sp}(N) = \mathbb{C}$ which <u>excludes</u> any resolvent tool to be used; this is what someone ought to take into account when trying to approach the theory.

Assorted topics on unbounded subnormals

Subnormality and its characterization. The defining formula (6) remains working also in the unbounded case; more precisely an operator S densely defined in a Hilbert space is called *subnormal* if there exists a normal operator N is a Hilbert space $\mathcal{K}$ containing isometrically $\mathcal{H}$ such that (6) holds true. Another way of expressing this is that $\mathcal{H}$ is invariant for N and $S \subset N\restriction_{\mathcal{H}}$.

The only characterization of subnormality which does not impose any constrain on behaviour of domains of the operator is that via semispectral measures [14] (see, [5] or [14]) or its versions (like in [44] and [55]).

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[11] Such measures are called in [16] *ultradeterminate*. By the way, a measure is ultradeterminate if the polynomials in $\mathcal{P}(\mu)$ are dense in some $\mathcal{L}^p(\mu)$, $p > 2$ (see [16], p. 61).

[12] It is tempting to call it rather *graph $*$–cyclic* as graph topology is behind this. Regrettably, we have to abandon this appeal; also because present term includes trivially that for bounded operators.

[13] The remark made in footnote 2 applies here as well.

[14] A semispectral measure differs from a spectral one by dropping the assumption its values are orthogonal projections; it is also known under the name 'positive operator valued measure'.

THEOREM 15. *An operator S is subnormal if and only if there is a semispectral measure F on Borel sets of $\mathbb{C}$ such that* [15, 16]

$$\langle S^m f, S^n f\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \langle F(\mathrm{d}z) f, g\rangle, \quad m,n = 0,1, \quad f,g \in \mathcal{D}(S). \tag{16}$$

Notice that semispectral measures related to a subnormal operator may not be uniquely determined, see [43] for an explicit example. As spectral measures of normal extensions come via dilating semispectral measure, according Naimark's dilation theorem, cf. [24], we may have quit a number of them as well. This foretells somehow the problem with uniqueness (and minimality) we are going to expose a little bit later. So far we turn Theorem 15 into an equivalent form involving scalar spectral measures, cf. [55].

Call a family $\{\mu_f\}_{f\in\mathcal{H}}$ of positive measures on $\mathbb{C}$, a family of *elementary spectral measures* of S such that for $f,g \in \mathcal{H}$

$$\mu_{\lambda f}(\sigma) = |\lambda|^2 \mu_f(\sigma) \text{ for } \lambda \in \mathbb{C}, \quad \mu_f(X) = \|f\|^2 \tag{17}$$

$$\mu_{f+g}(\sigma) + \mu_{f-g}(\sigma) = 2(\mu_f(\sigma) + \mu_g(\sigma)) \tag{18}$$

and

$$\langle S^m f, S^n f\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \mu_f(\mathrm{d}z), \quad m,n = 0,1, \quad f \in \mathcal{D}(S).$$

THEOREM 16. (A version of Theorem 15) *An operator S is subnormal if and only if it has a family of elementary spectral measures.*

As an immediate consequence of Theorem 15 we get a slight extension (no domain invariance required) of Proposition 18 in [54]. Though this very much wanted observation looks trivially no direct way of getting it from the definition of subnormality seems to be available. This is so because, unlike normality, the definition of subnormality 'exceeds the underlying Hilbert space'. Here a kind of exception is Ando's construction of the universal extension space in which the unitary equivalence can be placed in. However, in the unbounded case this construction does not look it to work at the full, cf. [36].

COROLLARY 17. *Let S be an operator $\mathcal{H}$ let $V\colon \mathcal{H} \to \mathcal{H}_1$ be a bounded operator such that $V^*VS = S$. If S is subnormal in $\mathcal{H}$, then so is VSV^* in $\mathcal{H}_1$ provided it is densely defined. More exactly, if F is a semispectral measure of S then $VF(\cdot)V^*$ is such for VSV^*.*

Minimality and uniqueness. Minimality in the unbounded case becomes a very sensitive issue. Let us start with a definition: call N *minimal of spectral type* if (M_1) on page 488 is satisfied. It turns out that, cf. Proposition 1 in [36], it is equivalent to (M_2)

[15] Condition (16) corresponds to those in Proposition 6.

[16] If (16) holds only for $m = 1$ and $n = 1$ ($m = n = 0$ is a triviality) then S has a normal dilation exclusively and *vice versa*. In that case the fourth condition encoded in (16) downgrades to the inequality $\langle Sf, Sg\rangle \leqslant \int_{\mathbb{C}} |z|^2 \langle F(\mathrm{d}z) f, g\rangle$, $f,g \in \mathcal{D}(S)$.

in the sense that $\mathcal{S}_{\mathcal{D}(S)} = \mathcal{K}$. The sad news is that minimal normal extensions of spectral type may not be $\mathcal{H}$–equivalent [17], see Example 1 in [36] much further developed in [8]; therefore no uniqueness can expected at this stage. The good news is the welcomed spectral inclusion

$$\operatorname{sp}(N) \subset \operatorname{sp}(S) \tag{19}$$

is preserved; as a consequence of (19) notify $\operatorname{sp}(S) \neq \varnothing$. A list of further spectral properties is in Theorem 1 of [36].

The third kind of minimality appearing in (M_3) though well defined cannot be well developed in this general setting. It does when S has an invariant domain; we come to this latter on.

Tightness of normal extensions. Assume for a little while A in $\mathcal{H}$ and B in $\mathcal{K}$ are arbitrary operators, $\mathcal{K}$ contains isometrically $\mathcal{H}$. If $A \subset B$ then

$$\mathcal{D}(A) \subset \mathcal{D}(B) \cap \mathcal{H} \text{ and } P\mathcal{D}(B^*) \subset \mathcal{D}(A^*) \text{ with } A^*Px = PB^*x \text{ for } x \in \mathcal{D}(B^*) \tag{20}$$

with P being apparently the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. If N is formally normal extension of S then both inclusions in (20) merge in one

$$\mathcal{D}(S) \subset \mathcal{D}(N) \cap \mathcal{H} \subset \mathcal{D}(N^*) \cap \mathcal{H} \subset P\mathcal{D}(N^*) \subset \mathcal{D}(S^*). \tag{21}$$

This implies immediately that

$$\mathcal{D}(S) \subset \mathcal{D}(S^*) \text{ and } \|S^*f\| \leqslant \|Sf\| \text{ for } f \in \mathcal{D}(S).$$

Hence S is closable and $\mathcal{D}(\bar{S}) \subset \mathcal{D}(S^*)$; the latter has to be compared with (15).

Call the extension N *tight* if $\mathcal{D}(\bar{S}) = \mathcal{D}(N) \cap \mathcal{H}$ and $*$*-tight* if $P\mathcal{D}(N^*) = \mathcal{D}(S^*)$, cf. [49], the topic was taken up in [19]. Notice that tight extendibility was one of the condition involved in the definition of subnormal operators given in [27]. It was proved in [38] that symmetric and analytic Toeplitz operator have tight extension. The question in [38] asks if this is always the case, which would give subnormality of [27] the same meaning as ours. It turns out they two different notions according to the example in [29]. Therefore the preference is the present one.

Cartesian decomposition. If A has $\mathcal{D}(A) \cap \mathcal{D}(A^*)$ dense then

$$\mathfrak{Re}\, A \overset{\text{def}}{=} \frac{1}{2}(A + A^*), \quad \mathfrak{Im}\, A \overset{\text{def}}{=} \frac{1}{2\mathrm{i}}(A - A^*), \quad \mathcal{D}(\mathfrak{Re}\, A) = \mathcal{D}(\mathfrak{Im}\, A) = \mathcal{D}(A) \cap \mathcal{D}(A^*)$$

leads to the Cartesian decomposition of A

$$A = \mathfrak{Re}\, A + \mathrm{i}\, \mathfrak{Im}\, A$$

with $\mathfrak{Re}\, A$ and $\mathfrak{Im}\, A$ symmetric on $\mathcal{D}(A) \cap \mathcal{D}(A^*)$.

[17] It is right time to give the definition: two extensions B_1 and B_2 in spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of A in $\mathcal{H}$ are called $\mathcal{H}$*–equivalent* if there is a unitary operator $U \colon \mathcal{K}_1 \to \mathcal{K}_2$ such that $U\mathcal{K}_1 = \mathcal{K}_2 U$ and $U\restriction_{\mathcal{H}} = I_{\mathcal{H}}$.

PROPOSITION 18. *A formally normal operator N is essentially normal if and only if the operators $\mathfrak{Re}N$ and $\mathfrak{Im}N$ are essentially selfadjoint and spectrally commute, that is there spectral measures commute. An operator S is subnormal if an only if it has such a formally normal extension.*

Proof. If N is formally normal and $\overline{\mathfrak{Re}N}$ and $\overline{\mathfrak{Im}N}$ commute spectrally then $\overline{\mathfrak{Re}N} + \mathrm{i}\,\overline{\mathfrak{Im}N}$ is normal. Furthermore, $N \subset \overline{\mathfrak{Re}N} + \mathrm{i}\,\overline{\mathfrak{Im}N} \subset \overline{\mathfrak{Re}N + \mathrm{i}\,\mathfrak{Im}N} = \overline{N}$ and Proposition 18 concludes with N to be essentially normal. The rest follows easily. □

This is a parallel to Theorem 15. It show that famous Nelson's example from [25] can be adopted as an alternative one to Coddington's [9].

Polar decomposition and quasinormal operators. For a closed densely defined operator $A : \mathcal{H} \supset \mathcal{D}(A) \longrightarrow \mathcal{K}$ there exists a unique partial isometry $U \in \boldsymbol{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{N}(U) = \mathcal{N}(A)$ and $A = U|A|$, where $|A| \stackrel{\text{def}}{=} (A^*A)^{1/2}$; $U|A|$ is called the *polar decomposition* of A (cf. [59, p. 197]). If $U|A|$ is the polar decomposition of A, then $\overline{\mathcal{R}(|A|)} = \overline{\mathcal{R}(A^*)}$ is the initial space of U and $\overline{\mathcal{R}(A)}$ is the final space of U.

One of the equivalent definitions of *quasinormal* operators says they are those for which in their polar decomposition $U|A| = |A|U$. These operators are <u>subnormal</u> (cf. [35, Theorem 2]), even more, they have a kind of Wold-von Neumann decomposition, see [7] for bounded operators and its version adapted to the unbounded case [41]. In a sense they become an intermediate step between subnormal and normal operators. Normal operators are those quasinormals for which $\mathcal{N}(A^*) \subset \mathcal{N}(A)$.

Because $\mathcal{N}(N) = \mathcal{N}(N^*)$ for a normal N, both factor in its polar decomposition can be extended properly so as to get the following result.

PROPOSITION 19. *N is normal if and only if $N = UP$ with U unitary and P a positive operator, U and P commuting. This decomposition is not unique.*

Old friends in the new environment. Because selfadjoint operators are apparently normal, symmetric operators are both formally normal and subnormal. The following draft shows how all the notions interplay; all the inclusions may[18] become proper. Notice the formally normal are somehow apart, formally normal operators may not be normal, see [9] for an explicit example.

$$\begin{array}{ccccc} \textit{selfadjoint} & \subset & \textit{symmetric} & \subset & \textit{formally normal} \\ \cap & & \cap & & \\ \text{normal} & \subset & \textbf{subnormal} & & \\ \cap & & & & \\ \textit{formally normal} & & & & \end{array}$$

Let us mention that Coddington characterizes in [3, 10] those formally normal operators which are subnormal.

[18] In the finite dimensional case subnormals, formally normals and normals coincide.

Subnormality of operators with invariant domain

From now onwards we declare

$$\boxed{S\mathcal{D}(S) \subset \mathcal{D}(S).}$$

This means we have to resign the temptation to consider an operator S to be closed unless we want deliberately exclude operators which not bounded.

Under these circumstances we have supplementing results to Theorems 15 and 16 at once.

THEOREM 20. *If S is subnormal and F is a semispectral measure such that* (16) *holds then* (16) *holds for all m,n, that is*

$$\langle S^m f, S^n f\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \langle F(\mathrm{d}z)f, f\rangle, \quad m,n = 0,1,\ldots \quad f \in \mathcal{D}(S).$$

Alternatively, the elementary spectral measures of S satisfy

$$\langle S^m f, S^n f\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \mu_f(\mathrm{d}z), \quad m,n = 0,1,\ldots \quad f \in \mathcal{D}(S). \tag{22}$$

Back to Halmos' positive definiteness or what has survived from Bram's theorem. Under the current circumstances Halmos' positive definiteness takes the form

$$\sum_{m,n} \langle S^m f_n, S^n f_m\rangle \geqslant 0, \quad \text{for any finite sequence } (f_k)_k \subset \mathcal{D}(S). \tag{PD}$$

What we still have in the flavour of Bram's characterization is in the following, see [35] or [45] for another techniques of building the proof up.

THEOREM 21. *An operator S in $\mathcal{H}$ satisfies* (PD) *if and only if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ isometrically, and a formally normal operator N in $\mathcal{K}$ such that $S \subset N$ as well as*

$$N\mathcal{D}(N) \subset \mathcal{D}(N) \text{ and } N^*\mathcal{D}(N) \subset \mathcal{D}(N) \tag{23}$$

If this happens, N can be chosen to satisfy

$$\mathcal{D}(N) = \operatorname{lin}\{N^{*k} f\colon\ k = 0,1,\ldots,\ f \in \mathcal{D}(S)\}. \tag{24}$$

Remark 22. Suppose S and N are as in Theorem 21. If S is cyclic with a cyclic vector e then N is $*$–cyclic with the same vector e. Indeed, if S is cyclic with a cyclic vector e then, by (24),

$$\mathcal{D}(N) = \operatorname{lin}\{N^{*k} N^l e\colon\ k,l = 0,1,\ldots\}$$

and the first conclusion follows.

COROLLARY 23. *If S is subnormal then it satisfies* (PD).

COROLLARY 24. *N in* Theorem 21 *is essentially normal if and only if*

$$x \in \mathcal{D}(N^*) \ \& \ \langle x,y\rangle + \langle x, N^*Ny\rangle = 0 \ \forall \ y \in \mathcal{D}(N) \implies x = 0.$$

Proof. Notice first that N is essentially normal if and only if $\mathcal{D}(N)$ is a core of N^*. Now use (13) and (23). □

We separate the uniqueness result because of its importance.

THEOREM 25. *If two pairs $(N_1, \mathcal{K}_1)$ and $(N_2, \mathcal{K}_2)$ satisfy the conclusion of* Theorem 21 *then they are $\mathcal{H}$-equivalent, that is there is a unitary operator between $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $U\restriction_{\mathcal{H}} = I_{\mathcal{H}}$ and $UN_1 = N_2U$.*

Proof. For $(f_k)_k \subset \mathcal{D}(S)$ we have, due to (24),

$$\begin{aligned}\|\sum_n N_1^{*n} f_n\|_1^2 = \sum_{k,l}\langle N_1^k f_l, N_1^l f_k\rangle_1 &= \sum_{k,l}\langle S^k f_l, S^l f_k\rangle \\ &= \sum_{k,l}\langle N_2^k f_l, N_2^l f_k\rangle_2 = \|\sum_n N_2^{*n} f_n\|_2^2\end{aligned}$$

which establishes the unitary operator between two dense subspaces. The next step is standard as well. □

COROLLARY 26. *Suppose S is subnormal in $\mathcal{H}$. If $\widetilde{N}$ is any normal extension of S and N is a formally normal extension of S as in* Theorem 21, *for which* (24) *holds, then there is a formally normal operator N_1 which is $\mathcal{H}$-equivalent to N and such that $S \subset N_1 \subset \widetilde{N}$.*

Proof. If $\widetilde{N}$ is normal in $\widetilde{\mathcal{K}}$ say, then the subspace

$$\mathcal{D} \stackrel{\text{def}}{=} \operatorname{lin}\{\widetilde{N}^{*n} f \colon \ n = 0,1,\ldots, \ f \in \mathcal{D}(S)\} \tag{25}$$

of $\mathcal{D}(\widetilde{N})$ is invariant for $\widetilde{N}$ and $\widetilde{N}^*$. The operator $N_1 \stackrel{\text{def}}{=} \widetilde{N}|_{\mathcal{D}}$ is formally normal. Indeed, because, due to (20), $\widetilde{N}^*|_{\mathcal{D}} \subset (\widetilde{N}\restriction_{\overline{\mathcal{D}}})^* \subset (\widetilde{N}|_{\mathcal{D}})^*$ we can write for $x \in \mathcal{D}$

$$\|N_1 x\| = \|\widetilde{N}x\| = \|\widetilde{N}^* x\| = \|(\widetilde{N}|_{\mathcal{D}})^* x\| = \|N_1^* x\|.$$

Comparing (25) with (24) suggests the definition of the required unitary $\mathcal{H}$-equivalence. □

COROLLARY 27. *If S is a cyclic and subnormal operator then the formally normal operator determined by* Theorem 21 *can be realized as the operator $M_Z|_{\mathcal{P}(\mu)}$ in the $\mathcal{L}^2(\mu)$–closure of the polynomials $\mathcal{P}(\mu)$, where $\mu \stackrel{\text{def}}{=} \langle F(\cdot)1,1\rangle$ and F is an arbitrary semispectral measure of S. According to* Theorem 25 *any two such models are $\mathcal{H}$–equivalent. Finally, N itself is subnormal*[19].

[19] Look at Corollary 17.

In general, Theorem 21 is nothing but an intermediate step toward subnormality. Because N is just formally normal the uncertainty is still ahead. The only known result which reminds that of Bram is as follows, cf. [35].

THEOREM 28. *Suppose S is a weighted shift*[20]*. Then S is subnormal if and only if it satisfies the positive definiteness condition* (PD).

Cyclicity and related matters. Getting experienced already with $*$–cyclicity we can say that a closable operator A with invariant domain is *cyclic* with a *cyclic vector* e if

$$\{p(A)e\colon\ p \in \mathbb{C}[Z]\}$$

is a core of A. On the other hand, given a vector $f \in \mathcal{D}(A)$ set

$$\mathcal{D}_f \stackrel{\mathsf{def}}{=} \{p(A)f\colon\ p \in \mathbb{C}[Z]\}, \quad \mathcal{H}_f \stackrel{\mathsf{def}}{=} \overline{\mathcal{D}_f}, \quad A_f \stackrel{\mathsf{def}}{=} A|_{\mathcal{D}_f}.$$

The definition of A_f is in accordance with what is on p. 489 and means an operator acting in the Hilbert space $\mathcal{H}_f$; call A_f the *cyclic portion* of A *at* f. Therefore A is cyclic if and only if $\overline{A} = \overline{A_f}$ for some $f \in \mathcal{D}(A)$.

Notice that if $g \in \mathcal{D}_f$ then $\mathcal{D}_g \subset \mathcal{D}_f$. However, if $f \neq g$ we can not say anything reasonable about dislocation of the spaces $\mathcal{D}_f$ and $\mathcal{D}_g$ unless they both are reducing.

The complex moment problem. Given a bisequence $(c_{m,n})_{m,n=0}^{\infty}$, call it a *complex moment sequence* it there exists a positive Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \,\mathrm{d}\mu, \quad m,n = 0,1,\ldots \tag{26}$$

The *complex moment problem* related to a bisequence $(c_{m,n})_{m,n=0}^{\infty}$ consists in finding a measure *representing* the bisequence via (26)[21]. The measure μ, thus the moment sequence $(c_{m,n})_{m,n=0}^{\infty}$, is called *determinate* it there is no other measure representing the sequence by (26). Another, stronger concept, introduced in [16], calls the measure μ, as well as the related bisequence[22], *ultradeterminate*, cf. footnote 23, if the operator $M_Z|_{\mathcal{P}(\mu)}$ of multiplication by the independent variable is essentially normal. If this happens, M_Z is normal in $\mathcal{L}^2(\mu)$, cf. Fact 13.

The moment problem version of Corollary 23 determines a kind of positive definiteness that a complex moment bisequence $(c_{m,n})_{m,n=0}^{\infty}$ has necessarily to satisfy.

PROPOSITION 29. *If $(c_{m,n})_{m,n=0}^{\infty}$ is a complex moment bisequence then*

$$\sum_{m,n} c_{m+q,n+p} \lambda_{m,n} \bar{\lambda}_{p,q} \text{ for all finite bisequences } (\lambda_{k,l})_{k,l} \subset \mathbb{C}. \tag{MPD}$$

[20] S in $\mathcal{H}$ is a *weighted shift* if there is an orthonormal basis $(e_n)_n$ in $\mathcal{H}$ such that $Se_n = \sigma_n e_{n+1}$ with some positive weights $(\sigma_n)_n$.

[21] It happens people carelessly mix up those concepts.

[22] The definition in [16] is stated for a bisequence, that for a measure comes from searching through the paper.

In the other direction again we stop halfway.

THEOREM 30. *A bisequence* $(c_{m,n})_{m,n=0}^{\infty}$ *satisfies* (MPD) *if and only if there is a Hilbert space* $\mathcal{K}$ *containing the* 1*–dimensional Hilbert space* $\mathbb{C}$ *isometrically, and a formally normal operator* N *in* $\mathcal{K}$ *such that* $\mathcal{D}(N) = \mathrm{lin}\{N^{*m}N^{n}1\colon\ m,n=0,1,\ldots\}$ *and*

$$c_{m,n} = \langle N^m 1, N^n 1\rangle, \quad m,n = 0,1,\ldots$$

Now everything depends on if the formally normal operator N has a normal extension or not.

The explicitly defined bisequence in [46], p.259, which in turn is an adapted to the present circumstances version of that in [15], satisfies (MPD) and is not a complex moment one.

Subnormality and the complex moment problem. Here we have the fundamental result which continues Theorem 30 and goes towards our open problem.

THEOREM 31. (*a*) *If* S *is subnormal then* $(\langle S^m f, S^n f\rangle)_{m,n=0}^{\infty}$ *is a complex moment bisequence for every* $f \in \mathcal{D}(S)$*;*

(*b*) *If* S *is cyclic with a cyclic vector* e *and* $(\langle S^m e, S^n e\rangle)_{m,n=0}^{\infty}$ *is a complex moment bisequence then* S *is subnormal. Moreover, if* μ *is a measure which represents* $(\langle S^m e, S^n e\rangle)_{m,n=0}^{\infty}$ *by* (26) *and* $f \in \mathcal{D}(S)$ *is of the form* $p(S)e$ *for some* $p \in \mathbb{C}[Z]$ *then* $(\langle S^m f, S^n f\rangle)_{m,n=0}^{\infty}$ *is a moment bisequence and* $|p|^2\mu$ *is its representing measure.*

Referring to Theorem 30 and Fuglede's classification of determinacy we have two relevant notions: call a vector $f \in \mathcal{D}(S)$ a *vector of determinacy* of S if the moment bisequence $(\langle S^m f, S^n f\rangle)_{m,n=0}^{\infty}$ is determinate; if this bisequence is ultradeterminate call f the *vector of ultradeterminacy* of S [23]. It is clear these two kinds of determinacy <u>require</u> the operator S_f to be already subnormal.

ADVICE. In the discussion which follows there are two alternating situations: they concern either a cyclic <u>operator</u> or a cyclic <u>portion</u> of an operator. A reader may chose to think of any of these two without any side effect.

THEOREM 32. *Suppose* S *is cyclic and* e *is its cyclic vector. Then the following two conclusions hold.*

(α) *If* e *is the vector of determinacy then the formally normal operator determined by* Theorem 21 *can be realized as the operator* $M_Z|_{\mathcal{P}(\mu)}$ *in the* $\mathcal{L}^2(\mu)$*–closure of polynomials* $\mathcal{P}(\mu)$*, where* μ *is the unique measure representing* $(\langle S^m e, S^n e\rangle)_{m,n=0}^{\infty}$.

(β) *If* e *is a vector of ultradeterminacy of* S *then the formally normal operator* N *constructed as in* Theorem 21 *is essentially normal.* $\overline{N}$ *can be realized as the normal operator* M_Z *in* $\mathcal{L}^2(\mu)$ *with* μ *as in* (α).

---

[23] The term 'vector of uniqueness' as in [26], which is more appropriate for symmetric operators and real one dimensional moment problems, splits here in two. Notice that in [55] we use the term 'vector of uniqueness with a slightly different meaning.

Proof. Apply Corollary 27 to get (α). Now use (α) and the fact that e is the vector of ultradeterminacy of S to come to (β). □

Notice (β) says that if e is a vector of ultradeterminacy for S then it is so for the formally normal operator N constructed as in Theorem 21. In other words, the property of a vector to be that of ultradeterminacy can be lifted to the extending space; this is a rough comment rather then a precise statements.

The next two results can be viewed as a <u>global</u> version of Theorem 32; the latter to be though of as a <u>local</u> one.

THEOREM 33. *The two following two conclusions hold.*

(α') *If every $f \in \mathcal{D}(S)$ is a vector of determinacy of S and for the (unique) family $(\mu_f)_{f\in\mathcal{D}(S)}$ of measures representing the complex moment bisequence $\langle S^m f, S^n f\rangle$, $f \in \mathcal{D}(S)$, one has*

$$\mu_{f+g} + \mu_{f-g} - 2\mu_g \geqslant 0, \quad f, g \in \mathcal{D}(S), \tag{27}$$

then S is subnormal and has a <u>unique</u> normal extension which is minimal of spectral type, and conversely.

Therefore, a formally normal operator N can be constructed as in Theorem 21 *and it is subnormal as well.*

(β') *If the set $\mathcal{U}(S)$ of vectors of ultradeterminacy of S is total in $\mathcal{H}$ then the formally normal operator N constructed as in* Theorem 21 *is essentially normal.*

Proof. Proof of (α'). It is clear there a unique family of measures μ_f, $f \in \mathcal{D}(S)$ such that (22) and (17) holds. The only condition missing so far to end up with the conclusion is (18).

Take $f, g \in \mathcal{D}(S)$ and with $m, n = 0, 1, \ldots$ write

$$\begin{aligned}\int_{\mathbb{C}} z^m \bar{z}^n \,\mathrm{d}\big(\tfrac{1}{2}(\mu_{f+g} - \mu_{f-g}) - \mu_f\big) &= \tfrac{1}{2}\langle S^m(f+g), f+g\rangle + \tfrac{1}{2}\langle S^m(f-g), f-g\rangle \\ &\quad - \langle S^m f, f\rangle = \langle S^m g, g\rangle = \int_{\mathbb{C}} z^m \bar{z}^n \,\mathrm{d}\mu_g.\end{aligned}$$

Now (27) and determinacy at g makes (18) hold. Therefore, S is subnormal due to Theorem 16. Corollary 26 establishes the final conclusion in (α') concerning N.

Proof of (β'). Let N be the formally normal operator constructed as Theorem 21. Set

$$\mathcal{D}_f(N) \overset{\mathsf{def}}{=} \operatorname{lin}\{p(N^*, N)f \colon\ p \in \mathbb{C}[Z, \overline{Z}]\}, \quad N_f \overset{\mathsf{def}}{=} N\restriction_{\overline{\mathcal{D}_e(N)}} \tag{28}$$

and denote by P_f the orthogonal projection on $\overline{\mathcal{D}_e(N)}$. Notice that for $f \in \mathcal{U}(S)$ the operator $\overline{N}$ is normal. According to Lemma 2 of [37] the subspace $\mathcal{H}_f$ reduces N. Because $\mathcal{U}(S)$ is total,

$$\operatorname{lin}\{\mathcal{D}_f \colon\ f \in \mathcal{U}(S)\} = \mathcal{D}(N).$$

Adapting arguments used in the proof of Theorem of [37] we can check that N is essentially normal. $\square$

Minimality and uniqueness again. Now is a right time to come back the minimality problem of extensions of S. An extension N of S *minimal of cyclic type* if $\mathcal{C}_{\mathcal{D}(S)}$ is a core of N; this definition works regardless what class of operators the extensions belong to, the only requirement is (9) to have sense. Theorem 21 provides us with formally normal extensions of cyclic type of an operator satisfying (PD).

Minimal <u>normal</u> extensions of cyclic type may <u>not</u> exist – see [36] and [8], the latter is for further, much broader development of the former; an example of another type is in [43]. In this matter quote Theorem 3 and Corollary 3, both in [36], in one.

THEOREM 34. *Let S be a subnormal operator. Suppose that it <u>has</u> at least one minimal normal extension of cyclic type. Then an arbitrary normal extension of S is minimal of <u>spectral</u> type if and only if it is minimal of <u>cyclic</u> type.*

If S has at least one minimal normal extension of cyclic type, then all its minimal normal extensions of spectral type (*hence those of cyclic type too*) *are $\mathcal{H}$–equivalent.*

This settles the question of uniqueness. Let us say carefully that a subnormal operator S has the *uniqueness extension property* if the circumstances of Theorem 34 happen [24]. Part (β') of Theorem 33 implies at once

COROLLARY 35. *S has the uniqueness extension property if its domain is composed of vectors of ultradeterminacy.*

Remark 36. It is clear that for a cyclic subnormal operator S with a cyclic vector e the following statements are equivalent:

- e is a vector of ultradeterminacy of S,
- the formally normal extension N of S constructed via Theorem 30 is essentially normal, hence it is minimal of cyclic type.

The other way around, it seems to be worthy to realize how minimality of cyclic type can be inherited by cyclic subspaces. More precisely, If N is a minimal normal extension of S acting in $\mathcal{K}$ then for $e \in \mathcal{D}(S)$ the closure $\overline{\mathcal{D}_e(N)}$ of $\mathcal{D}_e(N)$ defined in (28) reduces N (indeed, because $\overline{\mathcal{D}_e(N)}$ is invariant for both N and N^* we get it, cf. footnote). Therefore, N_e is a minimal normal extension of S_e of cyclic type and, consequently, e is a vector of ultradeterminacy of S.

Subnormality trough $\mathcal{C}^\infty$–vectors. The following can be regarded as what corresponds to Halmos' characterization of subnormality. Notice that his boundedness condition (7) takes now a more subtle form of a growth condition.

[24] Uniqueness extension property (and subnormality itself) has been characterized in [45], Theorems 4, 4',5 and 5'. When specialized to the complex moment problem it matters ultradeterminacy resembling the characterization of Hamburger of determinacy in the real case, cf. [31] p. 70.

THEOREM 37. *If S satisfies* (PD) *and* $\mathcal{D}(S) \in \{\mathcal{B}(S), \mathcal{A}(S), \operatorname{lin} \mathcal{Q}(S)\}$ *then it is subnormal and has the uniqueness extension property.*

We refer to [34], [35] and also to [41] for proofs. They consist in showing the vectors in $\mathcal{D}(S)$ are in fact vectors of ultradeterminacy of S, cf. Corollary 35.

This result is a sort of standard if one restricts an interest to essential selfadjointness of symmetric operators.

Complete characterization of subnormality by positive definiteness. What differs Theorem 37 from Bram's result is some oversupply, the presence of an additional conclusion. The characterization we give below does not have this defect. In [39] one can find actually two kinds of characterizations: the first makes use of extending the positive definiteness condition (PD) so as to get a spatial extension, the second is just a test, rather complicated to state it in this paper. Here we describe the first approach. Instead of stating it formally we explain the idea behind the result by a sequence of three drawings, they refer to the cyclic case when (PD) can be though of as (MPD). Only Picture 2 needs some comment. It refers to the situation of positive definiteness (MPD) defined on $\mathbb{Z} \times \mathbb{Z}$. In this case we get a solution and the extra conclusion that the measure involved does not have 0 in its support.

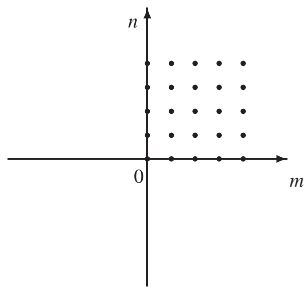

Figure 1. Halmos positive definiteness: *too little*

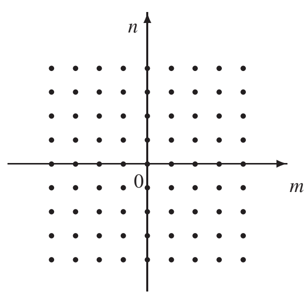

Figure 2. Very extended positive definiteness: *too much*

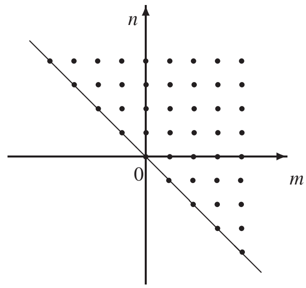

Figure 3. ‘Half plane’ extended positive definiteness: *that’s it!*

The example

The most spectacular example of the theory is the creation operator of the quantum harmonic oscillator; this was notified explicitly for the first time in [20]. This operator has many faces. Before we describe them here let us mention that from the abstract point of view they are indistinguishable: precisely any of them is the weighted shift with the weights $\sigma_n = \sqrt{n+1}$ in a particular Hilbert space and with respect to particular orthonormal basis; the adjoint acts as a backward shift according the usual rule. The creation operator is not only so prominent example but also belongs to the family of the best behaving subnormal operators. Among the pleasant features of the creation operator S we mention:

1° $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$, this rounds up (21);

2° S has ‘enough’ analytic vectors;

3° S enjoys the uniqueness property;

4° S has a ‘full’ analytic model;

5° it is determined by its selfcommutator.

The collection of models we are going to present in brief below shows on how many diverse and concrete ways this abstractly defined operator can be realized, look also at [53].

$\mathcal{L}^2(\mathbb{R})$ *model.* The oldest model of the quantum harmonic oscillator couple, the creation and the annihilation operator, is

$$S = \frac{1}{\sqrt{2}}(x - \frac{\mathrm{d}}{\mathrm{d}x}), \quad S^\times = \frac{1}{\sqrt{2}}(x + \frac{\mathrm{d}}{\mathrm{d}x})$$

considered in $\mathcal{L}^2(\mathbb{R})$ with $\mathcal{D}(S) = \mathcal{D}(S^\times) = \mathrm{lin}(h_n)_{n=0}^\infty$ where h_n is the n-th Hermite function

$$h_n = 2^{-n/2}(n!)^{-1/2}\pi^{-1/4}\mathrm{e}^{-x^2/2}H_n$$

with H_n, the n-Hermite polynomial, defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{29}$$

Analytic model: multiplication in the Segal–Bargmann space. An analytic model of the quantum oscillator is in $\mathcal{A}^2(\exp(-|z|^2\,dxdy)$, called the Bargmann-Segal space – cf. [30, 2], which is composed of all entire functions in $\mathcal{L}^2(\exp(-|z|^2\,dxdy))$ and is, in fact, a reproducing kernel Hilbert space with the kernel $(z,w) \mapsto \exp(z\overline{w})$. The standard orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in the space $\mathcal{A}^2(\exp(-|z|^2)\,dxdy)$ is composed of monomials

$$e_n = \frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}, \quad n = 0, 1, \ldots$$

Set $\mathcal{D}_0 = \operatorname{lin}(e_n)_{n=0}^{\infty}$. Then the operators S and $S^\times$ defined as

$$Sf(z) = zf(z), z \in \mathbb{C}, \quad S^\times f = \frac{d}{dz} f, \quad f \in \mathcal{D}(S) = \mathcal{D}(S^\times) = \mathcal{D}_0$$

are the creation and the annihilation operators.

Notice that $\mathcal{L}^2(\exp(-|z|^2\,dxdy)$ is the natural extension of $\mathcal{A}^2(\exp(-|z|^2)\,dxdy)$ and the creation operator S, which is the operator of multiplication by the independent variable, extends to the operator which acts in the same way in the larger space. Because the latter operator is normal, the creation operator is a subnormal operator. The annihilation operator, is the projection of the operator of multiplication by $\bar{z}$ in $\mathcal{L}^2(\exp(-|z|^2\,dxdy)$ to the Segal-Bargmann space.

The unitary equivalence between $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{A}^2(\exp(-|z|^2\,dxdy)$ and its inverse can be implemented by integral transforms, called Bargmann transform, whose kernels comes from the generating function of the Hermite polynomials, see [17] for more details on this and for a little piece of history. The Bargmann transform can be used also, via Corollary 17, to argue that the creation in $\mathcal{L}^2(\mathbb{R})$ is subnormal, this is parallel to other arguments.

Analytic model: not very classical. The Hermite polynomials, defined as in (29), are now considered as those in a complex variable. Let $0 < A < 1$. Then

$$\int_{\mathbb{R}^2} H_m(x+iy)H_n(x-iy)\exp\big[-(1-A)x^2 - \big(\frac{1}{A}-1\big)y^2\big]\,dxdy = b_n(A)\delta_{m,n}$$

where

$$b_n(A) = \frac{\pi\sqrt{A}}{1-A}\big(2\frac{1+A}{1-A}\big)^n n!.$$

Introducing the Hilbert space $\mathcal{X}_A$ of entire functions f such that

$$\int_{\mathbb{R}^2} |f(x+iy)|^2 \exp\big[Ax^2 - \frac{1}{A}y^2\big]\,dxdy < \infty$$

and defining

$$h_n^A(z) = b_n(A)^{-1/2} e^{-z^2/2} H_n(z), \quad z \in \mathbb{C}$$

it was shown in [13] that $\{h_n^A\}_{n=0}^\infty$ is an orthonormal basis in $\mathcal{X}_A$. From the algebraic relation $H_{n+1} = 2zH_n - H_n'$ we get directly

$$\sqrt{n+1}\,h_{n+1}^A = \sqrt{\frac{1-A}{2(1+A)}}\,[zh_n^A - (h_n^A)'].$$

Set $\mathcal{D}_A = \operatorname{lin}(h_n^A)_{n=0}^\infty$. The operators S_A and $S_A^\times$ defined as

$$S_A f(z) = \sqrt{\frac{1-A}{2(1+A)}}\,[zf(z) - f'(z)], \quad S_A^\times f(z) = \sqrt{\frac{1+A}{2(1-A)}}\,[zf(z) + f'(z)], \qquad z \in \mathbb{C}, \quad f \in \mathcal{D}_A$$

are the creation and the annihilation operator in $\mathcal{X}_A$, cf. [48].

It is interesting to notice that this model realizes a kind of 'homotopy' for the quantum harmonic oscillator between the $\mathcal{L}^2(\mathcal{B})$ model and that in the space $\mathcal{A}^2(\exp(-|z|^2)\,\mathrm{d}x\mathrm{d}y)$ which are both achieved as $0 < A < 1$ tends to its two extremities, cf. [48].

Discrete model. The Charlier polynomials $\{C_n^{(a)}\}_{n=0}^\infty$, $a > 0$, are determined by

$$\mathrm{e}^{-az}(1+z)^x = \sum_{n=0}^\infty C_n^{(a)}(x)\frac{z^n}{n!}.$$

They are orthogonal with respect to a nonnegative integer supported measure according to

$$\sum_{x=0}^\infty C_m^{(a)}(x)C_n^{(a)}(x)\frac{\mathrm{e}^{-a}a^x}{x!} = \delta_{mn}a^n n!, \quad m,n = 0,1,\dots .$$

Define the Charlier functions (or, rather, the *Charlier sequences*) $c_n^{(a)}$, $n = 0,1,\dots$ in discrete variable x as

$$c_n^{(a)}(x) = a^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}C_n^{(a)}(x)\mathrm{e}^{-\frac{a}{2}}a^{\frac{x}{2}}(x!)^{-\frac{1}{2}}, \quad \text{for } x \geqslant 0.$$

As we know from [47] so defined Charlier sequences satisfy

$$\sqrt{n+1}\,c_{n+1}^{(a)}(x) = \begin{cases} \sqrt{x}c_n^{(a)}(x-1) - \sqrt{a}c_n^{(a)}(x) & x \geqslant 1 \\ -\sqrt{a}c_n^{(a)}(x) & x = 0 \end{cases}$$

Therefore, the operator S_a defined as

$$(S_a f)(x) \stackrel{\mathsf{def}}{=} \begin{cases} \sqrt{x}f(x-1) - \sqrt{a}f(x) & x \geqslant 1 \\ -\sqrt{a}f(x) & x = 0 \end{cases}$$

with domain $\mathcal{D}(S_a) \stackrel{\mathsf{def}}{=} \operatorname{lin}\{c_n^{(a)} : n = 0,1,\dots\}$ is the creation. The annihilation operator is defined again as a finite difference operator

$$(S_a^\times f)(x) = \sqrt{x+1}f(x+1) - \sqrt{a}f(x), \quad x = 0,1,\dots$$

In [47] one can find an analog of Bargmann transform for this model as well.

Plays with the commutation relation. Remark at 5^{o}, p. 504, has to be developed a little bit more. It suggests the creation operator is in sense exceptional. It is clear the creation operator S and its formal adjoint $S^{\times}$, the annihilation operator, satisfy the canonical commutation relation of the quantum harmonic oscillator

$$S^{\times}S - SS^{\times} = I. \tag{30}$$

This relation has a rather formal appearance but after giving it a proper meaning makes the way back possible, cf. [49]. Roughly, an operator S in a separable Hilbert space is a creation operator if (and only if) it satisfies (30) properly understood, is subnormal and has the uniqueness extension property.

Another unprecedented feature the creation operator may be proud of is that it is uniquely determined as the only operator within the class of weighted shifts for which its translate(s) is still there, cf. [40] and also [51] where the role of the discrete model in ℓ^2 is fully explained.

The question

It is clear that a 'suboperator' of a subnormal operator is by definition subnormal too. The problem is to what extend the converse holds true. More specifically,

$$\text{if every } S_f \text{, for } f \in \mathcal{D}(S)\text{, is subnormal, is so } S? \tag{$\clubsuit$}$$

It is so in ($\clubsuit$) for bounded operators, see [21, 60]. In the unbounded case this is true if $\mathcal{D}(S) = \mathcal{A}(S)$, the analytic vectors of S, see [33]. Replacing analytic vectors by vectors of determinacy, as in part (α') of Theorem 33 leads to the positive answer provided (27) holds; here extra conclusion appears. However, condition (27) itself is sufficient for ($\clubsuit$) to be true, see [44] and Theorem 4 in [55]. It is also answered in positive when cyclic portions S_f are replace by, so to speak, 2–cyclic ones, see [39] and [55]. Our believe is the problem is a kind of selection one, see [55] for more discussion in this matter. All this supports the conjecture that it is 'yes' at large. Who knows?

The end

Missing topics

As always happens when one wants to write a survey of moderate length and the material is of considerable size the problem of selection becomes unavoidable. This has happened here as well. Among the topics which are absent we mention two.

Lifting commutant. The only thing we can do right now is to direct to [32], [23] and [22] where further references can be found.

Analytic models. Analytic models for unbounded operators are exhaustively presented in [36]; their relation to subnormality is also there. Analytic models are intimately associated with reproducing kernel Hilbert spaces, cf. [50] and [52]. Let us mention that from this point of view the question of subnormality can be roughly rephrased as the problem of integrability of those space. More precisely, when a reproducing kernel Hilbert space composed of analytic functions can be isometrically imbedded in an $\mathcal{L}^2$ space. It is clear that the Dirichlet space is not such.

Some final words

This is a story of unbounded subnormality as it has been more or less developed until now. This is also an open invitation to take part in its continuation.

Impressionism as understood in painting[25] and music at the turn of the 19th and 20th century does not happen too often in mathematical writing. My venture here is to make it happen somewhat.

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