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LINEAR MAPS PRESERVING THE MINIMUM AND SURJECTIVITY MODULI OF OPERATORS

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Abstract. Let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H, and denote by $\mathrm{m(T)}$ and $\mathrm{q(T)}$ respectively the minimum modulus and the surjectivity modulus for every $T \in \mathscr{B}(H)$. In this paper, we prove that if ϕ is a surjective unital linear map on $\mathscr{B}(H)$, then $\mathrm{m}(\phi(\mathrm{T})) = \mathrm{m(T)}$ for every $T \in \mathscr{B}(H)$ if and only if $\mathrm{q}(\phi(\mathrm{T})) = \mathrm{q(T)}$ for every $T \in \mathscr{B}(H)$ if and only if there exists an unitary operator $U \in \mathscr{B}(H)$ such that $\phi(T) = UTU^*$ for all $T \in \mathscr{B}(H)$.

1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras over the complex field. A bijective linear map $\phi : \mathscr{A} \to \mathscr{B}$ is called a *Jordan isomorphism* if $\phi(a^2) = (\phi(a))^2$ for every $a \in \mathscr{A}$, or equivalently $\phi(ab+ba) = \phi(a)\phi(b)+\phi(b)\phi(a)$ for all a and b in $\mathscr{A}$. It is obvious that every isomorphism and every anti-isomorphism, that is ϕ is bijective and $\phi(ab) = \phi(b)\phi(a)$ for all a and b in $\mathscr{A}$, is a Jordan isomorphism.

Over the last decade, the so-called linear preserver problems has attracted several mathematicians which have studied in some survey articles ([3], [4], [6], [13], [11], [20], [18]) linear maps ϕ between two Banach algebras such that ϕ preserves a given class of elements of algebras (such as the class of invertible elements, regular elements, nilpotents elements, idempotents elements, algebraic elements, finite rank operators, spectral radius).

One of the most famous problems in this direction is Kaplansky's problem [14]: Let ϕ be a surjective linear map between two semi-simple Banach algebras $\mathscr{A}$ and $\mathscr{B}$ such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathscr{A}$, where $\sigma(.)$ denotes as usual the spectrum. Is it true that ϕ is a Jordan isomorphism ? This problem has been first solved in the finite-dimensional case. J. Dieudonné [9], Marcus and Purves [16] have proved that every unital invertibility preserving linear map on a complex matrix algebra is either an inner automorphism, or an inner anti-automorphism. This result has been later extended to the algebra of all bounded linear operators on a Banach space by A.R. Sourour [25] and to von Neumann algebras by B. Aupetit [2].

Recently, M. Mbekhta, L. Rodman and P. Šemrl [17] studied bijective linear maps on $\mathscr{B}(H)$ that preserve generalized invertibility in both directions. Observe that every

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$n \times n$ complex matrix has a generalized inverse, and therefore every linear map on a matrix algebra preserves generalized invertibility in both directions. So, we have here an example of a linear preserver problem which makes sense only in the infinite-dimensional case.

The aim of the present paper is to study the linear maps preserving the minimum modulus and surjectivity modulus of operators on Hilbert space.

The paper is organized as follows. In the next section we recall some notation and known results concerning the minimum modulus and surjectivity modulus of operators. This is used in Section 3 to prove the main results.

2. Minimum and surjectivity moduli

Let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H and $\mathscr{K}(H)$ be its closed ideal of compact operators. For an operator $T \in \mathscr{B}(H)$ we write T^* for its adjoint, $|T| = (T^*T)^{1/2}$ for its positive square root, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range.

Let T be a bounded operator on X. The *minimum modulus* (called also *injectivity modulus*) of T is defined by

$$\mathrm{m(T)} = \inf\{\|\mathrm{Tx}\|;\ \|\mathrm{x}\| = 1\},$$

and the *surjectivity modulus* of T by

$$\mathrm{q(T)} = \sup\{\varepsilon \geqslant 0;\ \varepsilon\mathrm{B}(0,1) \subset \mathrm{T}(\mathrm{B}(0,1))\}.$$

We refer the reader to [10, 19, 21] for the properties of these quantities. It is well known that $\mathrm{m(T)} = \inf\{\sigma(|\mathrm{T}|)\}$ and $\mathrm{q(T)} = \inf\{\sigma(|\mathrm{T}^*|)\}$, and consequently $\mathrm{m(T)} = \mathrm{q(T^*)}$.

It is straightforward that $\mathrm{m(T)} > 0$ if and only if T is bounded below, or equivalently, there exists an operator $S \in \mathscr{B}(H)$ such that $ST = I$. In this case, $\mathrm{m(T)} \geqslant \|\mathrm{S}\|^{-1}$, and we have equality when the operator S is the Moore-Penrose left inverse (i.e the unique operator T^+ that satisfies $T^+T = I$ and $(TT^+)^* = TT^+$).

Analogously, $\mathrm{q(T)} > 0$ if and only if T is surjective, or equivalently, there exists an operator $S \in \mathscr{B}(H)$ such that $TS = I$. In this case, $\mathrm{q(T)} \geqslant \|\mathrm{S}\|^{-1}$, and we have equality when S is the Moore-Penrose right inverse (i.e the unique operator T^+ that satisfies $TT^+ = I$ and $(T^+T)^* = T^+T$). In particular, if T is invertible, then we obtain

$$\mathrm{m(T)} = \mathrm{q(T)} = \frac{1}{\|\mathrm{T}^{-1}\|}. \tag{2.1}$$

Let A and B be an invertibles operators on H, then it is clear that $\mathrm{m(T)} > 0$ (resp. $\mathrm{q(T)} > 0$) if and only if $\mathrm{m(ATB)} > 0$ (resp. $\mathrm{q(ATB)} > 0$). Further, if the operators U and V are unitaries, then we have

$$\mathrm{m(UTV)} = \mathrm{m(T)} \quad \text{and} \quad \mathrm{q(UTV)} = \mathrm{q(T)}. \tag{2.2}$$

Let us recall the following useful inequalities that will be often used in the sequel: Let A and B be an arbitrary operators, then

$$\mathrm{m(A).m(B)} \leqslant \mathrm{m(AB)} \leqslant \|\mathrm{A}\|.\mathrm{m(B)}; \tag{2.3}$$

$$|\mathrm{m(A)} - \mathrm{m(B)}| \leqslant \|\mathrm{A} - \mathrm{B}\|; \tag{2.4}$$

and by duality we have also

$$\mathrm{q(A).q(B)} \leqslant \mathrm{q(AB)} \leqslant \|\mathrm{B}\|.\mathrm{q(A)}; \tag{2.5}$$

$$|\mathrm{q(A)} - \mathrm{q(B)}| \leqslant \|\mathrm{A} - \mathrm{B}\|. \tag{2.6}$$

Let us remark that (2.4) and (2.6) imply that the maps

$$\lambda \mapsto m(T-\lambda) \quad \text{and} \quad \lambda \mapsto q(T-\lambda) \tag{2.7}$$

are continuous on $\mathbb{C}$.

For an operator $T \in \mathscr{B}(H)$, we recall that the *left spectrum* is given by

$$\sigma_\ell(T) = \{\lambda \in \mathbb{C};\ \mathrm{m(T-\lambda)} = 0\},$$

and the *right spectrum* by

$$\sigma_\mathrm{r}(T) = \{\lambda \in \mathbb{C};\ \mathrm{q(T-\lambda)} = 0\}.$$

Notice that it follows by (2.7) that $\sigma_\ell(T)$ and $\sigma_\mathrm{r}(T)$ are closed, and that

$$\partial\sigma(T) \subseteq \sigma_\ell(T) \cap \sigma_\mathrm{r}(T) \tag{2.8}$$

where $\partial\sigma(T)$ is the boundary of the spectrum of T. In fact, for the last assertion, suppose that $0 \in \partial\sigma(T)$ and $0 \notin \sigma_\ell(T)$. Then there exists a sequence $\{\lambda_n\}_n$ in $\mathbb{C} \setminus \sigma(T)$ that converges to zero. Hence by the continuity of the minimum modulus we obtain that $\lim_{n\to+\infty} \mathrm{m(T-\lambda_n)} = \mathrm{m(T)}$, and since by assumption we have $\mathrm{m(T)} > 0$, the inequality (2.4) ensures that $M = \sup_n \|(T-\lambda_n)^{-1}\|$ is finite. Now, if $|\lambda_n|.M < 1$, then $T = (T-\lambda_n)[I + (T-\lambda_n)^{-1}\lambda_n]$ is invertible, the desired contradiction. The other inclusion follows by duality.

3. Main results

The main result of the paper is the following:

THEOREM 3.1. *Let $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ be a surjective unital linear map. Then the following conditions are equivalent:*

(*i*) $\mathrm{m}(\phi(\mathrm{T})) = \mathrm{m(T)}$ *for every* $T \in \mathscr{B}(H)$*;*

(*ii*) $\mathrm{q}(\phi(\mathrm{T})) = \mathrm{q(T)}$ *for every* $T \in \mathscr{B}(H)$*;*

(*iii*) *there is* $U \in \mathscr{B}(H)$ *unitary such that*

$$\phi(T) = UTU^* \quad \textit{for every} \quad T \in \mathscr{B}(H).$$

The proof of this theorem which requires establishing others results will be given at the end of this section.

DEFINITION 3.2. Let $\mathscr{A}$ be a Banach algebra. A map $\Lambda : \mathscr{A} \to \{\text{closed set of } \mathbb{C}\}$ is said to be a ∂*-spectrum* provided that for all $x \in \mathscr{A}$,

$$\partial\sigma(x) \subseteq \Lambda(x) \subseteq \sigma(x). \tag{3.1}$$

REMARK 3.3. If $\mathscr{A}$ be a Banach algebra and $\Lambda(.)$ is a ∂-spectrum. Then for every $x \in \mathscr{A}$,

$$\Lambda(x) \text{ is countable} \iff \sigma(x) \text{ is countable}, \tag{3.2}$$

and in this case we have $\sigma(x) = \Lambda(x)$.

In the proof of the following proposition, we use a method introduced by B. Aupetit [2].

PROPOSITION 3.4. *Let $\mathscr{A}$, $\mathscr{B}$ be two unital semi-simple Banach algebras and $\Lambda(.)$ be a ∂-spectrum. If $\phi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map that satisfies $\Lambda(\phi(x)) = \Lambda(x)$ for all $x \in \mathscr{A}$, then ϕ is a unital continuous bijection that maps idempotents elements of $\mathscr{A}$ into idempotents elements of $\mathscr{B}$.*

Proof. Let us show first that ϕ is injective. Let $x \in \ker(\phi)$ and q be an arbitrary quasi-nilpotent element of $\mathscr{A}$, then

$$\Lambda(x+q) = \Lambda(\phi(x+q)) = \Lambda(\phi(q)) = \Lambda(q) = \{0\}.$$

Hence, by (3.2), we obtain that $\sigma(x+q) = \{0\}$, and consequently the spectral radius $r(x+q) = 0$ for all quasi-nilpotent $q \in \mathscr{A}$. Now, [1, Theorem 5.3.1] implies that x belongs to the radical of $\mathscr{A}$, and since $\mathscr{A}$ is a semi-simple Banach algebra, we get that $x = 0$. Thus ϕ is injective.

For the continuity of ϕ, it suffices to observe that if $x \in \mathscr{A}$, then

$$\partial\sigma(\phi(x)) \subseteq \Lambda(\phi(x)) = \Lambda(x) \subseteq \sigma(x), \tag{3.3}$$

and therefore $r(\phi(x)) \leqslant r(x)$.Now the continuity of ϕ follows from [1, Theorem 5.5.2].

To show that ϕ maps idempotents into idempotents, we will argue as in [2, page 922]. Since ϕ is a continuous bijection, the open mapping theorem implies that ϕ^{-1} is continuous and so there are two positive constants α, β such that

$$\alpha\|x\| \leqslant \|\phi(x)\| \leqslant \beta\|x\| \quad \text{for every } x \in \mathscr{A}. \tag{3.4}$$

Now let e be an idempotent, without loss of generality we can assume that $\phi(e)$ is non-zero. Since $\sigma(e) \subseteq \{0,1\}$, (3.1) and (3.2) imply that

$$\partial\sigma(\phi(e)) \subseteq \Lambda(\phi(e)) = \Lambda(e) = \sigma(e) \subseteq \{0,1\}.$$

Hence $\sigma(\phi(e)) \subseteq \{0,1\}$. On the other hand, the necessary condition of [2, Theorem 1.1] provides the existence of two positive real numbers r and $C > 0$ such that

$$\sigma(x) \subseteq \{0,1\} + C\|x-e\|, \quad \text{for } \|x-e\| < r.$$

Thus, using (3.3), we get that for $\|x-e\| < r$,

$$\sigma(x) \subseteq \{0,1\} + \frac{C}{\alpha}\|\phi(x)-\phi(e)\|,$$

and so that

$$\partial\sigma(\phi(x)) \subseteq \Lambda(\phi(x)) = \Lambda(x) \subseteq \sigma(x) \subseteq \{0,1\} + \frac{C}{\alpha}\|\phi(x)-\phi(e)\|.$$

Hence the polynomial convex hull of $\sigma(\phi(x))$ is included in $\{0,1\} + \frac{C}{\alpha}\|\phi(x)-\phi(e)\|$, and consequently

$$\sigma(\phi(x)) \subseteq \{0,1\} + \frac{C}{\alpha}\|\phi(x)-\phi(e)\|, \quad \text{for } \|x-e\| < r.$$

Again, by the open mapping theorem, there is $r' > 0$ such that

$$\sigma(y) \subseteq \{0,1\} + \frac{C}{\alpha}\|y-\phi(e)\|, \quad \text{for } \|y-\phi(e)\| < r'.$$

Finally, by the sufficient condition of [2, Theorem 1.1], $\phi(e)$ is idempotent.

To complete the proof, it remains to verify that ϕ is unital. Indeed we have

$$\{1\} = \sigma(I) = \Lambda(I) = \Lambda(\phi(I)) = \sigma(\phi(I)),$$

and since $\phi(I)^2 = \phi(I)$, we get that $\phi(I) = I$. $\square$

Recall that an algebra $\mathscr{A}$ is called a *prime algebra* if for every pair a,b in $\mathscr{A}$, $a\mathscr{A}b = \{0\}$ implies that $a=0$ or $b=0$. It is well-known that $\mathscr{B}(H)$ is a prime algebra.

A linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is said to *preserve the minimum* (resp. *surjectivity*) *modulus in both directions* if it satisfies the following equivalence

$$\mathrm{m(T)} = 0 \text{ (resp. } \mathrm{q(T)} = 0) \iff \mathrm{m}(\phi(\mathrm{T})) = 0 \text{ (resp. } \mathrm{q}(\phi(\mathrm{T})) = 0),$$

for every $T \in \mathscr{B}(H)$.

THEOREM 3.5. *Let $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ be a surjective linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves the minimum modulus in both directions and it is unital;*

(ii) *$\sigma_\ell(\phi(T)) = \sigma_\ell(T)$ for all $T \in \mathscr{B}(H)$;*

(iii) *ϕ preserves the surjectivity modulus in both directions and it is unital;*

(iv) *$\sigma_{\mathrm{r}}(\phi(T)) = \sigma_r(T)$ for all $T \in \mathscr{B}(H)$;*

(v) *ϕ is an automorphism;*

(vi) *there is $A \in \mathscr{B}(H)$ invertible such that $\phi(T) = ATA^{-1}$ for every $T \in \mathscr{B}(H)$.*

Proof. The implications $(i) \Longrightarrow (ii)$; $(iii) \Longrightarrow (iv)$ and $(vi) \Longrightarrow (i)$ and (iii) are obvious.

Let us establish that (ii) or (iv) implies (v). Since $\sigma_\ell(.)$ and $\sigma_r(.)$ are ∂-spectra (see (2.8)), the above proposition ensures that ϕ preserves the set of idempotents in $\mathscr{B}(H)$. Further, $\mathscr{B}(H)$ is a C^*-algebra of real rank zero (i.e, the set formed by all the finite real linear combinations of orthogonal projections is dense in the set of all Hermitian elements of $\mathscr{B}(H)$), [7], therefore ϕ is a Jordan automorphism by [11, Theorem 4.1]. Now from the fact that a Jordan automorphism of a prime algebra is either an automorphism or an anti-automorphism, [12], we obtain that ϕ is either an automorphism or an anti-automorphism. Let us show that ϕ cannot be an anti-automorphism. Assume, to the contrary, that ϕ is an anti-automorphism and suppose $0 \in \sigma(T) \setminus \sigma_\ell(T)$. Then there is $S \in \mathscr{B}(H)$ such that $ST = I$ but $TS \neq I$. Hence $I = \phi(ST) = \phi(T)\phi(S)$, and consequently $\phi(T)$ is invertible. Therefore $\phi(I) = I = \phi(S)\phi(T) = \phi(TS)$, and since ϕ is injective, we get that $TS = I$. This implies that T is invertible, desired contradiction. Thus ϕ is an automorphism and so (ii) $\Longrightarrow$ (v) is proved. Arguing as above we establish that (iv) $\Longrightarrow$ (v). The implication (v) $\Longrightarrow$ (vi) follows from the fundamental isomorphism theorem ([22, Theorem 2.5.19]). $\square$

REMARK 3.6. We mention that the equivalences (ii) $\Longleftrightarrow$ (vi) and (iv) $\Longleftrightarrow$ (vi) appear also in the paper of P. Šemrl [23] (see also [8]).

The following Lemma which will be needed in the sequel is due to R. V. Kadison.

LEMMA 3.7. [15, Lemma 8]. *Let $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ be a unital linear map. If ϕ is isometric on normal operators then it preserves adjoints (i.e., $\phi(T^*) = \phi(T)^*$ for all $T \in \mathscr{B}(H)$).*

In the next theorem we characterize unitary operators in terms of minimum surjective and injectivity modulus.

THEOREM 3.8. *Let $A \in \mathscr{B}(H)$ be invertible, then the following conditions are equivalent:*

- (i) *A is unitary multiplied by a nonzero real number;*
- (ii) $\mathrm{m}(\mathrm{ATA}^{-1}) = \mathrm{m}(\mathrm{T})$ *for every $T \in \mathscr{B}(H)$;*
- (iii) $\mathrm{q}(\mathrm{ATA}^{-1}) = \mathrm{q}(\mathrm{T})$ *for every $T \in \mathscr{B}(H)$.*

Proof. The implications $(i) \Longrightarrow (ii)$ and (iii) follow from (2.2).

Now assume that (ii) holds and consider the polar decomposition $A = U|A|$, where U is unitary and $|A| = (A^*A)^{1/2}$. Then from (2.2), we have

$$\mathrm{m}(\mathrm{T}) = \mathrm{m}(\mathrm{ATA}^{-1}) = \mathrm{m}(|\mathrm{A}|\mathrm{T}|\mathrm{A}|^{-1}) \quad \text{for all } \mathrm{T} \in \mathscr{B}(\mathrm{H}).$$

Consider the map $\psi : \mathscr{B}(H) \to \mathscr{B}(H)$ defined by $\psi(T) = |A|T|A|^{-1}$ for all $T \in \mathscr{B}(H)$. Then ψ is unital, and by (2.1) it follows that ψ is an isometry on the group of invertible elements of $\mathscr{B}(H)$. Now, let T be a normal operator and $T = V|T|$ be the polar decomposition with V an unitary operator. Put $T_n = V(|T| + \frac{1}{n}I)$, then T_n is invertible and $T_n \to T$. Therefore, $\|\psi(T)\| = \|T\|$ for every normal operators $T \in \mathscr{B}(H)$, and hence by Lemma 3.7, ψ preserves the adjoint, i.e., $\psi(T^*) = \psi(T)^*$ for every $T \in \mathscr{B}(H)$. Thus $|A|T|A|^{-1} = |A|^{-1}T|A|$ and so $|A|^2 T = T|A|^2$ for all $T \in \mathscr{B}(H)$. Hence, $A^*A = |A|^2$ is central in $\mathscr{B}(H)$. Since A is invertible, $A^*A = \lambda I$, clearly $\lambda \in \mathbb{R}^+$, and, consequently, A is unitary multiplied by a nonzero real number.

For (iii) $\Longrightarrow$ (ii), suppose that $\mathrm{q}(\mathrm{ATA}^{-1}) = \mathrm{q}(\mathrm{T})$ for every $T \in \mathscr{B}(H)$ and consider the polar decomposition $A = U|A|$. Then from (2.2), we have

$$\mathrm{q}(\mathrm{T}) = \mathrm{q}(\mathrm{ATA}^{-1}) = \mathrm{q}(|\mathrm{A}|\mathrm{T}|\mathrm{A}|^{-1}) \quad \text{for all } \mathrm{T} \in \mathscr{B}(\mathrm{H}).$$

Hence, since $\mathrm{m}(\mathrm{T}) = \mathrm{q}(\mathrm{T}^*)$ we get

$$\mathrm{m}(\mathrm{T}) = \mathrm{q}(|\mathrm{A}|\mathrm{T}^*|\mathrm{A}|^{-1}) = \mathrm{m}(|\mathrm{A}|^{-1}\mathrm{T}|\mathrm{A}|) \quad \text{for all } \mathrm{T} \in \mathscr{B}(\mathrm{H}).$$

Now, by replacing T by $S = |A|T|A|^{-1}$ in the above formula, we obtain that $\mathrm{m}(\mathrm{ATA}^{-1}) = \mathrm{m}(|\mathrm{A}|\mathrm{T}|\mathrm{A}|^{-1}) = \mathrm{m}(\mathrm{S}) = \mathrm{m}(|\mathrm{A}|^{-1}\mathrm{S}|\mathrm{A}|) = \mathrm{m}(\mathrm{T})$, which completes the proof. $\square$

Proof of Theorem 3.1. The implications $(iii) \Longrightarrow (i)$ and (ii) are clear. Let us prove that (i) or (ii) implies (iii). Assume that (i) (resp. (ii)) holds, then ϕ preserves the minimum injectivity (resp. surjectivity) modulus in both directions, and hence by Theorem 3.5, there exists an invertible operator $A \in \mathscr{B}(H)$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathscr{B}(H)$. Now Theorem 3.8 implies that A is unitary multiplied by a nonzero real number. Thus there is a unitary $U \in \mathscr{B}(H)$ such that $\phi(T) = UTU^*$ for every $T \in \mathscr{B}(H)$ as desired. $\square$

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