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NON–COMMUTATIVE INDEPENDENCE OF ALGEBRAS AND APPLICATIONS TO PROBABILITY

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Abstract. We present the notions of independence, which appear in non-commutative probability. The basic ones are free, boolean and monotonic independences, formulated for families of algebras indexed by totally ordered set. A generalization of the latter two is the bm-independence, defined for partially ordered index sets. For each independence there is an analogue of the classical central limit theorem. In the case of bm-independence this depends also on the index set. Examples of such partially ordered index sets are discrete lattices in symmetric positive cones.

1. Introduction

The non-commutative generalizations of the classical probability depend on replacing the classical objects, such as the probability space, the notion of probability, the expectation, the distribution of a random variable and the independence of random variables, with non-commutative objects and versions of these notions.

The non-commutative probability space is a unital *-algebra $\mathscr{A}$ with a given state (positive, normalized functional) φ on it. The non-commutative random variables are the self-adjoint elements $a = a^* \in \mathscr{A}$, the role of expectation is played by the state φ in the sence that the distribution of a random variable a is a probability measure μ which is defined by the moments

$$\varphi((a)^n) = \int_{-\infty}^{+\infty} t^n \mu(dt).$$

The existence of the measure μ is guaranteed by the positive definitness of the sequence $(\varphi(a^n))_{n=0}^{\infty}$, via the Hamburger's theorem.

In classical probability the notion of independence is a tool to compute "mixed moments", i.e. expressions of the form

$$\mathbb{E}(f_1(X)g_1(Y)\dots f_n(X)g_n(Y)) = \mathbb{E}\left(\prod_{i=1}^{n} f_i(X)\right) \cdot \mathbb{E}\left(\prod_{i=1}^{n} g_i(Y)\right)$$

if X, Y are independent random variables, and f_i, g_i are Borel functions. Here one uses the fact that $f_i(X)$ commutes with $g_j(Y)$ for every $1 \leqslant i, j \leqslant n$.

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In the non-commutative setting there is a need to find another property, which can be used to compute such mixed moments, instead of commutativity. Several such properties have been invented and used succesfully, and these are: free, boolean and monotonic independences. They give different procedures for the computation of mixed moments, i.e. expressions of the form

$$\varphi(a_1 \dots a_n)$$

for "independent" random variables $a_1, \dots, a_n$ in a non-commutative probability space $(\mathscr{A}, \varphi)$. For such random variables one can consider analogues of the classical central limit theorem, and it turns out that the limit measures are different from the classical gaussian one.

Another notion of independence arises when one replaces the index set $\mathbb{N}$ (which numerates the random variables) with a partially ordered set $\mathbf{I}$. There are several possible generalizations of the four notions of independence mentioned above, and in this note we present one which combines the monotonic and the boolean independences. This notion we call the *bm-independence*.

The paper is organized as follows. In Sections 2–4 we present the definitions, basic properties and constructions related to the free, boolean and monotonic independences. Section 5 presents the applications of these independences to probability, in particular we show what are the related Central Limit Theorems. Section 6 presents natural examples of partial orders, which come from positive cones in Euclidian spaces. Then, in Section 7, we define the bm-independence and explain its properties. In particular, we show the construction of operators (algebras) which are bm-independent. Finally, in Section 8, we present the bm-Central Limit Theorems for each positive symmetric cone (according to their classification given by Faraut and Koranyi in [3]). The limits we get are sequences of moments of probability measures, which satisfy various generalizations of the recurence for the Catalan numbers. In most cases finding the explicit formula for the associated measure is an open problem.

2. Free independence

As far as the notions of independence is concerned, there have been several related constructions in the non-commutative setting. Firstly, D. Avitzour in [1] and D. Voiculescu in [8] (1983) introduced the `free independence`, which is a generalization of the following property of the free groups.

Let $\mathscr{A} = \mathbb{C}(\mathbb{F}_N)$ be the group algebra of the free group $\mathbb{F}_N$ on N free generators $S := \{s_1, \dots, s_N\}$. Let $\varphi : \mathscr{A} \to \mathbb{C}$ be defined as $\varphi(f) := f(e)$, the value of $f \in \mathscr{A}$ on the neutral element $e \in \mathbb{F}_N$. Then the condition $\varphi(f) = 0$ is equivalent to $\mathrm{supp}(f) \subset \mathbb{F}_N \setminus \{e\}$. Let $\star$ denote the convolution in $\mathscr{A}$ defined as

$$(f \star g)(x) = \sum_y f(xy^{-1})g(y).$$

For $j = 1, \dots, N$ let us consider a function f_j supported on the abelian subgroup G_j generated by $\{s_j, s_j^{-1}\}$. Then

$$e \notin \mathrm{supp}(f_1) \cup \dots \cup \mathrm{supp}(f_N) \quad \Rightarrow \quad e \notin \mathrm{supp}(f_1 \star \dots \star f_N)$$

This can be generalized as follows. For an arbitrary sequence $i_1 \neq i_2 \neq \ldots \neq i_m$ let us assume that f_j is supported on G_{i_j}. Then

$$\varphi(f_1) = \ldots = \varphi(f_m) = 0 \quad \Rightarrow \quad \varphi(f_1 \star \ldots \star f_m) = 0.$$

This expresses the fact that the subgroups $G_1, \ldots, G_N$ are free in the sense that there is no relations between elements of different subgroups. On the level of group algebras it says that the algebras $\mathscr{A}_j := \mathbb{C}(G_j)$, for $1 \leqslant j \leqslant N$ are independent in the following sense.

DEFINITION 2.1. For a given unital algebra $\mathscr{A}$ and a linear functional φ on it, we say that subalgebras $A_1, \ldots, A_r \subset \mathscr{A}$ are `freely independent` with respect to φ, if for arbitrary elements $a_1 \in A_{j_1}, \ldots, a_n \in A_{j_n}$, such that $j_1 \neq \ldots \neq j_n$, the following holds:

$$\varphi(a_1 \cdot \ldots \cdot a_n) = 0 \quad \text{if} \quad \varphi(a_1) = \ldots = \varphi(a_n) = 0 \tag{2.1}$$

The construction of freely independent algebras follows the idea of free product of groups. In general, it is the free product of algebras with given functionals (A_j, φ_j), $j = 1, \ldots, N$, described in [8]. Here we shall show it for this instructional case of free product of groups.

Let us assume that $G_1, \ldots, G_N$ are given discrete groups and that $\varphi(f_j) = f_j(e_j)$ is the functional on the group algebra $\mathbb{C}[G_j]$, given by the evaluation on the unit element $e_j \in G_j$. Let

$$G := *_{j=1}^{N} G_j$$

be the free product of the groups. The unit $e \in G$ is obtained by the identification of all the units e_j, and the other elements are *words*, i.e. the products of the form $g_1 \cdot \ldots \cdot g_m$, where $g_k \in G_{j_k} \setminus \{e\}$, $k = 1, \ldots m$, and $j_1 \neq \ldots \neq j_m$. Thus in such a word the neighbours are from different groups, and the letters are different from the unit.

Now the group algebra $\mathbb{C}[G]$ is the free product of the algebras $\mathbb{C}[G_j]$.

3. Boolean independence

Another notion of non-commutative independence was invented by M. Bożejko [2], and later got the name `boolean independence`. It is a multiplicativity property of a functional on a product of elements.

DEFINITION 3.1. A family of subalgebras $\{A_j\}$ of a given algebra $\mathscr{A}$, is called `boolean independent` with respect to a given functional φ on $\mathscr{A}$, if it satisfies the following condition: for any $a_1 \in A_{j_1}, \ldots, a_n \in A_{j_n}$, if $j_1 \neq \ldots \neq j_n$, then

$$\varphi(a_1 \cdot \ldots \cdot a_n) = \varphi(a_1) \cdot \ldots \cdot \varphi(a_n). \tag{3.2}$$

The standard example of such independence is given by the following construction by M. Bożejko in [2]. We consider a pair H_1, H_2 of Hilbert spaces with a common one-dimensional subspace (the intersection), spanned by a unit vector Ω. These spaces have then the direct sum decomposition $H_j^0 = H_j \ominus \mathbb{C}\Omega$ with $j = 1,2$. Let $\mathscr{H} = H_1^0 \oplus \mathbb{C}\Omega \oplus H_2^0$ be the direct sum, and let us consider algebras $A_j \subset \mathscr{B}(H_j)$, $j = 1,2$, of bounded operators. The extension $\mathbf{a}_j$ of an operator $a_j \in A_j$ onto $\mathscr{H}$ is defined as:

$$\mathbf{a}_1 : H_1^0 \oplus \mathbb{C}\Omega \oplus H_2^0 \ni h_1 \oplus c\Omega \oplus h_2 \mapsto h_1 \oplus c\Omega \in H_1^0 \oplus \mathbb{C}\Omega \oplus H_2^0,$$

$$\mathbf{a}_2 : H_1^0 \oplus \mathbb{C}\Omega \oplus H_2^0 \ni h_1 \oplus c\Omega \oplus h_2 \mapsto c\Omega \oplus h_2 \in H_1^0 \oplus \mathbb{C}\Omega \oplus H_2^0.$$

The functional we consider is given by Ω:

$$\varphi(a) = \langle a\Omega|\Omega\rangle.$$

Let $\mathbf{A}_j \subset \mathscr{B}(\mathscr{H})$ be the algebra of extensions of elements from A_j, for $j = 1,2$, onto $\mathscr{H}$. Then the following holds:

THEOREM 3.2. *The extension algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ are boolean independent with respect to the state φ on $\mathscr{B}(\mathscr{H})$.*

Proof. Since the role of both algebras can be treated symmetrically, it suffices to show that for all $x_1,\dots,x_m \in \mathbf{A}_1$ and $y_1,\dots,y_m \in \mathbf{A}_2$, $m \in \mathbb{N}$, it holds

$$\varphi(x_1y_1\dots x_my_m) = \prod_{j=1}^m \varphi(x_j)\prod_{j=1}^m \varphi(y_j).$$

This can be written as

$$\langle x_1y_1\dots x_my_m\Omega|\Omega\rangle = \prod_{j=1}^m \langle x_j\Omega,\Omega\rangle \cdot \prod_{j=1}^m \langle y_j\Omega,\Omega\rangle.$$

Let us observe that for $x \in \mathbf{A}_1$ and $y \in \mathbf{A}_2$, if $y\Omega = c_2\Omega + h_2 \in \mathbb{C}\Omega \oplus H_2^0$ and $x\Omega = h_1 + c_1\Omega \in H_1^0 \oplus \mathbb{C}\Omega$, then

$$xy\Omega = c_2x\Omega = \varphi(y)\cdot x\Omega \quad \text{and} \quad yx\Omega = c_1y\Omega = \varphi(x)\cdot y\Omega$$

Therefore, by induction, we get

$$\langle x_1y_1\dots x_my_m\Omega|\Omega\rangle = \prod_{j=2}^m \varphi(x_j)\prod_{j=1}^m \varphi(y_j)\cdot\langle x_1\Omega|\Omega\rangle = \prod_{j=1}^m \varphi(x_j)\prod_{j=1}^m \varphi(y_j).$$

This finishes the proof. □

4. Monotonic independence

Third most important notion of independence in non-commutative probability was invented by N. Muraki [7].

DEFINITION 4.1. A family of subalgebras $\{A_i : i \in \mathbb{N}\}$ of a given algebra $\mathscr{A}$, indexed by the set of positive integers $\mathbb{N}$, is called *monotonically independent* with respect to a given functional φ on $\mathscr{A}$, if the following two conditions are satisfied:

(M1) $a_i a_j a_k = \varphi(a_j) \cdot a_i a_k$,
if $a_i \in A_i,\ a_j \in A_j,\ a_k \in A_k$, and $i < j > k$,

(M2) $\varphi(a_{i_r} \ldots a_{i_1} a_j a_{k_1} \ldots a_{k_s}) = \prod_{p=1}^{r} \varphi(a_{i_p}) \cdot \varphi(a_j) \cdot \prod_{t=1}^{s} \varphi(a_{k_t})$,
if $i_r > \ldots > i_1 > j < k_1 < \ldots < k_s$ and $a_{i_p} \in A_{i_p}$, $1 \leqslant p \leqslant r$, $a_{k_t} \in A_{k_t}$, $1 \leqslant t \leqslant s$, $a_j \in A_j$.

The definition works as follows: for the computation of the expression of the form $\varphi(a_1 \ldots a_m)$, where neighbouring elements are from different algebras: $a_j \in A_{i_j}$ with $i_1 \neq \ldots \neq i_m$, one first looks for the "local maxima" – the triples $j-1, j, j+1$ which satisfy $i_{j-1} < i_j > i_{j+1}$ (the i_j is such a local maximum) and apply the (M1) condition to get $a_{i_{j-1}} a_{i_j} a_{i_{j+1}} = \varphi(a_{i_j}) \cdot a_{i_{j-1}} a_{i_{j+1}}$. After finite number of such steps one gets several scalar factors of this form, multiplied by expression of the form $\varphi(b_1 \ldots b_t)$ with $b_j \in A_{k_j}$, $k_1 \neq \ldots \neq k_t$, with the indexes satisfying $k_1 > \ldots > k_s < \ldots < k_t$ (for some $1 \leqslant s \leqslant t$). To such expression one applies the condition (M2) to get the final factorization $\varphi(b_1 \ldots b_t) = \varphi(b_1) \ldots \varphi(b_t)$. In this way one can reduce computation of the mixed moments $\varphi(a_1 \ldots a_m)$ to the marginals $\varphi\restriction_{A_i}$. An example of the monotonically independent operators (algebras) is realized on the monotonic Fock space (cf. [9]).

Example: The monotonic Fock space.

Let $\{H_j : j \in \mathbb{N}\}$ be a given family of Hilbert spaces, with a common unit (vacuum) vector $\Omega \in H_j$. We have the natural direct sum decomposition $H_j^0 := H_j \ominus \mathbb{C}\Omega$. Let $\mathscr{H}$ be the associated full Fock space:

$$\mathscr{H} := \mathbb{C}\Omega \oplus \bigoplus_{n \geqslant 1} \bigoplus_{j_1, \ldots, j_n \in \mathbb{N}} H_{j_1}^0 \otimes \ldots \otimes H_{j_n}^0$$

We define the subspace $\mathscr{H}_m \subset \mathscr{H}$ as spanned by Ω and tensors of the form:

$$h_{j_n} \otimes \ldots \otimes h_{j_1} \in H_{j_n}^0 \otimes \ldots \otimes H_{j_1}^0 \quad \text{with} \quad j_n > \ldots > j_1.$$

If $A_j \in B(H_j)$ is a bounded operator, then we define its monotonic extension a_j onto $\mathscr{H}_m$ as follows: $a_j \Omega = A_j \Omega$ and

$$a_j(h_{j_n} \otimes \ldots \otimes h_{j_1}) = \begin{cases} (A_j \Omega) \otimes h_{j_n} \otimes \ldots \otimes h_{j_1} & \text{if } j > j_n \\ (A_j h_{j_n}) \otimes \ldots \otimes h_{j_1} & \text{if } j = j_n \\ 0 & \text{if } j < j_n \end{cases}.$$

In this definition the first case ($j > j_n$) should be understood as follows. If $A_j \Omega = \varphi(A_j)\Omega + h_j$ then $a_j(h_{j_n} \otimes \ldots \otimes h_{j_1}) = \varphi(A_j) \cdot h_{j_n} \otimes \ldots \otimes h_{j_1} + h_j \otimes h_{j_n} \otimes \ldots \otimes h_{j_1}$. In the second case $j = j_n$ we use the notation: if $v = h_{j_{n-1}} \otimes \ldots \otimes h_{j_1}$ with $j_{n-1} < j$ and if $A_j(h_j) = \beta_j \Omega + g_j$, then $a_j(h_j \otimes v) = \beta_j \cdot v + g_j \otimes v$.

THEOREM 4.2. *The monotonic extension operators $a_j \in B(\mathscr{H}_m)$, with $j = 1, 2, \ldots$, are monotonically independent with respect to the vacuum state $\varphi(\mathtt{a}) = \langle \mathtt{a}\Omega|\Omega\rangle$.*

Proof. For the proof let us observe that for each $k \in \mathbb{N}$ the range $a_k(\mathscr{H}_m)$ is contained in the subspace $\mathscr{H}_m^{(\leqslant k)}$, spanned by Ω and the simple tensors $h_{j_n} \otimes \ldots \otimes h_{j_1}$ with $j_1 < \ldots < j_n \leqslant k$.

To prove (M1) it suffices to check that the equality

$$a_i a_j a_k v = \varphi(a_j) a_i a_k v \tag{4.3}$$

holds for all $v \in \mathscr{H}_m$ of the form $v = \Omega$ or $v = h_{j_n} \otimes \ldots \otimes h_{j_1}$ with $j_n < k$ or $v = h_k \otimes h_{j_n} \otimes \ldots \otimes h_{j_1}$ with $j_n < k$. We shall use the notation $\varphi(a_s) = \beta_s = \varphi(A_s)$ and $A_s\Omega = \beta_s\Omega + h_s$ for $s \in \{i, j, k\}$.

- Case: $v = \Omega$. We have

$$a_i a_j a_k \Omega = \beta_i \beta_j \beta_k \Omega + \beta_j \beta_k h_i + \beta_j a_i h_k$$

and

$$a_i a_k \Omega = \beta_k(\beta_i \Omega + h_i) + a_i h_k,$$

so the equality in (4.3) holds.

- Case: $v = h_{j_n} \otimes \ldots \otimes h_{j_1}$ with $j_n < k$. In this case we have

$$a_i a_j a_k v = \beta_k \beta_j (a_i v) + \beta_j \cdot a_i(h_k \otimes v)$$

and

$$a_i a_k v = \beta_k(a_i v) + a_i(h_k \otimes v),$$

which gives (4.3).

- Case: $v = h_k \otimes h_{j_n} \otimes \ldots \otimes h_{j_1}$ with $j_n < k$. Here we need additional notation: $A_k h_k = \gamma_k \Omega + g_k$ and $h_{j_n} \otimes \ldots \otimes h_{j_1} = w$. Then

$$a_i a_j a_k(h_k \otimes w) = \gamma_k \beta_j (a_i w) + \beta_j \cdot a_i(g_k \otimes w)$$

and

$$a_i a_k(h_k \otimes w) = a_i(\gamma_k w + g_k \otimes w),$$

so (4.3) also holds.

For the proof of (M2) let us observe that

$$a_p a_q \Omega = \varphi(a_q) a_p \Omega = \beta_q a_p \Omega, \tag{4.4}$$

if $p < q$. Therefore, for $j < i_1 < \ldots < i_m$ and $j < k_1 < \ldots < k_n$, by induction, we obtain

$$a_j a_{k_1} \ldots a_{k_n} \Omega = \beta_{k_m} \ldots \beta_{k_1} \cdot a_j \Omega, \quad a_{i_1}^* \ldots a_{i_m}^* \Omega = \overline{\beta_{i_m}} \ldots \overline{\beta_{i_2}} \cdot a_{i_1}^* \Omega.$$

Since

$$\varphi(a_{i_m} \ldots a_{i_1} a_j a_{k_1} \ldots a_{k_n}) = \langle a_j a_{k_1} \ldots a_{k_n} \Omega | a_{i_1}^* \ldots a_{i_m}^* \Omega\rangle,$$

it follows that

$$\varphi(a_{i_m} \ldots a_{i_1} a_j a_{k_1} \ldots a_{k_n}) = \beta_{k_m} \ldots \beta_{k_1} \cdot \beta_{i_n} \ldots \beta_{i_2} \cdot \langle \Omega | a_j^* a_{i_1}^* \Omega\rangle,$$

from which the (M2) follows. $\square$

5. Applications to probability: classical and non-commutative Central Limit Theorems

In classical probability we consider a probability space $(\mathfrak{X},\mathfrak{F},\mathbb{P})$ consisting of a topological space $\mathfrak{X}$, a σ-field $\mathfrak{F}$ and the probability function $\mathbb{P}$. We say that random variables ($\mathfrak{F}$-measurable functions $X,Y : \mathfrak{X} \to \mathbb{R}$) are *independent* if for every Borel subsets $A,B \subset \mathbb{R}$ the condition

$$\mathbb{P}(X \in A \wedge Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$$

holds. For the classical Central Limit Theorem (CLT) one considers a sequence $\{X_i : \mathfrak{X} \to \mathbb{R} | i \in \mathbb{N}\}$ of independent, identically distributed random variables, which satisfy: $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = 1$. Here $\mathbb{E}$ is the expectation:

$$\mathbb{E}(X) := \int_{\mathfrak{X}} X(\omega)\mathbb{P}(d\omega).$$

Then, for the normalized partial sums:

$$S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i,$$

there exists the gaussian limit (in the sense of moments, but also in probability, distribution etc.)

$$\mathbb{E}\left((S_N)^n\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^n e^{\frac{-t^2}{2}} dt.$$

5.1. Non-commutative Central Limit Theorems

The formulation of non-commutative CLT is as follows. Let $b_i = b_i^*$, for $i \in \mathbb{N}$, be self-adjoint elements of a unital $*$-algebra $\mathscr{B}$, which satisfy $\varphi(b_i) = 0$, and $\varphi(b_i^2) = 1$ for a given state φ. Moreover, we assume that the elements $\{b_i : i \in \mathbb{N}\}$ are independent (in a non-commutative sense), with respect to φ. Let

$$S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} b_i \tag{5.5}$$

be the partial sums. Then there exists a limit measure μ such that for all $n \in \mathbb{N}$:

$$\lim_{N\to\infty} \varphi\left((S_N)^n\right) = \int_{-\infty}^{+\infty} x^n d\mu(x).$$

The limits are the moments of a symmetric probability measure on the real line $\mathbb{R}$. Here we list the exampes of these measures.

(1) free CLT: *semi-circle law* $\mu(dx) := \frac{1}{2\pi}\sqrt{4-x^2}dx,$

(2) boolean CLT: *Bernoulli law* $\mu = \frac{1}{2}(\delta_{-1} + \delta_1),$

(3) monotonic CLT: *arcsine law* $\mu(dx) := \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}.$

6. Partial orders

The notion of bm-independence is a generalization of the monotonic independence for an index set $\mathbf{I}$ which is partially ordered by a relation $\preceq$. The monotonic independence was defined for totally ordered sets (like positive integers $\mathbb{N}$), so the difference is in possible existence of incomparable pairs of elements. We shall write $\xi \nsim \eta$ if elements $\xi, \eta \in \mathbf{I}$ are incomparable.

The most natural examples of partial orders arise in vector spaces, and are related to positive cones. If V is a real (or complex) vector space, then a subset $\Pi \subset V$ is a *positive cone* if it contains the zero vector and if it is closed under addition of vectors and multiplication by positive scalars. A positive cone defines partial order $\preceq$ on V as follows. For $u, v \in V$ we write $u \preceq v$ if $v - u \in \Pi$. Examples of such partially ordered sets, which are of main interests for us, are the following.

1. $V = \mathbb{R}^d$, $\Pi = (\mathbb{R}_+ \cup \{0\})^d = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : a_1, \ldots, a_d \geqslant 0\}$. The partial order defined here is as follows. If $\xi = (a_1, \ldots, a_d), \eta = (b_1, \ldots, b_d) \in V$, then $\xi \preceq \eta$ if $a_j \leqslant b_j$ for every $1 \leqslant j \leqslant d$. In this example, if $a_1 > b_1$ and $a_2 < b_2$ then the elements ξ and η are incomparable.

2. $V = \mathbb{R} \times \mathbb{R}^d$, $\Pi = \left\{ (x; a_1, \ldots, a_d) \in \mathbb{R}_+ \times \mathbb{R}^d : x \geqslant \sqrt{a_1^2 + \ldots + a_d^2} \right\}$ is the Lorentz light cone in the Minkowski spacetime. The partial order is defined by the future cone of a vector: if $\xi = (x; a_1, \ldots, a_d), \eta = (y; b_1, \ldots, b_d) \in V$, then $\xi \preceq \eta$ if
$$y - x \geqslant \sqrt{(b_1 - a_1)^2 + \ldots + (b_d - a_d)^2}$$
In this example if, $x = y$ then $\xi \nsim \eta$ are incomparable (unless $\xi = \eta$).

3. $V = \mathbb{S}\mathrm{ymm}_d(\mathbb{R})$ is the real space of symmetric $d \times d$ matrices with real entries and Π is the cone of positive definite matrices in V. Explicitely,
$$\xi = (a_{jk})_{j,k=1}^d \in \Pi \quad \text{if} \quad \sum_{j,k=1}^d a_{jk} x_j x_k \geqslant 0$$
for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$, (equivalently, every minor of ξ is non-negative). For such $\xi \in \Pi$ the diagonal entries must be non-negative: $a_{jj} \geqslant 0$ for $1 \leqslant j \leqslant d$. In particular, if $\xi, \eta \in V$ have the same diagonals, then they are incomparable.

4. $V = \mathbb{H}\mathrm{erm}_d(\mathbb{C})$ is the complex vector space of hermitian $d \times d$ matrices with complex entries, and $\Pi \subset V$ is the subset of all positive definite matrices in V.

7. bm-independence

The definition of bm-independence is this.

DEFINITION 7.1. A family of subalgebras $\{\mathscr{B}_\xi : \xi \in \mathbf{I}\}$ of a given algebra $\mathscr{B}$, indexed by a partially ordered set $\mathbf{I}$, is called `bm-independent` with respect to a given functional φ on $\mathscr{B}$, if:

BM1. If $\xi \prec \rho \succ \eta$ or $\xi \nsim \rho \succ \eta$ or $\xi \prec \rho \nsim \eta$, then

$$b_\xi b_\rho b_\eta = \varphi(b_\rho) \cdot b_\xi b_\eta \qquad \forall b_\eta \in \mathscr{B}_\eta, b_\xi \in \mathscr{B}_\xi, b_\rho \in \mathscr{B}_\rho.$$

BM2. If $\xi_1 \succ \ldots \succ \xi_m \nsim \ldots \nsim \xi_k \prec \ldots \prec \xi_n$ for $1 \leqslant m \leqslant k \leqslant n$, then

$$\varphi(b_{\xi_1} \ldots b_{\xi_m} \ldots b_{\xi_k} \ldots b_{\xi_n}) = \prod_{j=1}^{n} \varphi(b_{\xi_j})$$

This definition generalizes the monotonic and boolean independences in the following way.

- If the index set $\mathbf{I}$ is totally ordered, i.e. if every two elements are comparable, then **BM1** and **BM2** are exactly the Muraki's conditions for the monotonic independence.
- If the index set $\mathbf{I}$ is totally disordered, i.e. every two elements are incomparable, then the condition **BM1** is void, and the condition **BM2** simplifies to the boolean independence condition.

The definition does not say that such objects exists, but we shall give examples in what follows. The first is a general construction of bm-independent algebras – the bm-product of algebras, with given functionals.

DEFINITION 7.2. Let $(\mathscr{B}_\xi, \varphi_\xi)_{\xi \in \mathbf{I}}$ be a family of algebras, indexed by a partially ordered set $\mathbf{I}$, with given functional φ_ξ on each algebra. The `bm-product` $\mathscr{B}_\mathbf{I} := \left(*_{\xi \in \mathbf{I}} \mathscr{B}_\xi\right) / \mathfrak{J}_1$ is the quotient of the free product algebra $*_{\xi \in \mathbf{I}} \mathscr{B}_\xi$, generated by this family, by the left ideal $\mathfrak{J}_1$ generated by the set

$$\{b_\xi b_\rho b_\eta - \varphi_\rho(b_\rho) b_\xi b_\eta \,|\, \xi \prec \rho \succ \eta \text{ or } \xi \nsim \rho \succ \eta \text{ or } \xi \prec \rho \nsim \eta\}.$$

We define the functional φ on $\mathscr{B}_\mathbf{I}$ by putting

$$\varphi\left(\widetilde{b_{\xi_1}} \ldots \widetilde{b_{\xi_n}}\right) := \prod_{j=1}^{n} \varphi_{\xi_{i_j}}(b_{\xi_j})$$

for elements $\widetilde{b_{\xi_j}} := b_{\xi_j} + \mathfrak{J}_1$ with $b_{\xi_j} \in \mathscr{B}_{\xi_{i_j}}$ for every $1 \leqslant j \leqslant n$, with $i_1 \neq \ldots \neq i_n$ and with $\xi_1 \succ \ldots \succ \xi_m \nsim \ldots \nsim \xi_k \prec \ldots \prec \xi_n$ for some $1 \leqslant m \leqslant k \leqslant n$.

REMARK 7.3. The relation defining the ideal is taken from the **BM1.**

REMARK 7.4. The functional φ is well defined, and the algebras $\mathscr{B}_\xi$ are bm-independent in $\mathscr{B}_\mathbf{I}$ with respect to φ. This functional, restricted to $\mathscr{B}_\xi$, equals φ_ξ.

This definition gives a general procedure for obtaining the bm-independent algebras. A more concrete example is given by the following construction of operators on a Hilbert space, called the bm-extension operators on the bm-product of Hilbert spaces.

Let $\{\mathbf{H}_\xi : \xi \in \mathbf{I}\}$ be a family of Hilbert spaces, indexed by a partially ordered set $\mathbf{I}$. We shall assume that these spaces have a common unit vector $\Omega \in \mathbf{H}_\xi$.

DEFINITION 7.5. By

$$\mathscr{H} = \circledast_{\xi\in\mathbf{I}}\mathbf{H}_\xi$$

we shall denote the *bm-product* Hilbert space, i.e. the subspace of the full Fock space, spanned by Ω and simple tensors $h_{\rho_j} \otimes \ldots \otimes h_{\rho_1}$ with $\rho_j \succ \ldots \succ \rho_1$ and $h_\rho \in \mathbf{H}_\rho, h_\rho \perp \Omega$.

On the bm-product Hilbert space we define the bm-extension operators. Such bm-extension is an extension of an operator $\mathbf{A}_\xi$, bounded on $\mathbf{H}_\xi$, onto $\mathscr{H}$.

DEFINITION 7.6. For $\xi, \rho_1, \ldots, \rho_j \in \mathbf{I}$, $\mathbf{H}_\rho \ni h_\rho \perp \Omega$ we define the bm-extension $A_\xi \in \mathscr{B}(\mathscr{H})$ of $\mathbf{A}_\xi \in \mathbf{B}(\mathbf{H}_\xi)$ by the following formulas:

$$A_\xi\left(h_{\rho_j} \otimes \ldots \otimes h_{\rho_1}\right) = \begin{cases} 0 & \text{if } \xi \prec \rho_j \text{ or } \xi \nsim \rho_j \\ (\mathbf{A}_\xi h_\xi) \otimes h_{\rho_{j-1}} \otimes \ldots \otimes h_{\rho_1} & \text{if } \rho_j = \xi \\ (\mathbf{A}_\xi \Omega) \otimes h_{\rho_j} \otimes \ldots \otimes h_{\rho_1} & \text{if } \rho_j \prec \xi \end{cases} \tag{7.6}$$

The second case is understood as follows. If $\xi = \rho_j$ and $\mathbf{A}_\xi h_\xi = \alpha\Omega + g_\xi$ then

$$A_\xi\left(h_{\rho_j} \otimes \ldots \otimes h_{\rho_1}\right) = \alpha \cdot h_{\rho_{j-1}} \otimes \ldots \otimes h_{\rho_1} + g_\xi \otimes h_{\rho_{j-1}} \otimes \ldots \otimes h_{\rho_1}$$

Similarly, if $\rho \prec \xi$ and $\mathbf{A}_\xi \Omega = \beta\Omega + h_\xi$ then

$$A_\xi\left(h_{\rho_j} \otimes \ldots \otimes h_{\rho_1}\right) = \beta \cdot h_{\rho_j} \otimes \ldots \otimes h_{\rho_1} + h_\xi \otimes h_{\rho_j} \otimes \ldots \otimes h_{\rho_1}$$

This construction provides an example of bm-independent operators (or algebras, in general). Let us notice that, since the "vacuum vector" Ω is in $\mathscr{H}$, we can consider the vacuum state $\varphi(X) := \langle X\Omega|\Omega\rangle$ on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$.

THEOREM 7.7. *Let $\mathbf{I}$ be a partially ordered set and for each $\xi \in \mathbf{I}$ let $\mathbf{B}_\xi \subset \mathbf{B}(\mathbf{H}_\xi)$ be an algebra of operators bounded on a given Hilbert spaces $\mathbf{H}_\xi$. Let $\mathscr{B}_\xi \subset \mathscr{B}(\mathscr{H})$ be the bm-extension algebra, which consists of the bm-extensions of the operators from $\mathbf{B}_\xi$ onto the bm-product Hilbert space $\mathscr{H} = \circledast_{\xi\in\mathbf{I}}\mathbf{H}_\xi$. Then the family $\{\mathscr{B}_\xi : \xi \in \mathbf{I}\}$ is bm-independent (with respect to the vacuum state φ).*

The proof of the Theorem can be found in [10]. It is based on the same considerations as the proof of the Theorem 4.2.

8. bm-Central Limit Theorems

For the bm-independence we can also consider the analogues of the CLT. The general setup is similar. We consider a partially ordered sets $\mathbf{I}$, which will be of special form (roughly speaking: discrete lattices in positive symmetric cones). Moreover, let us assume that we are given a family $\{\mathscr{B}_\xi : \xi \in \mathbf{I}\}$ of unital $*$-algebras, bm-independent in $\mathscr{B}$ with respect to a state φ. For a family of non-commutative random variables $b_\xi = b_\xi^* \in \mathscr{B}_\xi$, with $\varphi(b_\xi) = 0$ and $\varphi(b_\xi^2) = 1$ we consider the partial sums

$$S_{\mathbf{N}} := \frac{1}{\sqrt{|\mathbf{J_N}|}} \sum_{\xi \in \mathbf{J_N}} b_\xi \tag{8.7}$$

Here an important difference appears (comparing to the classical or the previously mentioned non-commutative cases): the summation intervals $\{1,2,\dots,N\} \subset \mathbb{N}$ are replaced by appropriate finite subsets $\mathbf{J_N} \subset \mathbf{I}$. These sets will have the properties: $\mathbf{J_N} \subset \mathbf{J_{N'}}$ if $\mathbf{N} \leqslant \mathbf{N}'$ and $\bigcup_{\mathbf{N}} \mathbf{J_N} = \mathbf{I}$. Here the index $\mathbf{N}$ will be either an integer or a d-tuple of integers. If $\mathbf{N} = (n_1,\dots,n_d)$ and $\mathbf{N}' = (n'_1,\dots,n'_d)$ (for $d \in \mathbb{N}$ with $d \geqslant 2$) then

$$\mathbf{N} \leqslant \mathbf{N}' \quad \Leftrightarrow \quad n_1 \leqslant n'_1, \dots, n_d \leqslant n'_d.$$

The bm-Central Limit Theorems have the same formulation, but we shall see that the limit measure depends on the choice of the positive cone, and the set $\mathbf{I}$ in it.

THEOREM 8.1. *For each non-negative integer $n \in \mathbb{N}$ there exist the limits*

$$g_n = \lim_{\mathbf{N}\to\infty} \varphi\left((S_{\mathbf{N}})^{2n}\right) \tag{8.8}$$

The sequence $(g_n)_{n=0}^{\infty}$ is a moment sequence of a (symmetric) probability measure $\mu = \mu_{\mathbf{I}}$ (depending on $\mathbf{I}$) on the real line:

$$g_n = \int_{-\infty}^{+\infty} t^{2n} \mu(dt), \quad 0 = \int_{-\infty}^{+\infty} t^{2n+1} \mu(dt)$$

Moreover, the sequence satisfies a generalization of the Catalan numbers' recurrence:

$$g_0 = g_1 = 1, \qquad g_n = \sum_{m=1}^{n} \gamma(m) \cdot g_{m-1} \cdot g_{n-m} \tag{8.9}$$

The coefficients $\gamma(m)$ (and hence the numbers g_n) depend on the index set $\mathbf{I}$ and can be computed from the following combinatorial formula:

$$\gamma(m) = \lim_{\mathbf{N}\to\infty} \sum_{\mathbf{k} \leqslant \mathbf{N}} \frac{|\mathbf{I_k}|}{|\mathbf{J_N}|} \left(\frac{|\mathbf{J_{N-k}}|}{|\mathbf{J_N}|}\right)^{m-1}$$

Here $\bigcup_{\mathbf{k}\leqslant\mathbf{N}} \mathbf{I_k} = \mathbf{J_N}$ is a special disjoint decomposition of the summation sets $\mathbf{J_N}$.

We shall present now the list of examples of this theorem, for various positive symmetric cones. We shall use the classification of these cones, given in [3]. This classification says that, in general, a positive symmetric cone is a set of positive definite symmetric matrices over a Jordan-Hurwitz algebra. A Jordan-Hurwitz algebras are also calssified, and it turns out that there are only 4 possible cases: real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$, and Cayley's octonions $\mathbb{O}$. In the classification of positive symmetric cones the octonions appear only in the case of 3×3 matrices. In our study we shall not consider this case, so the algebra $\mathbb{O}$ is not going to appear here.

The first two examples below are related to the symmetric cones which are not of the above form. In most cases the limit measure in unknown.

EXAMPLE 1. Let $V := \mathbb{R}^d$ be the d-dimensional real Euclidian space, and let $\Pi := \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1,\ldots,x_d \geqslant 0\}$ be the positive cone of vectors with non-negative coordinates. The index set in this case is $\mathbf{I}_d = \mathbb{N}^d$ and the summation sets are of the form $\mathbf{J}_\mathbf{N} = \{(k_1,\ldots,k_d) \in \mathbf{I} : k_1 \leqslant N_1,\ldots,k_d \leqslant N_d\}$, where $\mathbf{N} = (N_1,\ldots,N_d)$. In this case the bm-CLT gives the recurence

$$\gamma(m) = m^{-d}, \quad g_n = \sum_{m=1}^{n} \frac{1}{m^d} g_{m-1} g_{n-m},$$

particular cases of which are the following.

case $d = 0$. The CLT measures is the semi-circle law, since in the limit (8.8) we get the Catalan numbers $g_n = \frac{1}{n+1}\binom{2n}{n}$.

case $d = 1$. In this case the measure is the arcsine law, since the recurrence gives $g_n = \frac{1}{2^n}\binom{2n}{n}$.

case $d = 2$. Numbers $h_n := g_n \cdot (n!)^2$ are positive integers and describe number of *heap ordered labeled rooted trees* (the measure with the moments g_n is not known).

EXAMPLE 2. In the second example we consider the Minkowski spacetime $V := \{(x;y_1,\ldots,y_d) \in \mathbb{R}\times\mathbb{R}^d\}$ with the positive Lorentz light cone $\Pi_d := \{(x;y_1,\ldots,y_d) \in V : x \geqslant \left(y_1^2+\ldots y_d^2\right)^{\frac{1}{2}}\}$. The index set is $\mathbf{I}_d := \{(k;m_1,\ldots,m_d) \in \mathbb{N}\times\mathbb{Z}^d : k^2 \geqslant m_1^2 + \ldots + m_d^2\}$ and the summation sets are $\mathbf{J}_N := \{(k;m_1,\ldots,m_d) \in \mathbf{I}_d : k \leqslant N\}$ for $N \in \mathbb{N}$. Then the recurence is

$$\gamma(m) = \binom{m(d+1)}{d+1}^{-1}, \quad g_n = \sum_{m=1}^{n} \frac{1}{\binom{m(d+1)}{d+1}} g_{m-1} g_{n-m}.$$

Particular cases here are:

case $d = 0$. Arcsine law, since $\gamma(m) = \frac{1}{m}$

case $d = 1$. $\gamma(m) = \frac{1}{m(2m-1)}$, g_n's are Taylor expansion coefficients of the *inverse error function*.

The next examples are related to the cones of positive definite symmetric matrices.

EXAMPLE 3. Let us consider the real vector space $V := \mathbb{S}\mathrm{ymm}_d(\mathbb{R})$ of real symmetric $d \times d$ matrices, and the positive cone Π_d of all real symmetric positive definite $d \times d$ matrices. Then for the index set $\mathbf{I}_d := \left\{(a_{ij})_{i,j=1}^d \in \Pi_d : a_{ij} \in \mathbb{Z}\right\}$ and for the summation sets $\mathbf{J_N} := \left\{(a_{ij}) \in \mathbf{I}_d : 1 \leqslant a_{11} < N_1, \dots, 1 \leqslant a_{dd} < N_d\right\}$ we get

$$\gamma(m) = \left[\frac{d+1}{2} B\left(\frac{d+1}{2}; \frac{(m-1)(d+1)}{2}\right)\right]^d.$$

Here $B(a+1;b+1) := \int_0^1 x^a(1-x)^b dx$ is the Euler β-function of the first kind. A particular case here is $\mathbf{d} = \mathbf{1}$ for which the limit measure is the arcsine law.

EXAMPLE 4. (*Arbitrary symmetric cone.*) Let $V := \mathbb{H}\mathrm{erm}_d(\mathbb{F})$ be the algebra of all hermitian $d \times d$ matrices with $d \geqslant 3$; over a Jordan-Hurwitz algebra $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, with $p := \dim_{\mathbb{R}} \mathbb{F} \in \{2, 4\}$ (see [3]). Let $\mathbb{Z}(\mathbb{F}) := \{\xi = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{F},\ a, b, c, d \in \mathbb{Z}\}$ be the set of "integers" in $\mathbb{F}$. Here the quaternionic units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$. In particular, $\xi \in \mathbb{C}$ iff $c = d = 0$. In V we consider the positive cone $\Pi_d(\mathbb{F})$ of all hermitian positive definite $d \times d$ matrices over $\mathbb{F}$ and the index set $\mathbf{I}_d := \left\{(a_{ij})_{i,j=1}^d \in \Pi_d : a_{ij} \in \mathbb{Z}(\mathbb{F})\right\}$. The summation sets $\mathbf{J_N} := \left\{(a_{ij}) \in \mathbf{I}_d : 1 \leqslant a_{11} < N_1, \dots, 1 \leqslant a_{dd} < N_d\right\}$ have the asymptotical behaviour $|\mathbf{J_N}| \approx c_d \cdot (N_1 \dots N_d)^{1+\frac{(d-1)p}{2}}$, which allows to show that in these cases

$$\gamma(m) = \left[\frac{\alpha+1}{2} B\left(\frac{\alpha+1}{2}; \frac{(m-1)(\alpha+1)}{2}\right)\right]^d, \quad \text{with} \quad \alpha := \frac{(d-1)p}{2}.$$

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