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ON L^2–EIGENFUNCTIONS OF TWISTED LAPLACIAN ON CURVED SURFACES AND SUGGESTED ORTHOGONAL POLYNOMIALS

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Abstract. We show in a unified manner that the factorization method describes completely the L^2-eigenspaces associated to the discrete part of the spectrum of the twisted Laplacian on constant curvature Riemann surfaces. Subclasses of two variable orthogonal polynomials are then derived and arise by successive derivations of elementary complex valued functions depending on the geometry of the surface.

1. Introduction and preliminaries

Let M_κ be a given simply connected Riemann surface of constant scalar curvature $\kappa \in \mathbb{R}$ (SCRS for short). Precisely M_κ is the disc of radius $1/\sqrt{-\kappa}$ for $\kappa < 0$, the Euclidean plane for $\kappa = 0$ and the sphere in $\mathbb{R}^3$ of radius $1/\sqrt{\kappa}$ identified with the extended complex plane $\mathbb{C} \cup \{\infty\}$ for $\kappa > 0$. The corresponding Bergman-Kähler geometry is the one described by the Hermitian metric $ds^2_\kappa := (1+\kappa z\bar{z})^{-2} dz \otimes d\bar{z}$, $z = x+iy \in \mathbb{C}$, whose associated volume measure is

$$d\mu_\kappa = (1+\kappa z\bar{z})^{-2} d\lambda,$$

where $d\lambda$ denotes the usual Lebesgue measure on M_κ. Also let θ_κ be the differential one form $\theta_\kappa := (1+\kappa z\bar{z})^{-1}(\bar{z}dz - zd\bar{z})$. Associated to M_κ, ds^2_κ, $d\mu_\kappa$ and θ_κ, we consider the Laplacian $\mathfrak{L}^\nu_\kappa$, $\nu > 0$, realized as magnetic Schrödinger operator through

$$\mathfrak{L}^\nu_\kappa = (d + i\nu\theta_\kappa)^*(d + i\nu\theta_\kappa),$$

and acting on the Hilbert space $\mathscr{H}_\kappa := L^2(M_\kappa; d\mu_\kappa)$. It is an elliptic self-adjoint second order differential operator describing a single non relativistic spineless particle constrained to move on the two-dimensional analytic surface M_κ in the presence of the external constant magnetic field $B = \nu d\theta_\kappa$. The explicit expression of $\mathfrak{L}^\nu_\kappa$ in the z-complex variable is given (up to a multiplicative constant) by

$$\mathfrak{L}^\nu_\kappa = -\{(1+\kappa|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} + \nu(1+\kappa|z|^2)(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) - \nu^2|z|^2\}. \tag{1.1}$$

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Its concrete spectral theory is well known in the literature, see for example [8, 1] for $\kappa = -1, 0, +1$ and [2, 4] for arbitrary κ. In particular, the discrete part of the spectrum of $\mathfrak{L}_\kappa^\nu$ acting on $\mathscr{H}_\kappa$ is given by the eigenvalues

$$E_{\kappa,m}^\nu := \nu(2m+1) + m(m+1)\kappa$$

with m is a positive integer such that $0 \leqslant m < (2\nu+\kappa)/(|\kappa|-\kappa)$ and the conditions $2\nu+\kappa > 0$ for $\kappa \leqslant 0$ and $2\nu/\kappa$ is integer for $\kappa > 0$. Moreover, we have the following (see [3, 11, 4] for example):

PROPOSITION 1.1. *An orthogonal basis of the L^2-eigenspace*

$$A_m^{2,\nu}(M_\kappa) := \{\Phi \in \mathscr{H}_\kappa;\ \ \mathfrak{L}_\kappa^\nu \Phi = E_{\kappa,m}^\nu \Phi\}$$

is given in terms of the real Jacobi Polynomials $\mathrm{P}_l^{(\alpha,\beta)}(x)$ by

$$(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)} z^p\, \bar{z}^q \mathrm{P}_{m-q}^{(p+q,-2(\frac{\nu}{\kappa}+m)-1)}(1+2\kappa|z|^2)$$

with $q \leqslant m$ and the convention that $pq = 0$.

In the other hand, we know that the factorization algebraic method [13, 6, 2] allows one to construct L^2-eigenfunctions of second order differential operator like $\mathfrak{L}_\kappa^\nu$. Our goal here is to discuss the converse. Precisely, we have to show that such method describes completely the L^2-eigenspaces associated to the discrete part of the twisted Laplacians $\mathfrak{L}_\kappa^\nu$ (following the terminology of [1, 9]). This will be done in a unified manner taking into account the curvature of the considered SCRS. In particular, we recover the case $\kappa = +1$ discussed in [2]. The suggested subclass of two variable orthogonal polynomials $P_{m,n}^{\nu;\kappa}(z,\bar{z})$, i.e., such that the functions

$$(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)} P_{m,n}^{\nu;\kappa}(z,\bar{z})$$

are L^2-eigenfunctions of $\mathfrak{L}_\kappa^\nu$ with $E_\kappa^\nu(m) := \nu(2m+1) + m(m+1)\kappa$ as corresponding L^2-eigenvalue, are derived and satisfy the following Rodrigues type formula

$$P_{m,n}^{\nu;\kappa}(z,\bar{z}) = (-1)^m (1+\kappa|z|^2)^{2\frac{\nu}{\kappa}+m} [(1+\kappa|z|^2)^2 \frac{\partial}{\partial z}]^m (1+\kappa|z|^2)^{-2(\frac{\nu}{\kappa}+m)} z^n,$$

where the intertwining invariant operator $[(1+\kappa|z|^2)^2 \frac{\partial}{\partial z}]^m$ depends only on the geometry of the considered SCRS and not on the magnetic field.

2. Main result

We begin by recalling briefly the factorization method for curved surfaces. Indeed, one considers the first order differential operator ∇_α and its formal adjoint ∇_α^* given respectively by

$$\nabla_\alpha = -(1+\kappa|z|^2)\frac{\partial}{\partial z} + \alpha\bar{z}, \qquad \nabla_\alpha^* = (1+\kappa|z|^2)\frac{\partial}{\partial \bar{z}} + (\alpha-\kappa)z.$$

Thus, by direct computation one gets the algebraic relationships:

$$\nabla_{\nu+\kappa}\nabla^*_{\nu+\kappa} = \mathfrak{L}^{\nu}_{\kappa} - \nu \quad \text{and} \quad \nabla^*_{\nu+\kappa}\nabla_{\nu+\kappa} = \mathfrak{L}^{\nu+\kappa}_{\kappa} + (\nu+\kappa),$$

which gives rise to the following one

$$\mathfrak{L}^{\nu}_{\kappa}\nabla_{\nu+\kappa} = \nabla_{\nu+\kappa}\mathfrak{L}^{\nu+\kappa}_{\kappa} + (2\nu+\kappa)\nabla_{\nu+\kappa}.$$

Hence, for Ψ_0 being a nonzero L^2-eigenfuncion associated to the lowest Landau level $E^{\nu+m\kappa}_{\kappa,0} = \nu + m\kappa$ of $\mathfrak{L}^{\nu+m\kappa}_{\kappa}$, one can show that $\nabla_{\nu+m\kappa}\Psi_0$ is also an L^2-eigenfunction but this time for $\mathfrak{L}^{\nu+(m-1)\kappa}_{\kappa}$ with $E^{\nu+(m-1)\kappa}_{\kappa,1}$ as corresponding eigenvalue. Doing so, it follows that

$$\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}\Psi_0 \in A^{2,\nu}_m(M_\kappa) \tag{2.1}$$

i.e., it is an L^2-eigenfunction of $\mathfrak{L}^{\nu}_{\kappa}$ associated to the eigenvalue $\nu(2m+1)+m(m+1)\kappa =: E^{\nu}_{\kappa,m}$. Conversely, we show that every L^2-eigenfunction $\Phi \in A^{2,\nu}_m(M_\kappa)$ can be obtained from (2.1), that is each $\Phi \in A^{2,\nu}_m(M_\kappa)$ arises by successive derivations of an elementary complex valued function. More precisely, we have

MAIN THEOREM 2.1. *Fix $\nu > 0$ such that $2\nu+\kappa > 0$ for $\kappa \leqslant 0$ and $2\nu/\kappa \in \mathbb{Z}^+$ for $\kappa > 0$, and let m be a fixed positive integer satisfying $0 \leqslant m < (2\nu+\kappa)/(|\kappa|-\kappa)$. Then, the L^2-eigenfunctions*

$$\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}[(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)}z^n], \quad n = 0,1,2,\cdots, \tag{2.2}$$

constitute an orthogonal basis of the L^2-eigenspace $A^{2,\nu}_m(M_\kappa)$.

REMARK 2.2. The above result says that the factorization method determines completely all L^2-eigenfunctions of $\mathfrak{L}^{\nu}_{\kappa}$, i.e., solutions of the eigenvalue problem $\mathfrak{L}^{\nu}_{\kappa}\Psi = E^{\nu}_{\kappa,m}\Psi$ associated to the discrete part of the L^2-spectrum. The planar case ($\kappa = 0$) is classic. For $\kappa = +1$ (i.e., for the sphere $S^2 \cong \mathbb{C}\cup\{\infty\}$ equipped with the Fubini-Study metric on the chart $\mathbb{C}$) the result has been established by Ferapontov and Veselov [2, Theorem 3]. While when $\kappa = -1$ the result we obtain can be considered as its analogue for the non compact hyperbolic unit disc.

The proof of the above theorem relies essentially on Proposition 1.1 and the following result giving closed explicit expressions of (2.1) or also (2.2). Namely, we have

PROPOSITION 2.3. *Fix ν and m as in the theorem above and define $P^{\nu;\kappa}_{m,n}(z,\bar{z})$ by*

$$P^{\nu;\kappa}_{m,n}(z,\bar{z}) := (1+\kappa|z|^2)^{\frac{\nu}{\kappa}+m}\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}[(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)}z^n].$$

Let $m\wedge n := Min(m,n)$ and set

$$C^{m,n}_{\kappa,\nu} = (-1)^{m+n}\frac{\Gamma(2(\frac{\nu}{\kappa}+m)-(m+n)+1)}{\kappa^n\Gamma(2(\frac{\nu}{\kappa}+m)-m+1)}.$$

Then, we have

$$P^{\nu;\kappa}_{m,n}(z,\bar z) = (-1)^m(1+\kappa|z|^2)^{2\frac{\nu}{\kappa}+m}[(1+\kappa|z|^2)^2\frac{\partial}{\partial z}]^m\Big((1+\kappa|z|^2)^{-2(\frac{\nu}{\kappa}+m)}z^n\Big) \quad (2.3)$$

$$= C^{m,n}_{\kappa,\nu}\cdot(1+\kappa|z|^2)^{2(\frac{\nu}{\kappa}+m)+1}\frac{\partial^{m+n}}{\partial z^m\partial\bar z^n}(1+\kappa|z|^2)^{-2(\frac{\nu}{\kappa}+m)+m+n-1} \quad (2.4)$$

$$= (-1)^m(m\wedge n)!|z|^{|m-n|}e^{i[(n-m)\arg z]}\mathrm{P}^{(|m-n|,-2(\frac{\nu}{\kappa}+m)-1)}_{m\wedge n}(1+2\kappa|z|^2). \quad (2.5)$$

Sketched proof of Proposition 2.3. The identity (2.3) holds by observing that the first order differential operator $\nabla_\alpha = -(1+\kappa|z|^2)\frac{\partial}{\partial z}+\alpha\bar z$ can be rewritten as

$$\nabla_\alpha f = -(1+\kappa|z|^2)^{\frac{\alpha}{\kappa}+1}\frac{\partial}{\partial z}[(1+\kappa|z|^2)^{-\frac{\alpha}{\kappa}}f]$$

for every smooth function f on M_κ. Thus, we have

$$\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}f = (-1)^m(1+\kappa z\bar z)^{\frac{\nu}{\kappa}}[(1+\kappa z\bar z)^2\frac{\partial}{\partial z}]^m((1+\kappa z\bar z)^{-(\frac{\nu}{\kappa}+m)}f).$$

This yields (2.3) when specifying $f(z) = (1+\kappa z\bar z)^{-(\frac{\nu}{\kappa}+m)}z^n$.

The identity (2.4) is deduced from (2.3) using

CLAIM 1. For every fixed positive integer m and every smooth complex valued function f on M_κ, we have

$$[(1+\kappa|z|^2)^2\frac{\partial}{\partial z}]^m f = (1+\kappa|z|^2)^{m+1}\frac{\partial^m}{\partial z^m}((1+\kappa|z|^2)^{m-1}f)$$

combined with the fact that

$$z^n(1+\kappa|z|^2)^\alpha = \frac{1}{(\alpha+1)_n\kappa^n}\frac{\partial^n}{\partial\bar z^n}((1+\kappa|z|^2)^{\alpha+n}),$$

keeping in mind that $(-a)_n = (a-n+1)_n$. Here $(a)_n = a(a+1)\cdots(a+n-1)$.

The proof of (2.5) can be handled by induction together with the use of the following result satisfied by the real Jacobi Polynomials $\mathrm{P}^{(a,b)}_j(x)$:

CLAIM 2. For arbitrary real number a,b, we have

$$(x^2-1)\frac{d}{dx}P^{(a,b)}_j(x)+[(a-b)+(a+b)x]P^{(a,b)}_j(x) = 2(j+1)P^{(a-1,b-1)}_{j+1}(x). \qquad \square$$

Below, we give the proofs of Claims 1 and 2.

Proof of Claim 1. This is proved by induction, where $m=0$ and $m=1$ is obvious. Next, set $\not{D}_\kappa f = h^2\frac{\partial}{\partial z}f = (1+\kappa z\bar z)^2\frac{\partial}{\partial z}f$, let $\not{D}^m_\kappa$ stand for

$$\not{D}^m_\kappa = \underbrace{\not{D}_\kappa\circ\not{D}_\kappa\circ\cdots\circ\not{D}_\kappa}_{\text{m-times}}$$

and assume that

$$h^{m+1}\frac{\partial^m}{\partial z^m}(h^{m-1}f) = (h^2\frac{\partial}{\partial z})^m f = \not{D}_\kappa^m f$$

is satisfied for a given positive integer m. Hence, using direct computation and the fact that $\not{D}_\kappa^m((\frac{\partial h}{\partial z})f) = (\frac{\partial h}{\partial z})\not{D}_\kappa^m f$ for $\frac{\partial h}{\partial z}$ being an holomorphic function, we get

$$h^{m+2}\frac{\partial^{m+1}}{\partial z^{m+1}}(h^m f) = mh\frac{\partial}{\partial z}(h\not{D}_\kappa^m f) + h\not{D}_\kappa^m(h^{-1}\not{D}_\kappa f). \tag{2.6}$$

Next, note that we have

$$h\not{D}_\kappa^m(h^{-1}\not{D}_\kappa f) = h\not{D}_\kappa^{m-1}(h^{-1}\not{D}_\kappa{}^2 f) - h\frac{\partial h}{\partial z}\not{D}_\kappa^{m-1}(\not{D}_\kappa f). \tag{2.7}$$

Repeated application of (2.7) gives

$$h\not{D}_\kappa^m(h^{-1}\not{D}_\kappa f) = h\not{D}_\kappa^{m-j}(h^{-1}\not{D}_\kappa^{j+1} f) - jh\frac{\partial h}{\partial z}\not{D}_\kappa^{m-1}(\not{D}_\kappa f)$$

for every given positive integer j such that $0 \leqslant j \leqslant m$. In particular, for $j = m$ it follows

$$mh\frac{\partial h}{\partial z}\not{D}_\kappa{}^m(f) + h\not{D}_\kappa^m(h^{-1}\not{D}_\kappa f) = \not{D}_\kappa^{m+1}. \tag{2.8}$$

Finally, by combining (2.6) and (2.8), we get the desired result of Claim 1. $\square$

Proof of Claim 2. The assertion of Claim 2 is an immediate consequence of the facts that the classical Jacobi polynomial $P_k^{(\alpha,\beta)}$ is a solution of the second order differential equation [7, page 214]

$$(1-x^2)y'' + [(\alpha-\beta)+(\alpha+\beta+2)x]y' - k(k+\alpha+\beta+1)y = 0.$$

Indeed, by making the changes $\alpha = a-1, \beta = b-1$ and $k = j+1$, we get

$$(x^2-1)\frac{d^2}{dx^2}P_{j+1}^{(a-1,b-1)}(x) - [(a-b)+(a+b)x]\frac{d}{dx}P_{j+1}^{(a-1,b-1)}(x) + \\ +(j+1)(j+a+b)P_{j+1}^{(a-1,b-1)}(x) = 0.$$

Next, using the fact that [7, page 213]

$$\frac{d}{dx}P_{j+1}^{(a-1,b-1)}(x) = \frac{j+a+b}{2}P_j^{(a,b)}(x),$$

it follows

$$(x^2-1)\frac{d}{dx}P_j^{(a,b)} - [(a-b)+(a+b)x]P_j^{(a,b)} + 2(j+1)P_{j+1}^{(a-1,b-1)} = 0. \qquad \square$$

We conclude this section by giving a sketched proof of Theorem 2.1.

Proof of Theorem 2.1. We use the fact that the L^2-eigenspace of $\mathfrak{L}^b_\kappa$ associated to its first eigenvalue b coincides with the null space of ∇^*_b and is spanned in the Hilbert space $\mathscr{H}_\kappa$ as follows

$$Span\Big\{(1+\kappa|z|^2)^{-\frac{b}{\kappa}}z^n, \ \ n=0,1,2,\cdots\Big\}.$$

Hence, for $b=\nu+m\kappa$ and according to (2.1) it turns out to compute

$$\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}[(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)}z^n] =: (1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)}P^{\nu;\kappa}_{m,n}(z,\bar{z})$$

which is given by (2.5) in Proposition 2.3 and that we can rewrite also as follow

$$P^{\nu;\kappa}_{m,n}(z,\bar{z}) = (-1)^m(m\wedge n)!z^p\,\bar{z}^q\mathrm{P}^{(p+q,-2(\frac{\nu}{\kappa}+m)-1)}_{m-q}(1+2\kappa|z|^2)$$

with $p=n-m$ and $q=0$ if $m\leqslant n$ and $p=0$ and $q=m-n$ if $m\geqslant n$. Next, by applying Proposition 1.1, we obtain the desired result as asserted in Theorem 2.1. $\square$

3. Concluding remarks

According to Proposition 2.3 above, the class of two variable orthogonal polynomials $P^{\nu;\kappa}_{m,n}(z,\bar{z})$, suggested by the twisted Laplacians $\mathfrak{L}^\nu_\kappa$ are obtained in a unified manner taking into account the curvature of the considered SCRS M_κ. Indeed the functions

$$(1+\kappa|z|^2)^{-(\frac{\nu}{\kappa}+m)}P^{\nu;\kappa}_{m,n}(z,\bar{z})$$

are L^2-eigenfunctions of $\mathfrak{L}^\nu_\kappa$ with $E^\nu_{\kappa,m} := \nu(2m+1)+m(m+1)\kappa$ as corresponding L^2-eigenvalue. The obtained two variable polynomials satisfy (2.3), where the involved intertwining invariant operator

$$[(1+\kappa|z|^2)^2\frac{\partial}{\partial z}]^m$$

depends only on the geometry of M_κ and not on the magnetic field. Also, they satisfy the Rodrigues formula (2.4), up to a given multiplicative constant $C^{m,n}_{\kappa,\nu}$. Such polynomials $P^{\nu;\kappa}_{m,n}(z,\bar{z})$ are connected to the real Jacobi polynomials $\mathrm{P}^{(\alpha,\beta)}_l(x)$. Note that the identity (2.5) above for $\kappa\neq 0$ can be rewritten in the following form

$$P^{\nu;\kappa}_{m,n}(z,\bar{z}) = (-1)^m\begin{cases} m!z^{n-m}\mathrm{P}^{(n-m,-2(\frac{\nu}{\kappa}+m)-1)}_m(1+2\kappa z\bar{z}) & \text{if } m\leqslant n\\ n!\,\bar{z}^{m-n}\mathrm{P}^{(m-n,-2(\frac{\nu}{\kappa}+m)-1)}_n(1+2\kappa z\bar{z}) & \text{if } m\geqslant n\end{cases}$$

Therefore, one sees that the polynomials $P^{\nu,\kappa}_{m,n}(z,\bar{z})$ for $\kappa=-1$ are exactly the so-called disc polynomials [10, 14]. Here they appear, up to multiplicative functions, as L^2-eigenfunctions of the twisted Laplacian $\mathfrak{L}^\nu_{-1}$ on the hyperbolic disc. While for the limit case $\kappa=0$, the polynomials $P^{\nu;0}_{m,n}(z,\bar{z})$ are exactly the complex Hermite polynomials [12, 5] defined by

$$H^\nu_{m,n}(z,\bar{z}) := \frac{(-1)^{m+n}}{(2\nu)^n}e^{2\nu|z|^2}\frac{\partial^{m+n}}{\partial z^m\partial\bar{z}^n}e^{-2\nu|z|^2}, \tag{3.1}$$

which form a complete orthogonal system in $L^2(\mathbb{C};e^{-2\nu|z|^2}dxdy)$. The associated functions $e^{-\nu|z|^2}H^{\nu}_{m,n}(z,\bar{z})$ are L^2-eigenfunctions of the usual twisted Laplacian on the Euclidean plane,

$$\mathfrak{L}^{\nu}_0 = -\{\frac{\partial^2}{\partial z\partial\bar{z}} + \nu(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}) - \nu^2|z|^2\},$$

with $\nu(2m+1)$ as corresponding L^2-eigenvalue.

Furthermore, added to the facts that $\nu(2m+1) = \lim_{\kappa\longrightarrow 0} E^{\nu}_{\kappa,m}$ and the operator $\mathfrak{L}^{\nu}_0$ appears also as the formal limit of the unbounded differential operators $\mathfrak{L}^{\nu}_{\kappa}$ by letting κ goes to 0, one gets the limiting transition of $P^{\nu;\kappa}_{m,n}(z,\bar{z})$ to the complex Hermite polynomials $H^{\nu}_{m,n}(z,\bar{z})$:

$$\lim_{\kappa\longrightarrow 0} P^{\nu;\kappa}_{m,n}(z,\bar{z}) = H^{\nu}_{m,n}(z,\bar{z})$$

for every fixed $z\in\mathbb{C}$. This can be checked easily from (2.4) or also from (2.5) using some known useful transformations on special functions. But below, we give an alternative proof and we will see how this can be handled using the background related to the factorization method without knowing explicit expression of $P^{\nu;\kappa}_{m,n}(z,\bar{z})$. Indeed, by making formal limit, keeping in mind that

$$\lim_{\kappa\longrightarrow 0}(1+\kappa z\bar{z})^{\frac{\nu}{\kappa}+m} = e^{\nu z\bar{z}},$$

we get

$$\begin{aligned}\lim_{\kappa\longrightarrow 0} P^{\nu;\kappa}_{m,n}(z,\bar{z}) &= \lim_{\kappa\longrightarrow 0}(1+\kappa z\bar{z})^{\frac{\nu}{\kappa}+m}\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+m\kappa}[(1+\kappa z\bar{z})^{-(\frac{\nu}{\kappa}+m)}z^n]\\ &= e^{\nu z\bar{z}}\underbrace{\nabla_{\nu}\circ\nabla_{\nu}\circ\cdots\circ\nabla_{\nu}}_{\text{m-times}}[e^{-\nu z\bar{z}}z^n].\end{aligned}$$

Next, by rewriting ∇_{ν} in the following form $\nabla_{\nu}f = -e^{\nu z\bar{z}}\frac{\partial}{\partial z}(e^{-\nu z\bar{z}}f)$, it follows

$$\lim_{\kappa\longrightarrow 0} P^{\nu;\kappa}_{m,n}(z,\bar{z}) = (-1)^m e^{2\nu z\bar{z}}\frac{\partial^m}{\partial z^m}[e^{-2\nu z\bar{z}}z^n].$$

But since $\frac{\partial^n}{\partial\bar{z}^n}(e^{-2\nu z\bar{z}}) = (-2\nu)^n e^{-2\nu z\bar{z}}z^n$, we conclude easily that

$$\lim_{\kappa\longrightarrow 0} P^{\nu;\kappa}_{m,n}(z,\bar{z}) = \frac{(-1)^{m+n}}{(2\nu)^n}e^{2\nu z\bar{z}}\frac{\partial^{m+n}}{\partial z^m\partial\bar{z}^n}(e^{-2\nu z\bar{z}}) \overset{(3.1)}{=} H^{\nu}_{m,n}(z,\bar{z}).$$

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