REDUCTION OPERATOR ALGEBRAS AND
GENERALIZED SIMILARITY PROBLEM

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Abstract. In this paper, we give new results on reduction operator algebras. We investigate the relationship between the generalized similarity problem and the total reduction property. We also give some sufficient and necessary conditions for a reduction operator algebra to be self-adjoint or similar to a C^*-algebra.

1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra. A Banach $\langle$left, bi$\rangle$-module on $\mathscr{A}$ is a Banach space $\mathscr{X}$ which is an algebraic $\langle$left, bi$\rangle$-module on $\mathscr{A}$ such that the module actions are continuous. Note that $\mathscr{X}^*$ becomes a Banach $\langle$left, bi$\rangle$-module with respect to the dual actions $\langle a, f \rangle(b) = f(ba)$ and $\langle f, a \rangle(b) = f(ab)$ identically for every $a, b \in \mathscr{A}$ and $f \in \mathscr{X}^*$. A derivation from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $\mathscr{X}$ is a bounded map $D : \mathscr{A} \longrightarrow \mathscr{X}$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathscr{A}$. A derivation D is inner if there is $x \in \mathscr{X}$ such that $D(a) = a.x - x.a$ for $a \in \mathscr{A}$. A Banach algebra $\mathscr{A}$ is said to be amenable if for each Banach $\mathscr{A}$-bimodule $\mathscr{X}$, every derivation $D : \mathscr{A} \longrightarrow \mathscr{X}^*$ is inner. Amenable Banach algebras were introduced by B. N. Johnson in [13] and they were well investigated in [3, 10, 14]. It was established in [1] by A. Connes that every amenable C^*-algebra $\mathscr{A}$ (i.e., every closed self-adjoint of $\mathscr{B}(\mathscr{H})$, the algebra of bounded operators on a given Hilbert space $\mathscr{H}$) is nuclear; i.e., for all C^*-algebra $\mathscr{B}$, there exists a unique C^*-norm for which the completion of the (algebraic) tensor product $\mathscr{A} \otimes \mathscr{B}$ is a C^*-algebra. The converse has been shown by U. Haagerup in [11].

Assume now that $\mathscr{A}$ is an operator algebra, that is a Banach algebra which acts as an algebra of bounded operators on a Hilbert space $\mathscr{K}$ (in fact, $\mathscr{A}$ is a norm-closed subalgebra of the algebra $\mathscr{B}(\mathscr{K})$ of bounded operators on $\mathscr{K}$). Identifying the matrix space $M_n(\mathscr{B}(\mathscr{K}))$ over $\mathscr{B}(\mathscr{K})$ with $\mathscr{B}(\mathscr{K}^n)$, we let $M_n(\mathscr{A})$ have the induced norm in $\mathscr{B}(\mathscr{K}^n)$. A bounded homomorphism $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ is completely bounded (c.b. for short) if for every n, $\phi_n : M_n(\mathscr{A}) \longrightarrow M_n(\mathscr{B})$, $(a_{ij}) \to (\phi(a_{ij}))$ is bounded such that

$$\|\phi\|_{cb} = \sup \|\phi_n\|_{M_n(\mathscr{A}) \to M_n(\mathscr{B})} < \infty.$$

A representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, i.e., a bounded homomorphism $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$, is said to be non-degenerate if $\overline{\pi(\mathscr{A})\mathscr{H}} = \mathscr{H}$. It is called irreducible when the only closed invariant subspaces of $\pi(\mathscr{A})$ are $\{0\}$ and $\mathscr{H}$. A Hilbert

$\mathscr{A}$-module $\mathscr{H}$ is defined to be a left Banach module for $\mathscr{A}$ which is isomorphic to a Hilbert space. This is equivalent to the existence of a representation $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ of $\mathscr{A}$ on $\mathscr{H}$. A Hilbert $\mathscr{A}$-module is said to have the reduction property if for every closed submodule $\mathscr{V}$ of $\mathscr{H}$, there is a closed submodule $\mathscr{W}$ such that $\mathscr{H} = \mathscr{V} \oplus \mathscr{W}$. A operator algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ is called a reduction algebra if K has the reduction property. It is called a complete reduction algebra if the amplified module $\mathscr{K}^\infty$ $\left(= \mathscr{K} \otimes_2 l^2(\mathbf{N}) \right)$ has the reduction property for $\mathscr{A}^{(\infty)}$. Note that we can define a natural action of $\mathscr{A}^{(\infty)}$ on $\mathscr{K}^\infty$ via $(A_n)_n.(x_n)_n = (A_n.x_n)_n$. An operator algebra $\mathscr{A}$ is said to have the total reduction property if every Hilbert $\mathscr{A}$-module has the reduction property. It $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ is an operator algebra, then the total reduction property for $\mathscr{A}$ implies the complete reduction for $\mathscr{A}$, which in turn implies the reduction property. If $\mathscr{A}$ is an operator algebra and $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H}))$ is a bounded representation, then $\mathscr{H}$ has the reduction property if and only if every submodule of $\mathscr{H}$ is an idempotent operator in $\pi(\mathscr{A})'$, the commutant of $\pi(\mathscr{A})$. Such idempotents are called idempotent operator module projections. The total reduction property for an operator algebra was introduced in J.A. Gifford's thesis [8]. This property is satisfied by amenable operator algebras. In fact, many problems raised in the amenable case have natural extensions to operator algebras with the total reduction property. In [19, page 13], Pisier asks which unital Banach algebras $\mathscr{A}$ have the similarity property (SP)?, i.e., such that for each bounded unital representation $\phi : \mathscr{A} \subseteq \mathscr{B}(\mathscr{K}) \longrightarrow \mathscr{B} \subseteq \mathscr{B}(\mathscr{H})$, $\exists S \in \mathscr{B}(\mathscr{H})$ invertible such that $\phi_S : a \to S^{-1}\phi(a)S$ is a contraction. He then gave several results answering partially this question. In his paper [20], he also raised the Generalized Similarity Problem (GSP): Which unital operator algebras have the property that any representation $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ ($\mathscr{H}$ arbitrary Hilbert space) is c.b.

In the C*-algebras case, (SP) is equivalent to (GSP) and in [15] R. Kadison conjectured that every unital C*-algebra satisfies (SP). J. Gifford in [9] has shown that a C*-algebra $\mathscr{B}$ has the total reduction property if and only if it satisfies (SP), i.e., each representation of $\mathscr{A}$ is similar to a *-homomorphism. In the case of non-selfadjoint operator algebra, it is natural to look for similar results including the eventual connection of the total reduction property to (GSP) or (SP). On the other hand, it has been conjectured in [9] that:

Conjecture 1: Every non self-adjoint operator algebra with the total reduction property is isomorphic (as a Banach algebra) to a C*-algebra.

Conjecture 2: Every weakly closed complete reduction non self-adjoint operator algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ is similar to C*-algebra $\mathscr{B}$. i.e $\exists S \in \mathscr{B}(\mathscr{K})$ invertible such that $\mathscr{A} = S^{-1}\mathscr{B}S$.

As partial results, Gifford has proven that every operator algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ with the total reduction property, which is a closed subalgebra of an abelian C*-algebra, is a C*-algebra and every operator algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ with the total reduction property, which a closed subalgebra of the algebra of compact operators on $\mathscr{K}$, is similar to a C*-algebra. It is clear that if Conjecture 2 is true, then the statement of Conjecture 1 holds for weakly closed operator algebras.

In this paper, the operator algebras with the reduction property are investigated and some partial answers to both conjectures 1 and 2 are established. We shall give some sufficient and necessary conditions for a reduction operator algebra to be self-adjoint or similar to a C*-algebra or finite dimensional semi-simple algebra. For example, we give in Section 4 a partial answer to conjecture 2. Precisely, we show that: if $\mathscr{A}$ is weakly closed complete reduction operator algebra such that the idempotent operators are in the bidual of $\mathscr{A}$, then $\mathscr{A}$ is similar to a C*-algebra. In Section 3, we will see that the total reduction property plays a crucial role for answering (GSP). Thus, we show in particular that if the operator algebra is isomorphic to a C*-algebra, then the total reduction property is equivalent to (GSP).

2. Preliminaries

In this section, we fix notations and recall some standard results that we will use throughout the present paper. We start by recalling well known results on the amenability.

Let $\mathscr{A}$ be a Banach algebra. If $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ are $\mathscr{A}$-modules and $f : \mathscr{X} \to \mathscr{Y}$ and $g : \mathscr{Y} \to \mathscr{Z}$ are module morphisms, then the sequence

$$\sum : 0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

is exact if f is one-to-one, $\operatorname{Im} g = \mathscr{Z}$ and $\operatorname{Im} g = \ker g$. The exact sequence is admissible if $\operatorname{Im} f$ has a Banach space complement in $\mathscr{Y}$, equivalently, there is a bounded linear map $h : \mathscr{Z} \to \mathscr{Y}$ such that $hg = I$ on $\mathscr{Z}$. The sequence splits if there is a $\mathscr{A}$-module morphism $h : \mathscr{Z} \to \mathscr{Y}$ such that $hg = I$ on $\mathscr{Z}$. This means that $\operatorname{Im} f$ has a Banach space complement in $\mathscr{Y}$ which is a submodule.

THEOREM 2.1. ([3], [14]) *A Banach algebra $\mathscr{A}$ is amenable if and only if*
(i) $\mathscr{A}$ has a bounded approximate identity, and
(ii) For each essential $\mathscr{A}$-bimodule $\mathscr{X}$, any admissible short exact sequence

$$\sum : 0 \longrightarrow \mathscr{X}^* \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

of $\mathscr{A}$-bimodules splits. In this case, every two sided ideal $\mathscr{J}$ with an approximate identity is amenable Banach algebra.

We require some results of J.A. Gifford from his paper [9] and from his 1997 Ph.D. thesis (see [8]) that we know of no other published reference for. We will collect these in a single proposition and provide brief proofs because they are needed in this article.

PROPOSITION 2.1. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra.*
1. If $\mathscr{A}$ has the complete reduction property. Then the set P of central projections of $\mathscr{A}''$ is bounded. Moreover, there exists a similarity S of $\mathscr{K}$ which makes all the central projections self-adjoint.
2. If $\mathscr{A}$ is a subalgebra of the algebra of compact operators in $\mathscr{B}(\mathscr{K})$, then $\mathscr{A}$ satisfies the total reduction property if and only if $\mathscr{A}$ is similar to a C$*$-algebra. In particular $\mathscr{A}$ is amenable if and only if $\mathscr{A}$ is similar to a nuclear C*-algebra.

PROPOSITION 2.2. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra. Then*

1. *If $\mathscr{A}$ is a reduction algebra, and if $\mathscr{J} \subseteq \mathscr{A}$ is a two-sided ideal. Then $\mathscr{J}$ acts non degenerate on $\overline{\mathscr{J}\mathscr{K}}$*

2. *If $\mathscr{A}$ is a complete reduction operator algebra and if $\mathscr{J} \subseteq \mathscr{A}$ a two-sided ideal such that $\mathscr{K} = \overline{\mathscr{J}\mathscr{K}}$. Then, we have $h \in \overline{\mathscr{J}^{(n)}h}$ for every positive integer n and every $h \in \mathscr{K}^{(n)}$*

3. *If $\mathscr{A}$ is a complete reduction operator algebra and $\mathscr{J} \subseteq \mathscr{A}$ a two-sided ideal such that $\mathscr{K} = \overline{\mathscr{J}\mathscr{K}}$. Then the stronger closure of $\mathscr{J}$ is the bicommutant $\mathscr{J}''$.*

Proof.
2) Since $\mathscr{A}$ has the reduction property, there is a submodule $\mathscr{V}$ of $\mathscr{K}$ such that $\mathscr{K} = \mathscr{V} \oplus \overline{\mathscr{J}\mathscr{K}}$. We can easily check that $\overline{\mathscr{J}\mathscr{V}} = 0$ and therefore

$$\overline{\mathscr{J}\mathscr{K}} = \overline{\mathscr{J}\left(\overline{\mathscr{J}\mathscr{K}}\right)} \oplus \overline{\mathscr{J}\mathscr{V}} = \overline{\mathscr{J}\left(\overline{\mathscr{J}\mathscr{V}}\right)}.$$

3) Since $\mathscr{K}^{(n)}$ has the reduction property for $\mathscr{A}^{(n)}$ and $\overline{\mathscr{J}^{(n)}h}$ is $\mathscr{A}^{(n)}$-invariant, then there exists an idempotent operator module projection $p \in (\mathscr{A}^{(n)})'$ such that $p\mathscr{K} = \overline{\mathscr{J}^{(n)}h}$ and $h = ph + (1-p)h$. Note that $\mathscr{J}(1-p)h = 0$ and $\mathscr{V} = \{h \in \mathscr{K}$ such that $\mathscr{J}h = 0\} = 0$. Indeed, let q be an idempotent operator module projection such that $q\mathscr{K} = \mathscr{V}$. Then

$$\mathscr{K} = \overline{\mathscr{J}\mathscr{K}} \subseteq \overline{\mathscr{J}q\mathscr{K}} \oplus \overline{\mathscr{J}(1-q)\mathscr{K}} \subseteq \overline{\mathscr{J}(1-q)\mathscr{K}} \subseteq (1-q)\mathscr{K}.$$

This implies that $\mathscr{V} = 0$. It follows that $(1-p)h = 0$ and $h \in \overline{\mathscr{J}^{(n)}h}$.

4) Note first that Since $\mathscr{J}''$ is strong closed and $\mathscr{J} \subseteq \mathscr{J}''$, then the strong closure of $\mathscr{J}$ is contained in $\mathscr{J}''$. Next, let $A \in \mathscr{J}''$ and fix $n \geqslant 1$, then the norm-closure $\overline{\mathscr{J}^{(n)}h}$ is an invariant subspace of $\mathscr{A}^{(n)}$ for every fixed $h = (h_1, h_2, \ldots, h_n) \in \mathscr{K}^{(n)}$. Now, since $\mathscr{A}^{(n)}$ is a reduction algebra, it follows that $\overline{\mathscr{J}^{(n)}h}$ is the range of idempotent operator module projection $p \in (\mathscr{A}^{(n)})'$. According to the facts that $A^{(n)} \in (\mathscr{A}'')^{(n)}$ and $(\mathscr{A}'')^{(n)} = (\mathscr{A}^{(n)})''$ together with Proposition 12.2 in [2], we have $(\mathscr{A}^{(n)})' = ((\mathscr{A}'')^{(n)})'$. It follows that $pA^{(n)} = A^{(n)}p$ and $\overline{\mathscr{J}^{(n)}h}$ is an invariant subspace for A^n. Since $h \in \overline{\mathscr{J}^{(n)}h}$, we have $A^n h$ is in $\overline{\mathscr{J}^{(n)}h}$. Finally, using Proposition 8.6 in [2], we see that A is in the strong-closure of $\mathscr{A}$. This completes the proof.

3. Reduction operator algebras

We start with the following.

LEMMA 3.1. *Let $\mathscr{A}$ be an operator algebra. The following are equivalents:*
1. *$\mathscr{A}$ has the total reduction property.*
2. *Every exact short sequence of Hilbert $\mathscr{A}$-module is split.*
3. *For each representation $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$, $\mathscr{H}$ has the reduction property for $\pi(\mathscr{A})$.*

Proof.

1) $\Rightarrow$ 2) Consider an exact short sequence of Hilbert $\mathscr{A}$-module

$$\sum : \; 0 \longrightarrow \mathscr{V} \longrightarrow \mathscr{H} \longrightarrow \mathscr{W} \longrightarrow 0.$$

Since $\mathscr{H}$ has the reduction property for $\mathscr{A}$, there exists a Hilbert $\mathscr{A}$-module $\mathscr{V}'$ such that $\mathscr{H} = \mathscr{V} \oplus \mathscr{V}'$. This show that $\mathscr{W}$ is isomorphic to $\mathscr{V}'$ as a Hilbert $\mathscr{A}$-module.

2) $\Rightarrow$ 3) Consder any representation $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ of $\mathscr{A}$. Let $\mathscr{M}$ a closed invariant subspace of $\pi(\mathscr{A})$. Then the short sequence

$$\sum : \; 0 \longrightarrow \mathscr{M} \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}/\mathscr{M} \longrightarrow 0$$

is exact and thus it is split. There is a Hilbert $\mathscr{A}$-module $\mathscr{N}'$ such that $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$. This compltes this sens.

3) $\Rightarrow$ 1) Now, let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module and $\mathscr{V}$ be a submodule. Consider the map $\pi : \mathscr{A} \longrightarrow \mathscr{B}$ defined by $\pi(a)h = ah$ for all $h \in \mathscr{H}$. It is easy to see that π is a bounded homomorphism and $\mathscr{V}$ is a closed invariant subspace of $\pi(\mathscr{A})$. Then there is an other closed invariant subspace $\mathscr{W}$ which is a submodule of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{V} \oplus \mathscr{W}$. This complete the proof.

PROPOSITION 3.1. *Any amenable operator algebra has the total reduction property.*

Proof. By applying Theorem 2.1 combined with Lemma3.1, we conclude that every exact short Hilbert $\mathscr{A}$-module is split.

As examples of amenable operator algebras, we cite the abelian C*-algebras, the algebra of compact operators on a Hilbert space, Cuntz algebras, AF-algebras. Note that $\mathscr{B}(\mathscr{K})$ is not amenable when the Hilbert space $\mathscr{K}$ is of infinite dimension but it is a total reduction algebra.

It is important to see that the total reduction property is preserved under continuous homomorphisms. Therefore any operator algebra similar to a total reduction operator algebra must be a total reduction algebra.

PROPOSITION 3.2. *Let $\mathscr{A}$ be a total reduction (respectively amenable) operator algebra and $\psi : A \longrightarrow B$ a continuous homomorphism into an other operator algebra $\mathscr{B}$ such that $\overline{\psi(\mathscr{A})} = \mathscr{B}$. Then $\mathscr{B}$ is a total reduction (respectively amenable) operator algebra.*

Proof. It is enough to check that if $\pi : \mathscr{B} \longrightarrow \mathscr{B}(\mathscr{H})$ is a representation of $\mathscr{B}$, then a subspace $\mathscr{M} \in \mathscr{B}(\mathscr{H})$ is a closed invariant subspace of $\pi(\mathscr{B})$ if and only if it a is a closed invariant subspace of $\phi(\pi(\mathscr{A}))$.

PROPOSITION 3.3. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be a complete reduction operator algebra containing the identity. Then the stronger closure of $\mathscr{A}$ is $\mathscr{A}^{''}$.*

Proof. The result is a particular case of property (3) in Proposition 2.3.

COROLLARY 3.1. *For any operator algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$, we have $\mathscr{A}$ is a reduction algebra if and only if $\mathscr{A}''$ is a reduction algebra.*

The proof of this result is a straightforward exercise. It can be handled by writing down the definition of the reduction property and applying the last proposition.

COROLLARY 3.2. *If $\mathscr{A}$ is an operator algebra containing the identity with the total reduction property, then for every representation $\pi : A \longrightarrow \mathscr{B}(\mathscr{H})$, $\pi(\mathscr{A})''$ is a reduction operator algebra. Furthermore, the stronger closure of $\pi(\mathscr{A})$ is $(\pi(\mathscr{A}))''$.*

Proof. Denote by $\mathscr{U}$ the norm-closure of $\pi(\mathscr{A})$ and note that both the bicommutant and the strong-closure of $\pi(\mathscr{A})$ are equal to $\mathscr{U}$. Applying the last proposition to $\mathscr{U}$, we get the desired result.

It is a known fact that any Banach algebra $\mathscr{A}$ is weak* dense in its topological bidual $\mathscr{A}^{**}$. Hence for given $u, v \in \mathscr{A}^{**}$, we set $u ._1 v = \lim_\beta (\lim_\alpha a_\alpha b_\beta)$ and $u ._2 v = \lim_\alpha (\lim_\beta a_\alpha b_\beta)$ where a_α and b_β are two sequences in $\mathscr{A}$ which converge weakly* respectively to u and v. $\mathscr{A}$ is called Arens regular if $u . v = u ._1 v = u ._2 v$ for all $u, v \in \mathscr{A}^{**}$. Thus, it is well know that every C*-algebras is Arens regular. Assume now that $\mathscr{A} \subseteq \mathscr{B}(K)$ is a non self-adjoint operator algebra and let $\pi : \mathscr{B}(K) \longrightarrow \mathscr{B}(H)$ denotes the Gelfand-Naimark-Segal GNS-representation of $\mathscr{B}(K)$. Then $\mathscr{A}^{**}$ is isometrically isomorphic to $\overline{\pi(\mathscr{A})}^{w*}$ and therefore $\mathscr{A}$ is Arens regular. From the paper of P.G. Dixon [5], we can see that $\mathscr{A}$ has a bounded approximate identity if and only if the bidual $\mathscr{A}^{**}$ has an identity.

As example the algebra of bounded operators $\mathscr{B}(H)$ and the algebra of compact operators $\mathscr{K}(\mathscr{H})$ have both the total reduction property. It is shown in [10] that if $\mathscr{A}^{**}$ is amenable, then $\mathscr{A}$ is amenable. For the total reduction property case, we have the same result. Namely

COROLLARY 3.3. *Let $\mathscr{A} \in \mathscr{B}(\mathscr{K})$ be an operator algebra such that the bidual $\mathscr{A}^{**}$ has the total reduction property. Then $\mathscr{A}$ has a total reduction property.*

Proof. Assume that $\mathscr{A}^{**}$ has the total reduction property and let $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ be a representation of $\mathscr{A}$. Let $\mathscr{M}$ be a closed invariant subspace of $\pi(\mathscr{A})$. Note that the map π can be extended to representation $\tilde{\pi} : \mathscr{A}^{**} \longrightarrow \mathscr{B}(\mathscr{H})$. Since $\tilde{\pi}(\mathscr{A}^{**}) \subseteq \overline{\pi(\mathscr{A})}^{w*} \subseteq (\pi(\mathscr{A}))''$ and from Corollary 3.2, the stronger closure of $\pi(\mathscr{A})$ is $(\pi(\mathscr{A}))''$, $\mathscr{M}$ is also a closed invariant subspace of $\tilde{\pi}(\mathscr{A}^{**})$. Hence there is a closed invariant subspace $\mathscr{N}$ of $\tilde{\pi}(\mathscr{A}^{**})$ such that $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$. Notice that $\mathscr{N}$ is also a closed invariant subspace of $\pi(\mathscr{A})$. This completes the proof.

Notice that the converse is not true for the amenability's case. Indeed $\mathscr{B}(H)$ is not amenable while $\mathscr{K}(\mathscr{H})$ is amenable. But it seems to be true for the case of the total reduction property.

LEMMA 3.2. *Let* $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ *be a representation of a operator algebra* $\mathscr{A}$ *and let* $\mathscr{V}$ *an a closed invariant subspace of* $\pi(\mathscr{A})$. *If* $\mathscr{H}$ *has the reduction property (respectively the complete reduction property) for* $\mathscr{A}$ *then* $\mathscr{V}$ *has the reduction property (respectively the complete reduction property) for* $\mathscr{A}$.

Proof. Let $\mathscr{M}$ a closed invariant subspace of $\pi(\mathscr{A})$ included in $\mathscr{V}$. Since $\mathscr{H}$ has the reduction property, there is a closed invariant subspace $\mathscr{N}$ of $\pi(\mathscr{A})$ such that $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$. It is clair that $\mathscr{V} \bigcap \mathscr{N}$ is a closed invariant subspace of $\pi(\mathscr{A})$ included in $\mathscr{V}$ with $\mathscr{V} = \mathscr{M} \oplus \mathscr{V} \bigcap \mathscr{N}$.

LEMMA 3.3. *Let* $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ *be a weak* * *closed reduction operator algebra. Then* $\mathscr{A}$ *is weak* * *closed in* $\mathscr{B}(\overline{\mathscr{A}\mathscr{K}})$.

Proof. Consider the restriction weak * continuous contractive operator σ defined by
$$\sigma(T) = T/\overline{\mathscr{A}\mathscr{K}}, \, T \in \mathscr{B}(\mathscr{K}).$$
Since $\mathscr{A}$ is a reduction algebra, there is an $\mathscr{A}$-invariant subspace $\mathscr{V}$ such that $\mathscr{K} = \overline{\mathscr{A}\mathscr{K}} \oplus \mathscr{V}$. Then $\mathscr{A}\mathscr{V} = 0$ and so, σ is an isomorphism from $\mathscr{A}$ onto $\sigma(\mathscr{A}) = \mathscr{A}/\overline{\mathscr{A}\mathscr{K}}$. Assume that a_α is a net in $\mathscr{A}/\overline{\mathscr{A}\mathscr{K}}$ converging weakly * to $a \in \mathscr{B}(\overline{\mathscr{A}\mathscr{K}})$. It is easy to check that $a_\alpha \oplus 0 \longrightarrow a \oplus 0$ in the weak * topology in $\mathscr{B}(\mathscr{K})$. This shows the result.

For any two-sided ideal $\mathscr{J}$ of a total reduction (respectively amenable) operator algebra $\mathscr{A}$, the quotient $\mathscr{A}/\mathscr{J}$ is a total reduction (respectively. amenable) operator algebra. From Corollary 3.2.8 and Proposition 3.3.3 in [8], we deduce the following results:

PROPOSITION 3.4. *Let* $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ *be a unital total reduction (or amenable) operator algebra and let* $\mathscr{J}$ *be a closed two sided ideal of* $\mathscr{A}$. *Then* $\mathscr{J}$ *has a bounded approximation identity* (*b.a.i*). *Moreover, it is associated to* $\mathscr{J}$ *a central projection (central self-adjoint idempotent)* $p \in \mathscr{A}^{**}$ *such that* $\mathscr{A}^{**} = p\mathscr{A}^{**} \oplus (1 - p)\mathscr{A}^{**}$ *and* $p\mathscr{A}^{**} = J^{**}$.

Proof. First, consider $\pi : \mathscr{B}(\mathscr{K}) \longrightarrow \mathscr{B}(\mathscr{H})$, the GNS representation of $\mathscr{B}(\mathscr{K})$. Then $\mathscr{A}^{**}$ and $\mathscr{J}^{**}$ are isometrically isomorphic to $\overline{\pi(\mathscr{A})}^{w^*}$ and $\overline{\pi(\mathscr{J})}^{w^*}$, respectively. We have to show that $\overline{\pi(\mathscr{J})}^{w^*}$ has an identity which proves also that $\mathscr{J}^{**}$ has also an identity. Therefore, $\mathscr{J}$ has a bounded approximate identity.

Using Corollary 3.2, the strong closure of $\overline{\pi(\mathscr{A})}$ is the bicommutant $\pi(\mathscr{A})''$. Since the weak * topology is weaker than the strong topology, we have $\pi(\mathscr{A})'' = \overline{\pi(\mathscr{A})^{w^*}}$. Therefore, $\overline{\mathscr{J}^{w^*}}$ is a weak * a closed two-sided ideal of $\pi(\mathscr{A})''$. In view of Proposition 3.2 and Proposition 3.3, $\pi(\mathscr{A})''$ is a complete reduction algebra. Now, denote by $\mathscr{I}$ the tow-sided ideal $\overline{\pi(\mathscr{J})^{w^*}}$. From (1) in Proposition 2.2, $\mathscr{I}$ acts non degenerate on $\overline{\mathscr{I}\mathscr{H}}$. Thus, the property (3) of Proposition 2.2, infers that the weak * closure of $\mathscr{I}$ in $\mathscr{B}(\overline{\mathscr{I}\mathscr{H}})$ is the bicommutant $\mathscr{I}''$. Therefore, $\overline{\pi(\mathscr{J})^{w^*}}$ has an bounded

approximate identity as a subalgebra of $\mathscr{B}(\overline{\mathscr{I}\mathscr{H}})$. Next, by Lemma 3.3, we conclude that $\mathscr{I}$ is weak* closed in $\mathscr{B}(\overline{\mathscr{I}\mathscr{H}})$. Whence the restriction $\mathscr{I}$ to $\overline{\mathscr{I}\mathscr{H}}$ has an identity $e = 1_{\overline{\pi(\mathscr{I})\mathscr{H}}}$. On the other hand, there is a submodule V of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{V} \oplus \overline{\mathscr{I}\mathscr{H}}$. Notice that $e\mathscr{H} \subseteq \overline{\mathscr{I}\mathscr{H}}$ and $e\mathscr{V} = 0$. Therefore e is a projection onto the $\mathscr{A}''$-invariant $\overline{\mathscr{I}\mathscr{H}}$ along an $\mathscr{A}''$-invariant kernel $\mathscr{V}$ and so, e belongs to the commutant of $\mathscr{A}''$. This yields that $e \in \mathscr{A}'' \cap \mathscr{A}'$ and so, e is a central projection. This proves the first part.

Now, since $\mathscr{J}$ has an approximate identity $(p_\alpha)_{\alpha \in \Delta}$ and $\mathscr{A}$ is Arens regular, then $\mathscr{J}$ is Arens regular and therefore $\mathscr{J}^{**}$ is a weak*-closed two sided ideal of $\mathscr{A}^{**}$. Passing to a subset, we may assume that p_α converges in the weak* topology to an element $p \in \mathscr{J}^{**}$ with $\|p\| \leqslant 1$. It is clear that $jp = j$ and $pj = j$ for any $j \in \mathscr{J}$. Let now $j \in J^{**}$. Then there is a net $j_\beta \in \mathscr{J}$ which converges to j. Since $\mathscr{J}$ is Arens regular, i.e., the left and the right Arens product coincide, we have $jp = pj = j$ and so, p is a central projection in $\mathscr{A}^{**}$. Finally, p is self-adjoint projection for $\|p\| \leqslant 1$. This completes the second part.

COROLLARY 3.4. *If $\mathscr{A}$ is a total reduction (respectively amenable) operator algebra, then $\mathscr{J}$ is a total reduction (respectively amenable) operator algebra.*

Proof. $\mathscr{J}$ has a bounded appriximate identity according to Proposition 3.4 and also Proposition 3.1 in the amenability case. Then if $\mathscr{A}$ is amenable then from Theorem 2.1 Then if $\mathscr{A}$ is amenable then from Theorem 2.1, $\mathscr{J}$ is amenable. Now assume that $\mathscr{A}$ is a total reduction algebra, we let $\pi : \mathscr{J} \longrightarrow \mathscr{B}(\mathscr{H})$ be a representation. π can be extended to $\tilde{\pi} : \mathscr{J}^{**} \longrightarrow \mathscr{B}(\mathscr{H})$. Thus $\mathscr{J}^{**}$ has a identity j. This induces a representation $\bar{\pi} : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ defined by $\overline{\pi}(a) = \tilde{\pi}(aj)$. Let now $\mathscr{M}$ be a closed invariant subspace of $\pi(\mathscr{J})$. Then $\mathscr{M}$ is a closed invariant subspace of $\tilde{\pi}(\mathscr{A}j)$ and also of $\overline{\pi}(\mathscr{A})$. Since $\mathscr{A}$ has the total reduction property, there exists a closed invariant subspace subspace $\mathscr{N}$ of $\overline{\pi}(\mathscr{A})$ such that $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$. Using the fact that π is the restriction of $\bar{\pi}$ on $\mathscr{J}$, we see that $\mathscr{N}$ is a closed invariant subspace of $\overline{\pi(\mathscr{J})}$, and $\mathscr{J}$ is then a total reduction algebra.

It is natural to ask if the condition that the quotient $\mathscr{A}/\mathscr{J}$ is a total reduction (resp. amenable) operator algebra, where $\mathscr{J}$ is assumed to be a two sided ideal which is a total reduction (resp. amenable) algebra, implies that $\mathscr{A}$ is total reduction (resp. amenable) operator algebra. In fact, this problem is in fact still open for the amenability's case. The following is the positive answer for the case of total reduction algebras.

THEOREM 3.1. *Let $\mathscr{A}$ be a total reduction operator algebra and $\mathscr{J}$ a closed two sided ideal, which is a total reduction operator algebra, such that the quotient $\mathscr{A}/\mathscr{J}$ be total reduction operator algebra. Then $\mathscr{A}$ is a total reduction operator algebra.*

Proof. Let p be the central projection associated to $\mathscr{J}$ in $\mathscr{J}^{**}$. Consider $\pi : \mathscr{J} \longrightarrow \mathscr{B}(\mathscr{H})$ and let $\mathscr{V}$ be an invariant subspace of $\pi(\mathscr{A})$. Then $\mathscr{V}$ is also an invariant subspace of $\tilde{\pi}(\mathscr{A}^{**})$, where $\tilde{\pi} : \mathscr{A}^{**} \longrightarrow \mathscr{B}(\mathscr{H})$ is the weak* continuous extension of π to $\mathscr{A}^{**}$. Put $q = \pi(p)$ and note that it is easy to check that $q\mathscr{V}$ is an

invariant subspace of $\tilde{\pi}(\mathscr{A})$, and so it is an invariant subspace of $\pi(\mathscr{J})$. By Lemma 3.3, $\pi(\mathscr{J})$ has the total reduction property on $q\mathscr{H}$. Then, there exists an invariant subspace $\mathscr{W}_q$ of $\tilde{\pi}(\mathscr{J})$ such that $q\mathscr{H} = q\mathscr{V} \oplus \mathscr{W}_q$. On the other hand, we have $\mathscr{A}/\mathscr{J}^{**} \cong \mathscr{A}^{**}(1-p)$ and $(1-q)\mathscr{V}$ is a submodule of Hilbert $\mathscr{A}^{**}(1-p)$-module and so it is a submodule of the Hilbert $\mathscr{A}/\mathscr{J}$-module $(1-q)\mathscr{H}$. Thus there is a submodule Hilbert $\mathscr{W}_{(1-q)}$ of $(1-q)\mathscr{H}$ such that $(1-q)\mathscr{H} = (1-q)\mathscr{V} \oplus \mathscr{W}_{(1-q)}$. Note here that $\mathscr{V} = q\mathscr{V} \oplus (1-q)\mathscr{V}$ and $\mathscr{H} = q\mathscr{H} \oplus (1-q)\mathscr{H}$. Furthermore, it is obvious that $\mathscr{V} \cap \mathscr{W}_q \cap \mathscr{W}_{(1-q)} = 0$ and $\mathscr{H} = \mathscr{V} \oplus \mathscr{W}_q \oplus \mathscr{W}_{(1-q)}$. It remains to show that $\mathscr{W}_q \oplus \mathscr{W}_{(1-q)}$ is a submodule of the Hilbert $\mathscr{A}$-module $\mathscr{H}$. Indeed, we have $\tilde{\pi}(\mathscr{A}^{**}p)\mathscr{W}_{(1-q)} = 0$ and $\tilde{\pi}(\mathscr{A}^{**})(1-p))\mathscr{W}_q = 0$. This gives rise to the following

$$\tilde{\pi}(\mathscr{A}^{**})(\mathscr{W}_q \oplus \mathscr{W}_{(1-q)}) \subseteq \tilde{\pi}(\mathscr{A}^{**}p) \oplus \left(\tilde{\pi}(\mathscr{A}^{**})(1-p)\right)\left(\mathscr{W}_q \oplus \mathscr{W}_{(1-q)}\right) \subseteq \mathscr{W}_q \oplus \mathscr{W}_{(1-q)}.$$

This completes the proof.

COROLLARY 3.5. *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{J}$ any closed two-sided ideal of $\mathscr{A}$. Then $\mathscr{A}$ satisfies (GSP) if and only if $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ satisfies (GSP).*

In the sequel, we make use of the following notations. We define $\mathscr{A}^{(n)}$, for given $\mathscr{A} \in \mathscr{B}(\mathscr{K})$ and positive integer n, to be the direct sum of n copies of $\mathscr{A}$ acting on $\mathscr{K}^{(n)}$, the direct sum of n copies of $\mathscr{K}$. If $\mathscr{A}$ is a collection of operators, we denote by $\mathrm{Lat}\,\mathscr{A}$ the set of closed invariant subspaces of $\mathscr{A}$.

PROPOSITION 3.5. *Let $\mathscr{A}$ be a reduction operator algebra. Then $\mathscr{A}^{(n)}$ is a reduction operator algebra for every positive integer n.*

Proof. Let $\mathscr{M} \in \mathrm{Lat}\,\mathscr{A}^{(n)}$ and consider $\mathscr{M}_k$, the set of all elements $(x_1, x_2, ..., x_n)$ of $\mathscr{M}$ for which $x_i = 0$ for $i \geqslant k$. Then, we assert that $\mathrm{P}_k(\mathscr{M}_k) \in \mathrm{Lat}\,\mathscr{A}$, where P_k is the orthogonal projection of $\mathscr{K}^{(n)}$ on $\mathscr{K}^{(k)}$. Since $\mathscr{A}^{(k)}$ is a reduction algebra, there is $\mathscr{N}_k$ such that $\mathscr{K} = \mathrm{P}_k(\mathscr{M}_k) \oplus \mathscr{N}_k$. It is easy to check that $\mathscr{M} = \sum_{k=1}^{n} \oplus \mathrm{P}_k(\mathscr{M}_k)$ and $\mathscr{N} = \sum_{k=1}^{n} \oplus \mathscr{N}_k$ belong to $\mathrm{Lat}\,\mathscr{A}^{(n)}$. This proves that $\mathscr{A}^{(n)}$ is a reduction operator algebra for every positive integer n.

It is conjectured by Galé, Ransford and White in [7] that a reflexive amenable Banach algebra is finite dimensional and semi-simple. The following results answer partially to this conjecture in the case of operator algebras.

THEOREM 3.2. *Let $\mathscr{A}$ be a total reduction operator algebra whose underling Banach space is reflexive. Then it is a semi-simple operator algebra.*

Proof. We begin by noting that $\mathscr{A}$ is Arens regular and that the Jacobson Radical $\mathrm{Rad}(\mathscr{A})$ of $\mathscr{A}$ is a closed ideal of $\mathscr{A}$ which is also Arens regular. Assume that $\mathrm{Rad}(\mathscr{A}) \neq \{0\}$. By Proposition 3.4, $\mathrm{Rad}(\mathscr{A})$ has a b.a.i and so, the bidual $\mathrm{Rad}(\mathscr{A})^{**}$ has an identity. But $\mathrm{Rad}(\mathscr{A})$ is reflexive as a Banach space. This is a contradiction.

COROLLARY 3.6. *A reflexive amenable operator algebra $\mathscr{A}$ is semi-simple.*

COROLLARY 3.7. *An operator algebra $\mathscr{A}$ with the total reduction property, which is a Hilbert space is finite dimensional and semi-simple.*

Proof. Notice that $\mathscr{A}$ is semi-simple by the previous theorem. Now, for any maximal left ideal $\mathscr{M}$ of $\mathscr{A}$, the exact short sequence

$$0 \longrightarrow \mathscr{M}^* \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}/\mathscr{M} \longrightarrow 0$$

of Hilbert $\mathscr{A}$-modules is split according to Lemma 3.1. Then there is a left ideal $\mathscr{N}$ such that $\mathscr{A} = \mathscr{M} \oplus \mathscr{N}$. Finally, the result follows using similar argument as in the proof of Theorem 3.1 in [6].

THEOREM 3.3. *Let $\mathscr{A}$ be a reflexive abelian operator algebra. Then the following are equivalent:*

1. *$\mathscr{A}$ is a total reduction operator algebra.*

2. *$\mathscr{A}$ is finite dimensional and semi-simple.*

3. *$\mathscr{A}$ is amenable.*

Proof. We only need to show that $1 \Rightarrow 2$. Let $\mathscr{M}$ be a maximal two-sided ideal of $\mathscr{A}$. Since $\mathscr{M}$ is reflexive with a b.a.i, it has identity. Then there is a central self-adjoint projection p such that $\mathscr{A} = p\mathscr{A} \oplus (1-p)\mathscr{A}$ with $\mathscr{M} = p\mathscr{A}$. Let χ be a character on $\mathscr{A}$ and let p be the central self-adjoint projection associated to $\ker \chi$. If χ' is an other character such that $\chi'(p) = \chi'(p) = 0$, then we have $p\mathscr{A} \subseteq \ker(\chi') \bigcap \ker(\chi)$ and so, $\chi' = \chi$. This shows that for each character χ, $\{\chi\}$ is weak-closed set in the dual $\mathscr{A}^*$. Now, for $\chi' \neq \chi$ and $\chi(p) = 0$, we have $\chi'(p) \cong 0$. Also, we have $\chi'(p) = 1$ for $p^2 = p$. This proves that $\{\chi\}$ is a weak open set in $\mathscr{A}^*$. Consequently, the set X of all characters on $\mathscr{A}$ is discrete. Since $\mathscr{A}$ is reflexive with an approximate identity, it is unital and hence X is a compact set and so finite. The algebra $\mathscr{A}$ is then the finite direct sum of one dimensional two-side ideals $(1 - p_k)\mathscr{A}$ with p_k is the central self-adjoint projection of $\ker \chi_k$, where $\chi_k \in \mathscr{X}$.

COROLLARY 3.8. *A finite dimensional operator algebra with the total reduction property is isomorphic to a finite direct sum of full matrix algebras.*

THEOREM 3.4. *Let $\mathscr{A}$ be a total reduction operator algebra and assume that $\mathscr{A}$ is generated by its normal elements. Then $\mathscr{A}$ is self-adjoint.*

Proof. Note that $\mathscr{A}$ satisfies the conditions of Proposition in [4], and therefore demonstration of Theorem in the same paper gives the result.

We now focus on the abelian case. An abelian C^*-algebra is amenable and thus it is a total reduction operator algebra. For general operator algebras case, the amenability implies the total reduction property according to Proposition 3.1. It is natural to ask if the total reduction property implies the amenability for abelian operator algebras. From Theorem 3.4, the following result is a partial answer to this question.

PROPOSITION 3.6. *An abelian operator algebra with the total reduction property, which is isomorphic to a closed subalgebra of a C^*-algebra is similar to a C^*-algebra and so amenable.*

COROLLARY 3.9. [9] *An operator algebra with the total reduction property containing in an abelian C^*-algebra is self-adjoint.*

Let now $\mathscr{A}$ and $\mathscr{B}$ be two operator algebras with $\mathscr{B}$ semisimple and $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ an isomorphism map onto $\mathscr{B}$. If ϕ is bounded then open mapping theorem implies that the inverse map ϕ^{-1} is bounded. But ϕ^{-1} is not necessarily c.b even if ϕ is c.b. Indeed, consider the disc algebra $A(D)$ which can be described as the completion of the set of all polynomials for the norm

$$\|P\|_\infty = \sup\{|P(z)| \ / \ z \in D\}$$

or also as an operator algebra of the C^*-algebra $C(\partial D)$. Let $u : A(D) \longrightarrow \mathscr{B}(\mathscr{H})$ the homomorphism given by G. Pisier in [18] which is bounded but not c.b. (The disc algebra does not satisfy then (GSP)). Consider also $w : A \to \mathscr{B}(\mathscr{H}) \oplus C(\partial D)$ defined by $w(a) = u(a) \oplus a$. Then $w(A(D))$ is closed in $\mathscr{B}(\mathscr{H}) \oplus C(\partial D)$ and w is an isomorphism into $w(A(D))$. Furthermore, one can check that the inverse homomorphism $w^{-1} : w(A(D)) \longrightarrow A(D)$ is c.b. but w is not c.b.

Now assume that $\mathscr{B}$ is a C^*-algebra. Then ϕ is bounded. Moreover, ϕ is completely continuous by Pitts' theorem [21]. In addition, the inverse homomorphism ϕ^{-1} is bounded. It is c.b. if and only if $\mathscr{A}$ is similar to a C^*-algebra.

THEOREM 3.5. *An operator algebra $\mathscr{A} \in \mathscr{B}(\mathscr{K})$ with the total reduction property, which is isomorphic to a C^*-algebra $\mathscr{B}$ is similar to a C^*-algebra.*

Proof. Using Proposition 3.2, it follows that $\mathscr{B}$ is a total reduction operator algebra. Therefore, it satisfies (SP). Now, let $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ be a given isomorphism. By Corollary 3.6 of [9] , ϕ^{-1} is similar to *-homomorphism, i.e, there exists $S \in \mathscr{B}(\mathscr{K})$ and a *-representation $\rho : \mathscr{B} \longrightarrow \mathscr{B}(\mathscr{K})$ such that $\phi^{-1}(a) = S^{-1}\rho(a)S$ for $a \in \mathscr{B}$. This yields that $\rho(\mathscr{B})$ is a C^*-algebra and $\mathscr{A} = S^{-1}\rho(\mathscr{B})S$.

By using Paulsen's theorem in [17], (GSP) implies (SP) for general operator algebras and both of them is equivalent to the total reduction property for a C^*-algebra. It is natural to ask it is the same case for nonself-adjoint operator algebras.

THEOREM 3.6. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra isomorphic to a C^*-algebra $\mathscr{B}$ and let $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ the isomorphism. We assert the following:*

1. If $\mathscr{A}$ is a total reduction algebra, then $\mathscr{A}$ satisfies (GSP) and so (SP).

2. If ϕ^{-1} is c.b. and $\mathscr{A}$ satisfies (GSP), then $\mathscr{A}$ is a total reduction algebra.

3. If ϕ^{-1} is c.b. and if $\mathscr{J}$ is a closed two-sided ideal of $\mathscr{A}$, then $\mathscr{A}$ satisfies (GSP) if and only if $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ satisfy (GSP).

Proof. 1. Let $\pi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ be a bounded representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. Since $\mathscr{A}$ is isomorphic to C*-algebra $\mathscr{B}$ which is a total reduction algebra. Thus, C*-algebra $\mathscr{B}$ satisfies (GSP). Hence $\pi \circ \phi^{-1}$ is c.b. Now making use of Theorem 2.3 in [21], it follows that ϕ is c.b. and and thus π is c.b.

2. According to the fact that $\mathscr{A}$ is isomorphic to C*-algebra $\mathscr{B}$, it is enough to show that $\mathscr{B}$ is a total reduction algebra. Let $\pi : \mathscr{B} \longrightarrow \mathscr{B}(\mathscr{H})$ be a bounded representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. Since $\mathscr{A}$ satisfies (GSP), $\pi \circ \phi$ is c.b.. Hence π is c.b. so that $\mathscr{B}$ is C*-algebra satisfying (GSP). It is then a total reduction algebra.

3. Using property 2 above, $\mathscr{A}$ is a total reduction algebra. By Proposition 3.4 and Corollary 3.4, $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ are total reduction algebras. Notice that $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ are respectively isomorphic to C*-algebras $\phi(\mathscr{J})$ and $\mathscr{B}/\phi(\mathscr{J})$. Next, the first property shows that $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ satisfy (GSP). To prove the converse, we use the property 2 and Theorem 3.1.

An operator algebra $\mathscr{A}$ satisfies (CC) if each contractive representation $\phi : \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$ is completely contractive, i.e., it is c.b. and $\|\phi\|_{cb} \leqslant 1..$ Every C*-algebra satisfies (CC) and it is clear that for an operator algebra $\mathscr{A}$ which verifies (CC), the algebra $\mathscr{A}$ satisfies (SP) if and only if satisfies (GSP). For example, the disc algebra $A(D)$ satisfies (CC) but not the (GSP), then it does not satisfies (SP). It is also not a total reduction algebra.

4. Complete reduction operator algebras

Here, we deal with complete reduction algebras. We discuss some particular cases of complete reduction algebras which are similar to self-adjoint operator algebras.

PROPOSITION 4.1. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra containing the identity. Then $\mathscr{A}$ is a complete reduction operator algebra if and only if $\mathscr{A}''$ is a complete reduction operator algebra.*

Proof. According to Corollary 3.1, $\mathscr{A}^{(\infty)}$ has the reduction property for $\mathscr{K}^{(\infty)}$ if and only if $\mathscr{A}^{(\infty)''}$ has the reduction property for $\mathscr{K}^{(\infty)}$. Hence $\mathscr{A}^{(\infty)''} = (\mathscr{A}'')^{(\infty)}$, $\mathscr{A}$ is a complete reduction algebra if and only if $\mathscr{A}''$ is a complete reduction algebra.

COROLLARY 4.1. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra containing the identity. $\mathscr{A}$ has the complete reduction property if and only if the bidual $\mathscr{A}^{**}$ has the complete reduction property.*

THEOREM 4.1. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ be an operator algebra. $\mathscr{A}$ is a complete reduction operator algebra if and only if $\mathscr{A}^{(n)}$ is a complete reduction operator algebra for every positive integer n.*

Proof. By Proposition 3.2, $(\mathscr{A}^{(\infty)})^n$ has the reduction property on $(\mathscr{K}^{(\infty)})^n$. Then, it is easy to check that $(\mathscr{A}^{(\infty)})^n \cong (\mathscr{A}^{(n)})^{(\infty)}$ and $(\mathscr{K}^{(\infty)})^n \cong (\mathscr{K}^n)^{(\infty)}$. Whence, it follows that $(\mathscr{A}^{(n)})^{(\infty)}$ has the reduction property on $(\mathscr{K}^n)^{(\infty)}$.

An operator algebra is said to be reductive if it is weakly closed, contains the identity operator, and has the property that every closed subspace invariant under (all operators in) $\mathscr{A}$ reduces (all the operators in) $\mathscr{A}$. Von Neumann algebras are obviously reductive. The reductive algebra problem lifted by M. Radjavi and P. Rosenthal in [16] is the following question: *Is every reductive algebra a von Neumann algebra?* The conjecture 2 can be considered as a non-self-adjoint analogue of the reduction algebra problem.

THEOREM 4.2. *Let* $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ *be a weakly closed complete reduction operator algebra. Assume that each idempotent operator in* $\mathscr{A}'$ *is in* $\mathscr{A}''$. *Then* $\mathscr{A}$ *is similar to a* C^*-*algebra.*

Proof. For each positive integer n, let P_n denote the set of idempotent operators in $(\mathscr{A}^{(n)})'$. Since $(\mathscr{A}')^{(n)} = (\mathscr{A}^{(n)})'$, then P_n is exactly $\mathrm{P}^{(n)}$. Using Lemma 4.4 of [9], there exists a similarity S of $\mathscr{K}$ which makes all elements of P self-adjoints, i.e., $S^{-1}pS$ is self-adjoint projection for any $p \in \mathrm{P}$. Set $\mathscr{U} = S^{-1}\mathscr{A}S$ and note that $\mathscr{U}^{(n)} = S^{(n)^{-1}}\mathscr{A}^{(n)}S^{(n)}$. Therefore $S^{(n)^{-1}}pS^{(n)}$ is a self-adjoint projection for every fixed $p \in \mathrm{P}_n$. Now, since the similarity preserves the idempotent operators module projections, each $\mathscr{U}_n$ is a reduction operator algebra. Indeed, note that $S^{(n)}\mathscr{M} \in \mathrm{Lat}\,\mathscr{A}^{(n)}$ for $\mathscr{M}$ being in $\mathrm{Lat}\,\mathscr{U}_n$. Hence, there is $p^{(n)} \in \mathrm{P}_n$ such that $p^{(n)}\mathscr{K}^{(n)} = S^{(n)}\mathscr{M}$. Therefore $S^{(n)^{-1}}pS^{(n)}\mathscr{K}^{(n)} = \mathscr{M}$ and thus $\mathscr{M}$ reduces $\mathscr{U}_n$. It follows from principal result in [16] that $\mathscr{U}$ is self-adjoint.

Let mention here that the assumed condition of Theorem 4.2 occurs if $\mathscr{A}'$ is abelian. The latest one is satisfied only in some particular situations, for example when $\mathscr{A}$ contains a unilateral shift or a maximal abelian subalgebra of $\mathscr{B}(\mathscr{K})$.

COROLLARY 4.2. *A complete reduction maximal abelian operator algebra is similar to a maximal abelian (von-Neumann) self-adjoint algebra.*

THEOREM 4.3. *Let* $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ *be a complete reduction operator algebra and let* A *the weak-closed algebra generated by* $\mathscr{A}'$ *and* $\mathscr{A}''$. *Then* A *is similar to a* C^*-*algebra.*

Proof. In view of Proposition 4.1, $\mathscr{A}''$ is a weak closed complete reduction operator algebra. Also, let note that A' is clearly contained in $\mathscr{A}' \bigcap \mathscr{A}''$. Now, making use of Lemma 4.4 in [9], there exists a similarity S on $\mathscr{K}$ which makes all idempotent operator module projection p of in A self-adjoint. Using the technical argument in the proof of Theorem 4.2, we check that $S^{-1}\mathrm{A}S$ is C^*-algebra.

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