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TANNAKA–KREIN DUALITY FOR COMPACT GROUPOIDS II, DUALITY

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Abstract. We show that from the representation theory of a locally non-trivial compact groupoid whose irreducible representations separate its points, one can reconstruct the groupoid using a procedure similar to the Tannaka-Krein duality for compact groups. We study the Fourier and Fourier-Plancherel transforms and prove the Plancherel theorem for compact groupoids.

1. introduction

In this paper, we have generalized the Tannaka-Krein duality to compact groupoids. In [1] we studied the representation theory of compact groupoids. In particular, we showed that irreducible representations have finite dimensional fibres. We also proved the Schur's lemma, Gelfand-Raikov theorem and Peter-Weyl theorem for compact groupoids. Here we first study the Fourier and Fourier-Plancherel transforms on compact groupoids. In section two we develop the theory of Fourier transforms on sections of the Banach algebra bundle $L^1(\mathscr{G})$ of a compact groupoid $\mathscr{G}$. As in the group case, a parallel theory of Fourier-Plancherel transform on the Hilbert space bundle $L^2(\mathscr{G})$ is constructed. Section three considers the inverse Fourier and Fourier-Plancherel transforms. In this section we prove Plancherel theorem for compact groupoids. Section four introduces the Tannaka groupoid and finally section five is devoted to the duality theorem.

The theory of (topological) groupoids has a long history. We refer the interested reader to [11] for historical remarks and references. The first one who treated categories endowed with topological (or differential) structure was C. Ehresmann [4]. The next important work in this direction was the pioneering work of K. Seda on invariant measures on topological groupoids [10]. Recent monographs by R. Brown [3], K. C. H. Mackenzie [6], A. L. T. Paterson [7], and J. Renault [8] have considerably promoted the area.

Here we follow [8] in our treatment of topological groupoids, with a minor difference that we only work with continuous representations [1]. As for the Tannaka-Krein duality, we follow the classical treatment of duality for compact groups as presented in [5].

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All over this paper, $X = \mathscr{G}^{(0)}$. We assume that $\mathscr{G}$ is compact and the Haar system on $\mathscr{G}$ is normalized so that $\lambda_u(\mathscr{G}_u) = 1$, for each $u \in X$. In general $\mathscr{G}$ may be non-Hausdorff, but as usual we assume that $\mathscr{G}^{(0)}$ and $\mathscr{G}^u$, $\mathscr{G}_v$, for each $u, v \in \mathscr{G}^{(0)}$ are Hausdorff.

2. Fourier transform

Let $\mathscr{G}$ be a groupoid [8, 1.1]. The unit space of $\mathscr{G}$ and the range and source maps are denoted by $X = \mathscr{G}^{(0)}$, r and s, respectively. For $u, v \in \mathscr{G}^{(0)}$, we put $\mathscr{G}^u = r^{-1}\{u\}$, $\mathscr{G}_v = s^{-1}\{v\}$, and $\mathscr{G}^u_v = \mathscr{G}^u \cap \mathscr{G}_v$. Also we put $\mathscr{G}^{(2)} = \{(x,y) \in \mathscr{G} \times \mathscr{G} : r(y) = s(x)\}$.

We say that $\mathscr{G}$ is a topological groupoid [8, 2.1] if the inverse map $x \mapsto x^{-1}$ on $\mathscr{G}$ and the multiplication map $(x,y) \mapsto xy$ from $\mathscr{G}^{(2)}$ to $\mathscr{G}$ are continuous. This implies that the range and source maps r and s are continuous and the subsets $\mathscr{G}^u$, $\mathscr{G}_v$, and $\mathscr{G}^u_v$ are closed, and so compact, for each $u, v \in X$. We fix a left Haar system $\lambda = \{\lambda^u\}_{u \in X}$ and put $\lambda_u(E) = \lambda^u(E^{-1})$, for Borel sets $E \subseteq \mathscr{G}_u$, and let λ^v_u be the restriction of λ_u to the Borel σ-algebra of $\mathscr{G}^v_u$. The integrals against λ^u and λ^v_u are understood to be on $\mathscr{G}^u$ and $\mathscr{G}^v_u$, respectively. The functions on $\mathscr{G}^v_u$ are extended by zero, if considered as functions on $\mathscr{G}$. All over this paper, we assume that $\lambda_u(\mathscr{G}^v_u) \neq 0$, for each $u, v \in X$. This holds in transitive groupoids (see the discussion after Theorem 5.6). In this case, we say that $\mathscr{G}$ is *locally non-trivial*.

The convolution product of two measurable functions f and g on $\mathscr{G}$ is defined by

$$f * g(x) = \int f(y) g(y^{-1}x) d\lambda^{r(x)}(y) = \int f(xy^{-1}) g(y) d\lambda_{s(x)}(y).$$

A (continuous) *representation* of G is a double $(\pi, \mathscr{H}_\pi)$, where $\mathscr{H}_\pi = \{\mathscr{H}^\pi_u\}_{u \in X}$ is a continuous bundle of Hilbert spaces over X such that:

(i) $\pi(x) : \mathscr{B}(\mathscr{H}^\pi_{s(x)}, \mathscr{H}^\pi_{r(x)})$ is a unitary operator, for each $x \in \mathscr{G}$,

(ii) $\pi(u) = id_u : \mathscr{H}^\pi_u \to \mathscr{H}^\pi_u$, for each $u \in X$,

(iii) $\pi(xy) = \pi(x)\pi(y)$, for each $(x,y) \in \mathscr{G}^{(2)}$,

(iv) $\pi(x^{-1}) = \pi(x)^{-1}$, for each $x \in \mathscr{G}$,

(v) $x \mapsto \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle$ is continuous on $\mathscr{G}$, for each $\xi, \eta \in C_0(G^{(0)}, \mathscr{H}_\pi)$.

Two representations π_1, π_2 of $\mathscr{G}$ are called (unitarily) *equivalent* if there is a (continuous) bundle $U = \{U_u\}_{u \in X}$ of unitary operators $U_u \in \mathscr{B}(\mathscr{H}^\pi_u, \mathscr{H}^\pi_u)$ such that

$$U_{r(x)} \pi_1(x) = \pi_2(x) U_{s(x)} \quad (x \in \mathscr{G}).$$

We use $\mathscr{R}ep(\mathscr{G})$ to denote the category consisting of (equivalence classes of continuous) representations of $\mathscr{G}$ as objects and intertwining operators as morphisms [1, Notation 2.5]. Let $\pi \in \mathscr{R}ep(\mathscr{G})$, the mappings

$$x \mapsto \langle \pi(x)\xi_{s(x)}, \eta_{r(x)} \rangle,$$

where ξ,η are continuous sections of $\mathscr{H}_\pi$ are called *matrix elements* of π. This terminology is based on the fact that if $\{e_u^i\}$ is a basis for $\mathscr{H}_u^\pi$, then $\pi_{ij}(x)=\langle\pi(x)e^j_{s(x)},e^i_{r(x)}\rangle$ is the (i,j)-th entry of the (possibly infinite) matrix of $\pi(x)$. We denote the linear span of matrix elements of π by $\mathscr{E}_\pi$. By continuity of representations, $\mathscr{E}_\pi$ is a subspace of $C(\mathscr{G})$. It is clear that $\mathscr{E}_\pi$ depends only on the unitary equivalence class of π. For $u,v\in X$, $\mathscr{E}^\pi_{u,v}$ consists of restrictions of elements of $\mathscr{E}_\pi$ to $\mathscr{G}_u^v$. Also we put $\mathscr{E}_{u,v}=span\big(\cup_{\pi\in\hat{\mathscr{G}}}\mathscr{E}^\pi_{u,v}\big)$ and $\mathscr{E}=span\big(\cup_{\pi\in\hat{\mathscr{G}}}\mathscr{E}_\pi\big)$.

It follows from the Peter-Weyl theorem [1, Theorem 3.10] (note that there is a typo in [1, Theorem 3.10], and the orthonormal basis elements $\sqrt{d_u^\pi/\lambda_u(\mathscr{G}_u^v)}\pi^{ij}_{u,v}$ is wrongly inscribed as $\sqrt{d_u^\pi\lambda_u(\mathscr{G}_u^v)}\pi^{ij}_{u,v}$) that, for $u,v\in X$, if $\lambda_u(\mathscr{G}_u^v)\neq 0$, then for each $f\in L^2(\mathscr{G}_u^v,\lambda_u^v)$,

$$f=\sum_{\pi\in\hat{\mathscr{G}}}\sum_{i=1}^{d_v^\pi}\sum_{j=1}^{d_u^\pi}c^{ij}_{u,v,\pi}\pi^{ij}_{u,v}\ ,$$

where

$$c^{ij}_{u,v,\pi}=\frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)}\int_{\mathscr{G}_u^v}f(x)\overline{\pi^{ij}_{u,v}(x)}d\lambda_u^v(x)\quad(1\leqslant i\leqslant d_v^\pi,\ 1\leqslant j\leqslant d_u^\pi).$$

This is a local version of the classical non commutative Fourier transform. As in the classical case, the main drawback is that it depends on the choice of the basis (which in turn gives the choice of the coefficient functions). The trick is similar to the classical case, that's to use the continuous decomposition using integrals. This is the content of the next definition. As usual, all the integrals are supposed to be on the support of the measure against which they are taken.

DEFINITION 2.1. Let $u,v\in X$ and $f\in L^1(\mathscr{G}_u^v,\lambda_u^v)$, then the Fourier transform of f is $\mathfrak{F}_{u,v}(f):\mathscr{R}ep(\mathscr{G})\to\mathscr{B}(\mathscr{H}_v^\pi,\mathscr{H}_u^\pi)$ defined by

$$\mathfrak{F}_{u,v}(f)(\pi)=\int f(x)\pi(x^{-1})d\lambda_u^v(x).$$

To better understand this definition, let us go back to the group case for a moment. Let's start with a locally compact abelian group G. Then the Pontryagin dual $\hat{G}$ of G is a locally compact abelian group and for each $f\in L^1(G)$, its Fourier transform $\hat{f}\in C_0(\hat{G})$ is defined by

$$\hat{f}(\chi)=\int_G f(x)\overline{\chi(x)}dx\quad(\chi\in\hat{G}).$$

The continuity of $\hat{f}$ is immediate and the fact that it vanishes at infinity is the so called *Riemann-Lebesgue lemma*. For non abelian compact groups, a similar construction exists, namely, with an slight abuse of notation, for each $f\in L^1(G)$ one has $\hat{f}\in C_0(\hat{G},\mathscr{B}(\mathscr{H}))$, where $\hat{G}$ is the set of (unitary equivalence classes of) irreducible representations of G endowed with the Fell topology. In the groupoid case, one has a similar local interpretation. Each $f\in L^1(\mathscr{G}_u^v,\lambda_u^v)$ has its Fourier transform $\mathfrak{F}_{u,v}(f)$

in $C_0(\hat{\mathscr{G}},\mathscr{B}_{u,v}(\mathscr{H}))$, where $\hat{\mathscr{G}}$ is the set of (unitary equivalence classes of) irreducible representations of $\mathscr{G}$ endowed again with the Fell topology, and $\mathscr{B}_{u,v}(\mathscr{H})$ is a bundle of operator spaces over $\hat{\mathscr{G}}$ whose fiber at π is $\mathscr{B}(\mathscr{H}_v^\pi,\mathscr{H}_u^\pi)$, and $C_0(\hat{\mathscr{G}},\mathscr{B}_{u,v}(\mathscr{H}))$ is the set of all continuous sections vanishing at infinity.

Now let us discuss the properties of the Fourier transform. If we choose (possibly infinite) orthonormal bases for $\mathscr{H}_u^\pi$ and $\mathscr{H}_v^\pi$ and let each $\pi(x)$ be represented by the (possibly infinite) matrix with components $\pi_{u,v}^{ij}(x)$, then $\mathfrak{F}_{u,v}(f)$ is represented by the matrix with components $\mathfrak{F}_{u,v}(f)(\pi)^{ij}=\frac{\lambda_u(\mathscr{G}_u^v)}{d_u^\pi}c_{u,v,\pi}^{ji}$. When $f\in L^2(\mathscr{G}_u^v,\lambda_u^v)$, summing up over all indices i,j, we get the following.

PROPOSITION 2.2. (Fourier inversion formula) *For each $u,v\in X$ and $f\in L^2(\mathscr{G}_u^v,\lambda_u^v)$,*

$$f=\sum_{\pi\in\hat{\mathscr{G}}}\frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)}\operatorname{Tr}\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(\cdot)\big),$$

where the sum converges in the L^2 norm and

$$\|f\|_2^2=\sum_{\pi\in\hat{\mathscr{G}}}\frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)}\operatorname{Tr}\big(\mathfrak{F}_{u,v}(f)(\pi)\mathfrak{F}_{u,v}(f)(\pi)^*\big).$$

We collect the properties of the Fourier transform in the following lemma. The proof is routine and is omitted. Note that in part (iii), $f^*(x)=\overline{f(x^{-1})}$, for $x\in\mathscr{G}_u^v$ and $f\in L^1(\mathscr{G}_u^v,\lambda_u^v)$.

LEMMA 2.3. *Let $u,v,w\in X$, $a,b\in\mathbb{C}$, and $f,f_1,f_2\in L^1(\mathscr{G}_u^v,\lambda_u^v)$, $g\in L^1(\mathscr{G}_v^w,\lambda_v^w)$, then for each $\pi\in\mathscr{R}ep(\mathscr{G})$,*

(i) $\mathfrak{F}_{u,v}(af_1+bf_2)=a\mathfrak{F}_{u,v}(f_1)+b\mathfrak{F}_{u,v}(f_2)$,

(ii) $\mathfrak{F}_{u,w}(f*g)(\pi)=\mathfrak{F}_{u,v}(f)(\pi)\mathfrak{F}_{v,w}(g)(\pi)$,

(iii) $\mathfrak{F}_{v,u}(f^*)(\pi)=\mathfrak{F}_{u,v}(f)(\pi)^*$,

(iv) $\mathfrak{F}_{u,w}(\ell_x(f))(\pi)=\mathfrak{F}_{u,v}(f)(\pi)\pi(x^{-1})$ *and* $\mathfrak{F}_{w,v}(r_y(f))(\pi)=\pi(y)\ \mathfrak{F}_{u,v}(f)(\pi)$, *whenever $x\in\mathscr{G}_v^w, y\in\mathscr{G}_u^w$.*

As in the group case, there is yet another way of introducing the Fourier transform. For each finite dimensional continuous representation π of $\mathscr{G}$, let the *character* χ_π of π be the bundle of functions χ_π whose fiber χ_u^π at $u\in X$ is defined by $\chi_u^\pi(x)=\operatorname{Tr}(\pi(x))$, for $x\in\mathscr{G}_u^u$, where Tr is the trace of matrices. Note that one can not have these as functions defined on $\mathscr{G}_u^v$, since when $x\in\mathscr{G}_u^v$, $\pi(x)$ is not a square matrix in general. Also note that the values of the above character functions depend only on the unitary equivalence class of π, as similar matrices have the same trace. Now if $\pi\in\hat{\mathscr{G}}$, $x\in\mathscr{G}_u^v$, and $f\in L^1(\mathscr{G}_u^v,\lambda_u^v)$, then

$$\operatorname{Tr}\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(x)\big)=\int f(y)\operatorname{Tr}(\pi(y^{-1}x))d\lambda_u^v(y)=f*\chi_u^\pi(x),$$

where in the last equality f is understood to be extended by zero to $\mathscr{G}_u$. Hence, it follows from Proposition 2.2 that

COROLLARY 2.4. *The map $P^{\pi}_{u,v} : L^2(\mathscr{G}^v_u, \lambda^v_u) \to \mathscr{E}^{\pi}_{u,v}$, $f \mapsto d^{\pi}_u f * \chi^{\pi}_u$ is a surjective orthogonal projection and for each $f \in L^2(\mathscr{G}^v_u, \lambda^v_u)$, we have the decomposition*

$$f = \sum_{\pi \in \hat{\mathscr{G}}} d^{\pi}_u f * \chi^{\pi}_u,$$

which converges in the L^2 norm.

Applying the above decomposition to the case where $u = v$ and $f = \chi^{\pi}_u$, we get

COROLLARY 2.5. *For each $u \in X$ and $\pi, \pi' \in \hat{\mathscr{G}}$,*

$$\chi^{\pi}_u * \chi^{\pi'}_u = \begin{cases} d^{\pi\,-1}_u & \text{if } \pi \sim \pi', \\ 0 & \text{otherwise.} \end{cases}$$

3. inverse Fourier and Fourier-Plancherel transforms

Next we are aiming at the construction of the inverse Fourier transform. This is best understood if we start with yet another interpretation of the local Fourier transform. It is clear from the definition of $\mathfrak{F}_{u,v}$ that if $u, v \in X$, $\pi_1, \pi_2 \in \mathscr{R}ep(\mathscr{G})$, and $f \in L^1(\mathscr{G}^v_u, \lambda^v_u)$, then

$$\mathfrak{F}_{u,v}(f)(\pi_1 \oplus \pi_2) = \mathfrak{F}_{u,v}(f)(\pi_1) \oplus \mathfrak{F}_{u,v}(f)(\pi_2),$$

and the same is true for any number (even infinite) of continuous representations, so it follows from Theorem 2.16 in [1] that $\mathfrak{F}_{u,v}(f)$ is uniquely characterized by its values on $\hat{\mathscr{G}}$, namely we can regard

$$\mathfrak{F}_{u,v} : L^1(\mathscr{G}^v_u, \lambda^v_u) \to \prod_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u),$$

where the Cartesian product is the set of all choice functions $g : \mathscr{G} \to \bigcup_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u)$ with $g(\pi) \in \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u)$, for each $\pi \in \hat{\mathscr{G}}$. Consider the ℓ^{∞}-direct sum

$$\sum_{\pi \in \hat{\mathscr{G}}} \bigoplus \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u).$$

The domain of our inverse Fourier transform then would be the algebraic sum

$$\sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u),$$

consisting of those elements of the direct sum with only finitely many nonzero components. An element $g \in \mathscr{D}(\mathfrak{F}^{-1}_{u,v})$ is a choice function such that $g(\pi) \in \mathscr{B}(\mathscr{H}^{\pi}_v, \mathscr{H}^{\pi}_u)$, for each $\pi \in \hat{\mathscr{G}}$ is zero, except for finitely many π's.

DEFINITION 3.1. Let $u, v \in X$. The inverse Fourier transform

$$\mathfrak{F}_{u,v}^{-1} : \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi) \to C(\mathscr{G}_u^v)$$

is defined by

$$\mathfrak{F}_{u,v}^{-1}(g)(x) = \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\big(g(\pi)\pi(x)\big) \quad (x \in \mathscr{G}_u^v).$$

To show that this is indeed the inverse map of the (local) Fourier transform we need a version of the Schur's orthogonality relations [1, Theorem 3.6]. We only give a sketch of the proof.

PROPOSITION 3.2. (Orthogonality relations) *Let* $\tau, \rho \in \hat{\mathscr{G}}$, $u, v \in X$, $T \in \mathscr{B}(\mathscr{H}_\tau)$, $S \in \mathscr{B}(\mathscr{H}_\rho)$, $A \in B(\mathscr{H}_\rho, \mathscr{H}_\tau)$, *and* $\xi \in \mathscr{H}_\tau$, $\eta \in \mathscr{H}_\rho$, *then*

(i)

$$\int \tau(x^{-1}) A_{r(x)} \rho(x) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathscr{G}_u^v)}{d_u^\tau} \operatorname{Tr}(A_u) id_{\mathscr{H}_u^\tau} & \text{if } \tau = \rho, \\ 0 & \text{otherwise}, \end{cases}$$

(ii)

$$\int \tau(x^{-1}) \xi_{r(x)} \otimes \rho(x) \eta_{s(x)} d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathscr{G}_u^v)}{d_u^\tau} \eta_u \otimes \xi_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise}, \end{cases}$$

(iii)

$$\int \operatorname{Tr}(T_{s(x)} \tau(x^{-1})) \operatorname{Tr}(S_{r(x)} \rho(x)) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathscr{G}_u^v)}{d_u^\tau} \operatorname{Tr}(T_u S_u) & \text{if } \tau = \rho, \\ 0 & \text{otherwise}, \end{cases}$$

(iv)

$$\int \operatorname{Tr}(T_{s(x)} \tau(x^{-1})) \rho(x) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathscr{G}_u^v)}{d_u^\tau} T_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. (i) For a fixed $v \in X$, call the left hand side $\tilde{A}_u$. As in [1, Lemma 3.4], the following equality shows that $\tilde{A} \in Mor(\rho, \tau)$.

$$\begin{aligned} \tilde{A}_{r(x)} \rho(x) &= \int \tau(y^{-1}) A_{r(y)} \rho(y) \rho(x) d\lambda_{r(x)}^v(y) \\ &= \int \tau(y^{-1}) A_{r(y)} \rho(yx) d\lambda_{r(x)}^v(y) \\ &= \int \tau(xy^{-1}) A_{r(y)} \rho(y) d\lambda_{s(x)}^v(y) \\ &= \tau(x) \tilde{A}_{s(x)}. \end{aligned}$$

Hence by Schur's lemma [1, Theorem 2.14] it is $c.id_{\mathscr{H}_u^\tau}$, if $\tau=\rho$, and 0, otherwise. Now

$$\operatorname{Tr}\Big(\int \tau(x)A_{r(x)}\tau(x^{-1})d\lambda_u^v(x)\Big)=\operatorname{Tr}\Big(\int \tau(x)\tau(x^{-1})A_{s(x)}d\lambda_u^v(x)\Big)$$
$$=\lambda_u(\mathscr{G}_u^v)\operatorname{Tr}(A_u),$$

where as $\operatorname{Tr}\left(c.id_{\mathscr{H}_u^\tau}\right)=cd_u^\tau$, so c is what it should be.

(ii) Take any $\phi,\psi\in\mathscr{H}_\tau^*$ and apply (i) to A defined by

$$A_u(\zeta_u)=\phi_u(\zeta_u)\xi_u \quad (u\in X)$$

and then calculate both sides of the resulting operator equation at η_u to get

$$\int \tau(x^{-1})\xi_{r(x)}\phi(\rho(x)\eta_{s(x)})d\lambda_u^v(x)=\begin{cases}\frac{\lambda_u(\mathscr{G}_u^v)\phi_u(\xi_u)}{d_u^\tau}\eta_u & \text{if } \tau=\rho,\\ 0 & \text{otherwise.}\end{cases}$$

The result now follows if we apply ψ_u to both sides of the above equality and use the fact that $\phi_u(\xi_u)\psi_u(\eta_u)=(\psi\otimes\phi)_u(\eta_u\otimes\xi_u)$.

(iii) Note that all the involved Hilbert spaces are finite dimensional [1, Theorem 2.16]. In particular, rank one operators generate all operators on these spaces. Also the required relation is linear in T and S. Hence we may assume that T and S have rank one fibers, say $S=\phi(.)\xi, T=\psi(.)\eta$, where ϕ,ψ are as above. Now applying $(\phi\otimes\psi)_u$ to both sides of (ii), we get (iii).

(iv) Let L and R be the left and right hand sides of (iv), respectively. We need only to show that $\operatorname{Tr}((L-R)S)=0$, for each $S\in\mathscr{B}(\mathscr{H}_\rho)$. But $\operatorname{Tr}(LS)$ is clearly the right hand side of (iii), which is in turn equal to $\operatorname{Tr}(RS)$. □

In some applications we need to use the orthogonality relations over $\mathscr{G}_u$ (not $\mathscr{G}_u^v$). In this case, using the normalization $\lambda_u(\mathscr{G}_u)=1$, and essentially by the same argument we get the following result.

PROPOSITION 3.3. (Orthogonality relations) *Let* $\tau,\rho\in\hat{\mathscr{G}}$, $u\in X$, $T\in\mathscr{B}(\mathscr{H}_\tau)$, $S\in\mathscr{B}(\mathscr{H}_\rho)$, $A\in B(\mathscr{H}_\rho,\mathscr{H}_\tau)$, *and* $\xi\in\mathscr{H}_\tau$, $\eta\in\mathscr{H}_\rho$, *then*

(i)

$$\int \tau(x^{-1})A_{r(x)}\rho(x)d\lambda_u(x)=\begin{cases}\frac{\operatorname{Tr}(A_u)}{d_u^\tau}id_{\mathscr{H}_u^\tau} & \text{if } \tau=\rho,\\ 0 & \text{otherwise,}\end{cases}$$

(ii)

$$\int \tau(x^{-1})\xi_{r(x)}\otimes\rho(x)\eta_{s(x)}d\lambda_u(x)=\begin{cases}\frac{1}{d_u^\tau}\eta_u\otimes\xi_u & \text{if } \tau=\rho,\\ 0 & \text{otherwise,}\end{cases}$$

(iii)

$$\int \operatorname{Tr}(T_{s(x)}\tau(x^{-1}))\operatorname{Tr}(S_{r(x)}\rho(x))d\lambda_u(x)$$
$$=\begin{cases}\frac{1}{d_u^\tau}\operatorname{Tr}(T_uS_u) & \text{if } \tau=\rho,\\ 0 & \text{otherwise,}\end{cases}$$

(iv)

$$\int \operatorname{Tr}(T_{s(x)}\tau(x^{-1}))\rho(x)d\lambda_u(x) = \begin{cases} \frac{1}{d_u^\tau}T_u & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}. \end{cases}$$

Now we are ready to prove the properties of the local inverse Fourier transform. But let us first introduce the natural inner products on its domain and range. For $f, g \in C(\mathscr{G}_u^v)$ and $h, k \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$ put $\langle f, g\rangle = \int \bar{f}.g d\lambda_u^v$, and

$$\langle h, k\rangle = \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}(k(\pi)h^*(\pi)),$$

where the right hand side is a finite sum as h and k are of finite support. Also note that if $\varepsilon_u : C(\mathscr{G}_u^u) \to \mathbb{C}$ is defined by $\varepsilon_u(f) = f(u)$, then for each $f, g \in C(\mathscr{G}_u^v)$, we have $g * f^* \in C(\mathscr{G}_u^u)$ and $\langle f, g\rangle = \varepsilon_u(g * f^*)$, where $f^* \in C(\mathscr{G}_v^u)$ is defined by $f^*(x) = \overline{f(x^{-1})}$, for $x \in \mathscr{G}_v^u$. Similarly, $h^* \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$ is defined by $h^*(\pi) = \bar{h}(\check{\pi})$, where $\bar{h}(\pi) = h(\pi)^*$, $\bar{\pi}(x) = \pi(x)^*$, and $\check{\pi}(x) = \pi(x^{-1})^*$, for each $\pi \in \hat{\mathscr{G}}$ and $x \in \mathscr{G}$. The star superscript denotes the conjugation of Hilbert space operators.

PROPOSITION 3.4. *For each $u, v \in X$ and $h, k \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$, $\ell \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_w^\pi, \mathscr{H}_v^\pi)$ we have*

(i) $\mathfrak{F}_{u,v}\mathfrak{F}_{u,v}^{-1}(h) = h$,

(ii) $\lambda_u(\mathscr{G}_u^v)\mathfrak{F}_{u,w}^{-1}(hk) = \mathfrak{F}_{u,v}^{-1}(h) * \mathfrak{F}_{v,w}^{-1}(k)$,

(iii) $\mathfrak{F}_{v,u}^{-1}(h^*) = (\mathfrak{F}_{u,v}^{-1}(h))^*$,

(iv) $\langle \mathfrak{F}_{u,v}^{-1}(h), \mathfrak{F}_{u,v}^{-1}(k)\rangle = \langle h, k\rangle$.

Proof. (i) By (iv) of Proposition 3.2, for each $\tau \in \hat{\mathscr{G}}$,

$$\mathfrak{F}_{u,v}\mathfrak{F}_{u,v}^{-1}(h)(\tau) = \sum_{\pi \in \hat{\mathscr{G}}} \int \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}(h(\pi)\pi(x^{-1}))\tau(x)d\lambda_u^v(x) = h(\tau).$$

(ii) By (iii) of Proposition 3.2, for each $x \in \mathscr{G}_u^v$,

$$\begin{aligned}
(\mathfrak{F}_{u,v}^{-1}(h) * \mathfrak{F}_{u,v}^{-1}(k))(x) &= \int \mathfrak{F}_{u,v}^{-1}(h)(xy^{-1})\mathfrak{F}_{u,v}^{-1}(k)(y)d\lambda_u^v(y) \\
&= \sum_{\tau,\rho \in \hat{\mathscr{G}}} \int \frac{d_u^\tau d_u^\rho}{\lambda_u(\mathscr{G}_u^v)^2} \operatorname{Tr}\left(h(\tau)\tau(yx^{-1})\right) \operatorname{Tr}\left(k(\rho)\rho(y^{-1})\right) d\lambda_u^v(y) \\
&= \sum_{\tau,\rho \in \hat{\mathscr{G}}} \frac{d_u^\tau d_u^\rho}{\lambda_u(\mathscr{G}_u^v)^2} \int \operatorname{Tr}\left(h(\tau)\tau(x^{-1})\tau(y)\right) \operatorname{Tr}\left(k(\rho)\rho(y^{-1})\right) d\lambda_u^v(y) \\
&= \sum_{\tau \in \hat{\mathscr{G}}} \frac{d_u^\tau}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\left(h(\tau)k(\tau)\tau(x^{-1})\right) \\
&= \mathfrak{F}_{u,v}^{-1}(hk)(x).
\end{aligned}$$

(iii) For each $x \in \mathscr{G}_v^u$

$$\mathfrak{F}_{v,u}^{-1}(h^*)(x) = \sum_{\tau \in \hat{\mathscr{G}}} \frac{d_u^\tau}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\left(h^*(\tau)\tau(x^{-1})\right) = \sum_{\tau \in \hat{\mathscr{G}}} \frac{d_u^\tau}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\left(\bar{h}(\tau)\check{\tau}(x^{-1})\right)$$
$$= \sum_{\tau \in \hat{\mathscr{G}}} \frac{d_u^\tau}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\left(\bar{h}(\tau)\bar{\tau}(x)\right) = (\mathfrak{F}_{u,v}^{-1}(h))^*(x).$$

(iv) By the above observation about ε_u,

$$\begin{aligned}\langle \mathfrak{F}_{u,v}^{-1}(h), \mathfrak{F}_{u,v}^{-1}(k)\rangle &= \varepsilon_u((\mathfrak{F}_{u,v}^{-1})(k) * \mathfrak{F}_{u,v}^{-1}(h^*)) \\ &= \varepsilon_u(\mathfrak{F}_{u,v}^{-1}(kh^*)) = \mathfrak{F}_{u,v}^{-1}(kh^*)(u) \\ &= \sum_{\tau \in \hat{\mathscr{G}}} \frac{d_u^\tau}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}\left(k(\tau)h^*(\tau)\right) = \langle h, k\rangle.\end{aligned}$$

□

Next we define a norm on the domain of the inverse Fourier transform in order to get a Plancherel type theorem. Let $u,v \in X$, for $h \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$ we put $\|h\|_2 = \langle h,h\rangle^{\frac{1}{2}}$. This is the natural norm on the algebraic direct sum, when one endows each component $\mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$ with the Hilbert space structure given by $\langle T,S\rangle = \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \operatorname{Tr}(ST^*)$. We denote the completion of $\sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$ with respect to this norm by $\mathscr{L}_{u,v}^2(\mathscr{G})$.

THEOREM 3.5. (Plancherel Theorem) *For each $u,v \in X$ such that $\lambda_u(\mathscr{G}_u^v) \neq 0$, $\mathfrak{F}_{u,v}$ extends to a unitary $\mathfrak{F}_{u,v} : L^2(\mathscr{G}_u^v, \lambda_u^v) \to \mathscr{L}_{u,v}^2(\mathscr{G})$.*

Proof. By Proposition 3.2 (iii), $\mathfrak{F}_{u,v}^{-1} : \mathscr{L}_{u,v}^2(\mathscr{G}) \to L^2(\mathscr{G}_u^v, \lambda_u^v)$ is an isometric embedding. It is also surjective, since $\operatorname{Im}(\mathfrak{F}_{u,v}^{-1})$ is complete and so closed, and also it clearly includes $\mathscr{E}_{u,v}$ which is dense in $L^2(\mathscr{G}_u^v, \lambda_u^v)$. □

The above map is called the (local) *Fourier-Plancherel transform*. Before we end this section, let us show that how one can use characters of representations in $\hat{\mathscr{G}}$ and the orthogonality relations of the beginning of this section to prove statements about subsets of $\mathscr{R}ep(\mathscr{G})$.

LEMMA 3.6. *For each $u \in X$ and $\pi \in \hat{\mathscr{G}}$, $\chi_u^\pi \in \mathscr{E}_{u,u}$.*

Proof. Let $\{e_u^i\}_{1 \leqslant i \leqslant d_u^\pi}$ be a basis for $\mathscr{H}_u^\pi$, then

$$\chi_u^\pi = \operatorname{Tr}(\pi(\cdot)) = \sum_{i=1}^{d_u^\pi} \langle \pi(\cdot)e_u^\pi, e_u^\pi\rangle.$$

□

DEFINITION 3.7. A subset Σ of $\mathscr{R}ep(\mathscr{G})$ is called closed if it contains

(i) π_1 if π_1 is unitary equivalent to some $\pi_2 \in \Sigma$,

(ii) π_1 if π_1 is weakly contained in some $\pi_2 \in \Sigma$,

(iii) $\pi_1 \oplus \pi_2$ if π_1, π_2 are in Σ,

(iv) $\pi_1 \otimes \pi_2$ if π_1, π_2 are in Σ,

(v) $\bar{\pi}_1$ if π_1 is in Σ,

(vi) the trivial representation tr.

PROPOSITION 3.8. *If $\Sigma \subseteq \mathscr{R}ep(\mathscr{G})$ is closed and separates the points of $\mathscr{G}$, then $\Sigma = \mathscr{R}ep(\mathscr{G})$.*

Proof. If not, by condition (iii) of the definition of closedness and Theorem 2.16 of [1], there is $\tau \in \hat{\mathscr{G}}$ which is not in Σ. Let $\mathscr{E}^{\Sigma}_{u,v} = \oplus_{\pi\in\Sigma}\mathscr{E}^{\pi}_{u,v}$, for $u, v \in X$. By Proposition 3.2, elements of each $\mathscr{E}^{\tau}_{u,v}$ is orthogonal to $\mathscr{E}^{\Sigma}_{u,v}$. In particular, by the above lemma, $\chi^{\tau}_{u} \in (\mathscr{E}^{\Sigma}_{u,v})^{\perp}$. But by conditions (iii)-(v) of the definition of closedness, $\mathscr{E}^{\Sigma}_{u,v}$ is a subalgebra of $C(\mathscr{G})$ which is closed under conjugation, and by condition (vi), it contains the constants, and finally by assumption, it separates the points of $\mathscr{G}$. Hence, by Stone-Weierstrass Theorem, $\mathscr{E}^{\Sigma}_{u,v}$ is dense in $C(\mathscr{G})$. Therefore χ^{τ}_{u} is orthogonal to $C(\mathscr{G})$ and so it is zero, which is a contradiction. $\square$

4. Tannaka groupoid

Let $\mathscr{R}ep(\mathscr{G})$ be the category of continuous representations of $\mathscr{G}$ and sections of intertwining bundles [1, Notation 2.5] and $\mathscr{H}il_X$ be the category of Hilbert bundles over X and operator bundles. There is a forgetful functor $\mathscr{U} : \mathscr{R}ep(\mathscr{G}) \to \mathscr{H}il_X$ [1]. A *natural transformation* $a : \mathscr{U} \to \mathscr{U}$ is a family of bundle maps $a_\pi : \mathscr{H}_\pi \to \mathscr{H}_\pi$ indexed by $\mathscr{R}ep(\mathscr{G})$ such that for each $\pi_1, \pi_2 \in \mathscr{R}ep(\mathscr{G})$ and $h \in Mor(\pi_1, \pi_2)$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{H}_{\pi_1} & \xrightarrow{a_{\pi_1}} & \mathscr{H}_{\pi_1} \\ h\Big\downarrow & & \Big\downarrow h \\ \mathscr{H}_{\pi_2} & \xrightarrow[a_{\pi_2}]{} & \mathscr{H}_{\pi_2} \end{array}$$

One should understand this as each a_π being a bundle $a_\pi = \{a^{\pi}_{u,v}\}$ of bounded linear operators $a^{\pi}_{u,v} \in \mathscr{B}(H^{\pi}_{u}, \mathscr{H}^{\pi}_{v})$ (possibly zero) indexed by $X \times X$ such that for each $u, v \in X$ the following diagrams commute

$$\begin{array}{ccc} \mathscr{H}^{\pi_1}_{u} & \xrightarrow{a^{\pi_1}_{u,v}} & \mathscr{H}^{\pi_1}_{v} \\ h_u\Big\downarrow & & \Big\downarrow h_v \\ \mathscr{H}^{\pi_2}_{u} & \xrightarrow[a^{\pi_2}_{u,v}]{} & \mathscr{H}^{\pi_2}_{v} \end{array}$$

To justify the name natural transformation for a, one might compose the forgetful functor $\mathscr{U}$ with evaluation at $u \in X$ to get a functor $\mathscr{U}_u : \mathscr{R}ep(\mathscr{G}) \to \mathscr{H}il$. Then each $a_{u,v} : \mathscr{U}_u \to \mathscr{U}_v$; $\pi \mapsto a^{\pi}_{u,v}$ is a natural transformation in the classical sense.

Given $x \in \mathscr{G}$ there is a natural transformation $\mathscr{T}_x : \mathscr{U} \to \mathscr{U}$ defined by

$$(\mathscr{T}_x)^{\pi}_{u,v} = \begin{cases} \pi(x) & \text{if } u = s(x), v = r(x), \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Another interesting example of a natural transformation is the global Fourier transform. Recall that we looked at $L^1(\mathscr{G})$ as a bundle of Banach algebras over $\mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$, whose fiber at (u,v) is $L^1(\mathscr{G}^v_u, \lambda^v_u)$, and then each section f of $L^1(\mathscr{G})$ had its Fourier transform $\mathfrak{F}(f)$ in $C_0(\hat{\mathscr{G}}, \mathscr{B}(\mathscr{H}))$, where $\mathfrak{F}(f)(\pi)_{(u,v)} = \mathfrak{F}_{u,v}(f_{(u,v)})(\pi)$. Now we need a flip in the order of u, v, when we consider $\mathfrak{F}(f)$ as a natural transformation, namely we put $\mathfrak{F}(f)^{\pi}_{u,v} = \mathfrak{F}(f)(\pi)_{(v,u)}$. Then we have $\mathfrak{F}(f)^{\pi}_{u,v} \in \mathscr{B}(\mathscr{H}^{\pi}_u, \mathscr{H}^{\pi}_v)$. To see that this is indeed a natural transformation, note that for each $u, v \in X$, $x \in \mathscr{G}^u_v$, $\pi_1, \pi_2 \in \mathscr{R}ep(\mathscr{G})$, and $h \in Mor(\pi_1, \pi_2)$ we have

$$\begin{array}{ccc} \mathscr{H}^{\pi_1}_u & \xrightarrow{\pi_1(x^{-1})} & \mathscr{H}^{\pi_1}_v \\ {\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle h_v} \\ \mathscr{H}^{\pi_2}_v & \xrightarrow[\pi_2(x^{-1})]{} & \mathscr{H}^{\pi_2}_v \end{array}$$

Multiplying both sides with $f_{(v,u)}(x)$ and integrating against λ^u_v we get

$$\begin{array}{ccc} \mathscr{H}^{\pi_1}_u & \xrightarrow{\mathfrak{F}(f)^{\pi_1}_{u,v}} & \mathscr{H}^{\pi_1}_v \\ {\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle h_v} \\ \mathscr{H}^{\pi_2}_v & \xrightarrow[\mathfrak{F}(f)^{\pi_2}_{u,v}]{} & \mathscr{H}^{\pi_2}_v \end{array}$$

This means that $\mathfrak{F}(f) : \mathscr{U} \to \mathscr{U}$ is a natural transformation. Let $\mathscr{E}nd(\mathscr{U})$ be the set of all natural transformations : $\mathscr{U} \to \mathscr{U}$ with the coarsest topology making all maps $a \mapsto a^{\pi}_{u,v}$ continuous. We define an involution on $\mathscr{E}nd(\mathscr{U})$ by

$$\bar{a}^{\pi}_{u,v}(\xi) = \overline{a^{\bar{\pi}}_{u,v}(\bar{\xi})} \quad (u, v \in X, \pi \in \mathscr{R}ep(\mathscr{G}), \xi \in \overline{\mathscr{H}^{\pi}_u}).$$

The following is trivial.

LEMMA 4.1. *$\mathscr{E}nd(\mathscr{U})$ is a topological vector space with continuous involution.*

In the next proposition, $\mathscr{E}nd(\mathscr{H}_\rho)$ could be understood as a bundle over X with fiber $\mathscr{B}(\mathscr{H}^{\rho}_u)$ at $u \in X$.

PROPOSITION 4.2. *The map*

$$q : \mathscr{E}nd(\mathscr{U}) \to \prod_{\rho\in\hat{\mathscr{G}}} \mathscr{E}nd(\mathscr{H}_\rho)$$

$$a \mapsto (a_\rho)_{\rho\in\hat{\mathscr{G}}}$$

is an isomorphism of topological vector spaces.

Proof. The following commutative diagrams (with vertical maps being canonical imbeddings) illustrates that $a_{\pi_1\oplus\pi_2} = a_{\pi_1} \oplus a_{\pi_2}$, for each $\pi_1, \pi_2 \in \mathscr{R}ep(\mathscr{G})$ and $a \in \mathscr{E}nd(\mathscr{U})$.

$$\begin{tikzcd}
\mathscr{H}_{\pi_1} \arrow[r, "a_{\pi_1}"] \arrow[d, "\iota_1"'] & \mathscr{H}_{\pi_1} \arrow[d, "\iota_1"] \\
\mathscr{H}_{\pi_1} \oplus \mathscr{H}_{\pi_2} \arrow[r, "a_{\pi_1\oplus\pi_2}"'] & \mathscr{H}_{\pi_1} \oplus \mathscr{H}_{\pi_2} \\
\mathscr{H}_{\pi_2} \arrow[r, "a_{\pi_2}"'] \arrow[u, "\iota_2"] & \mathscr{H}_{\pi_2} \arrow[u, "\iota_2"']
\end{tikzcd}$$

This plus the fact that each representation of $\mathscr{G}$ is the direct sum of its irreducible subrepresentations [1, Theorem 2.16] shows that q is one-one. To show that it is onto, let $b = (b_\rho)$ with $b_\rho \in \mathscr{E}nd(\mathscr{H}_\rho)$ be given. Let $\pi \in \mathscr{R}ep(\mathscr{G})$ and $\mathscr{H}_\pi = \oplus_{\rho\in\hat{\mathscr{G}}} \mathscr{H}_{\pi_\rho}$ be the unique decomposition into isotropical components. For $\rho \in \hat{\mathscr{G}}$, the canonical map

$$\psi_\rho : \mathscr{H}_\rho \otimes Hom_{\mathscr{G}}(\mathscr{H}_\rho, \mathscr{H}_\rho) \to \mathscr{H}_\rho$$

$$\xi \otimes \varphi \mapsto \varphi(\xi)$$

is an isomorphism of $\mathscr{G}$-modules. Put $a_{\pi_\rho} = \psi_\rho \circ (b_\rho \otimes id) \circ \psi_\rho^{-1}$, $a_\pi = \oplus_{\rho\in\hat{\mathscr{G}}} a_{\pi_\rho}$, and $a = (a_\pi)_{\pi\in\mathscr{R}ep(\mathscr{G})}$. It is easy to see that $a : \mathscr{U} \to \mathscr{U}$ is a natural transformation and $a_\rho = b_\rho$, for each $\rho \in \hat{\mathscr{G}}$. Hence q is onto. The way we defined the topology of $\mathscr{E}nd(\mathscr{U})$ makes q^{-1} continuous. The fact that q is continuous is trivial. $\square$

DEFINITION 4.3. An element $a \in \mathscr{E}nd(\mathscr{U})$ is called monoidal (tensor preserving) if for each $\pi_1, \pi_2 \in \mathscr{R}ep(\mathscr{G})$, $a_{\pi_1\otimes\pi_2} = a_{\pi_1} \otimes a_{\pi_2}$ and a_{tr} is trivial, i.e. for each $u, v \in X$ the following diagram commutes

$$\begin{tikzcd}
\mathscr{H}_u^{\pi_1} \otimes \mathscr{H}_u^{\pi_2} \arrow[r, "a_{u,v}^{\pi_1}\otimes a_{u,v}^{\pi_2}"] \arrow[d, equal] & \mathscr{H}_v^{\pi_1} \otimes \mathscr{H}_v^{\pi_2} \arrow[d, equal] \\
\mathscr{H}_u^{\pi_1\otimes\pi_2} \arrow[r, "a_{u,v}^{\pi_1\otimes\pi_2}"'] & \mathscr{H}_v^{\pi_1\otimes\pi_2}
\end{tikzcd}$$

and $a_{u,v}^{tr} = id$, where tr is the trivial representation of $\mathscr{G}$ on $\mathbb{C}$. An element $a \in \mathscr{E}nd(\mathscr{U})$ is called Hermitian if $\bar{a} = a$.

DEFINITION 4.4. For each $u,v \in X$ and $a \in \mathscr{E}nd(\mathscr{U})$, consider the continuous section $a_{u,v}$ defined on $\mathscr{R}ep(\mathscr{G})$ by $a_{u,v}(\pi) = a^{\pi}_{u,v}$. The set $\mathscr{T}(\mathscr{G})$ of all triples $(u,v,a_{u,v})$, which we briefly denote by $a_{u,v}$, where $a \in \mathscr{E}nd(\mathscr{U})$ is monoidal and Hermitian and $u,v \in X$, is called the Tannaka groupoid of $\mathscr{G}$. For fixed $u,v \in X$, we denote the set of all $a_{u,v} \in \mathscr{T}(\mathscr{G})$ by $\mathscr{T}_{u,v}(\mathscr{G})$.

THEOREM 4.5. *$\mathscr{T}(\mathscr{G})$ is a compact groupoid.*

Proof. We define the product for the pairs of the form $(a_{w,v}, b_{u,w}) \in \mathscr{T}(\mathscr{G})^{(2)}$ by composition

$$(ab)^{\pi}_{u,v} = a^{\pi}_{w,v} \circ b^{\pi}_{u,w}.$$

This is clearly an associative partial operation on $\mathscr{T}(\mathscr{G})$.

It is easy to check that if $a,b \in \mathscr{E}nd(\mathscr{U})$ are monoidal and Hermitian, then so is ab. Indeed

$$\begin{aligned}(ab)^{\pi_1\otimes\pi_2}_{u,v} &= a^{\pi_1\otimes\pi_2}_{w,v} \circ b^{\pi_1\otimes\pi_2}_{u,w} = (a^{\pi_1}_{w,v} \otimes a^{\pi_2}_{w,v}) \circ (b^{\pi_1}_{u,w} \otimes b^{\pi_2}_{u,w}) \\ &= (a^{\pi_1}_{w,v} \circ b^{\pi_1}_{u,w}) \otimes (a^{\pi_2}_{w,v} \circ b^{\pi_2}_{u,w}) = (ab)^{\pi_1}_{u,v} \otimes (ab)^{\pi_2}_{u,v}.\end{aligned}$$

For each $\pi \in \mathscr{R}ep(\mathscr{G})$ let $\bar{\pi} \in \mathscr{R}ep(\mathscr{G})$ be its conjugate representation, and put

$$(a^{-1}_{u,v})^{\pi} := {}^{t}a^{\bar{\pi}}_{u,v} \quad (u,v \in X, \pi \in \mathscr{R}ep(\mathscr{G})).$$

For each $u \in X$, let j^{π} be the anti-unitary intertwiner from π to $\bar{\pi}$, and define $\varepsilon_u : \mathscr{H}^{\bar{\pi}}_u \otimes \mathscr{H}^{\pi}_u \to \mathbb{C}$ by

$$\varepsilon_u(j^{\pi}_u \eta \otimes \xi) = \langle \eta, \xi \rangle \quad (\xi, \eta \in \mathscr{H}^{\pi}_u).$$

We claim that $\varepsilon \in Mor(\bar{\pi} \otimes \pi, tr)$. Indeed for each $x \in \mathscr{G}$ and $\xi, \eta \in \mathscr{H}^{\pi}_{s(x)}$ we have

$$\begin{aligned}\varepsilon_{r(x)}(\bar{\pi} \otimes \pi)(x)(\eta \otimes \xi) &= \varepsilon_{r(x)}(\bar{\pi}(x)\eta \otimes \pi(x)\xi) = \langle j^{\pi}_{r(x)} \bar{\pi}(x)\eta, \pi(x)\xi \rangle \\ &= \langle j^{\pi}_{s(x)} \eta, \xi \rangle = \varepsilon_{s(x)}(\eta \otimes \xi) \\ &= tr(x)\varepsilon_{s(x)}(\eta \otimes \xi).\end{aligned}$$

Therefore, for each $u,v \in X$ and $a \in \mathscr{E}nd(\mathscr{U})$, we have $\varepsilon_v a^{\bar{\pi}\otimes\pi}_{u,v} = a^{tr}_{u,v}\varepsilon_u$. In particular for each $a_{u,v} \in \mathscr{T}(\mathscr{G})$ and $\xi, \eta \in \mathscr{H}^{\pi}_v$, we have

$$\begin{aligned}\langle a^{\bar{\pi}}_{u,v}(j^{\pi}_u \eta), a^{\pi}_{u,v}(\xi) \rangle &= \varepsilon_v(a^{\bar{\pi}}_{u,v}(j^{\pi}_u \eta) \otimes a^{\pi}_{u,v}(\xi)) \\ &= \varepsilon_v(a^{\bar{\pi}\otimes\pi}_{u,v}(j^{\pi}_u \eta \otimes \xi)) = a^{tr}_{u,v}\varepsilon_u(j^{\pi}_u \eta \otimes \xi) = \langle \eta, \xi \rangle.\end{aligned}$$

Put $j^{\pi}_u \eta = b^{\bar{\pi}}_{w,u}(\zeta) \in \mathscr{H}^{\bar{\pi}}_u$, where $\zeta \in \mathscr{H}^{\bar{\pi}}_w$, then

$$\langle a^{\bar{\pi}}_{u,v}(b^{\bar{\pi}}_{w,u}(\zeta)), a^{\pi}_{u,v}(\xi) \rangle = \langle b^{\bar{\pi}}_{w,u}(\zeta), \xi \rangle,$$

for each ζ, ξ as above. Hence, changing π to $\bar{\pi}$, we get

$$^{t}a^{\bar{\pi}}_{u,v} \circ a^{\pi}_{u,v} \circ b^{\pi}_{w,u} = b^{\pi}_{w,u},$$

that is $a^{-1}_{u,v}a_{u,v}b_{w,u} = b_{w,u}$. Similarly $b_{v,w}a^{-1}_{v,u}a_{v,u} = b_{v,w}$. This shows that $\mathscr{T}(\mathscr{G})$ is a groupoid.

Next we show that $\mathscr{T}(\mathscr{G})$ is a closed subset of a compact groupoid. Recall that isotropy groups $\mathscr{G}^u_u$ are compact groups and the restriction of the invariant measure λ_u to $\mathscr{G}^u_u$ is a left (and so right) Haar measure. For each $\pi \in \mathscr{R}ep(\mathscr{G})$ and $u \in X$, let $g_u : \mathscr{H}^{\pi}_u \otimes \mathscr{H}^{\pi}_u \to \mathbb{C}$ be defined by

$$g_u(\xi, \eta) = \int \langle \pi(x)\xi, \eta\rangle d\lambda^u_u(x) \quad (\xi, \eta \in \mathscr{H}^{\pi}_u).$$

Also define $h_u : \overline{\mathscr{H}^{\pi}_u} \otimes \mathscr{H}^{\pi}_u \to \mathbb{C}$ by $h_u(\xi, \eta) = g_u(\bar{\xi}, \eta)$. We claim that $h \in Mor(\bar{\pi} \otimes \pi, tr)$. Indeed for each $\xi, \eta \in \mathscr{H}^{\pi}_u$ and $x \in \mathscr{G}$ we have

$$\begin{aligned}
h_{r(x)}(\bar{\pi} \otimes \pi)(x)(\xi \otimes \eta) &= h_{r(x)}(\bar{\pi}(x)\xi \otimes \pi(x)\eta) = g_{r(x)}(\pi(x)\bar{\xi} \otimes \pi(x)\eta) \\
&= \int \langle \pi(y)\pi(x)\bar{\xi}, \pi(x)\eta\rangle d\lambda^{r(x)}_{r(x)}(y) \\
&= \int \langle \pi(x^{-1}yx)\bar{\xi}, \eta\rangle d\lambda^{r(x)}_{r(x)}(y) \\
&= \int \langle \pi(y)\bar{\xi}, \eta\rangle d\lambda^{s(x)}_{s(x)}(y) \\
&= g_{s(x)}(\bar{\xi} \otimes \eta) = h_{s(x)}tr(x)(\xi \otimes \eta).
\end{aligned}$$

Therefore, for each $u, v \in X$ and $a \in \mathscr{E}nd(\mathscr{U})$, we have $h_v a^{\bar{\pi}\otimes\pi}_{u,v} = a^{tr}_{u,v}h_u$. In particular, for each $a_{u,v} \in \mathscr{T}(\mathscr{G})$, using the monoidal property, we get

$$h_v(\overline{a^{\pi}_{u,v}(\xi)}, a^{\pi}_{u,v}(\eta)) = h_u(\bar{\xi}, \eta),$$

that is

$$g_v(a^{\pi}_{u,v}(\xi), a^{\pi}_{u,v}(\eta)) = g_u(\xi, \eta),$$

for each $\xi, \eta \in \mathscr{H}^{\pi}_u$. Now we can view g_u and g_v as new inner products on $\mathscr{H}^{\pi}_u$ and $\mathscr{H}^{\pi}_v$, respectively, and look at the unitary elements in $\mathscr{B}(\mathscr{H}^{\pi}_u, \mathscr{H}^{\pi}_v)$. If we denote the set of unitaries from $(\mathscr{H}^{\pi}_u, g_u)$ onto $(\mathscr{H}^{\pi}_v, g_v)$ by $\mathscr{U}(\mathscr{H}^{\pi}_u, \mathscr{H}^{\pi}_v)$, then $a^{\pi}_{u,v} \in \mathscr{U}(\mathscr{H}^{\pi}_u, \mathscr{H}^{\pi}_v)$, whence

$$\mathscr{T}(\mathscr{G}) \subseteq \prod_{\pi} \bigsqcup_{u,v} \mathscr{U}(\mathscr{H}^{\pi}_u, \mathscr{H}^{\pi}_v),$$

where the right hand side is a product of compact groupoids. The fact that $\mathscr{T}(\mathscr{G})$ is a closed subset of this groupoid follows immediately from the definition of the topology on $\mathscr{E}nd(\mathscr{U})$. $\square$

PROPOSITION 4.6. *$\mathscr{G}^{(0)} \subseteq \mathscr{T}(\mathscr{G})^{(0)}$ and $\mathscr{T}(\mathscr{G})^v_u \subseteq \mathscr{T}_{u,v}(\mathscr{G})$, for each $u, v \in X$.*

Proof. Given $u \in X$, it is easy to see that

$$(\mathscr{T}_u)^{\pi}_{v,w} = (\mathscr{T}_u^{-1})^{\pi}_{v,w} = \begin{cases} id_u & \text{if } v = w = u, \\ 0 & \text{otherwise} \end{cases}$$

for each $v, w \in X$. In particular $\mathscr{T}_u^{-1}\mathscr{T}_u = \mathscr{T}_u$ and so $u \in \mathscr{T}(\mathscr{G})^{(0)}$. Also if $t = (y, z, b_{y,z}) \in \mathscr{T}(\mathscr{G})$ and $s(t) = u, r(t) = v$, then

$$(y, y, b_{y,z}^{-1} b_{y,z}) = (u, u, id_u),$$

hence $y = u$. Similarly we get $z = v$ and so $t \in \mathscr{T}_{u,v}(\mathscr{G})$. $\square$

5. Tannaka duality

Let us consider the natural transformations $\mathscr{T}_x \in \mathscr{E}nd(\mathscr{U})$, $x \in \mathscr{G}$. It is clear that for each $x \in \mathscr{G}$, $\mathscr{T}_x \in \mathscr{T}(\mathscr{G})$ and

$$\mathscr{T}_{xy} = \mathscr{T}_x \mathscr{T}_y \quad (x, y \in \mathscr{G}^{(2)}).$$

In particular the image of $\mathscr{G}$ under $\mathscr{T}$ is a subgroupoid of $\mathscr{T}(\mathscr{G})$. We identify $\mathscr{G}$ with its image in $\mathscr{T}(\mathscr{G})$. For each $u, v \in X$, let $\mathscr{T}_{u,v} : \mathscr{G}_u^v \to \mathscr{T}_{u,v}(\mathscr{G})$ be defined by $\mathscr{T}_{u,v}(x)(\pi) = \pi(x)$, for $x \in \mathscr{G}_u^v$. Consider two adjoint maps

$$\mathscr{T}^* : \mathscr{R}ep(\mathscr{T}(\mathscr{G})) \to \mathscr{R}ep(\mathscr{G})$$

defined by

$$\mathscr{T}^*(\Pi)(x) = \Pi(\mathscr{T}_x) \quad (x \in \mathscr{G}, \Pi \in \mathscr{R}ep(\mathscr{T}(\mathscr{G}))),$$

and

$$\mathscr{T}_* : \mathscr{E}(\mathscr{T}(\mathscr{G})) \to \mathscr{E}(\mathscr{G})$$

defined by

$$\mathscr{T}_*(f)(x) = f(\mathscr{T}_x) \quad (x \in \mathscr{G}, f \in \mathscr{E}(\mathscr{T}(\mathscr{G}))),$$

where $\mathscr{E}(\mathscr{G})$ and $\mathscr{E}(\mathscr{T}(\mathscr{G}))$ are the representation algebras of $\mathscr{G}$ and $\mathscr{T}(\mathscr{G})$, respectively [1, Proposition 3.8].

LEMMA 5.1. *The restriction map*

$$\mathscr{T}^* : \mathscr{R}ep(\mathscr{T}(\mathscr{G})) \to \mathscr{R}ep(\mathscr{G})$$

is a bundle isomorphism.

Proof. We define the extension bundle map $\mathfrak{E} : \mathscr{R}ep(\mathscr{G}) \to \mathscr{R}ep(\mathscr{T}(\mathscr{G}))$ as follows. Given $u, v \in X$, $a_{u,v} \in \mathscr{T}(\mathscr{G})$, and $\pi \in \mathscr{R}ep(\mathscr{G})$, the map $P_\pi : a_{u,v} \mapsto a^{\pi}_{u,v}$ is a representation of $\mathscr{T}(\mathscr{G})$ on $\mathscr{H}_\pi$ and we have the commutative diagram

$$\begin{array}{ccc} \mathscr{G}_{u,v} & \xrightarrow{\pi} & \mathscr{B}(\mathscr{H}_u^\pi, \mathscr{H}_v^\pi) \\ \mathscr{T}_{u,v}\Big\downarrow & & \Big\uparrow P_\pi \\ \mathscr{T}_{u,v}(\mathscr{G}) & \xrightarrow[=]{} & \mathscr{T}_{u,v}(\mathscr{G}) \end{array}$$

Therefore $\mathscr{T}^*(P_\pi) = \pi$. We put $\mathfrak{E}(\pi) = P_\pi$. If $h \in Mor_{\mathscr{G}}(\pi_1, \pi_2)$ then clearly $h \in Mor_{\mathscr{T}(\mathscr{G})}(P_{\pi_1}, P_{\pi_2})$. Also it is easy to check that $\mathfrak{E}$ preserves direct sums, tensor products, and conjugation of representations. Moreover the above commutative diagram shows that if π is irreducible, then so is P_π. Hence $\mathrm{Im}(\mathfrak{E})$ is a closed subset of $\mathscr{R}ep(\mathscr{T}(\mathscr{G}))$ in the sense of Definition 3.7. It also separates the points of $\mathscr{T}(\mathscr{G})$. Indeed If $a_{u,v}$ and $b_{w,z}$ are distinct elements of $\mathscr{T}(\mathscr{G})$, there is a representation $\pi \in \mathscr{R}ep(\mathscr{G})$ such that $a^\pi_{u,v} \neq b^\pi_{w,z}$, which means that P_π separates $a_{u,v}$ and $b_{w,z}$. By Proposition 3.8, $\mathfrak{E}$ is surjective. Now $\mathscr{T}^* \circ \mathfrak{E} = id$, so $\mathscr{T}^*$ is a bundle isomorphism. $\square$

For the next two lemmas, we assume that $\mathscr{G}$ is locally non-trivial, that is $\lambda_u(\mathscr{G}_u^v) \neq 0$, for each $u, v \in X$.

LEMMA 5.2. *The restriction map*

$$\mathscr{T}_* : \mathscr{E}(\mathscr{T}(\mathscr{G})) \to \mathscr{E}(\mathscr{G})$$

is a bundle isomorphism.

Proof. We define the extension bundle map $\mathfrak{E} : \mathscr{E}(\mathscr{G}) \to \mathscr{E}(\mathscr{T}(\mathscr{G}))$ as follows. Given $u, v \in X$, by Proposition 2.2, any $f \in \mathscr{E}^\pi_{u,v}$ has a unique representation in the form

$$f = \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \mathrm{Tr}\big(g(\pi))\pi(\cdot)\big),$$

where $g = \mathfrak{F}_{u,v}(f) \in \sum_{\pi \in \hat{\mathscr{G}}} \mathscr{B}(\mathscr{H}_v^\pi, \mathscr{H}_u^\pi)$. Define $\mathfrak{E}_{u,v}(f)$ on $\mathscr{T}_{u,v}(\mathscr{G})$ by

$$\mathfrak{E}_{u,v}(f)(a_{u,v}) = \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\lambda_u(\mathscr{G}_u^v)} \mathrm{Tr}\big(g(\pi)P_\pi(a_{u,v})\big) \quad (a_{u,v} \in \mathscr{T}_{u,v}(\mathscr{G})).$$

By the proof of the above lemma, $\mathfrak{E}$ is injective, so $\mathscr{T}_*$ is bijective, and we have $\mathscr{T}_* \circ \mathfrak{E} = id$. $\square$

LEMMA 5.3. *For each $f \in C(\mathscr{T}(\mathscr{G}))$ and $u, v \in X$,*

$$\int_{\mathscr{T}(\mathscr{G})_u^v} f(t)\, d\tilde{\lambda}_u^v(t) = \int_{\mathscr{G}_u^v} f(\pi(x))\, d\lambda_u^v(x),$$

where $\tilde{\lambda}$ is the Haar system of $\mathscr{T}(\mathscr{G})$. In particular, $\tilde{\lambda}_u(\mathscr{T}(\mathscr{G})_u^v) = \lambda_u(\mathscr{G}_u^v)$.

Proof. Let $f \in \mathscr{E}(\mathscr{T}(\mathscr{G}))$. As in Proposition 2.2, we may represent f on $\mathscr{T}(\mathscr{G})_u^v$ as

$$f(t) = \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\tilde{\lambda}_u(\mathscr{T}(\mathscr{G})_{u^v})} \operatorname{Tr}\big(g(\pi)P_\pi(t)\big) \quad (t \in \mathscr{T}(\mathscr{G})_u^v),$$

where $g = \mathfrak{F}(\mathscr{T}_*(f))$. In particular, for each $x \in \mathscr{G}_u^v$,

$$\begin{aligned} f(\mathscr{T}_x) &= \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\tilde{\lambda}_u(\mathscr{T}(\mathscr{G})_{u^v})} \operatorname{Tr}\big(g(\pi)P_\pi(\mathscr{T}_x)\big) \\ &= \sum_{\pi \in \hat{\mathscr{G}}} \frac{d_u^\pi}{\tilde{\lambda}_u(\mathscr{T}(\mathscr{G})_{u^v})} \operatorname{Tr}\big(g(\pi)P_\pi(x)\big). \end{aligned}$$

By Proposition 3.2 (iii), we have

$$\operatorname{Tr}\big(g(\pi)\pi(x)\big)d\lambda_u^v(x) = \begin{cases} \lambda_u(\mathscr{G}_u^v)g(tr) & \text{if } \pi = tr, \\ 0 & \text{otherwise,} \end{cases}$$

where tr is the trivial representation. Similarly, one can show that

$$\int \operatorname{Tr}\big(g(\pi)P_\pi(t)\big)d\tilde{\lambda}_u^v(t) = \begin{cases} \tilde{\lambda}_u(\mathscr{T}(\mathscr{G})_u^v)g(tr) & \text{if } \pi = tr, \\ 0 & \text{otherwise,} \end{cases}$$

hence the result. $\square$

If $\{\lambda^u\}$ is a (left) Haar system on $\mathscr{G}$ and $\{\lambda_u^u\}$ is a (left) Haar system on the isotropy groupoid $\mathscr{I}_\mathscr{G} := \cup_u \mathscr{G}_u^u$, then there is a (left) Haar system $\{\nu^u\}$ on the equivalence relation $\mathscr{R}_\mathscr{G} = X \times X$ such that

$$\lambda^u = \int \lambda_u^v d\nu^u(v,u) \quad (u \in X).$$

Conversely, if $\{\lambda_u^v\}$ and $\{\nu^u\}$ are (left) Haar systems on $\mathscr{I}_\mathscr{G}$ and $\mathscr{R}_\mathscr{G}$, the above formula gives a (left) Haar system on $\mathscr{G}$ [9]. Using $\{\lambda_u^v\}$ one can construct the continuous field of Banach spaces $(L^p(\mathscr{G}), \Delta^p(\mathscr{G}))$ over $\mathscr{I}_\mathscr{G}$ [2, 4.2.5]. This is a continuous field of Hilbert spaces for $p = 2$. Let $1 \leqslant p < \infty$. For $f \in C_c(\mathscr{G})$, let $f_u \in L^p(\mathscr{G}_u, \lambda_u)$ be the restriction of f to $\mathscr{G}_u$ and put

$$\|f\|_p = \sup_{u \in X} \Big(\int |f_u|^p d\lambda_u\Big)^{\frac{1}{p}}.$$

Let $\Delta^p(\mathscr{G})$ be the closure of $C_c(G)$ in this norm. This is a Banach $C_0(X)$-module. Let $\mathscr{E}(\mathscr{G}) \subseteq \Delta^2(\mathscr{G})$ be the $C_0(\mathscr{R}_\mathscr{G})$-submodule spanned by the coefficient functions of irreducible continuous representations of $\mathscr{G}$. Note that irreducibility defined in [1] is stronger than the same concept in [2], however the results quoted here from [2] also hold in our setting. Let $u \in X$. The restriction map

$$Res_u : \mathscr{R}ep(\mathscr{G}) \to \mathscr{R}ep(\mathscr{G}_u^u)$$

is called *dominant* if for each continuous representation $(\sigma,\mathscr{K})$ of $\mathscr{G}_u^u$, there is a continuous representation $(\pi,\mathscr{H})$ of $\mathscr{G}$ such that $(\sigma,\mathscr{K})$ is equivalent to a subrepresentation of $(\pi_u^u,\mathscr{H}_u)$, where π_u^u is the restriction of π to $\mathscr{G}_u^u$ [2, 6.4.1]. If this happen to hold for each $u \in X$ we say that $\mathscr{G}$ is *dominant*. Examples of dominant Hausdorff groupoids include $\mathscr{G} = \mathscr{H} \ltimes \mathscr{M}$, where $\mathscr{H}$ is a compact connected Lie group acting on a manifold $\mathscr{M}$ [2, 6.4.4]. When $\mathscr{M}$ is also compact, $\mathscr{G}$ is a compact groupoid. An example of a non dominant (non Hausdorff) groupoid is $\mathscr{G} = (\mathbb{R}\times\mathbb{Z}_2)/\sim$ with quotient topology, where $(x,0)\sim(x,1)$ for $x \neq 0$. Here $\mathscr{G}^{(0)} = \mathbb{R}$, $\mathscr{G}_x^y = \emptyset$, unless $x = y$, $\mathscr{G}_0^0 = \mathbb{Z}_2$, whereas $\mathscr{G}_x^x = \mathscr{G}^x$ is a singleton consisting of the equivalence class of $(x,0)$, for $x \neq 0$. In particular, $\mathscr{G}$ is not transitive. The discrete group $\mathbb{Z}_2$ has a non trivial representation which is not image of any representation of $\mathscr{G}$ under Res_0 (all of whose images are trivial) [2, 6.4.5]. If we replace $\mathbb{R}$ with $\mathbb{T}$ in this example, we get a compact (non Hausdorff) groupoid which is not dominant. It is shown in [2, 6.4.6] that if $\mathscr{G}$ is dominant then $\mathscr{E}(\mathscr{G})$ is dense in $\Delta^2(\mathscr{G})$ in L^∞ norm. A similar argument shows that in this case, $\mathscr{E}(\mathscr{G})$ is dense in $C(\mathscr{G})$ in L^∞ norm.

We have shown in [1, Theorem 3.9] that a weak version of the Gelfand-Raikov Theorem holds for compact Hausdorff groupoids. We say that $\hat{\mathscr{G}}$ separates the points of $\mathscr{G}$ if for each $x,y \in \mathscr{G}$ with $x \neq y$, there is $\pi \in \hat{\mathscr{G}}$ and $\xi,\eta \in \mathscr{H}_\pi$ such that

$$\langle \pi(x)\xi_{s(x)},\eta_{r(x)}\rangle \neq \langle \pi(y)\xi_{s(y)},\eta_{r(y)}\rangle.$$

We then (formally) write $\pi(x) \neq \pi(y)$. This is just an abbreviation, as $\pi(x)$ and $\pi(y)$ are acting on different Hilbert spaces. If $\mathscr{G}$ is Hausdorff and $\mathscr{E}(\mathscr{G})$ is dense in $C(\mathscr{G})$ in the L^∞ norm, then $\hat{\mathscr{G}}$ separates the points of $\mathscr{G}$ [1, Theorem 3.9]. We have also the converse.

LEMMA 5.4. *If $\mathscr{G}$ is Hausdorff and $\hat{\mathscr{G}}$ separates the points of $\mathscr{G}$, then $\mathscr{E}(\mathscr{G})$ is dense in $C(\mathscr{G})$ in both L^∞ and L^2 norms.*

Proof. $\mathscr{E}(\mathscr{G})$ is a subalgebra of $C(\mathscr{G})$ [1, Proposition 3.8] which is closed under conjugation (consider the conjugate representations) and vanishes nowhere (consider the trivial representation). By assumption it also separates the points. Therefore it is dense by Stone-Weierstrass Theorem. Since $\mathscr{G}$ is compact and the Haar system is normalized, $\|.\|_2 \leqslant \|.\|_\infty$ on $C(\mathscr{G})$, hence it is also dense in the L^2-norm. □

LEMMA 5.5. *If $\mathscr{E}(\mathscr{G})$ is dense in $C(\mathscr{G})$ in L^2 norm, then $\mathscr{E}(\mathscr{T}(\mathscr{G}))$ is dense in $C(\mathscr{T}(\mathscr{G}))$ in L^2 norm.*

Proof. Let $f \in C(\mathscr{T}(\mathscr{G}))$, then by restriction we get $f_0 \in C(\mathscr{G})$. Given $\varepsilon > 0$, there is $g_0 \in C(\mathscr{G})$ such that $\|f_0 - g_0\| < \varepsilon$. By Lemma 5.2, choose $g \in \mathscr{E}(\mathscr{T}(\mathscr{G}))$ such that $\mathscr{T}_*(g) = g_0$. Then $\|f-g\| = \|\mathfrak{C}(f_0 - g_0)\| \leqslant \|f_0 - g_0\| < \varepsilon$. □

In particular the conclusions of the last two lemmas hold for any Hausdorff dominant compact groupoid. Now we are ready to prove the main result of these series of papers, the *Tannaka-Krein duality theorem* for compact groupoids.

THEOREM 5.6. (Tannaka-Krein Duality Theorem) *For a locally non-trivial, compact Hausdorff groupoid whose dual object separates the points is isomorphic to its Tannaka groupoid.*

Proof. Let $\mathscr{G}$ be a compact Hausdorff groupoid such that $\hat{\mathscr{G}}$ separates the points of $\mathscr{G}$. We show that $\mathscr{T}:\mathscr{G}\to\mathscr{T}(\mathscr{G})$ is an isomorphism of topological groupoids. By Lemmas 5.3 and 5.5, the compact groupoid $\mathscr{T}(\mathscr{G})$ satisfies conditions of the Peter-Weyl Theorem [1, Theorem 3.10]. The injectivity of $\mathscr{T}$ follows from this theorem applied to $\mathscr{T}(\mathscr{G})$. For the surjectivity, assume on the contrary that $\mathrm{Im}(\mathscr{T})$ is a proper subset of $\mathscr{T}(\mathscr{G})$. This is a closed subset. Let $f\in C(\mathscr{T}(\mathscr{G}))$ be a positive function such that $\mathrm{supp}(f)$ is contained in the complement of $\mathrm{Im}(\mathscr{T})$. Then from the two integrals in Lemma 5.3, the one on the right hand side is 0, where as the one on the left hand side is strictly positive, a contradiction. $\square$

In Peter-Weyl Theorem [1, Theorem 3.10] we have required $\mathscr{E}_{u,v}$ to be dense in $C(\mathscr{G}_u^v)$ and $\lambda_u^v(\mathscr{G}_u^v)\neq 0$, for each $u,v\in X$. Each $\mathscr{E}_{u,u}$ is the set of trigonometric polynomials on the (Hausdorff) compact group $\mathscr{G}_u^u$ and so it is dense in $C(\mathscr{G}_u^u)$, by Stone-Weierstrass Theorem. Now if $\mathscr{G}$ is transitive and we choose $y\in\mathscr{G}_v^u$, the map $x\mapsto yx$ is a (measure preserving) homeomorphism from $\mathscr{G}_u^v$ onto $\mathscr{G}_u^u$. This induces an isometric isomorphism from $C(\mathscr{G}_u^v)$ onto $C(\mathscr{G}_u^u)$ which sends $\mathscr{E}_{u,v}$ onto $\mathscr{E}_{u,u}$. Hence $\mathscr{E}_{u,v}$ is dense in $C(\mathscr{G}_u^v)$. Also, under this homeomorphism, $\lambda_u^v(\mathscr{G}_u^v)=\lambda_u^u(\mathscr{G}_u^u)\neq 0$, as the right hand side is the norm of the Haar measure of (non empty) compact group $\mathscr{G}_u^u$. This argument replaces part of the proof of the above theorem, and proves the following.

COROLLARY 5.7. *Each compact, Hausdorff, transitive groupoid is isomorphic to its Tannaka groupoid.*

A possible way to avoid the local non-triviality assumption in Theorem 5.6 could be to use a local Fourier-Plancherel transform $\mathfrak{F}_u$ on $L^2(\mathscr{G}_u,\lambda_u)$ (instead of $\mathfrak{F}_u^v$ on $L^2(\mathscr{G}_u^v,\lambda_u)$) and use Proposition 3.3 (instead of proposition 3.2) in a modified version of Lemma 5.3 in which integrals are taken against λ_u and $\tilde{\lambda}_u$.

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