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BOUNDARY INTEGRAL METHODS FOR WEDGE DIFFRACTION PROBLEMS: THE ANGLE $2\pi/n$, DIRICHLET AND NEUMANN CONDITIONS

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Abstract. In this paper we use analytical methods for boundary integral operators (more precisely, pseudodifferential operators) together with symmetry arguments in order to treat harmonic wave diffraction problems in which the field does not depend on the third variable and the wave incidence is perpendicular. These problems are formulated as two-dimensional, mixed elliptic boundary value problems in a non-rectangular wedge.

We solve explicitly a number of reference problems for the Helmholtz equation regarding particular wedge angles, boundary conditions, and space settings, which can be modified and generalized in various ways. The solution of these problems in Sobolev spaces was open for some fifty years.

1. Introduction

We consider a class of so-called canonical [19, 23] boundary value problems where we look for all weak solutions of the Helmholtz equation $(\Delta+k^2)u=0$ in a plane cone $\Omega=\Omega_{0,\alpha}$, where $k\in\mathbb{C}$, $\mathrm{Im}(k)>0$, and

$$\Omega_{0,\alpha}=\{x=(x_1,x_2)\in\mathbb{R}^2 : 0<\arg(x_1+ix_2)<\alpha\},\qquad 0<\alpha\leqslant 2\pi,\tag{1}$$

which satisfy Dirichlet or Neumann conditions on the two half-line parts Γ_1,Γ_2 of the boundary, also admitting the mixed type (i.e., the cases DD, NN, DN). We will look from the very beginning for solutions with "small regularity", i.e., $u\in H^{1+\varepsilon}(\Omega), \varepsilon\in[0,1/2[$, while to a certain extent we will discuss the cases with $\varepsilon\in]-1/2,0[$ as well. The boundary data will be given in the corresponding trace spaces $H^{1/2+\varepsilon}(\Gamma_j)$ or $H^{-1/2+\varepsilon}(\Gamma_j)$, respectively. However, we have to identify the proper well-posed problems, i.e., we have to make a systematic normalization of the problem when necessary. In fact, the compatibility conditions in this paper correspond to the minimal image normalization in the sense of [25].

The most important questions consist in (a) the identification of well-posed problems or at least of those which have the Fredholm property, (b) identifying the convenient spaces if the problem is not well-posed in the original setting (e.g., normalization

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by inclusion of compatibility conditions for the data), (c) finding an explicit analytical solution, possibly in closed form, and (d) a discussion of its properties, particularly the singular behavior near the corner.

Various authors emphasized that a rigorous analytical solution of these problems will be helpful for a general understanding of elliptic boundary value problems in conical domains [12, 15, 26, 37]. This paper is aimed at solving *some* of them effectively (in the above-mentioned sense). The known results are either very limited to special situations such as the rectangular case [3, 4, 20] or rather complicated in what concerns the analytical methods [14, 39] or not describing appropriate function spaces, see, e.g., [17, 36]. For the historical background (from our point of view) and for further literature we refer to [2, 37, 39]. One of the most remarkable papers is the article of A.I. Komec [13]. In this paper general (linear) boundary value problems for second order elliptic differential equations are considered on manifolds with edges. Key results such as normal solvability and the construction of parametrices are obtained, in particular for the quarter-plane, by application of the Fourier transformation (after extension-by-zero to the full plane) and making use of the theory of analytic functions of several variables. However, the methods are completely different, restricted to the case of $s = 1 + \varepsilon > 3/2$ (which is crucial for stratification of 3D problems), and do not yield explicit solutions as in the present work.

It should also be mentioned that diffraction of waves on wedges of rational angles was treated already by more classical methods in obtaining analytic solutions for particular incoming waves, but not as a well-posed problem in Sobolev spaces constructing resolvent operators (see, e.g., a series of papers by A.D. Rawlins in the 1980's [29]). There is also a tight connection with the theory of mixed boundary value problems for the Laplacian and other elliptic equations, which need a similar operator and space reasoning for efficient analytical and numerical treatment, see for instance the fundamental paper [38] and the recent book [11].

The question of solving canonical diffraction problems in Sobolev spaces appeared with the development of pseudodifferential operators in domains with corners and, more generally, in Lipschitz domains, in the 1960's, see the introduction of the book of Vasil'ev [37] with a considerable list of references. It was popularized by Meister [18, 19, 21], Wendland [38], Ferreira dos Santos [30] and their collaborators in the 1980's and gave an impact on the advance of areas in operator theory, such as the constructive factorization of non-rational matrix functions [7] and the factorization theory for Toeplitz plus Hankel operators [6], to mention only a few.

The present method consists of a combination of our knowledge about the analytical solution of Sommerfeld and rectangular wedge diffraction problems [3, 4, 20, 23] with new symmetry arguments that relate the present to previously solved problems and yield the explicit analytical solution in a great number of cases. For this purpose we introduce here so-called "Sommerfeld potentials" (explicit solutions to special Sommerfeld problems) whose use turns out to be most efficient. It is surprising that the case where the angle is an *integer part* of 2π can be solved completely whilst the case of "rational" angles $\alpha = 2\pi m/n$ for $m \geqslant 2$ appears much harder and remains, in general, unsolved at present. An exception is the rectangular exterior wedge problem ($\alpha = 3\pi/2$) [2, 20] where it became evident that the continuation to general rational angles is not an easy enterprise.

As an interesting and very direct conclusion we obtain the result that, for the angles under consideration, the behavior of the field $\operatorname{grad} u$ shows the same singularity in the corner as (more precisely, is not worse than) in the corresponding half-plane or Sommerfeld potential case, namely we obtain that $\operatorname{grad} u$ behaves asymptotically like $r^{-1/2}$ in the DD and NN problems, as well as in DN problems, if n is even, and like $r^{-3/4}$ in the DN case, if n is odd.

The limiting absorption principle is not considered in this article, since it needs rather different techniques (variational formulation, Lax-Milgram lemma etc) such as presented in [1] which are promising, however can be carried out only in a separate publication.

2. Notation and basic results

We start by introducing some notation that will be useful for the formulation of our wedge diffraction problems. Let $\Omega^{\pm} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \gtrless 0\}$, and consider the rotation $\mathscr{R}_\alpha$ about the angle α given by

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathscr{R}_\alpha x = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We also think of $\mathscr{R}_\alpha$ as acting on subsets of $\mathbb{R}^2$. With the help of $\mathscr{R}_\alpha$ we define the half-planes (see Figure 1)

$$\Omega^{\pm}_\alpha = \mathscr{R}_\alpha \Omega^{\pm} = \{\mathscr{R}_\alpha x : x = (x_1, x_2) \in \Omega^{\pm}\}.$$

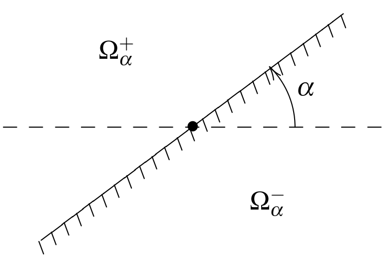

Figure 1: Rotation $\mathscr{R}_\alpha$

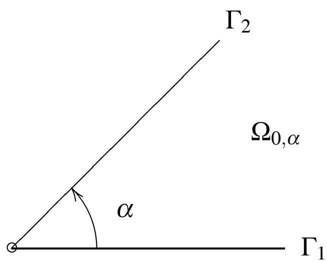

Figure 2: The cone $\Omega_{0,\alpha}$

The backward rotation is given by $\mathscr{R}_\alpha^{-1} = \mathscr{R}_{-\alpha} : \Omega^{\pm}_\alpha \to \Omega^{\pm}_0 = \Omega^{\pm}$, and we also need a rotation operator acting on suitable functions (or distributions) which is given by

$$(J_\alpha f)(x) = f(\mathscr{R}_\alpha^{-1} x), \quad x \in \mathbb{R}^2.$$

For convenience we also define the (rotated) half-lines

$$\Sigma = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 = 0\}, \qquad \Sigma_\alpha = \mathscr{R}_\alpha \Sigma. \tag{2}$$

Moreover, for given $0<\alpha<2\pi$, we put $\Gamma_j=\Sigma_{(j-1)\alpha}$. Then the boundary of the wedge $\Omega_{0,\alpha}$ defined in (1) can be decomposed into $\partial\Omega_{0,\alpha}=\overline{\Gamma}_1\cup\overline{\Gamma}_2=\Gamma_1\cup\Gamma_2\cup\{(0,0)\}$ (cf. Figure 2).

Furthermore, in order to make the connection with notation used in previous papers [2, 3, 20], let Q_j, $j=1,2,3,4$, denote the four open quadrants in $\mathbb{R}^2$, and consider the open half-planes denoted by $Q_{jk}=\operatorname{int}\operatorname{clos}(Q_j\cup Q_k)$, where $k=j+1$ for $j=1,2,3$ and $jk=41$ for $j=4$.

The Fourier transform acting on functions (or distributions) on $\mathbb{R}^n$ will be written as

$$\hat{u}(\vec{\xi})=(\mathscr{F}u)(\vec{\xi})=\int_{\mathbb{R}^n}u(\vec{x})\exp[i\vec{x}\cdot\vec{\xi}]\,dx_1\cdots dx_n,\quad \vec{\xi}\in\mathbb{R}^n.$$

In addition, we need the Fourier transform (in $\mathbb{R}^2$) acting only on the first variable,

$$\mathscr{F}_{x_1\mapsto\xi}u(x_1,x_2)=\int_{\mathbb{R}}u(x_1,x_2)\exp[i\xi x_1]\,dx_1,\quad \xi\in\mathbb{R}.$$

By $H^s=H^s(\mathbb{R}^n)\subset\mathscr{S}'$ we denote the usual Sobolev spaces equipped with the norm

$$\|u\|^2_{H^s}=\int_{\mathbb{R}^n}|\hat{u}(\vec{\xi})|^2(1+|\vec{\xi}|^2)^s\,d\xi_1\cdots d\xi_n<\infty.$$

For a non-empty, open subset $\Omega\subseteq\mathbb{R}^n$, let $H^s_\Omega=H^s_\Omega(\mathbb{R}^n)$ stand for the set of all distributions in H^s with support in the closure of Ω. Notice that H^s_Ω is a closed subspace of H^s. Any distribution in the Schwartz space $\mathscr{S}'$ can be restricted to Ω, i.e., it can be considered as a distribution in $\mathscr{D}'(\Omega)$. The restriction operator will be denoted by r_Ω. We define Sobolev spaces on Ω as images of the restriction operator as follows,

$$H^s(\Omega)=r_\Omega(H^s),\qquad \widetilde{H}^s(\Omega)=r_\Omega(H^s_\Omega).$$

A norm in $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$, resp., which make these spaces Hilbert spaces, is naturally defined as

$$\|u\|_{H^s(\Omega)}=\inf_{\ell}\|\ell u\|_{H^s},\qquad \|u\|_{\widetilde{H}^s(\Omega)}=\inf_{\ell_0}\|\ell_0 u\|_{H^s}.$$

Here ℓu and $\ell_0 u$ stand for any extension of a distribution on Ω to a distribution in H^s or H^s_Ω, respectively. The infimum is taken over all such extensions. It is clear that $\widetilde{H}^s(\Omega)$ is continuously embedded in $H^s(\Omega)$.

If Ω is a strong Lipschitz domain [11] or a special Lipschitz domain [34], then the mapping $r_\Omega:H^s_\Omega\to\widetilde{H}^s(\Omega)$ is injective if and only if $s\geqslant-1/2$. Hence in this case (and only in this case) one can define [9] an extension-by-zero operator

$$\ell_0:\widetilde{H}^s(\Omega)\to H^s_\Omega\subseteq H^s, \tag{3}$$

such that ℓ_0 is the inverse of r_Ω. Clearly, $\widetilde{H}^s(\Omega)$ can be identified with H^s_Ω, i.e.,

$$\widetilde{H}^s(\Omega)=r_\Omega H^s_\Omega,\qquad H^s_\Omega=\ell_0\widetilde{H}^s(\Omega)\qquad\text{for}\qquad s\geqslant-1/2,$$

and the norm in $\widetilde{H}^s(\Omega)$ is given by $\|u\|_{\widetilde{H}^s(\Omega)} = \|\ell_0 u\|_{H^s}$.

We shall need the above properties mainly for cones and for the half-axes $\Omega = \mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$. In the latter case, we will write

$$r_\pm = r_\Omega \quad \text{and} \quad H^s_\pm = H^s_\Omega$$

for brevity, see [11, Sec. 4.1]. Note that the δ-distribution does not belong to $H^s(\mathbb{R})$ if $s \geqslant -1/2$. Hence, for these values of s, $\widetilde{H}^s(\mathbb{R}_+)$ and H^s_+ are identified in some publications, cf. [8, 9, 11].

The spaces $\widetilde{H}^s(\mathbb{R}_\pm)$ and $H^s(\mathbb{R}_\pm)$ coincide exactly for $s \in]-1/2, 1/2[$. Hence in this case the extension-by-zero operator is well defined and bounded on $H^s(\mathbb{R}_\pm)$,

$$\ell_0 : H^s(\mathbb{R}_\pm) \to H^s_\pm(\mathbb{R}) \subseteq H^s(\mathbb{R}), \qquad s \in]-1/2, 1/2[.$$

For $s = \pm 1/2$, i.e., for the spaces of particular interest for the Dirichlet and Neumann problems, we have proper, dense embeddings

$$\widetilde{H}^{-1/2}(\mathbb{R}_\pm) \subset H^{-1/2}(\mathbb{R}_\pm), \qquad \widetilde{H}^{1/2}(\mathbb{R}_\pm) \subset H^{1/2}(\mathbb{R}_\pm),$$

while for $s \in]1/2, 3/2[$ the space $\widetilde{H}^s(\mathbb{R}_\pm)$ is a closed subspace of $H^s(\mathbb{R}_\pm)$ of co-dimension one. In fact, it consists of those (continuous) functions u with $u(0) = 0$.

Obviously, the even and odd extension operators ℓ^e and ℓ^o from $H^s(\mathbb{R}_+)$ into $H^s(\mathbb{R})$ can be defined via the operator ℓ_0 in the cases $s \in]-1/2, 1/2[$. However, much more is true. In fact, the even and odd extension operators are well defined and bounded in the following cases:

$$\begin{aligned} \ell^e &: H^s(\mathbb{R}_+) \to H^s(\mathbb{R}), \qquad s \in]-1/2, 3/2[, \\ \ell^o &: H^s(\mathbb{R}_+) \to H^s(\mathbb{R}) \qquad s \in]-3/2, 1/2[, \end{aligned}$$

which includes partially the cases $s = \pm 1/2$. On the other hand, $u \in H^{1/2}(\mathbb{R}_+)$ admits an odd extension if and only if $u \in \widetilde{H}^{1/2}(\mathbb{R}_+)$, i.e., it is extendable by zero, and $u \in H^{-1/2}(\mathbb{R}_+)$ admits an even extension if and only if $u \in \widetilde{H}^{-1/2}(\mathbb{R}_+)$. We refer to [35] or the Appendix of [33], which contains a careful description of the details needed here.

For any $s \in \mathbb{R}$ and any strong or special Lipschitz domain $\Omega \subseteq \mathbb{R}^2$, one can define $\mathscr{H}^s(\Omega)$ to be the set of all $u \in H^s(\Omega)$ such that u satisfies the Helmholtz equation

$$(\Delta + k^2)u = 0$$

in the distributional sense of $\mathscr{D}'(\Omega)$. Of usual interest is the case $s = 1 + \varepsilon$, $\varepsilon \in [0, 1/2[$, where one obtains weak solutions with small regularity [9, 11]. We will also consider (to some extent) the case of $s = 1 + \varepsilon$ with $\varepsilon \in]-1/2, 0[$.

In the case of the slit-plane $\Omega = \mathbb{R}^2 \setminus \overline{\Sigma}$, see (2), we will also use the notation $\mathscr{H}^s(\Omega)$, however, the definition will be different and given later on. Notice that the slit-plane is not a Lipschitz domain.

We will also make use of convolution type operators A_ϕ on the real line with $\phi \in L^\infty_{loc}(\mathbb{R})$ given by

$$A_\phi g(x) = \mathscr{F}^{-1}_{\xi \mapsto x}\, \phi(\xi) \widehat{g}(\xi), \qquad x \in \mathbb{R}, \tag{4}$$

which act between (possibly different) Sobolev spaces. In particular, we consider symbols $\phi(\xi)$ equal to

$$t(\xi) = \sqrt{\xi^2 - k^2}, \quad \xi \in \mathbb{R}, \tag{5}$$

or

$$t_+^{1/2}(\xi) = (\xi + k)^{1/2}, \quad t_-^{1/2}(\xi) = (\xi - k)^{1/2}, \quad \xi \in \mathbb{R}, \tag{6}$$

where $k \in \mathbb{C}$ with $\mathrm{Im}(k) > 0$. The branch cuts are chosen vertically via ∞ not crossing the real axis, and the choice of the square-roots are such that the asymptotics of these three functions as $\xi \to +\infty$ is $t(\xi) \sim \xi$ and $t_\pm^{1/2}(\xi) \sim \sqrt{\xi}$, respectively. One of the basic properties of the operator A_ϕ is that A_ϕ is invertible provided so is the symbol and that in this case $A_\phi^{-1} = A_{\phi^{-1}}$. Moreover, we have

$$A_t : H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R}), \qquad A_{t_\pm^{1/2}} : H^s(\mathbb{R}) \to H^{s-1/2}(\mathbb{R})$$

and

$$A_{t_+^{1/2}} : H_+^s(\mathbb{R}) \to H_+^{s-1/2}(\mathbb{R}), \qquad A_{t_-^{1/2}} : H_-^s(\mathbb{R}) \to H_-^{s-1/2}(\mathbb{R})$$

and similar statements for the inverses.

3. The Dirichlet problem

In this section we are going to consider the wedge diffraction problem with Dirichlet conditions on both parts of the boundary of $\Omega = \Omega_{0,\alpha}$. Let us start with some preliminary considerations, which are for the most part known [9, 11].

Let $\varepsilon \in [0, 1/2[$. The Dirichlet problem consists of finding the (general) solution $u \in H^{1+\varepsilon}(\Omega)$ of

$$\begin{aligned} (\Delta + k^2)u &= 0 \text{ in } \Omega, \\ T_{0,\Gamma_j} u &= g_j \text{ on } \Gamma_j, \end{aligned} \tag{7}$$

where $g_j \in H^{1/2+\varepsilon}(\Gamma_j)$, $j = 1, 2$, is given boundary data, and T_{0,Γ_j} stands for the trace operators onto the corresponding parts Γ_j of the boundary. As we will see below, in this formulation the problem is not well-posed because g_1 and g_2 have to satisfy a certain compatibility condition. Uniqueness is known in this case [2, 5, 11].

The above problem can also be formulated for $\varepsilon \in]-1/2, 0[$. However, we are not aware of general uniqueness results (apart from special cases, such as the half-plane).

Let us for a moment be more general and assume that $\varepsilon \in]-1/2, 1/2[$. (The statements we are going to make for $\varepsilon \leqslant 0$ will be needed in the next section, where we consider the Neumann problem.) Then the trace operator

$$T_{0,\partial\Omega} : H^{1+\varepsilon}(\Omega) \to H^{1/2+\varepsilon}(\partial\Omega) \tag{8}$$

is well defined, linear, bounded, and surjective. The trace space $H^{1/2+\varepsilon}(\partial\Omega)$ can be defined using the Slobodetski norm. Moreover, there exist natural restrictions,

$$r_{0,\Gamma_j} : H^{1/2+\varepsilon}(\partial\Omega) \to H^{1/2+\varepsilon}(\Gamma_j), \qquad j = 1, 2,$$

and T_{0,Γ_j} is, by definition, equal to the composition of r_{0,Γ_j} with $T_{0,\partial\Omega}$.

Throughout the paper, we will identify Γ_j with $\mathbb{R}_+$, i.e., we will think of g_j in (7) as taken from $H^{1/2+\varepsilon}(\mathbb{R}_+)$. Moreover, we will think of T_{0,Γ_j} and r_{0,Γ_j} as acting into $H^{1/2+\varepsilon}(\mathbb{R}_+)$.

Given $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2$ it will be useful to define a function $\imath g$ on $\mathbb{R}$ by

$$\imath g(x) = \begin{cases} g_1(x) & \text{if } x > 0 \\ g_2(-x) & \text{if } x < 0 \end{cases}, \tag{9}$$

taking any value in $x = 0$ if $\varepsilon \in [0, 1/2[$ (see Figure 3). For $\varepsilon \in]-1/2, 1/2[$ the formula might be replaced by $\imath g = \ell_0 g_1 + \mathscr{J} \ell_0 g_2$ (using the reflection operator $\mathscr{J} f(x) = f(-x)$, $x \in \mathbb{R}$), i.e., by a continuous extension of $\imath g$ to (small) negative values of ε.

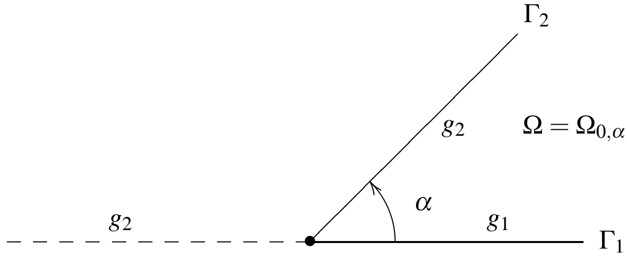

Figure 3: Identification of Dirichlet data on $\partial\Omega$ and functions on $\mathbb{R}$

The following fact characterizes the space $H^{1/2+\varepsilon}(\partial\Omega)$.

LEMMA 3.1. *Let $\varepsilon \in]-1/2, 1/2[$ and $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2$. Then the following conditions are equivalent:*

(DD-1) $r_{0,\Gamma_1} \mathfrak{g} = g_1$ *and* $r_{0,\Gamma_2} \mathfrak{g} = g_2$ *for some (unique)* $\mathfrak{g} \in H^{1/2+\varepsilon}(\partial\Omega)$,

(DD-2) $\imath g \in H^{1/2+\varepsilon}(\mathbb{R})$,

(DD-3) $g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$.

The last condition is the *compatibility condition* for the problem (7), which is redundant if and only if $\varepsilon \in]-1/2, 0[$. The above equivalencies suggest to define (for $\varepsilon \in]-1/2, 1/2[$) the Banach space

$$H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim} = \left\{ g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2 \text{ with } g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) \right\}$$

with the norm

$$\|g\|_{H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}} = \|\imath g\|_{H^{1/2+\varepsilon}(\mathbb{R})}.$$

Equivalent norms are given by $\|\mathfrak{g}\|_{H^{1/2+\varepsilon}(\partial\Omega)}$ and by

$$\|g_1+g_2\|_{H^{1/2+\varepsilon}(\mathbb{R}_+)}+\|g_1-g_2\|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)}.$$

Let us now return to the Dirichlet problem (7) and assume that $\varepsilon\in[0,1/2[$. The correct formulation of the problem (or, more precisely, the minimal image normalization in the sense of [25]) is to consider the trace operators restricted to the space $\mathscr{H}^{1+\varepsilon}(\Omega)$ of solutions of the Helmholtz equation, i.e., to consider the operator

$$\mathscr{T}^{\varepsilon}_{D,\Omega}:u\in\mathscr{H}^{1+\varepsilon}(\Omega)\mapsto(T_{0,\Gamma_1}u,T_{0,\Gamma_2}u)\in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}. \tag{10}$$

Obviously, this operator is linear, bounded, and injective, while it is not a-priori clear whether it is surjective.

Our goal is to show that this operator is surjective and to obtain an explicit representation formula for the inverse $\mathscr{K}^{\varepsilon}=(\mathscr{T}^{\varepsilon}_{D,\Omega})^{-1}$ in the case $\Omega=\Omega_{0,\alpha}$ with angles $\alpha=2\pi/n$. In fact, the surjectivity follows immediately form the representation formula, as does the fact that $\mathscr{T}^{\varepsilon}_{D,\Omega}$ and $\mathscr{K}^{\varepsilon}$ are linear homeomorphism between the appropriate spaces.

The construction of the representation formulas will be based on the special case of the upper-half plane $Q_{12}=\Omega^+$, where the solution to the Dirichlet problem is well known (see, e.g., [3, 23, 32]). Indeed the solution is given by

$$u(x)=(\mathscr{K}_{D,Q_{12}}f)(x)=\mathscr{F}^{-1}_{\xi\mapsto x_1}\,e^{-t(\xi)x_2}\widehat{f}(\xi),\qquad x=(x_1,x_2)\in Q_{12}.$$

Similarly, the solution in the lower-half plane $Q_{34}=\Omega^-$ is given by

$$u(x)=(\mathscr{K}_{D,Q_{34}}f)(x)=\mathscr{F}^{-1}_{\xi\mapsto x_1}\,e^{t(\xi)x_2}\widehat{f}(\xi),\qquad x=(x_1,x_2)\in Q_{34}.$$

By rotation we can generalize these formulas to the rotated half-planes $\Omega^{\pm}_{\alpha}$, which is precisely what we will need. Then these formulas read as follows:

$$\begin{aligned}&(\mathscr{K}_{D,\Omega^{\pm}_{\alpha}}f)(x)\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\exp[-i\xi(x_1\cos\alpha+x_2\sin\alpha)\mp t(\xi)(-x_1\sin\alpha+x_2\cos\alpha)]\widehat{f}(\xi)\,d\xi,\end{aligned} \tag{11}$$

where $x=(x_1,x_2)\in\Omega^{\pm}_{\alpha}$, $\alpha\in\mathbb{R}$, and $f\in H^{1/2+\varepsilon}(\mathbb{R})$ is given.

PROPOSITION 3.2. *For $\varepsilon\in[0,1/2[$, $\alpha\in\mathbb{R}$, and $f\in H^{1/2+\varepsilon}(\mathbb{R})$, the functions*

$$u^+=\mathscr{K}_{D,\Omega^+_{\alpha}}f,\qquad u^-=\mathscr{K}_{D,\Omega^-_{\alpha}}f$$

represent the unique solutions of the Helmholtz equation in $H^{1+\varepsilon}(\Omega^{\pm}_{\alpha})$, respectively, satisfying the Dirichlet boundary conditions

$$T_{0,\partial\Omega^+_{\alpha}}u^+=J_{\alpha}f,\qquad T_{0,\partial\Omega^-_{\alpha}}u^-=J_{\alpha}f.$$

(identifying f as a function on $\partial\Omega^{\pm}_0$). Moreover, the mappings

$$\mathscr{K}_{D,\Omega^{\pm}_{\alpha}}:H^{1/2+\varepsilon}(\mathbb{R})\to\mathscr{H}^{1+\varepsilon}(\Omega^{\pm}_{\alpha})$$

are linear homeomorphisms.

Proof. The Helmholtz equation is invariant for linear motions (rotation, reflection, translation). Thus we obtain the representation formulas by rotation from the solution in $\Omega^{\pm}$,

$$u^{+}(x) = (J_{\alpha}\mathscr{K}_{D,Q_{12}}f)(x) = (\mathscr{K}_{D,Q_{12}}f)(\mathscr{R}_{\alpha}^{-1}x), \qquad x \in \Omega_{\alpha}^{+}.$$

A similar formula holds for u^{-}. The fact that $\mathscr{K}_{D,\Omega_{\alpha}^{\pm}}$ are linear homeomorphisms follows also from the known case of the upper/lower-half plane. $\square$

We remark that Proposition 3.2 actually holds for all $\varepsilon \in]-1/2,+\infty[$.

The solutions obtained by the operators $\mathscr{K}_{D,\Omega_{\alpha}^{\pm}}$ enjoy a couple of basic properties, which can be verified straightforwardly. These facts will be used frequently in what follows and read as

$$\mathscr{K}_{D,\Omega_{\alpha}^{+}}f = \mathscr{K}_{D,\Omega_{\alpha\pm\pi}^{-}}\widetilde{f}, \qquad \widetilde{f}(x) = f(-x), \tag{12}$$

$$T_{0,\Sigma_{\alpha}}\mathscr{K}_{D,\Omega_{\alpha}^{\pm}}f = r_{+}f, \tag{13}$$

$$T_{0,\Sigma_{\gamma}}(\mathscr{K}_{D,\Omega_{\alpha}^{+}} - \mathscr{K}_{D,\Omega_{\beta}^{-}})f = 0 \tag{14}$$

if $\gamma = (\alpha+\beta)/2$ and $0 \leqslant \beta - \alpha \leqslant 2\pi$. Recall the definition (2) and notice that the trace operators $T_{0,\Sigma_{\gamma}} : H^{1+\varepsilon}(\Omega) \to H^{1/2+\varepsilon}(\mathbb{R}_{+})$, with the identification $\mathbb{R}_{+} \cong \Sigma_{\gamma}$, are well defined, bounded linear operators as long as $\Sigma_{\gamma} \subseteq \overline{\Omega}$.

3.1. The case $\alpha = 2\pi/n$, $n \equiv 2 \mod 4$

The special case $n = 2$ is already solved by Proposition 3.2, but it is also included in the following theorem.

THEOREM 3.3. *Let $\alpha = 2\pi/n$ with $n \equiv 2 \mod 4$, and let $\varepsilon \in [0,1/2[$. The Dirichlet problem for the Helmholtz equation in $\Omega = \Omega_{0,\alpha}$ in weak formulation with Dirichlet data $g \in H^{1/2+\varepsilon}(\mathbb{R}_{+})^{2}_{\sim}$ is uniquely solved by $u = \mathscr{K}^{\varepsilon}g$, where*

$$\mathscr{K}^{\varepsilon}g = r_{\Omega}\left(\mathscr{K}_{D,\Omega_{0}^{+}} + \sum_{j=1}^{\frac{n-2}{4}}\left(\mathscr{K}_{D,\Omega_{-2j\alpha}^{+}} - \mathscr{K}_{D,\Omega_{2j\alpha}^{-}}\right)\right)\imath g. \tag{15}$$

Moreover, the operator $\mathscr{K}^{\varepsilon} : H^{1/2+\varepsilon}(\mathbb{R})^{2}_{\sim} \to \mathscr{H}^{1+\varepsilon}(\Omega)$ is a linear homeomorphism.

Proof. It is obvious that given $g = (g_1,g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_{+})^{2}_{\sim}$, the function $u = \mathscr{K}^{\varepsilon}g$ is a solution, i.e., it belongs to $\mathscr{H}^{1+\varepsilon}(\Omega)$. Notice that Ω is a subset of all the half-planes that appear in (15). Moreover, the operator $\mathscr{K}^{\varepsilon}$ is bounded.

The boundary condition on $\Gamma_1 = \Sigma_0$ is satisfied because $T_{0,\Gamma_1}\mathscr{K}_{D,\Omega_{0}^{+}}\imath g = g_1$ due to (13), and $T_{0,\Gamma_1}\mathscr{K}_{D,\Omega_{-2j\alpha}^{+}} = T_{0,\Gamma_1}\mathscr{K}_{D,\Omega_{2j\alpha}^{-}}$ due to (14). Note that $0 < 4j\alpha < 2\pi$.

In order to get the corresponding result for the boundary condition on $\Gamma_2 = \Sigma_{\alpha}$ note that

$$\mathscr{K}^{\varepsilon}g = r_{\Omega}\left(\mathscr{K}_{D,\Omega_{-(n-2)\alpha/2}^{+}} + \sum_{j=1}^{\frac{n-2}{4}}\left(\mathscr{K}_{D,\Omega_{2(1-j)\alpha}^{+}} - \mathscr{K}_{D,\Omega_{2j\alpha}^{-}}\right)\right)\imath g$$

by a change of variables. Moreover, $\Omega^+_{-(n-2)\alpha/2} = \Omega^-_\alpha$ and $0 < 2(2j-1)\alpha < 2\pi$. Now the same kind of reasoning as before can be applied.

Hence we have shown that the (unique) solution of the Dirichlet problem with boundary data g is given by $u = \mathscr{K}^\varepsilon g$. Since the operators $\mathscr{T}^\varepsilon_{D,\Omega}$ and $\mathscr{K}^\varepsilon$ are bounded, we can directly conclude that $\mathscr{K}^\varepsilon = (\mathscr{T}^\varepsilon_{D,\Omega})^{-1}$ is a linear homeomorphism. □

EXAMPLE 3.4. Figure 4 illustrates the situation of Theorem 3.3 for $n = 6$, where the solution is composed of three terms defined in half-planes whose intersection is $\Omega_{0,\pi/3}$.

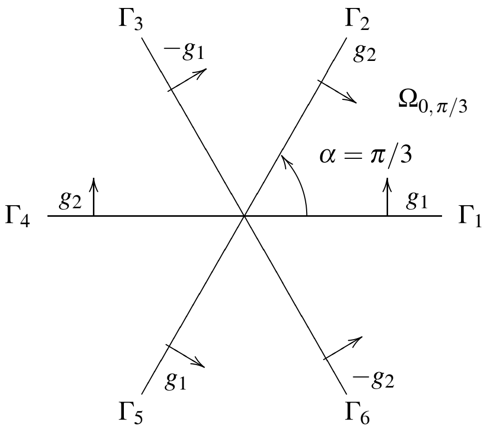

Figure 4: Solution of the DD problem for $\alpha = \pi/3$ by superposition of three half-plane solutions

3.2. The case $\alpha = 2\pi/n$, $n \equiv 0 \mod 4$

For sake of illustration, let us recall the simplest case. The Dirichlet problem for a quarter-plane ($n = 4$, $\varepsilon = 0$) is known to be uniquely solvable [20] by

$$u = \mathscr{K}_{D,Q_1}(g_1, g_2) = r_{Q_1}(\mathscr{K}_{D,\Omega^+}\ell^o g_1 - \mathscr{K}_{D,\Omega^+_{-\pi/2}}\ell^o g_2), \tag{16}$$

where the last term coincides with $+\mathscr{K}_{D,\Omega^-_{\pi/2}}\ell^o g_2$ due to (12).

We remark that $\ell^o g_j \in H^{1/2}(\mathbb{R})$ if and only if $g_j \in \widetilde{H}^{1/2}(\mathbb{R}_+)$, which is a dense subspace of $H^{1/2}(\mathbb{R})$. Hence, the two parts of (16) represent (possibly) unbounded and densely defined operators into $\mathscr{H}^1(Q_{12})$ or $\mathscr{H}^1(Q_{41})$, respectively, and into $\mathscr{H}^1(Q_1)$ after restriction, which add up to a bounded operator

$$\mathscr{K}_{D,Q_1} : H^{1/2}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^1(Q_1).$$

The following result is a direct generalization.

THEOREM 3.5. *Let* $\alpha = 2\pi/n$ *with* $n \equiv 0 \mod 4$, *and* $\Omega = \Omega_{0,\alpha}$. *For* $g = (g_1, g_2) \in \widetilde{H}^{1/2}(\mathbb{R}_+)^2$ *define the linear operator*

$$\widetilde{\mathscr{K}} g = r_\Omega \left(\sum_{j=1}^{\frac{n}{4}} \left(\mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}} \ell^o g_1 + \mathscr{K}_{D,\Omega^-_{(2j-1)\alpha}} \ell^o g_2 \right) \right). \tag{17}$$

Then this operator can be extended by continuity to a linear homeomorphism

$$\mathscr{K}^0 : H^{1/2}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^1(\Omega).$$

For $\varepsilon \in]0,1/2[$, *the restriction of* $\mathscr{K}^0$ *to* $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ *maps into* $\mathscr{H}^{1+\varepsilon}(\Omega)$, *and*

$$\mathscr{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

is also a linear homeomorphism.

Moreover, the Dirichlet problem for the Helmholtz equation in Ω *in weak formulation with Dirichlet data* $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, $\varepsilon \in [0,1/2[$, *is uniquely solved by* $u = \mathscr{K}^\varepsilon g$.

Before we proceed to the proof, let us remark that $\widetilde{H}^{1/2}(\mathbb{R}_+)^2$ is a dense subspace of $H^{1/2}(\mathbb{R}_+)^2_\sim$. Hence the continuous extension $\mathscr{K}^0$ of $\widetilde{\mathscr{K}}$ is unique (provided it exists). Notice, however, that $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$ is a proper closed subspace of $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ for $\varepsilon \in]0,1/2[$. Hence $\mathscr{K}^\varepsilon$ cannot be defined in the same way as $\mathscr{K}^0$.

The definition of $\mathscr{K}^0$ and $\mathscr{K}^\varepsilon$ via continuous continuation and restriction is somewhat cumbersome. It arises the natural question whether each solution $u = \mathscr{K}^\varepsilon g$ for $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ is given by formula (17) when we think of the operators appearing in this formula as acting between appropriately modified spaces. A moment's thought shows that this is true. Namely, think of $\widetilde{\mathscr{K}}$ as a bounded linear operator from $\widetilde{H}^{1/2-\delta}(\mathbb{R}_+)^2 = H^{1/2-\delta}(\mathbb{R}_+)^2_\sim \supset H^{1/2}(\mathbb{R}_+)^2_\sim$ into $H^{1-\delta}(\Omega)$ for some $\delta \in]0,1/2[$. With this re-interpretation of $\widetilde{\mathscr{K}}$ it is almost immediately clear that $\widetilde{\mathscr{K}} g = \mathscr{K}^\varepsilon g$ for all $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$.

Proof. For $\varepsilon \in [0,1/2[$, if we take $g \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$, then both $\ell^o g_1$ and $\ell^o g_2$ belong to $H^{1/2+\varepsilon}(\mathbb{R})$. Hence $\widetilde{\mathscr{K}}$ is a linear operator from $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$ into $H^{1+\varepsilon}(\Omega)$. In fact, $\widetilde{\mathscr{K}} g$ is a solution of the Helmholtz equation, and thus it belongs to $\mathscr{H}^{1+\varepsilon}(\Omega)$.

Next we are going to verify that $T_{0,\Gamma_1}\widetilde{\mathscr{K}} g = g_1$ and $T_{0,\Gamma_2}\widetilde{\mathscr{K}} g = g_2$ for $g \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$. Let us focus on the trace on the boundary Γ_1. The trace on the boundary Γ_2 can be computed analogously or deduced by a symmetry argument.

Notice that $\Gamma_1 = \Sigma_0$. Using (13) the contribution of the first term of the sum in (17) for $j = 1$ gives

$$T_{0,\Gamma_1} \mathscr{K}_{D,\Omega_0^+} \ell^o g_1 = g_1.$$

The remaining sum over the first terms can be expressed as

$$\sum_{j=2}^{\frac{n}{4}} T_{0,\Gamma_1} \mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}} \ell^o g_1 = \sum_{j=2}^{\frac{n}{4}} T_{0,\Gamma_1} \mathscr{K}_{D,\Omega^+_{2(j-1-\frac{n}{4})\alpha}} \ell^o g_1$$

by making a change of variables $j \mapsto \frac{n}{4}+2-j$. Using $\frac{n\alpha}{2}=\pi$, the fact that $\ell^o g_1$ is odd, and (12) we see that this equals

$$-\sum_{j=2}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^-_{2(j-1)\alpha}}\ell^o g_1 = -\sum_{j=2}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}}\ell^o g_1,$$

where we used (14) and $0<2(j-1)\alpha<\pi$ in order to derive the last identity. We thus can conclude that the sum with which we started must be zero.

The contribution involving the functions g_2 is also zero. Indeed, making a change of variables $j \mapsto \frac{n}{4}+1-j$ we obtain

$$\sum_{j=1}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^-_{(2j-1)\alpha}}\ell^o g_2 = \sum_{j=1}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^-_{(\frac{n}{2}+1-2j)\alpha}}\ell^o g_2.$$

By the same kind of arguments as before this equals

$$-\sum_{j=1}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^+_{(1-2j)\alpha}}\ell^o g_2 = -\sum_{j=1}^{\frac{n}{4}} T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^-_{(1-2j)\alpha}}\ell^o g_2,$$

where for the last equality we used (14) and $0<(2j-1)\alpha<\pi$.

So far we have shown that for $\varepsilon\in[0,1/2[$ the operator $\widetilde{\mathscr{K}}$ maps $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$ into $\mathscr{H}^{1+\varepsilon}(\Omega)$, and that $\mathscr{T}^\varepsilon\widetilde{\mathscr{K}}|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2} = I_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2}$, where $\mathscr{T}^\varepsilon=\mathscr{T}^\varepsilon_{D,\Omega}$ is the corresponding trace operator defined by (10).

Now we want to show that $\mathscr{T}^\varepsilon$ is surjective, i.e., that there exists a solution to the Dirchlet problem for all data $(g_1,g_2)\in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. We reduce the problem to a semi-homogeneous problem by the substitution $u=v+w$ where v is a solution of the Helmholtz equation in the half-plane Ω_0^+, which covers Ω and whose boundary is the union of Γ_1 and $-\Gamma_1$. We require that $v=g_1$ on Γ_1. A construction of v is easily done by Proposition 3.2, e.g., we may require in addition that $v=g_1$ on $-\Gamma_1$ noting that $\ell^e g_1\in H^{1/2+\varepsilon}(\mathbb{R})$. In other words, $v=r_\Omega\mathscr{K}_{D,\Omega_0^+}\ell^e g_1$. Now let ϕ be the trace of v on Γ_2. Because of the compatibility conditions it follows that $\phi-g_1$ is in $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$. The remaining problem is to find w such that $w=0$ on Γ_1 and $w=g_2-\phi$ on Γ_2. Notice that the compatibility condition for this problem is fulfilled because $g_2-\phi=(g_2-g_1)+(g_1-\phi)$ is in the tilde space. Hence using what we have shown in the first part of the proof, we can conclude that $\mathscr{T}^\varepsilon$ is surjective.

It follows that the inverse of $\mathscr{T}^\varepsilon$ exists and is bounded by the inverse mapping theorem. We obtain

$$\widetilde{\mathscr{K}}|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2} = (\mathscr{T}^\varepsilon)^{-1}|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2},$$

from which we conclude (for $\varepsilon=0$) that $\widetilde{\mathscr{K}}$ has a unique continuous extension $\mathscr{K}^0 : H^{1/2}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^1(\Omega)$.

Furthermore, we obtain now the identity $\mathscr{K}^0\mathscr{T}^0 = I_{\mathscr{H}^1(\Omega)}$. Restricting it to the space $\mathscr{H}^{1+\varepsilon}(\Omega)$ it follows that $\mathscr{K}^\varepsilon\mathscr{T}^\varepsilon = I_{\mathscr{H}^{1+\varepsilon}(\Omega)}$ by definition of $\mathscr{K}^\varepsilon$. Thus $\mathscr{K}^\varepsilon$

is the left inverse of $\mathscr{T}^{\varepsilon}$. Because $\mathscr{T}^{\varepsilon}$ is bounded and invertible, we conclude again by the inverse mapping theorem that $\mathscr{K}^{\varepsilon}$ is bounded and in fact a linear homeomorphism with inverse $\mathscr{T}^{\varepsilon}$. As a consequence the solution to the Dirichlet boundary value problem with data in $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ is uniquely given by $u = \mathscr{K}^{\varepsilon} g$. $\square$

3.3. The case $\alpha = 2\pi/n$, $n \equiv 1 \mod 2$

In order to deal with the case of odd n, we have to rely on the solutions of the Helmholtz equation in the (rotated) slit-plane with Dirichlet boundary conditions [3, 31]. These solutions are given by different surface potentials, and will be called Sommerfeld potentials with Dirichlet density.

Let us first focus on the usual (non-rotated) slit-plane $\Omega = \mathbb{R}^2 \backslash \overline{\Sigma}$, where $\Sigma = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 = 0\}$. The weak solutions of the Helmholtz equation in Ω can be composed of the solutions in the upper and lower half-plane $\Omega^{\pm}$, satisfying the appropriate compatibility (or jump) conditions on $-\Sigma$. More precisely, for $\varepsilon \in [-1/2, 1/2[$ we define

$$\begin{aligned}\mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma}) = \Big\{ & u \in L^2(\mathbb{R}^2) : u^{\pm} = u|_{\Omega^{\pm}} \in \mathscr{H}^{1+\varepsilon}(\Omega^{\pm}), \\ & u_0^+ - u_0^- \in H_+^{1/2+\varepsilon}(\mathbb{R}), \quad u_1^+ + u_1^- \in H_+^{-1/2+\varepsilon}(\mathbb{R}) \Big\}\end{aligned}$$

where

$$u_0^{\pm} = T_{0,\mathbb{R}} u^{\pm} \in H^{1/2+\varepsilon}(\mathbb{R}), \qquad u_1^{\pm} = T_{1,\mathbb{R}} u^{\pm} = \pm T_{0,\mathbb{R}} \frac{\partial u^{\pm}}{\partial x_2} \in H^{-1/2+\varepsilon}(\mathbb{R})$$

are the Dirichlet and Neumann data of $u^{\pm}$ on the boundary $\mathbb{R} = \partial\Omega^{\pm}$ of the upper and lower half-plane, respectively. Recall (8) for the definition of $T_{0,\mathbb{R}}$, and see (26) and (27) in the next section for $T_{1,\mathbb{R}}$.

It is well known that the jump conditions imply that $u \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$ satisfies the Helmholtz equation in the sense of a distribution in $\mathscr{D}'(\Omega)$ (more precisely, in any Lipschitz subdomain of Ω),

$$(\Delta + k^2) u = 0. \tag{18}$$

Now the Dirichlet problem for the slit-plane $\mathbb{R}^2 \backslash \overline{\Sigma}$ consists of finding $u \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$ such that

$$T_{0,\Sigma^+} u = f_1, \qquad T_{0,\Sigma^-} u = f_2, \tag{19}$$

where (f_1, f_2) are the given boundary data, and the operators $T_{0,\Sigma^{\pm}}$ are defined by

$$T_{0,\Sigma^{\pm}} u = r_+ u_0^{\pm} = r_+ T_{0,\mathbb{R}} u|_{\Omega^{\pm}} \in H^{1/2+\varepsilon}(\mathbb{R}_+). \tag{20}$$

for any given $u \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$.

It follows immediately from the jump conditions that (f_1, f_2) must be taken from $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. Hence we encounter the same compatibility conditions as for the cones $\Omega_{0,\alpha}$ with $0 < \alpha < 2\pi$.

As for the rotated slit-planes $\mathbb{R}^2\backslash\overline{\Sigma}_\alpha$, where $\Sigma_\alpha = \mathscr{R}_\alpha\Sigma$, we define

$$\mathscr{H}^{1+\varepsilon}(\mathbb{R}^2\backslash\overline{\Sigma}_\alpha) = J_\alpha\mathscr{H}^{1+\varepsilon}(\mathbb{R}^2\backslash\overline{\Sigma}),$$

and the corresponding trace operators by $T_{0,\Sigma_\alpha^\pm}u = T_{0,\Sigma^\pm}J_\alpha^{-1}u$.

Recall the definitions (3), (4) and (6), and define the bounded linear operator

$$\Pi_{1/2} = A_{t_-^{-1/2}}\ell_0 r_+ A_{t_-^{1/2}} : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \qquad s \in]0,1[.$$

Moreover, introduce the operator matrices (thought of as "symmetrization operators")

$$\Upsilon_D = \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \quad , \quad \Upsilon_D^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$$

acting on appropriate spaces. Finally, we will use $\ell\phi$ to denote any extension of $\phi \in H^s(\mathbb{R}_+)$ to $H^s(\mathbb{R})$. The symbol ℓ will be used when the term does not depend on the choice of the extension.

DEFINITION 3.6. For $\varepsilon \in [0,1/2[$, $\alpha \in \mathbb{R}$, and $f \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ the function

$$u(x) = (\mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} f)(x) = \begin{cases} (\mathscr{K}_{D,\Omega_\alpha^+} u_0^+)(x) & x \in \Omega_\alpha^+ \\ (\mathscr{K}_{D,\Omega_\alpha^-} u_0^-)(x) & x \in \Omega_\alpha^- \end{cases} \tag{21}$$

with

$$\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} = \Upsilon_D^{-1} \begin{pmatrix} I & 0 \\ 0 & \Pi_{1/2} \end{pmatrix} \begin{pmatrix} \ell_0 & 0 \\ 0 & \ell \end{pmatrix} \Upsilon_D f \tag{22}$$

is called a Sommerfeld potential with Dirichlet density f.

PROPOSITION 3.7. *Let $\varepsilon \in [0,1/2[$, $\alpha \in \mathbb{R}$, and $f = (f_1, f_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. The Dirichlet problem for the Helmholtz equation in $\mathbb{R}^2\backslash\overline{\Sigma}_\alpha$ is uniquely solved by the Sommerfeld potential $u = \mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} f$. Moreover, the operator*

$$\mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} : H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2\backslash\overline{\Sigma}_\alpha)$$

is a linear homeomorphism.

Proof. This is mainly a modification of formers results [23, 31]. Without loss of generality we can assume that $\alpha = 0$. The uniqueness of the Dirichlet problem in the slit-plane is well-known [31].

Suppose that u is given by (21) and (22). First let us rewrite (22) as

$$u_0^+ - u_0^- = \ell_0(f_1 - f_2)$$

and

$$u_0^+ + u_0^- = A_{t_-^{-1/2}}\ell_0 r_+ A_{t_-^{1/2}}\ell(f_1 + f_2).$$

Keeping track of the intermediated spaces between which the operators in the above expression act and using the fact that $\varepsilon \in [0,1/2[$, one can conclude that the map from $f \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ to $(u_0^+, u_0^-) \in H^{1/2+\varepsilon}(\mathbb{R})^2$ is well defined, linear, and bounded.

Moreover, $r_-(u_0^+ - u_0^-) = 0$ and

$$r_- A_t(u_0^+ + u_0^-) = r_- A_{t_+^{1/2}} \ell_0 r_+ A_{t_-^{1/2}} \ell(f_1 + f_2) = 0$$

because $A_{t_+^{1/2}} \ell_0$ acts from $H^\varepsilon(\mathbb{R}_+) = \widetilde{H}^\varepsilon(\mathbb{R}_+)$ into $H_+^{-1/2+\varepsilon}(\mathbb{R})$.

If we put $u^\pm = u|_{\Omega^\pm}$, then $u^\pm \in \mathscr{H}^{1+\varepsilon}(\Omega^\pm)$, and the Dirichlet and Neumann boundary data of u^+ and u^- (on $\mathbb{R} = \partial\Omega^\pm$) are just given by

$$T_{0,\mathbb{R}} u^\pm = u_0^\pm, \qquad T_{1,\mathbb{R}} u^\pm = -A_t u_0^\pm = u_1^\pm.$$

These facts follow from the representation formula (21) and the definition of $\mathscr{K}_{D,\Omega^\pm}$. From the statements made in the previous paragraph we obtain

$$\begin{aligned} u_0^+ - u_0^- &= T_{0,\mathbb{R}} u^+ - T_{0,\mathbb{R}} u^- \in H_+^{1/2+\varepsilon}(\mathbb{R}), \\ u_1^+ + u_1^- &= T_{1,\mathbb{R}} u^+ + T_{1,\mathbb{R}} u^- \in H_+^{-1/2+\varepsilon}(\mathbb{R}), \end{aligned}$$

and conclude that $u \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$. Because all mappings encountered are bounded, we infer that $\mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}}$ is bounded from $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ into $\mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$.

Finally, let us verify that u satisfies the correct boundary conditions on the two banks of Σ. We clearly have $r_+(u_0^+ - u_0^-) = f_1 - f_2$, and moreover

$$\begin{aligned} r_+(u_0^+ + u_0^-) &= r_+ A_{t_-^{-1/2}} \ell_0 r_+ A_{t_-^{1/2}} \ell(f_1 + f_2) = r_+ A_{t_-^{-1/2}} A_{t_-^{1/2}} \ell(f_1 + f_2) \\ &= r_+ \ell(f_1 + f_2) = f_1 + f_2. \end{aligned}$$

Here we used the fact that the image of $I - \ell_0 r_+$ in $H^\varepsilon(\mathbb{R})$ is $H_-^\varepsilon(\mathbb{R})$, which is sent by $A_{t_-^{-1/2}}$ into $H_-^{1/2+\varepsilon}(\mathbb{R})$. Consequently we obtain $r_+ u_0^+ = f_1$ and $r_+ u_0^- = f_2$, which are the correct Dirichlet boundary conditions.

Thus we have proved that $\mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}}$ has as its left inverse the trace operator, which acts from $\mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$ into $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ and is bounded and injective. From this we can conclude that both operators are inverse to each other and are linear homeomorphisms. □

Next we are going to study the superposition of Sommerfeld potentials with symmetry properties. For $f \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ consider the involution

$$f^\# = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^\# = \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} \tag{23}$$

(which corresponds to the flip operator when making the identification (9)). For convenience let

$$f^{\#j} = \begin{cases} f^\# & \text{if } j \text{ is odd} \\ f & \text{if } j \text{ is even.} \end{cases}$$

We proceed with an auxiliary result similar to formula (14).

LEMMA 3.8. *Let $\alpha,\beta\in\mathbb{R}$ be such that $0<|\beta-\alpha|<2\pi$, let $\varepsilon\in[0,1/2[$, and $f\in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. Then*

$$\left(\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}f-\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\beta}f^{\#}\right)(x)=0$$

for $x\in\Sigma_{\frac{\alpha+\beta}{2}}\cup\Sigma_{\frac{\alpha+\beta}{2}+\pi}$ in the sense of the trace theorem, i.e.,

$$T_{0,\Sigma_\gamma}\left(\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}f-\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\beta}f^{\#}\right)=0$$

for $\gamma=\frac{\alpha+\beta}{2}+j\pi,\ j\in\{0,1\}$.

Proof. The main observation is that if the operator (22) sends $f=(f_1,f_2)$ to $u_0=(u_0^+,u_0^-)$, then it sends $f^{\#}=(f_2,f_1)$ to $u_0^{\#}=(u_0^-,u_0^+)$. From that point on, one only needs to apply formula (14). We leave the details to the reader. □

THEOREM 3.9. *Let $\alpha=2\pi/n$ with $n\equiv 1 \mod 2$, $n\geqslant 3$, and let $\varepsilon\in[0,1/2[$. The Dirichlet problem for the Helmholtz equation in $\Omega=\Omega_{0,\alpha}$ in weak formulation with Dirichlet data $g\in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ is uniquely solved by $u=\mathscr{K}^\varepsilon g$, where*

$$\mathscr{K}^\varepsilon g=r_\Omega\left(\sum_{j=1}^{n}(-1)^{j+1}\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_{j\alpha}}g^{\#(j+1)}\right).\tag{24}$$

Moreover, the operator $\mathscr{K}^\varepsilon: H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim\to\mathscr{H}^{1+\varepsilon}(\Omega)$ is a linear homeomorphism.

Proof. We first observe that $u=\mathscr{K}^\varepsilon g$ is a solution of the Helmholtz equation in $\mathscr{H}^{1+\varepsilon}(\Omega)$. In order to verify that the Dirichlet conditions are satisfied we split the formula into

$$\mathscr{K}^\varepsilon g=r_\Omega\left(\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_0}g+\sum_{j=1}^{n-1}(-1)^{j+1}\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_{j\alpha}}g^{\#(j+1)}\right).$$

If we take the trace on $\Gamma_1=\Sigma_0$ we see that the first term gives g_1, while the term that consists of the sum from $j=1$ to $n-1$ gives zero. This can be seen by similar cancelation arguments as in the proof of Theorem 3.5 and by using Lemma 3.8. As for the trace on $\Gamma_2=\Sigma_\alpha$, we split the expression into

$$\mathscr{K}^\varepsilon g=r_\Omega\left(\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}g+\sum_{j=2}^{n}(-1)^{j+1}\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_{j\alpha}}g^{\#(j+1)}\right)$$

and proceed analogously. We conclude that $u=\mathscr{K}^\varepsilon g$ is the (unique) solution of the Dirichlet problem. Because $\mathscr{K}^\varepsilon$ is bounded, it follows that the operator is a linear homeomorphism. □

REMARK. The operators $\mathscr{K}^\varepsilon$ defined in Theorem 3.3, Theorem 3.5, and Theorem 3.9 can be defined also for values $\varepsilon\in]-1/2,0[$. In fact, they represent bounded linear operators

$$\mathscr{K}^\varepsilon: H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim\to\mathscr{H}^{1+\varepsilon}(\Omega),$$

and $u = \mathscr{K}^\varepsilon g$ with $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ represents a solution of the Helmholtz equation in $H^{1+\varepsilon}(\Omega)$ with trace given by $g = (g_1, g_2)$.

The crucial point here is that Proposition 3.2 and Proposition 3.7 also hold for $\varepsilon \in]-1/2, 0[$ without any changes. The statements corresponding to Theorem 3.3 and Theorem 3.9 can be proved in the very same way. The analogue of Theorem 3.5 and its proof is even easier because we do not have compatibility conditions, i.e., $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim = H^{1/2+\varepsilon}(\mathbb{R}_+)^2 = \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$. In particular, one can directly define $\mathscr{K}^\varepsilon$ by (17) without the detour of continuous extension. This is because the odd-extension operators ℓ^o are continuous on the appropriate spaces. Moreover, the boundedness of the various operators $\mathscr{K}^\varepsilon$ follows from the boundedness of the operators $\mathscr{K}_{D,\Omega^\pm_\alpha}$ and $\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}$.

An issue which we do not want to take up to discuss is the uniqueness of the Dirichlet problem in the case $\varepsilon \in]-1/2, 0[$, or, equivalently, the injectiveness of $\mathscr{T}^\varepsilon_{D,\Omega}$ and the surjectiveness of the operators $\mathscr{K}^\varepsilon$. In case of the half-plane and the slit-plane the Dirichlet problem is known to be unique also for $\varepsilon \in]-1/2, 0[$.

4. The Neumann problem

Let us now consider the Neumann problem for the Helmholtz equation [11, 20] on the wedge $\Omega = \Omega_{0,\alpha}$. Given data $g_j \in H^{-1/2+\varepsilon}(\Gamma_j)$ and $\varepsilon \in [0, 1/2[$, the Neumann problem consists of finding all $u \in H^{1+\varepsilon}(\Omega)$ such that

$$\begin{aligned} (\Delta + k^2)u &= 0 \text{ in } \Omega, \\ T_{1,\Gamma_j} u &= g_j \text{ on } \Gamma_j, \end{aligned} \tag{25}$$

where formally $T_{1,\Gamma_j} = T_{0,\Gamma_j} \frac{\partial}{\partial n_j}$ and the normal derivative n_j on Γ_j is directed into the interior of Ω. Note that $T_{1,\Gamma_j} u$ does not exist for arbitrary $u \in H^{1+\varepsilon}(\Omega)$, but it exists for each weak solution of the Helmholtz equation $u \in \mathscr{H}^{1+\varepsilon}(\Omega)$ in a strong or special Lipschitz domain [11].

More precisely, we can define

$$T_{1,\partial\Omega} : \mathscr{H}^{1+\varepsilon}(\Omega) \to H^{-1/2+\varepsilon}(\partial\Omega) \tag{26}$$

by

$$\langle T_{1,\partial\Omega} u \,|\, T_{0,\partial\Omega} v \rangle = \int_\Omega (k^2 u(x) v(x) - \nabla u(x) \cdot \nabla v(x))\, dx, \quad v \in H^{1-\varepsilon}(\Omega). \tag{27}$$

This definition is motivated by Green's formula and makes sense for $\varepsilon \in]-1/2, 1/2[$. Here the trace space is defined as a dual space,

$$H^{-s}(\partial\Omega) = (H^s(\partial\Omega))', \qquad s > 0.$$

There exist natural "restrictions" ($j = 1, 2$)

$$r_{1,\Gamma_j} : \mathfrak{g} \in H^{-1/2+\varepsilon}(\partial\Omega) \to g_j \in H^{-1/2+\varepsilon}(\Gamma_j) = (\widetilde{H}^{1/2-\varepsilon}(\Gamma_j))'$$

defined by

$$\langle g_j \,|\, \phi \rangle = \langle \mathfrak{g} \,|\, \ell_{0,\Gamma_j}\phi \rangle, \quad \phi \in \widetilde{H}^{1/2-\varepsilon}(\Gamma_j),$$

with $\ell_{0,\Gamma_j} : \widetilde{H}^{1/2-\varepsilon}(\Gamma_j) \to H^{1/2-\varepsilon}(\partial\Omega)$ denoting the extension-by-zero operator, whose definition is justified by the equivalencies (DD-1)–(DD-3) of Lemma 3.1. The operator T_{1,Γ_j} is, by definition, the composition of r_{1,Γ_j} with $T_{1,\partial\Omega}$.

The analogue of the mapping ι defined by (9) is now (as we are dealing with distributions) slightly more tedious to define. For

$$g = (g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2 \cong (\widetilde{H}^{1/2-\varepsilon}(\mathbb{R}_+))' \dotplus (\widetilde{H}^{1/2-\varepsilon}(\mathbb{R}_+))',$$

we put

$$\langle \iota g \,|\, \phi \rangle = \langle g_1 \,|\, \phi_+ \rangle + \langle g_2 \,|\, \phi_- \rangle,$$

where

$$\phi = \ell_0\phi_+ + \ell_0\phi_- \in H_+^{1/2-\varepsilon}(\mathbb{R}) + H_-^{1/2-\varepsilon}(\mathbb{R}) \subseteq H^{1/2-\varepsilon}(\mathbb{R}).$$

The functional ιg may or may not extend by continuity to a distribution in $H^{-1/2+\varepsilon}(\mathbb{R}) = (H^{1/2-\varepsilon}(\mathbb{R}))'$. However, due to density of the last inclusion, the extension is unique for $\varepsilon \in [0,1/2[$ if it exists. Notice that this fact is no longer true for $\varepsilon \in]-1/2,0[$. Hence for this and some other reasons, which will occasionally be mentioned, we will mostly restrict ourselves to the case $\varepsilon \in [0,1/2[$.

LEMMA 4.1. *For* $\varepsilon \in [0,1/2[$ *and* $g = (g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2$*, the following three statements are equivalent:*

(NN-1) $r_{1,\Gamma_1}\mathfrak{g} = g_1$ *and* $r_{1,\Gamma_2}\mathfrak{g} = g_2$ *for some (unique)* $\mathfrak{g} \in H^{-1/2+\varepsilon}(\partial\Omega)$*,*

(NN-2) $\iota g \in H^{-1/2+\varepsilon}(\mathbb{R})$*,*

(NN-3) $g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$*.*

The last condition is redundant for $\varepsilon \in]0,1/2[$. Thus we have only a *compatibility condition* in the case $\varepsilon = 0$. For $\varepsilon \in]-1/2,0[$ the above characterization fails. We define the Banach space

$$H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim = \Big\{ g = (g_1,g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2 \text{ with } g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+) \Big\},$$

with the norm $\|g\|_{H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim} = \|\iota g\|_{H^{-1/2+\varepsilon}(\mathbb{R})}$. Equivalent norms are given by $\|\mathfrak{g}\|_{H^{-1/2+\varepsilon}(\partial\Omega)}$ and $\|g_1 - g_2\|_{H^{-1/2+\varepsilon}(\mathbb{R}_+)} + \|g_1 + g_2\|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)}$.

The natural formulation for the Neumann problem (25), assuming $\varepsilon \in [0,1/2[$, is to consider the trace operator onto both parts of the boundary, analogous to (10),

$$\mathscr{T}^{\varepsilon}_{N,\Omega} : u \in \mathscr{H}^{1+\varepsilon}(\Omega) \mapsto (T_{1,\Gamma_1}u, T_{1,\Gamma_2}u) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim. \tag{28}$$

This operator is linear, bounded and injective, whereas its surjectiveness is not a-priori clear.

For $\varepsilon \in]-1/2, 0[$, however, this operator is not injective. In fact, the function

$$u(x_1, x_2) = H_0^{(1)}(kr), \qquad r = \sqrt{x_1^2 + x_2^2},$$

where $H_0^{(1)}(z)$ is a Hankel function (Bessel function of the third kind) with parameter zero, lies in the kernel of $\mathscr{T}^{\varepsilon}_{N,\Omega}$ for any wedge $\Omega = \Omega_{0,\alpha}$, $0 < \alpha < 2\pi$, or even the slit-plane.

The goal of this section is to construct explicitly the inverses of the operators $\mathscr{T}^{\varepsilon}_{N,\Omega}$, which will again be denoted by $\mathscr{K}^{\varepsilon}$, for wedges $\Omega = \Omega_{0,\alpha}$ and $\varepsilon \in [0, 1/2[$. As in the Dirichlet case, it will follow that the mappings $\mathscr{T}^{\varepsilon}_{N,\Omega}$ and $\mathscr{K}^{\varepsilon}$ are linear homeomorphism and that $\mathscr{K}^{\varepsilon} = (\mathscr{T}^{\varepsilon}_{N,\Omega})^{-1}$.

PROPOSITION 4.2. *For $\varepsilon \in [0, 1/2[$, $\alpha \in \mathbb{R}$, and $f \in H^{-1/2+\varepsilon}(\mathbb{R})$, the functions*

$$u^+ = \mathscr{K}_{N,\Omega^+_\alpha} f = -\mathscr{K}_{D,\Omega^+_\alpha} A_{t^{-1}} f, \qquad u^- = \mathscr{K}_{N,\Omega^-_\alpha} f = -\mathscr{K}_{D,\Omega^-_\alpha} A_{t^{-1}} f, \tag{29}$$

represent the unique solutions of the Helmholtz equation in $H^{1+\varepsilon}(\Omega^{\pm}_\alpha)$, respectively, satisfying the Neumann boundary conditions

$$T_{1,\partial\Omega^+_\alpha} u^+ = J_\alpha f, \qquad T_{1,\partial\Omega^-_\alpha} u^- = J_\alpha f.$$

Moreover, the mappings

$$\mathscr{K}_{N,\Omega^{\pm}_\alpha} : H^{-1/2+\varepsilon}(\mathbb{R}) \to \mathscr{H}^{1+\varepsilon}(\Omega^{\pm}_\alpha)$$

are linear homeomorphisms.

Proof. Again the result can be derived from the upper and lower half-plane case by rotation. Note that in the upper half-plane case we have the normal derivative $\frac{\partial}{\partial x_2}$ whereas in the lower half-plane case we have $-\frac{\partial}{\partial x_2}$. □

The previous proposition can be generalized to the case $\varepsilon \in]-1/2, 0[$, and with appropriate modifications regarding the definition of the operators $T_{1,\partial\Omega^{\pm}_\alpha}$ even to the case $\varepsilon \in]-1/2, +\infty[$.

By analogy with the formulas (12)–(14) we have the following equalities in the Neumann case:

$$\mathscr{K}_{N,\Omega^+_\alpha} f = \mathscr{K}_{N,\Omega^-_{\alpha\pm\pi}} \widetilde{f}, \qquad \widetilde{f}(x) = f(-x), \tag{30}$$

$$T_{1,\Sigma_\alpha} \mathscr{K}_{N,\Omega^{\pm}_\alpha} f = r_+ f, \tag{31}$$

$$T_{1,\Sigma^{\pm}_\gamma} (\mathscr{K}_{N,\Omega^+_\alpha} - \mathscr{K}_{N,\Omega^-_\beta}) f = 0 \tag{32}$$

if $\gamma = (\alpha + \beta)/2$ and $0 < \beta - \alpha < 2\pi$. Unlike the Dirichlet problem, there is a minor ambiguity in choosing the direction of the normal derivative for the operators T_{1,Σ_γ}. Therefore we use the notation $T_{1,\Sigma^{\pm}_\gamma}$ to indicate that the normal derivative should be taken into positive $(+)$ or negative $(-)$ direction when considering the ray $\Sigma_\gamma = \mathscr{R}_\gamma \Sigma$. The ambiguity will cause no trouble. The definition of the operators T_{1,Σ_α} and $T_{1,\Sigma_\gamma^\pm}$ is analogous to the definition of T_{1,Γ_j}.

In order to deal with the case of odd n, we need to define the analogue of the Sommerfeld potentials in the Neumann case, and prove a corresponding result about the operator $\mathscr{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}$ (to be defined below). In this case we need the operator

$$\Pi_{-1/2} = A_{t_-^{1/2}} \ell_0 r_+ A_{t_-^{-1/2}} : H^s(\mathbb{R}) \to H^s(\mathbb{R}),$$

with $s \in]-1,0[$ (or, actually only $s \in]-1/2,0]$ when assuming $\varepsilon \in [0,1/2[$), and the operator matrices

$$\Upsilon_N = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \qquad \Upsilon_N^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

DEFINITION 4.3. For $\varepsilon \in [0,1/2[$, $\alpha \in \mathbb{R}$, and $f \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, define

$$u(x) = (\mathscr{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha} f)(x) = \begin{cases} (\mathscr{K}_{N,\Omega_\alpha^+} u_1^+)(x) & x \in \Omega_\alpha^+ \\ (\mathscr{K}_{N,\Omega_\alpha^-} u_1^-)(x) & x \in \Omega_\alpha^- \end{cases} \tag{33}$$

with

$$\begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} = \Upsilon_N^{-1} \begin{pmatrix} I & 0 \\ 0 & \Pi_{-1/2} \end{pmatrix} \begin{pmatrix} \ell_0 & 0 \\ 0 & \ell \end{pmatrix} \Upsilon_N f. \tag{34}$$

The function u is called the Sommerfeld potential with Neumann density f.

PROPOSITION 4.4. *Let $\varepsilon \in [0,1/2[$, $\alpha \in \mathbb{R}$, and $f = (f_1, f_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. The Neumann problem for the Helmholtz equation in $\mathbb{R}^2\setminus\overline{\Sigma}_\alpha$ is uniquely solved by $u = \mathscr{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha} f$. Moreover, the operator*

$$\mathscr{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha} : H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2\setminus\overline{\Sigma}_\alpha)$$

is a linear homeomorphism.

Proof. The proof is very similar to the Dirichlet case. Again assume $\alpha = 0$ and note the uniqueness of the Neumann problem in the slit-plane.

Let u be given by (33) and (34). Rewrite (34) as

$$u_1^+ + u_1^- = \ell_0(f_1 + f_2) \tag{35}$$

and

$$u_1^+ - u_1^- = A_{t_-^{1/2}} \ell_0 r_+ A_{t_-^{-1/2}} \ell(f_1 - f_2).$$

Because $\varepsilon \in [0,1/2[$, the map from $f \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ to $(u_1^+, u_1^-) \in H^{-1/2+\varepsilon}(\mathbb{R})^2$ is well defined and bounded. Clearly, $r_-(u_1^+ + u_1^-) = 0$ and

$$r_- A_{t^{-1}}(u_1^+ - u_1^-) = r_- A_{t_+^{-1/2}} \ell_0 r_+ A_{t_-^{-1/2}} \ell(f_1 - f_2) = 0.$$

Let $u^{\pm} = u|_{\Omega^{\pm}}$. Then $u^{\pm} \in \mathscr{H}^{1+\varepsilon}(\Omega^{\pm})$, and the Dirichlet and Neumann boundary data of u^{+} and u^{-} are

$$T_{0,\mathbb{R}}u^{\pm} = -A_{t^{-1}}u_1^{\pm} = u_0^{\pm}, \qquad T_{1,\mathbb{R}}u^{\pm} = u_1^{\pm}.$$

From the above conclusions we obtain

$$\begin{aligned} u_0^{+} - u_0^{-} &= T_{0,\mathbb{R}}u^{+} - T_{0,\mathbb{R}}u^{-} \in H_{+}^{1/2+\varepsilon}(\mathbb{R}), \\ u_1^{+} + u_1^{-} &= T_{1,\mathbb{R}}u^{+} + T_{1,\mathbb{R}}u^{-} \in H_{+}^{-1/2+\varepsilon}(\mathbb{R}), \end{aligned}$$

and thus $u \in \mathscr{H}^{1+\varepsilon}(\mathbb{R}^2 \backslash \overline{\Sigma})$. The operator $\mathscr{K}_{N,\mathbb{R}^2\backslash\overline{\Sigma}}$ is bounded.

Let us check the Neumann boundary conditions. Obviously, $r_+(u_1^{+} + u_1^{-}) = f_1 + f_2$. Furthermore,

$$\begin{aligned} r_+(u_1^{+} - u_1^{-}) &= r_+ A_{t_-^{1/2}} \ell_0 r_+ A_{t_-^{-1/2}} \ell(f_1 - f_2) = r_+ A_{t_-^{1/2}} A_{t_-^{-1/2}} \ell(f_1 - f_2) \\ &= r_+ \ell(f_1 - f_2) = f_1 - f_2. \end{aligned}$$

It follows that $r_+ u_1^{+} = f_1$ and $r_+ u_1^{-} = f_2$, as desired. As in the Dirichlet case, it follows immediately that $\mathscr{K}_{N,\mathbb{R}^2\backslash\overline{\Sigma}}$ is a homeomorphism. □

If one tries to generalize this proposition to the case $\varepsilon \in]-1/2, 0[$, one encounters the difficulty that the operator ℓ_0 in (34) (see also (35)) is not well defined from $\widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$ into $H_{+}^{-1/2+\varepsilon}(\mathbb{R})$. In fact, one could arbitrarily add a δ-distribution at the point zero. This is related to the fact that the operator $\mathscr{T}_{N,\Omega}^{\varepsilon}$ is not injective for $\varepsilon \in]-1/2, 0[$.

The analogue of Lemma 3.8 is the following formula

$$T_{1,\Sigma_\gamma}\left(\mathscr{K}_{N,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} f - \mathscr{K}_{N,\mathbb{R}^2\backslash\overline{\Sigma}_\beta} f^{\#}\right) = 0 \tag{36}$$

which holds under the assumptions $0 < |\beta - \alpha| < 2\pi$, $\varepsilon \in [0, 1/2[$, $f \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ and $\gamma = (\alpha+\beta)/2 + j\pi$, $j \in \{0, 1\}$.

Our complete results for the Neumann case are now summarized in the following theorem. For the most part, the solution of the Neumann problems is analogous to what we got for the Dirichlet problems in Section 3.

THEOREM 4.5. *Let $\varepsilon \in [0, 1/2[$, $\alpha = 2\pi/n$. Then the weak Neumann problem for the Helmholtz equation in $\Omega = \Omega_{0,\alpha}$ for given data $g = (g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, is uniquely solved by the following functions:*

(i) <u>*case* $n \equiv 2 \mod 4$</u>

$$u = \mathscr{K}^{\varepsilon} g = r_\Omega \left(\mathscr{K}_{N,\Omega_0^+} + \sum_{j=1}^{\frac{n-2}{4}} \left(\mathscr{K}_{N,\Omega^+_{-2j\alpha}} + \mathscr{K}_{N,\Omega^-_{2j\alpha}} \right) \right) \iota g$$

(ii) <u>*case* $n \equiv 0 \mod 4$</u>

$$u = \mathscr{K}^{\varepsilon} g = r_{\Omega} \left(\sum_{j=1}^{\frac{n}{4}} \left(\mathscr{K}_{N,\Omega^{+}_{2(1-j)\alpha}} \ell^{e} g_{1} + \mathscr{K}_{N,\Omega^{-}_{(2j-1)\alpha}} \ell^{e} g_{2} \right) \right)$$

Here, for $\varepsilon = 0$, *the operator is defined on the dense subspace* $\widetilde{H}^{-1/2}(\mathbb{R}_{+})$ *and can be extended by continuity to all of* $H^{-1/2}(\mathbb{R}_{+})^{2}_{\sim}$.

(iii) <u>*case* $n \equiv 1 \mod 2$</u>

$$u = \mathscr{K}^{\varepsilon} g = r_{\Omega} \left(\sum_{j=1}^{n} \mathscr{K}_{N,\mathbb{R}^{2}\setminus\overline{\Sigma}_{j\alpha}} g^{\#(j+1)} \right)$$

Moreover, in all cases, the operators

$$\mathscr{K}^{\varepsilon} : H^{-1/2+\varepsilon}(\mathbb{R}_{+})^{2}_{\sim} \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

are linear homeomorphisms.

The proof of these facts is essentially the same as in the Dirichlet case. All the necessary auxiliary results have been provided above, and therefore we will leave the details to the reader. It amounts essentially to verifying that all the Neumann boundary conditions are satisfied.

The boundedness of $\mathscr{K}^{\varepsilon}$ is also immediately clear except for the case (ii) with $\varepsilon = 0$, where a similar argumentation as in the proof of Theorem 3.5 has to be applied. In case (ii) with $\varepsilon \in]0, 1/2[$ the operator $\mathscr{K}^{\varepsilon}$ can be defined directly since the even-extensions ℓ^{e} are bounded from $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_{+}) = H^{1/2+\varepsilon}(\mathbb{R}_{+})$ into $H^{1/2+\varepsilon}(\mathbb{R})$.

For $\varepsilon \in [0, 1/2[$ the injectiveness of $\mathscr{T}^{\varepsilon}_{N,\Omega}$ is standard (see (28) and Lemma 4.1), and the formulas for $\mathscr{K}^{\varepsilon}$ show its surjectiveness. It follows that both $\mathscr{K}^{\varepsilon}$ and $\mathscr{T}^{\varepsilon}_{N,\Omega}$ are linear homeomorphism and that $\mathscr{K}^{\varepsilon} = (\mathscr{T}^{\varepsilon}_{N,\Omega})^{-1}$.

Since the case $\varepsilon \in]-1/2, 0[$ causes some not completely trivial problems, we will refrain from discussing it in this paper.

5. The mixed Dirichlet/Neumann problem

The mixed problem (see Figure 5) can be solved completely by a somewhat modified reasoning. As we shall see, no compatibility conditions appear (if $\varepsilon \in [0, 1/2[$), as it was observed already in the cases $n = 2$ and $n = 4$ [3, 31].

The representation formulas need a bit more effort in the first and in the third case, while uniqueness is always guaranteed by a little modified, but basically the same reasoning [5] provided $\varepsilon \geqslant 0$.

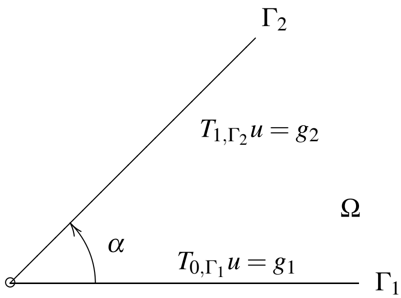

Figure 5: The DN problem in Ω

5.1. The case $\alpha = 2\pi/n$, $n \equiv 2 \mod 4$

Let us start with the case $n = 2$ solved by the method of [32]. Here we have $\Gamma_1 = \{(x_1, 0) : x_1 > 0\}$ and $\Gamma_2 = \{(x_1, 0) : x_1 < 0\}$. As before we will identify Γ_1 and Γ_2 with $\mathbb{R}_+$. In what follows, let $\mathscr{J}$ be the reflection operator on $\mathbb{R}$ given by $\mathscr{J} f(x) = \widetilde{f}(x) = f(-x)$, $x \in \mathbb{R}$.

PROPOSITION 5.1. *The mixed Dirichlet/Neumann boundary value problem for the Helmholtz equation in the upper half-plane*

$$\begin{aligned} (\Delta + k^2)u = 0 \quad \text{where} \quad u \in H^{1+\varepsilon}(\Omega^+) \\ T_{0,\Gamma_1}u = g_1 \in H^{1/2+\varepsilon}(\mathbb{R}_+) \\ T_{1,\Gamma_2}u = g_2 \in H^{-1/2+\varepsilon}(\mathbb{R}_+) \end{aligned} \tag{37}$$

is well-posed for $\varepsilon \in]-1/2, 1/2[$ *and uniquely solved by* $u = \mathscr{K}_{DN,\Omega^+} g$*, where*

$$\mathscr{K}_{DN,\Omega^+} g = \mathscr{K}_{D,\Omega^+} A_{t_-^{-1/2}} v = \mathscr{K}_{N,\Omega^+} A_{t_+^{1/2}} v = \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{-t(\xi) x_2} t_-^{-1/2}(\xi) \hat{v}(\xi) \tag{38}$$

with

$$v = \ell_0 r_+ A_{t_-^{1/2}} \ell g_1 - \ell_0 r_- A_{t_+^{-1/2}} \mathscr{J} \ell g_2, \qquad g = (g_1, g_2).$$

Herein, $\ell_0 : H^{\varepsilon}(\mathbb{R}_\pm) \to H^{\varepsilon}(\mathbb{R})$ *stand for the extension-by-zero operators, and* ℓ *for any extension to a distribution in* $H^{\pm 1/2+\varepsilon}(\mathbb{R})$*. Moreover, the potential operator*

$$\mathscr{K}_{DN,\Omega^+} : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega^+)$$

is a linear homeomorphism.

Proof. By Propositions 3.2 and 4.2 each solution $u \in \mathscr{H}^{1+\varepsilon}(\Omega^+)$ can be described by either the Dirichlet data $u_0 = T_{0,\partial\Omega^+} u$ or Neumann data $u_1 = T_{1,\partial\Omega^+} u$. Both are connected by the formula $u_1 = -A_t u_0$. Conversely, given $u_0 \in H^{1/2+\varepsilon}(\mathbb{R})$ and

$u_1 \in H^{-1/2+\varepsilon}(\mathbb{R})$ satisfying $u_1 = -A_t u_0$, we can find a unique solution u having these boundary data on the whole axis.

The mixed boundary conditions now read

$$r_+ u_0 = g_1, \qquad r_+ u_1 = g_2.$$

We can rewrite this, equivalently, as

$$r_+ u_0 = r_+ \ell g_1, \qquad r_+ u_1 = r_- \mathscr{J} \ell g_2.$$

Using the fact that $r_+ w = 0$ is equivalent to $r_+ A_{t_-^{1/2}} w = 0$, and a similar fact involving r_- and $A_{t_+^{-1/2}}$, the above can be expressed as

$$r_+ A_{t_-^{1/2}} u_0 = r_+ A_{t_-^{1/2}} \ell g_1, \qquad r_+ A_{t_+^{-1/2}} u_1 = r_- A_{t_+^{-1/2}} \mathscr{J} \ell g_2.$$

As $v = A_{t_-^{1/2}} u_0 = -A_{t_+^{-1/2}} u_1 \in H^{\varepsilon}(\mathbb{R})$ and $\varepsilon \in [0, 1/2[$ we see that

$$v = \ell_0 r_+ A_{t_-^{1/2}} \ell g_1 - \ell_0 r_- A_{t_+^{-1/2}} \mathscr{J} \ell g_2.$$

Conversely, any v given by the above formula yields, when eliminating for u_0 and u_1, a solution to the mixed boundary valued problem. $\square$

We remark that formula (38) can be equivalently written as

$$(\mathscr{K}_{DN,\Omega^+} g)(x) = \mathscr{F}^{-1}_{\xi \to x_1} e^{-t(\xi)x_2} t_-^{-1/2}(\xi) \Big(P_+ t_-^{1/2}(\xi) \widehat{\ell g_1}(\xi) - P_- t_+^{-1/2}(\xi) \widehat{\ell g_2}(\xi) \Big)$$

where $P_\pm = \mathscr{F} \ell_0 r_\pm \mathscr{F}^{-1}$ are projections on $H^\varepsilon(\mathbb{R})$ related with the Hilbert transform ($P_\pm = \frac{1}{2}(I \pm H_\mathbb{R})$, see [24]).

COROLLARY 5.2. *The mixed Dirichlet/Neumann boundary value problem for the Helmholtz equation in the lower half-plane is well-posed for* $\varepsilon \in]-1/2, 1/2[$ *and uniquely solved by* $u = \mathscr{K}_{DN,\Omega^-} g$, $g = (g_1, g_2)$, *where*

$$\mathscr{K}_{DN,\Omega^-} g = \mathscr{K}_{D,\Omega^-} A_{t_-^{-1/2}} v = \mathscr{K}_{N,\Omega^-} A_{t_+^{1/2}} v = \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{t(\xi)x_2} t_-^{-1/2}(\xi) \hat{v}(\xi) \tag{39}$$

with

$$v = \ell_0 r_+ A_{t_-^{1/2}} \ell g_1 - \ell_0 r_- A_{t_+^{-1/2}} \mathscr{J} \ell g_2, \qquad g = (g_1, g_2).$$

Moreover, the potential operator

$$\mathscr{K}_{DN,\Omega^-} : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega^-)$$

is a linear homeomorphism.

We remark that formula (39) can be equivalently written as

$$(\mathscr{K}_{DN,\Omega^+}g)(x) = \mathscr{F}^{-1}_{\xi\to x_1} e^{t(\xi)x_2} t_-^{-1/2}(\xi)\Big(P_+ t_-^{1/2}(\xi)\widehat{\ell g_1}(\xi) - P_- t_+^{-1/2}(\xi)\mathscr{J}\widehat{\ell g_2}(\xi)\Big).$$

Finally we define the operators

$$\mathscr{K}_{DN,\Omega^\pm_\alpha} = J_\alpha \mathscr{K}_{DN,\Omega^\pm}$$

and remark that these operators applied to the corresponding boundary data yield the solution in the rotated half-planes Ω^+_α and Ω^-_α.

Now, quite similarly to Theorem 3.3, we obtain the solution for the Helmholtz equation for the cases $n \equiv 2 \mod 4$. The proof is also similar and therefore left to the reader. It amounts to check the boundary conditions on Γ_1 and Γ_2.

THEOREM 5.3. *Let* $\alpha = 2\pi/n$ *with* $n = 2 \mod 4$ *and* $\varepsilon \in [0,1/2[$. *Then the Dirichlet/Neumann problem for the Helmholtz equation in* $\Omega = \Omega_{0,\alpha}$ *for given data* $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *is well-posed and uniquely solved by*

$$u = \mathscr{K}^\varepsilon\, h = r_\Omega \left(\mathscr{K}_{DN,\Omega_0^+} + \sum_{j=1}^{\frac{n-2}{4}} (-1)^j \left(\mathscr{K}_{DN,\Omega^+_{-2j\alpha}} - \mathscr{K}_{DN,\Omega^-_{2j\alpha}} \right) \right) h$$

where

(i) $h = (g_1(x), g_2(-x))$ *if* $n \equiv 2 \mod 8$,

(ii) $h = (g_1(x), -g_2(-x))$ *if* $n \equiv 6 \mod 8$.

Moreover, the operator

$$\mathscr{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

is a linear homeomorphism.

The previous result can be extended to the case $\varepsilon \in]-1/2, 0[$ apart from the statement about the uniqueness of the solution. The operator $\mathscr{K}^\varepsilon$ is well defined, linear, bounded, and (clearly) injective. We leave open the question of the surjectiveness of $\mathscr{K}^\varepsilon$.

EXAMPLE 5.4. Figures 6 and 7 illustrate the two different cases of Theorem 5.3 where the solution is composed of several terms defined in half-planes whose intersection is Ω. The letters D and N stand for Dirichlet conditions $T_{0,\Gamma_j}u = g_j$ and Neumann conditions $T_{1,\Gamma_j}u = g_j$, respectively.

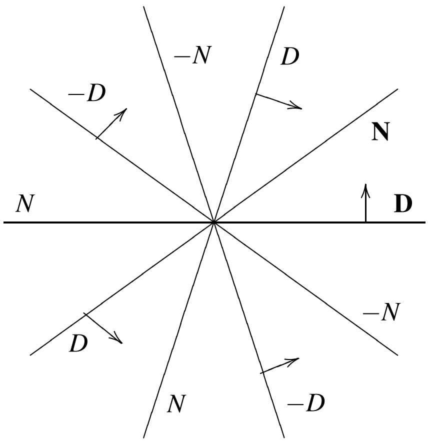

Figure 6: Case $n \equiv 2 \mod 8$

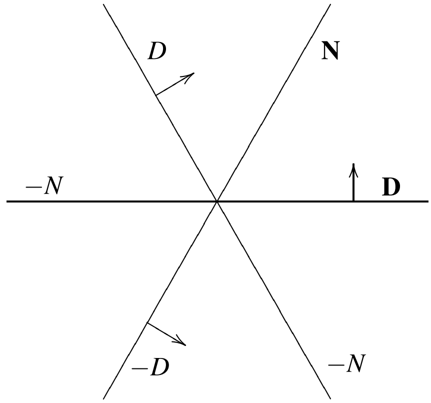

Figure 7: Case $n \equiv 6 \mod 8$

5.2. The case $\alpha = 2\pi/n$, $n \equiv 0 \mod 4$

In this case we can state our result immediately. We have to make use of the half-plane solutions of the pure Dirichlet and the pure Neumann problem. Notice that the formulas involve the even and odd extension operators ℓ^e and ℓ^o.

THEOREM 5.5. *Let* $\alpha = 2\pi/n$ *with* $n = 0 \mod 4$ *and* $\varepsilon \in [0, 1/2[$. *Then the Dirichlet/Neumann problem for the Helmholtz equation in* $\Omega = \Omega_{0,\alpha}$ *for given data* $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *is well-posed and uniquely solved by the following formulas,*

(i) *if* $n \equiv 0 \mod 8$,

$$u = \mathscr{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{\frac{n}{4}} (-1)^{j+1} \left(\mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}} \ell^o g_1 + \mathscr{K}_{N,\Omega^-_{(2j-1)\alpha}} \ell^e g_2 \right) \right),$$

(ii) *if* $n \equiv 4 \mod 8$,

$$u = \mathscr{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{\frac{n}{4}} (-1)^{j+1} \left(\mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}} \ell^e g_1 + \mathscr{K}_{N,\Omega^-_{(2j-1)\alpha}} \ell^o g_2 \right) \right).$$

Moreover, the operator

$$\mathscr{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

is a linear homeomorphism.

In case (i) the definition of $\mathscr{K}^\varepsilon$ has to be understood similarly as in Theorem 3.5 and Theorem 4.5(ii). To be more specific, the operator $\mathscr{K}^0$ is defined by the above formula on the dense subspace $\widetilde{H}^{1/2}(\mathbb{R}_+)\times\widetilde{H}^{-1/2}(\mathbb{R}_+)$ and extends by continuity to a bounded linear operator acting on $H^{1/2}(\mathbb{R}_+)\times H^{-1/2}(\mathbb{R}_+)$. For $\varepsilon\in]0,1/2[$, the operator $\mathscr{K}^\varepsilon$ is defined as the restriction of $\mathscr{K}^0$ to the subspace $H^{1/2+\varepsilon}(\mathbb{R}_+)\times H^{-1/2+\varepsilon}(\mathbb{R}_+)$. On the other hand, for $\varepsilon\in]0,1/2[$ the operator $\mathscr{K}^\varepsilon$ can also be defined directly by the above formulas with a slight reinterpretation. Indeed, we can just consider $g_2\in H^{-1/2+\varepsilon}(\mathbb{R}_+)\mapsto \ell^e g_2\in H^{-1/2+\varepsilon}(\mathbb{R})$ without any modification, while we consider $g_1\in H^{1/2+\varepsilon}(\mathbb{R}_+)\subset H^{1/2-\delta}(\mathbb{R}_+)\mapsto \ell^o g_1\in H^{1/2-\delta}(\mathbb{R}_+)$, where $\delta\in]0,1/2[$.

In case (ii) the direct definition works because the even and odd extension operators ℓ^e and ℓ^o are bounded on the appropriate spaces.

Proof. (i): Let us consider first the case $n\equiv 0 \mod 8$. Assume first that $\varepsilon\in[0,1/2[$ and $g=(g_1,g_2)\in\widetilde{H}^{1/2}(\mathbb{R}_+)\times\widetilde{H}^{-1/2}(\mathbb{R}_+)$, and notice that $u=\mathscr{K}^\varepsilon g$ is well-defined and in $H^{1+\varepsilon}(\Omega)$. Clearly, u satisfies the Helmholtz equation. The contributions to the trace on Γ_1 read as follows. As seen before in Subsection 3.1, we have

$$T_{0,\Gamma_1}\mathscr{K}_{D,\Omega^+}\ell^o g_1=g_1,$$

$$T_{0,\Gamma_1}\left(\sum_{j=2}^{\frac{n}{4}}(-1)^{j+1}\mathscr{K}_{D,\Omega^+_{2(1-j)\alpha}}\ell^o g_1\right)=0.$$

The terms containing g_2 do not contribute to the trace on Γ_1. For $j=1,\dots,\frac{n}{4}$ and considering $\beta=(2j-1)\alpha$, it holds that

$$T_{0,\Gamma_1}\left(\mathscr{K}_{N,\Omega^-_{(2j-1)\alpha}}-\mathscr{K}_{N,\Omega^-_{\left(\frac{n}{2}-2j+1\right)\alpha}}\right)\ell^e g_2=T_{0,\Gamma_1}\left(\mathscr{K}_{N,\Omega^-_\beta}-\mathscr{K}_{N,\Omega^-_{\pi-\beta}}\right)\ell^e g_2=0$$

since

$$\mathscr{K}_{N,\Omega^-_{\pi-\beta}}\ell^e g_2=-\mathscr{K}_{N,\Omega^+_{-\beta}}\ell^e g_2,$$

$$T_{0,\Gamma_1}\left(\mathscr{K}_{N,\Omega^-_\beta}+\mathscr{K}_{N,\Omega^+_{-\beta}}\right)=0.$$

Therefore, it follows that $T_{0,\Gamma_1}u=g_1$.

In what concerns the contributions to the trace of the normal derivative on Γ_2, we have $T_{1,\Gamma_2}\mathscr{K}_{N,\Omega^-_\alpha}\ell^e g_2=g_2$. Rewriting the second part of the sum as

$$\begin{aligned}&\sum_{j=1}^{\frac{n}{4}}(-1)^{j+1}\mathscr{K}_{N,\Omega^-_{(2j-1)\alpha}}\ell^e g_2\\&=\left(\mathscr{K}_{N,\Omega^-_\alpha}-\mathscr{K}_{N,\Omega^-_{3\alpha}}+\dots-\mathscr{K}_{N,\Omega^-_{\left(\frac{n}{2}-3\right)\alpha}}+\mathscr{K}_{N,\Omega^-_{\left(\frac{n}{2}-1\right)\alpha}}\right)\ell^e g_2\end{aligned}\tag{40}$$

one obtains

$$T_{1,\Gamma_2}\left(\mathscr{K}_{N,\Omega^-_{-3\alpha}}+\mathscr{K}_{N,\Omega^-_{-\left(\frac{n}{2}-1\right)\alpha}}\right)\ell^e g_2=0.$$

Further terms of (40) can be dealt with by the same arguments. There remains a “middle” term whose trace on Γ_2 is equal to zero:

$$T_{1,\Gamma_2}\mathscr{K}_{N,\Omega^-_{\frac{\pi}{2}+\alpha}}\ell^e g_2 = 0.$$

The terms containing g_1 result similarly to vanishing contributions on Γ_2 (consider the first plus the last term, etc). Consequently, $T_{1,\Gamma_2}u = g_2$.

Hence for g taken from the subspace

$$\widetilde{X}^{\varepsilon} = \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)\times\widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+),$$

we have that $u = \mathscr{K}^{\varepsilon}g$ belongs to $\mathscr{H}^{1+\varepsilon}(\Omega)$ and satisfies the correct boundary conditions. Introducing the operator

$$\mathscr{T}^{\varepsilon} : u \in \mathscr{H}^{1+\varepsilon}(\Omega) \mapsto (T_{0,\Gamma_1}u, T_{1,\Gamma_2}u) \in H^{1/2+\varepsilon}(\mathbb{R}_+)\times H^{-1/2+\varepsilon}(\mathbb{R}_+),$$

this means that $\mathscr{K}^{\varepsilon} : \widetilde{X}^{\varepsilon} \to \mathscr{H}^{1+\varepsilon}(\Omega)$ is well defined and that $\mathscr{T}^{\varepsilon}\mathscr{K}^{\varepsilon}|_{\widetilde{X}^{\varepsilon}} = I_{\widetilde{X}^{\varepsilon}}$. Notice that $\mathscr{T}^{\varepsilon}$ is linear bounded and injective.

We claim that $\mathscr{T}^{\varepsilon}$ is surjective. Here we split the problem into two problems and resort to previous results. First, there exists $u \in \mathscr{H}^{1+\varepsilon}(\Omega)$ such that $T_{0,\Gamma_1}u = g_1$ and $T_{1,\Gamma_2}u = 0$. This solution can be obtained from a solution defined in $\Omega_{0,2\alpha}$ with Dirichlet data on both boundaries (Γ_1 and Γ_3; see (2)) given by g_1. The existence of such a solution is guaranteed by Theorem 3.5. Because the Dirichlet data on both parts of the boundary is the same, the solution has some symmetry, which implies that the Neumann data on Γ_2 is zero, as desired. Secondly, one can find a solution $u \in \mathscr{H}^{1+\varepsilon}(\Omega)$ such that $T_{0,\Gamma_1}u = 0$ and $T_{1,\Gamma_2}u = g_2$. This solution can be obtained from a solution in $\Omega_{-\alpha,\alpha} = \mathscr{R}_{\alpha}^{-1}\Omega_{0,2\alpha}$ with Neumann conditions $-g_2$ and g_2 on the boundaries Γ_0 and Γ_2 (see Theorem 4.5(ii)). Again a symmetry argument implies that the Dirichlet data on Γ_1 is zero. Thus we can conclude that $\mathscr{T}^{\varepsilon}$ is surjective.

It follows that $\mathscr{T}^{\varepsilon}$ is a linear homeomorphism from $\mathscr{H}^{1+\varepsilon}(\Omega)$ to $H^{1/2+\varepsilon}(\mathbb{R}_+)\times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ for $\varepsilon \in [0,1/2[$. Now the argumentation is similar as in Theorem 3.5. First consider $\varepsilon = 0$, to conclude that $\mathscr{K}^0|_{\widetilde{X}^0} = (\mathscr{T}^0)^{-1}|_{\widetilde{X}^0}$ allows a continuous extension (denoted by $\mathscr{K}^0$). It follows that $\mathscr{K}^0\mathscr{T}^0 = I$ and now we restrict this identity to the appropriate space to obtain that $\mathscr{K}^{\varepsilon}\mathscr{T}^{\varepsilon} = I$. Hence $\mathscr{K}^{\varepsilon}$ is a linear homeomorphism and this concludes the proof of part (i).

(ii) The case $n \equiv 4 \mod 8$ runs analogously concerning the verification of the boundary conditions. It is also evident with respect to the definition of $\mathscr{K}^{\varepsilon}$ and the fact that it is a linear homeomorphism. □

Case (ii) of the previous theorem can be generalized for $\varepsilon \in]-1/2,0[$ in as far as that the operator $\mathscr{K}^{\varepsilon}$ is still well-defined, linear, and bounded. In case (i) again some not completely trivial problems occur and we refrain from discussing it here.

5.3. The case $\alpha = 2\pi/n$, $n \equiv 1 \mod 2$

The solution of the general case is based on the solution of a famous and non-trivial Sommerfeld problem [27] and [10, 18, 28].

This problem consists of finding the solution of the Helmholtz equation in the slit-plane $\Omega = \mathbb{R}^2 \backslash \overline{\Sigma}$ with mixed DN boundary conditions on Σ^+ and Σ^-, as shown in Figure 8. More precisely, we are looking for $u \in \mathscr{H}^{1+\varepsilon}(\Omega)$ (see Sec. 3.3) such that

$$\begin{aligned} T_{0,\Sigma^+} u &= g_1 \in H^{1/2+\varepsilon}(\mathbb{R}_+), \\ T_{1,\Sigma^-} u &= g_2 \in H^{-1/2+\varepsilon}(\mathbb{R}_+). \end{aligned} \tag{41}$$

Here we identify $\Sigma^{\pm}$ with $\mathbb{R}_+$. For the underlying definitions of T_{0,Σ^+} and T_{1,Σ^-} see Sections 3.3 and 4.

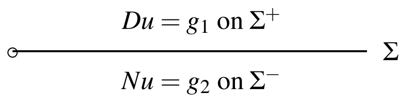

Figure 8: The Sommerfeld DN problem

The above Sommerfeld problem is known to be well-posed and uniquely solvable for $\varepsilon = 0$. As far as we are aware of, for $\varepsilon \neq 0$, this problem has not yet been examined explicitly, although one just needs to modify the original arguments. It should be mentioned that it turns out, perhaps somewhat surprisingly, that the problem is only well-posed for $\varepsilon \in]-1/4, 1/4[$. Before we are going to show this positive result, let us show that the problem is either not uniquely solvable or requires a compatibility condition in the cases where $\varepsilon \in]-1/2, -1/4[\cup]1/4, 1/2[$. For the cases $\varepsilon = \pm 1/4$ see the remark made after Proposition 5.7.

As before, let $H^{(1)}_{1/4}(z)$ stand for the Hankel function (Bessel function of the third kind) with parameter $1/4$. Recall that $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$.

PROPOSITION 5.6. *Let* $\Omega = \mathbb{R}^2 \backslash \overline{\Sigma}$.

(i) If $\varepsilon \in]-1/2, -1/4[$*, then the function*

$$v(x) = H^{(1)}_{1/4}(kr) \sin(\phi/4), \quad x = (r\cos(\phi), r\sin(\phi)), \; 0 < \phi < 2\pi, \; r > 0,$$

is a non-trivial solution to the homogeneous problem (41)*, i.e.,*

$$v \in \mathscr{H}^{1+\varepsilon}(\Omega), \quad T_{0,\Sigma^+} v = 0, \quad T_{1,\Sigma^-} v = 0.$$

(ii) If $\varepsilon \in]1/4, 1/2[$ *and if* (g_1, g_2) *are the boundary data of a solution of the problem* (41)*, then*

$$\int_0^\infty H^{(1)}_{1/4}(kt) g_2(t)\, dt = \int_0^\infty \frac{H^{(1)}_{1/4}(kt)}{4t} \Big(g_1(t) - g_1(0) \Big) dt + g_1(0) \frac{1 + i + i\sqrt{2}}{k^2}.$$

Proof. (i) By passing to polar coordinates it can be checked easily that v satisfies the Helmholtz equation in Ω. In fact, this solution can be obtained via the ansatz $v(x) = f(r)\cos(\phi/4)$, where $f(r)$ is seen to satisfy

$$r^2 f'' + r f' + (r^2 k^2 - (1/4)^2) f = 0, \tag{42}$$

which is (up to the factor k^2) Bessel's equation. One of its solutions, $f(r) = H^{(1)}_{1/4}(kr)$, is exponentially decaying as $r \to \infty$ since $\mathrm{Im}(k) > 0$. The behavior as $r \to 0$ is given by $f(r) \sim r^{-1/4}$. From this it follows (see, e.g., [9]) that $v \in H^s(\Omega)$ if and only if $1/4 < 1 - s$, i.e., if and only if $\varepsilon < -1/4$ in our setting where $s = 1 + \varepsilon$. It is also not hard to see that

$$\lim_{x_2 \to +0} v(x) = 0 \quad \text{and} \quad \lim_{x_2 \to -0} \frac{\partial v(x)}{\partial x_2} = 0 \quad \text{for } x_1 > 0.$$

Hence v is a (non-trivial) solution of the homogeneous DN Sommerfeld problem in the slit-plane, $T_{0,\Sigma^+} v = 0$, $T_{1,\Sigma^-} v = 0$.

(ii) Let v be defined as in (i) and let $u \in \mathscr{H}^{1+\varepsilon}(\Omega)$ be a solution of the Sommerfeld problem with boundary data $g_1 \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ and $g_2 \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$. As we have just shown, $v \in \mathscr{H}^{1-\varepsilon}(\Omega)$, and this implies that the following considerations make sense. We will not go through all the technical details and point out only the main steps.

For $\delta > 0$, let $\Sigma_{(\delta)} = \{x \in \mathbb{R}^2 : x_1 \geqslant \delta,\ x_2 = 0\}$ and use the same symbolic notation $\Sigma^{\pm}_{(\delta)}$ as for $\Sigma^{\pm}$ to distinguish between the limit values on $\Sigma_{(\delta)}$ from above and below. Moreover, put $D_\delta = \{x \in \mathbb{R}^2 : |x| < \delta\}$ and $\Omega_\delta = \Omega \setminus \overline{D_\delta}$. Then using Green's formula and referring to n as the normal vector pointing inwards, we obtain (because both u and v satisfy the Helmholtz equation)

$$\begin{aligned}
0 &= \lim_{\delta \to 0} \int_{\Omega_\delta} (\Delta v(x) u(x) - v(x) \Delta u(x))\, dx \\
&= \lim_{\delta \to 0} \int_{\partial \Omega_\delta} \left(-\frac{\partial v(x)}{\partial n} u(x) + v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) \\
&= -\lim_{\delta \to 0} \int_{\Sigma^+_{(\delta)} \cup \Sigma^-_{(\delta)} \cup \partial D_\delta} \frac{\partial v(x)}{\partial n} u(x)\, d\sigma(x) + \int_{\Sigma^+ \cup \Sigma^-} v(x) \frac{\partial u(x)}{\partial n}\, d\sigma(x) \\
&= -\lim_{\delta \to 0} \left(\int_{\Sigma^+_{(\delta)}} \frac{\partial v(x)}{\partial x_2} u(x)\, d\sigma(x) + \int_{\partial D_\delta} \frac{\partial v(x)}{\partial r} u(x)\, d\sigma(x) \right) \\
&\qquad + \int_{\Sigma^-} v(x) \frac{\partial u(x)}{\partial n}\, d\sigma(x).
\end{aligned}$$

The first integral in the last expression evaluates to

$$\int_{\Sigma^+_{(\delta)}} \frac{\partial v(x)}{\partial x_2} u(x)\, d\sigma(x) = \int_\delta^\infty \frac{f(t)}{4t} g_1(t)\, dt,$$

where $v(x) = f(r)\cos(\phi/4)$, $f(r) = H^{(1)}_{1/4}(kr)$, as in (i). The second integral equals

$$\begin{aligned}
\int_{\partial D_\delta} \frac{\partial v(x)}{\partial r} u(x)\, d\sigma(x) &= \delta \int_0^{2\pi} f'(\delta) \sin(\phi/4) u(\delta\cos\phi, \delta\sin\phi)\, d\phi \\
&= 4\delta f'(\delta) g_1(0) + o(1) \qquad \text{as } \delta \to 0.
\end{aligned}$$

Here we used the fact that $u(x)$ is Lipschitz continuous with an exponent greater than $1/4$ (since $\varepsilon > 1/4$). Finally, the third integral equals

$$\int_{\Sigma^-} v(x)\frac{\partial u(x)}{\partial n}\,d\sigma(x) = \int_0^\infty f(t)g_2(t)\,dt.$$

Now we deduce from (42) by integration that

$$\begin{aligned}\int_\delta^\infty \frac{f(t)}{4t}\,dt &= \int_\delta^\infty (4tf'(t))'\,dt + \int_\delta^\infty 4tf(t)\,dt\\ &= -4\delta f'(\delta) + \int_0^\infty 4tf(t)\,dt + o(1).\end{aligned}$$

Combining the foregoing results we obtain the condition

$$\begin{aligned}\int_0^\infty f(t)g_2(t)\,dt &= \lim_{\delta\to 0}\left(\int_\delta^\infty \frac{f(t)}{4t}g_1(t)\,dt + g_1(0)\int_0^\infty 4tf(t)\,dt - g_1(0)\int_\delta^\infty \frac{f(t)}{4t}\,dt\right)\\ &= \int_0^\infty \frac{f(t)}{4t}\Big(g_1(t)-g_1(0)\Big)dt + g_1(0)\frac{1+i+i\sqrt{2}}{k^2},\end{aligned}$$

which is the desired statement. □

Let us proceed with establishing the positive results. Assume that $\varepsilon \in]-1/2,1/2[$, let $\mathscr{A}$ stand for the block operator (defined even for $\varepsilon \in \mathbb{R}$)

$$\mathscr{A} = \frac{1}{2}\begin{pmatrix} I & -A_{t^{-1}} \\ A_t & I\end{pmatrix} : H^{1/2+\varepsilon}(\mathbb{R}) \times H^{-1/2+\varepsilon}(\mathbb{R}) \to H^{1/2+\varepsilon}(\mathbb{R}) \times H^{-1/2+\varepsilon}(\mathbb{R}),$$

and let W_{DN} stand for the Wiener-Hopf type operator

$$W_{DN} = r_+\mathscr{A} : H_+^{1/2+\varepsilon}(\mathbb{R}) \times H_+^{-1/2+\varepsilon}(\mathbb{R}) \to H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+).$$

PROPOSITION 5.7. *For $\varepsilon \in]-1/4,1/4[$, the operator W_{DN} is invertible and the mixed DN problem* (41) *in the slit-plane $\Omega = \mathbb{R}^2\backslash\overline{\Sigma}$ with boundary data $g = (g_1,g_2)$ is well-posed and uniquely solvable. Its solution is given by*

$$u(x) = (\mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}}\,g)(x) = \begin{cases} (\mathscr{K}_{D,\Omega^+}u_0^+)(x) & x\in\Omega^+\\ (\mathscr{K}_{N,\Omega^-}u_1^-)(x) & x\in\Omega^-\end{cases} \tag{43}$$

where

$$\begin{pmatrix} u_0^+\\ u_1^-\end{pmatrix} = \mathscr{A}W_{DN}^{-1}\begin{pmatrix} g_1\\ g_2\end{pmatrix}. \tag{44}$$

Moreover, the Sommerfeld potential operator

$$\mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}} : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

is a linear homeomorphism.

Proof. Each solution of the Helmholtz equation in the slit-plane Ω can be described by the Dirichlet or Neumann boundary values $u_0^\pm$ and $u_1^\pm$, where we have the relation $u_1^\pm = -A_t u_0^\pm$. The jump condition on $-\Sigma$ and the boundary conditions on Σ can be described by

$$\begin{aligned} u_0^+ - u_0^- &\in H_+^{1/2+\varepsilon}(\mathbb{R}), & r_+ u_0^+ &= g_1, \\ u_1^+ + u_1^- &\in H_+^{-1/2+\varepsilon}(\mathbb{R}), & r_+ u_1^- &= g_2. \end{aligned}$$

It is convenient (and possible) to eliminate u_0^- and u_1^+, and then the conditions read

$$\begin{aligned} u_0^+ + A_{t^{-1}} u_1^- &=: f_0 \in H_+^{1/2+\varepsilon}(\mathbb{R}), & r_+ u_0^+ &= g_1, \\ -A_t u_0^+ + u_1^- &=: f_1 \in H_+^{-1/2+\varepsilon}(\mathbb{R}), & r_+ u_1^- &= g_2. \end{aligned}$$

Here we introduced the functions f_0 and f_1, and it is possible to express also u_0^+ and u_1^- in terms of them,

$$\begin{pmatrix} I & A_{t^{-1}} \\ -A_t & I \end{pmatrix} \begin{pmatrix} u_0^+ \\ u_1^- \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \quad \Leftrightarrow \quad \frac{1}{2} \begin{pmatrix} I & -A_{t^{-1}} \\ A_t & I \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} u_0^+ \\ u_1^- \end{pmatrix}.$$

Here the block operator $\mathscr{A}$ occurs. A moment's though shows that the Helmholtz equation in the slit-plane is uniquely solvable if and only if so is the Wiener-Hopf system

$$W_{DN} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Indeed, combining the previous two equations just gives (44) and from there (43) follows.

Thus our concern is the invertibility of the operator W_{DN}. Its symbol (up to the factor $1/2$) is given by

$$\Phi(\xi) = \begin{pmatrix} 1 & -t^{-1}(\xi) \\ t(\xi) & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}.$$

However, in order to facilitate the analysis of W_{DN} we pass to another operator which acts on the spaces $H_+^0(\mathbb{R})^2 \cong H^0(\mathbb{R}_+)^2 = L^2(\mathbb{R}_+)^2$. We introduce the symbols

$$t_+^\mu(\xi) = (\xi + k)^\mu, \qquad t_-^\mu(\xi) = (\xi - k)^\mu, \qquad \xi \in \mathbb{R},$$

where the principle values are chosen (due to vertical branch cuts from $\mp k$ to ∞ not crossing the real line), and conclude that the operators

$$D_- = r_+ \begin{pmatrix} A_{t_-^{1/2+\varepsilon}} & 0 \\ 0 & A_{t_-^{-1/2+\varepsilon}} \end{pmatrix} \ell : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to H^0(\mathbb{R}_+)^2$$

$$D_+ = \begin{pmatrix} A_{t_+^{-1/2-\varepsilon}} & 0 \\ 0 & A_{t_+^{1/2-\varepsilon}} \end{pmatrix} : H_+^0(\mathbb{R})^2 \to H_+^{1/2+\varepsilon}(\mathbb{R}) \times H_+^{-1/2+\varepsilon}(\mathbb{R})$$

are well defined linear homeomorphisms. Now define an operator equivalent to W_{DN} by

$$\widetilde{W}_{DN} = D_- W_{DN} D_+$$

and observe that

$$\widetilde{W}_{DN} = r_+ A_\Psi : H^0_+(\mathbb{R})^2 \to H^0(\mathbb{R}_+)^2,$$

where A_Ψ is the block version of (4) and Ψ is the block symbol

$$\Psi(\xi) = \begin{pmatrix} t_-^{1/2+\varepsilon}(\xi) & 0 \\ 0 & t_-^{-1/2+\varepsilon}(\xi) \end{pmatrix} \begin{pmatrix} 1 & -t^{-1}(\xi) \\ t(\xi) & 1 \end{pmatrix} \begin{pmatrix} t_+^{-1/2-\varepsilon}(\xi) & 0 \\ 0 & t_+^{1/2-\varepsilon}(\xi) \end{pmatrix}$$
$$= \left(\frac{\xi-k}{\xi+k}\right)^{\varepsilon} \begin{pmatrix} \left(\frac{\xi-k}{\xi+k}\right)^{1/2} & -1 \\ 1 & \left(\frac{\xi+k}{\xi-k}\right)^{1/2} \end{pmatrix}.$$

The operator $\widetilde{W}_{DN}$ is a block Wiener-Hopf operator on $L^2(\mathbb{R}_+)$ with piecewise continuous matrix symbol. More specifically, the symbol $\Psi(\xi)$ is continuous on $\mathbb{R}$ and has limits as $\xi \to \pm\infty$ given by

$$\Psi(+\infty) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \Psi(-\infty) = e^{-2\pi i\varepsilon} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Moreover, all values of $\Phi(\xi)$ are invertible matrices. Applying the basic theory for Wiener-Hopf operators with such symbols, we can conclude that $\widetilde{W}_{DN}$ is a Fredholm operator if and only if the matrix

$$\Psi(-\infty)\Psi(+\infty)^{-1} = e^{-2\pi i\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{45}$$

has no real negative eigenvalues. As the eigenvalues are $e^{-2\pi i(\varepsilon\pm 1/4)}$, this is the case if $\varepsilon \in]-1/4, 1/4[$.

Hence it follows that $\widetilde{W}_{DN}$ (and thus also W_{DN}) is a Fredholm operator. Given this, the invertibility of $\widetilde{W}_{DN}$ follows from the existence of a weak generalized canonical factorization of Ψ is L^2. Recall that $\Psi(\xi) = \Psi_-(\xi)\Psi(\xi)$ is a weak generalized canonical factorization in L^2 if

$$(\xi - i)^{-1/2}\Psi_-(\xi)^{\pm 1} \in (L^2_-(\mathbb{R}))^{2\times 2}, \qquad (\xi + i)^{-1/2}\Psi_+(\xi)^{\pm 1} \in (L^2_+(\mathbb{R}))^{2\times 2}$$

where $L^2_\pm(\mathbb{R})$ stands for the set of all functions in $L^2(\mathbb{R})$ which admit an analytic continuation in the upper or lower, resp., complex half-plane. We refer to [16] for details.

This factorization of Ψ can be derived from that of Φ, for which the factorization is known [10, 18, 28]:

$$\Phi(\xi) = \frac{1}{\sqrt{4k}} \Phi_-(\xi)\Phi_+(\xi), \qquad \xi \in \mathbb{R}, \tag{46}$$

where

$$\Phi_-(\xi) = \begin{pmatrix} t_{+-}(\xi) & -t^{-1}(\xi)t_{--}(\xi) \\ t(\xi)t_{--}(\xi) & t_{+-}(\xi) \end{pmatrix}$$

$$\Phi_+(\xi) = \begin{pmatrix} t_{++}(\xi) & -t^{-1}(\xi)t_{-+}(\xi) \\ t(\xi)t_{-+}(\xi) & t_{++}(\xi) \end{pmatrix}$$

where $t(\xi) = -i\sqrt{k+\xi}\sqrt{k-\xi}$, which is consistent with (5), and $t_{\pm\pm}(\xi) = \left(\sqrt{2k} \pm \sqrt{k \pm \xi}\right)^{1/2}$ with the first/second index corresponding to the first/second sign. We also allow complex values for ξ, and, as to the choice of the square-roots, we stipulate that we always choose the principal values (i.e., the branch cut is on the set of negative real numbers). We remark that

$$t_{--}(\xi)t_{+-}(\xi) = \sqrt{k+\xi}, \qquad t_{-+}(\xi)t_{++}(\xi) = \sqrt{k-\xi}.$$

Using this and a couple of straightforward computations one can show the validity of (46) as well as the following facts:

(i) The matrix function $\Phi_-(\xi)$ is analytic for $\xi \in \mathbb{C}$, $\mathrm{Im}(\xi) < \mathrm{Im}(k)$.

(ii) The matrix function $\Phi_+(\xi)$ is analytic for $\xi \in \mathbb{C}$, $\mathrm{Im}(\xi) > -\mathrm{Im}(k)$.

(iii) The determinants $\det \Phi_\pm(\xi) = 2\sqrt{2k}$ are constant functions.

(iv) The asymptotics as $\xi \to \infty$ of the factors can be estimated as follows:

$$\Phi_\pm(\xi) = \begin{pmatrix} O(|\xi|^{1/4}) & O(|\xi|^{-3/4}) \\ O(|\xi|^{5/4}) & O(|\xi|^{1/4}) \end{pmatrix}$$

It follows that the factorization of $\Psi(\xi)$ is now given by $\Psi(\xi) = \frac{1}{\sqrt{4k}}\Psi_-(\xi)\Psi_+(\xi)$ with

$$\Psi_-(\xi) = \begin{pmatrix} t_-^{1/2+\varepsilon}(\xi) & 0 \\ 0 & t_-^{-1/2+\varepsilon}(\xi) \end{pmatrix} \Phi_-(\xi),$$

$$\Psi_+(\xi) = \Phi_+(\xi) \begin{pmatrix} t_+^{-1/2-\varepsilon}(\xi) & 0 \\ 0 & t_+^{1/2-\varepsilon}(\xi) \end{pmatrix}.$$

Using these fact it is easily seen that the above factorization is indeed a weak generalized factorization in L^2 if $\varepsilon \in]-1/4, 1/4[$.

Thus we can conclude the invertibility of $\widetilde{W}_{DN}$ and of W_{DN}. □

In connection with (45) it should be noticed that the Fredholm condition is not fulfilled if $\varepsilon = \pm 1/4$. Hence for those values, the mixed Sommerfeld problem in the slit-plane is not well-posed.

For the purpose of defining the operators $\mathscr{K}_{DN,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}$, which give the solution in the rotated slit-plane we state the following obvious corollary. We also define the operators $\mathscr{K}_{ND,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}$, where the location of the Dirichlet and Neumann data is interchanged.

COROLLARY 5.8. *Let* $\varepsilon \in]-1/4, 1/4[$. *The solution of the Sommerfeld DN problem in* $\mathbb{R}^2\backslash\overline{\Sigma}_\alpha$ *is given by*

$$u = J_\alpha\, \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}}\, g, \quad g \in H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+).$$

Additionally, for the same g, *we obtain by reflection the solution of the Rawlins ND problem in* $\mathbb{R}^2\backslash\overline{\Sigma}$

$$u = \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}}\, g^{\#} = R\, \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}}\, g$$

where the two conditions on $\Sigma^{\pm}$ *are exchanged,* $g^{\#}$ *is defined as in* (23), *and* R *is the reflection operator given by* $Rf(x_1,x_2) = f(x_1,-x_2)$, $(x_1,x_2) \in \mathbb{R}^2$.

In combination, the solution of the Sommerfeld ND problem in $\mathbb{R}^2\backslash\overline{\Sigma}_\alpha$ *is given by*

$$u = \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha}\, g^{\#} = R\, \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_{-\alpha}}\, g.$$

In order to state the solution of the Dirichlet/Neumann problem for the Helmholtz equation in $\Omega_{0,\alpha}$, let us first state some symmetry properties concerned with the superposition of the Sommerfeld potentials $\mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha}$ and $\mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_\beta}$ $(\alpha,\beta \in \mathbb{R})$.

LEMMA 5.9. *Let* $0 < \beta - \alpha < 2\pi$, $g = (g_1, g_2) \in H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)$, *then*

$$T_{0,\Sigma_\gamma^{\pm}}\left(\mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} g - \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_\beta} g^{\#}\right) = 0$$
$$T_{1,\Sigma_\gamma^{\pm}}\left(\mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_\alpha} g + \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_\beta} g^{\#}\right) = 0$$

for $\gamma = \frac{\alpha+\beta}{2} + j\pi$, $j \in \mathbb{Z}$.

Proof. Both formulas result from the symmetry properties of the formulas given in Corollary 5.8. □

THEOREM 5.10. *Let* $\varepsilon \in [0, 1/4[$, $\Omega = \Omega_{0,\alpha}$ *with* $\alpha = 2\pi/n$, $n = 1,3,5,\ldots$. *Then the Dirichlet/Neumann problem for the Helmholtz equation in* Ω *with given data* $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *is well-posed and uniquely solved by the following formulas:*

(i) if $n \equiv 1 \mod 4$,

$$u = \mathscr{K}^{\varepsilon} g = r_\Omega \left(\sum_{j=1}^{\frac{n+1}{2}} (-1)^{j-1} \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_{(2j-1)\alpha}} g + \sum_{j=1}^{\frac{n-1}{2}} (-1)^{j-1} \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_{2j\alpha}} g^{\#} \right),$$

(ii) if $n \equiv 3 \mod 4$, *and* $h = (g_1, -g_2)$,

$$u = \mathscr{K}^{\varepsilon} g = r_\Omega \left(\sum_{j=1}^{\frac{n+1}{2}} (-1)^{j} \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_{(2j-1)\alpha}} h + \sum_{j=1}^{\frac{n-1}{2}} (-1)^{j-1} \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_{2j\alpha}} h^{\#} \right).$$

In both cases, the operator

$$\mathscr{K}^{\varepsilon} : H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

is a linear homeomorphism.

Proof. This follows directly from Proposition 5.7, Corollary 5.8, and Lemma 5.9. □

Let us remark that the previous formulas can be extended to the case $\varepsilon \in]-1/4,0[$ and that the corresponding operator $\mathscr{K}^{\varepsilon}$ is bounded. We will not discuss the uniqueness of the Helmholtz problem in this case.

EXAMPLE 5.11. Figures 9 and 10 illustrate the two different cases of Theorem 5.10.

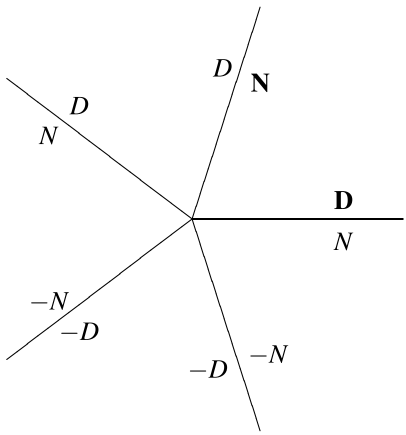

Figure 9: Case $n \equiv 1 \bmod 4$

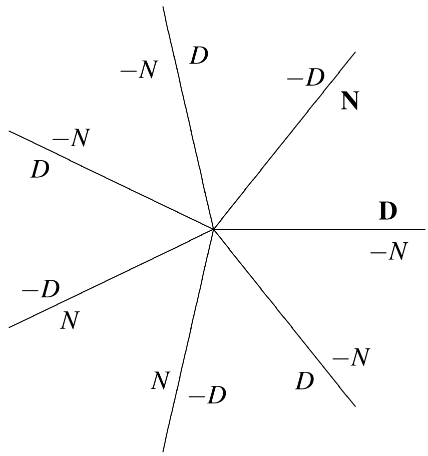

Figure 10: Case $n \equiv 3 \bmod 4$

6. Summary and asymptotic behavior of the solution

After having discussed the various problems, by distinguishing them into several cases, let us finally present some kind of summary. We will indicate on which types of operators the corresponding constructions were based, and for which values of ε we have obtained positive results.

We first consider the Dirichlet problem and the Neumann problem in $\Omega_{0,\alpha}$ with $\alpha = 2\pi/n$.

Case	Dirichlet problem	Neumann problem
$n \equiv 0 \mod 4$	D-half-plane operator and ℓ^o $\varepsilon \in]-1/2,1/2[$	N-half-plane operator and ℓ^e $\varepsilon \in [0,1/2[$
$n \equiv 2 \mod 4$	D-half-plane operator and ι $\varepsilon \in]-1/2,1/2[$	N-half-plane operator and ι $\varepsilon \in [0,1/2[$
$n \equiv 1 \mod 2$	D-slit-plane operator $\varepsilon \in]-1/2,1/2[$	N-slit-plane operator $\varepsilon \in [0,1/2[$

For the mixed DN problem in $\Omega_{0,\alpha}$ with $\alpha = 2\pi/n$ the situation is as follows.

Case	mixed Dirichlet/Neumann problem
$n \equiv 0 \mod 8$	D-half-plane operator and ℓ^o, N-half-plane operator and ℓ^e $\varepsilon \in [0,1/2[$
$n \equiv 4 \mod 8$	D-half-plane operator and ℓ^e, N-half-plane operator and ℓ^o $\varepsilon \in]-1/2,1/2[$
$n \equiv 2 \mod 4$	mixed DN-half-plane operator $\varepsilon \in]-1/2,1/2[$
$n \equiv 1 \mod 2$	mixed DN-slit-plane operator $\varepsilon \in]-1/4,1/4[$

It is known that the half-plane and Sommerfeld potentials $u = \mathscr{K} g$ studied before are bounded near the critical point $x = 0$ provided the data are sufficiently smooth, which is assumed in physically relevant situations. However $\operatorname{grad} u$ behaves asymptotically in a different way, see [18, 23, 31], namely for the DD and NN problems, like

$$\operatorname{grad} u = O(r^{-1/2}) \text{ if } r \to 0.$$

The same is true for the solution of the DN problem in a half-plane, see Lemma 5.1 and Corollary 5.2, but not for the solution in the slit-plane (see Lemma 5.7) where

$$\operatorname{grad} u = O(r^{-3/4}) \text{ if } r \to 0.$$

Since we have presented all solutions of Dirichlet, Neumann and mixed problems in rational angles by finite sums of half-plane and Sommerfeld potentials, we obtain the following conclusion:

COROLLARY 6.1. *Suppose that* $\Omega = \Omega_{0,\alpha}$, $\alpha = 2\pi/n$, $n \in \mathbb{N}$.
I. If $g = (g_1, g_2) \in H^{1/2}(\mathbb{R}_+)^2_\sim \cap C^\infty(\mathbb{R}_+)^2$*, then the solution of the DD problem in* Ω *satisfies*

$$\operatorname{grad} u = O(r^{-1/2}) \text{ if } r \to 0.$$

II. If $g = (g_1, g_2) \in H^{-1/2}(\mathbb{R}_+)^2_\sim \cap C^\infty(\mathbb{R}_+)^2$*, then the solution of the NN problem in* Ω *satisfies*

$$\operatorname{grad} u = O(r^{-1/2}) \text{ if } r \to 0.$$

III. If $g = (g_1, g_2) \in (H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)) \cap C^\infty(\mathbb{R}_+)^2$*, then the solution of the DN problem in* Ω *satisfies*

$$\operatorname{grad} u = O(r^{-1/2}) \text{ if } r \to 0 \text{ and } n = 0 \bmod 2,$$
$$\operatorname{grad} u = O(r^{-3/4}) \text{ if } r \to 0 \text{ and } n = 1 \bmod 2.$$

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