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INTEGRAL ESTIMATES FOR THE FAMILY OF B–OPERATORS

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Abstract. Let $\mathscr{P}_n$ be the class of polynomials of degree at most n. In 1969, Rahman introduced a class $\mathscr{B}_n$ of operators B that map $\mathscr{P}_n$ into itself and proved that

$$\|B[P(R\,\cdot)]\|_\infty \leqslant |B[E_n(R\,\cdot)]|\|P\|_\infty,\ \ R \geqslant 1,$$

for every $B \in \mathscr{B}_n$, where $E_n(z) := z^n$.

In this paper, we show that this inequality holds analogously for the norm $\|\cdot\|_q$ with $q \geqslant 1$ and for some of its refinements as well.

1. Introduction

Let $\mathscr{P}_n$ be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n with complex coefficients. For $P \in \mathscr{P}_n$, define

$$\|P\|_q := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \quad \text{and} \quad \|P\|_\infty := \max_{|z|=1}|P(z)|.$$

It is known that if $P \in \mathscr{P}_n$, then

$$\|P'\|_\infty \leqslant n\|P\|_\infty \tag{1}$$

$$\|P(R\,\cdot)\|_\infty \leqslant R^n\|P\|_\infty,\ \ R > 1. \tag{2}$$

Inequality (1) is an immediate consequence of a famous result due to Bernstein on the derivative of a trignometric polynomial (for reference see[4]), whereas inequality (2) is a simple deduction from the maximum modulus principle (see [15, p.346], [11, p.158 problem 269]).

Inequalities (1) and (2) can be obtained by letting $q \to \infty$ in

$$\|P'\|_q \leqslant n\|P\|_q,\ \ q > 0 \tag{3}$$

and

$$\|P(R\,\cdot)\|_q \leqslant R^n\|P\|_q,\ \ R > 1 \text{ and } q > 0. \tag{4}$$

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Inequality (3) for $q \geqslant 1$ is due to Zygmund [17], where as inequality (4) is a simple consequence of a result due to Hardy [8]. Arestov [2] proved that (3) remains true for $0 < q < 1$ as well.

For the class of polynomials $P \in \mathscr{P}_n$ such that $P(z) \neq 0$ in $|z| < 1$, inequalities (1) and (2) can be replaced by

$$\|P'\|_\infty \leqslant \frac{n}{2}\|P\|_\infty \tag{5}$$

and

$$\|P(R\,\cdot)\|_\infty \leqslant \frac{R^n+1}{2}\|P\|_\infty, \;\; R > 1. \tag{6}$$

Inequality (5) was conjectured by Erdös and later verified by lax [9], whereas Ankeny and Rivilin [1] used (5) to prove (6).

Inequalities (5) and (6) can be obtained by letting $q \to \infty$ in

$$\|P'\|_q \leqslant \frac{n}{\|1+E_n\|_q}\|P\|_q, \quad \text{for } q > 0, \tag{7}$$

and

$$\|P(R\,\cdot)\|_q \leqslant \frac{\|E_n(R\,\cdot)+1\|_q}{\|1+E_n\|_q}\|P\|_q \quad \text{for } R > 1 \text{ and } q > 0. \tag{8}$$

Inequality (7) was found out by de Brujin [6] for $q \geqslant 1$, whereas inequality (8) for $q \geqslant 1$ was proved by Boas and Rahman [5]. Rahman and Schmeisser [13] have shown that inequalities (7) and (8) remain true for $0 < q < 1$ as well.

Rahman [12] (see also Rahman and Schmeisser [14, p.538]) introduced a class $\mathscr{B}_n$ of operators B that map $P \in \mathscr{P}_n$ into itself. That is, the operator B carries $P \in \mathscr{P}_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{9}$$

where $\lambda_0,\ \lambda_1$ and λ_2 are real or complex numbers such that all the zeros of

$$\mathscr{U}(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \quad C(n,r) = \frac{n!}{r!(n-r)!}, \tag{10}$$

lie in the half plane

$$|z| \leqslant \left|z - \frac{n}{2}\right| \tag{11}$$

and observed:

THEOREM A. *If $P(z)$ is a polynomial of degree n, then*

$$|P(z)| \leqslant M, \quad |z| = 1$$

implies

$$|B[P](z)| \leqslant M\,|B[z^n]|, \quad |z| \geqslant 1. \tag{12}$$

As an improvement of (12), recently authors [16] proved the following:

THEOREM B. *If* $P \in \mathscr{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then*

$$|B[P](z)| \leqslant \frac{1}{2}\{\,|B[z^n]| + |\lambda_0|\}\max_{|z|=1}|P(z)|, \quad |z| \geqslant 1. \tag{13}$$

The result is sharp and equality holds for a polynomial whose all zeros lie on the unit disk.

For suitable choices of λ_0, λ_1 and λ_2 (see [16]) Theorem A yields inequalities (1) and (2), whereas Theorem B yields inequalities (5) and (6).

A natural question arises. Does there exist similar integral estimates which yield the compact generalizations of inequalities (3), (4) and (7), (8) respectively such that for $q \to \infty$, these inequalities reduce to Theorem A and Theorem B as well? As an answer to this question, we have been able to prove the following:

THEOREM 1. *If* $P \in \mathscr{P}_n$, *then for every* $R \geqslant 1$, $q \geqslant 1$ *and* $|z| = 1$,

$$\|B[P(R\,\cdot)]\|_q \leqslant |B[E_n(R\,\cdot)]|\|P\|_q, \tag{14}$$

where $B \in \mathscr{B}_n$ *and* $E_n(z) := z^n$. *Or, equivalently for* $0 \leqslant \theta < 2\pi$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\lambda_0 P(Re^{i\theta}) + \lambda_1\left(\frac{nRe^{i\theta}}{2}\right)P'(Re^{i\theta}) + \lambda_2\left(\frac{nRe^{i\theta}}{2}\right)^2\frac{P''(Re^{i\theta})}{2!}\right|^q d\theta\right\}^{1/q}$$

$$\leqslant R^n\left|\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right| \quad \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}, \tag{15}$$

where λ_0, λ_1, λ_2 *are defined above.*
The result is best possible and equality holds for $P(z) = \alpha z^n$, $\alpha \neq 0$.

Theorem A immediately follows from Theorem 1, if we let $q \to \infty$ in inequality (14).

REMARK 1. If we choose $\lambda_0 = 0 = \lambda_2$ in (15), which is possible, as it can be easily verified that in this case all the zeros of $\mathscr{U}(z)$ defined by (10) lie in (11), we get inequality (3) for every $q \geqslant 1$.

THEOREM 2. *Let* $P \in \mathscr{P}_n$ *be such that* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $R \geqslant 1$, $q \geqslant 1$ *and* $|z| = 1$,

$$\|B[P(R\,\cdot)]\|_q \leqslant \frac{|B[E_n(R\,\cdot)]| + |\lambda_0|}{\|1 + E_n\|_q} \quad \|P\|_q, \tag{16}$$

where $B \in \mathscr{B}_n$ *and* $E_n(z) := z^n$.
Or, equivalently for $0 \leqslant \theta < 2\pi$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\lambda_0 P(Re^{i\theta}) + \lambda_1\left(\frac{nRe^{i\theta}}{2}\right)P'(Re^{i\theta}) + \lambda_2\left(\frac{nRe^{i\theta}}{2}\right)^2\frac{P''(Re^{i\theta})}{2!}\right|^q d\theta\right\}^{1/q}$$

$$\leqslant \frac{\left|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\right|R^n+|\lambda_0|}{\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{in\theta}|^q\,d\theta\}^{1/q}} \quad \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}. \tag{17}$$

The result is best possible and equality holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$.

Theorem B easily follows from Theorem 2, if we make $q\to\infty$ in inequality (16). Further, if we choose $\lambda_0=0=\lambda_2,\ R=1$ in (17) which is possible, we get inequality (7) for every $q\geqslant 1$. On the otherhand, for $\lambda_1=\lambda_2=0,$ we have the following:

COROLLARY 1. *If $P\in\mathscr{P}_n$ be such that $P(z)\neq 0$ in $|z|<1$, then for every $R\geqslant 1,\ q\geqslant 1$ and $|z|=1$,*

$$\|P(R\,\cdot)\|_q\leqslant\frac{R^n+1}{\|1+E_n\|_q}\|P\|_q.$$

REMARK 2. Since inequalities (3), (4) and (7), (8) hold for every $q\geqslant 0$, we have a feeling that Theorem 1 and Theorem 2 hold true for $q\in(0,1)$ as well.

A polynomial $P(z)$ is said to be self-inversive if $P(z)=uQ(z),\ |u|=1$, where $Q(z)=z^n\overline{P(1/\overline{z})}$. It is known [7] that if $P\in\mathscr{P}_n$ is a self inversive polynomial, then for every $q\geqslant 1$,

$$\|P'\|_q\leqslant\frac{n}{\|1+E_n\|_q}\|P\|_q. \tag{18}$$

We next present the following more general result concerning self inversive polynomials, which includes inequality (18) as a special case. We prove.

THEOREM 3. *If $P\in\mathscr{P}_n$ is self inversive, then for every $q\geqslant 1,\ R\geqslant 1$ and $|z|=1$,*

$$\|B\left[P(R\,\cdot)\right]\|_q\leqslant\frac{|B[E_n(R\,\cdot)]|+|\lambda_0|}{\|1+E_n\|_q}\quad\|P\|_q, \tag{19}$$

where $B\in\mathscr{B}_n$ and $E_n(z):=z^n$.
Or, equivalently for $0\leqslant\theta<2\pi$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\lambda_0P(Re^{i\theta})+\lambda_1\left(\frac{nRe^{i\theta}}{2}\right)P'(Re^{i\theta})+\lambda_2\left(\frac{nRe^{i\theta}}{2}\right)^2\frac{P''(Re^{i\theta})}{2!}\right|^q d\theta\right\}^{1/q}$$

$$\leqslant \frac{\left|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\right|+|\lambda_0|}{\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{in\theta}|^q\,d\theta\}^{1/q}} \quad \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}. \tag{20}$$

The result is sharp and equality holds for $P(z)=z^n+1$.

Theorem 4 of [16] is a special case of this theorem, if we let $q\to\infty$.

For $\lambda_0=0=\lambda_2,\ R=1$ inequality (20) yields inequality (18) and for $\lambda_1=0=\lambda_2$, we also have the following:

COROLLARY 2. *If $P\in\mathscr{P}_n$ is self inversive, then for every $q\geqslant 1,\ R\geqslant 1$,*

$$\|P(R\,\cdot)\|_q\leqslant\frac{R^n+1}{\|1+E_n\|_q}\|P\|_q\ \ for\ \ |z|=1. \tag{21}$$

The result is sharp and equality holds for $P(z)=z^n+1$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

LEMMA 1. *Let $\mathscr{P}_n$ denote the linear space of polynomials*

$$P(z) = a_0 + \cdots + a_n z^n$$

of degree n with complex coefficients, normed by $\|P\| = \max|P(e^{i\theta})|,\ 0 < \theta \leqslant 2\pi$. Define the linear functional $\mathscr{L}$ on $\mathscr{P}_n$ as

$$\mathscr{L} : P \to l_0 a_0 + l_1 a_1 + \cdots + l_n a_n,$$

where l_j's are complex numbers. If the norm of the functional is $\mathscr{N}$ then

$$\int_0^{2\pi} \Theta\left(\frac{\left|\sum_{k=0}^{n} l_k a_k e^{ik\theta}\right|}{\mathscr{N}}\right) d\theta \leqslant \int_0^{2\pi} \Theta\left(\left|\sum_{k=0}^{n} a_k e^{ik\theta}\right|\right) d\theta, \tag{22}$$

where $\Theta(t)$ is a non-decreasing convex function of t.
The above lemma is due to Rahman [12].

The next lemma which we need follows from [10, Corollary 18.3], (see also [12]).

LEMMA 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leqslant 1$, then all the zeros of the polynomial $B[P](z)$ also lie in the circle $|z| \leqslant 1$.*

LEMMA 3. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $|z| \geqslant 1$,*

$$|B[P](z)| \leqslant |B[Q](z)|, \tag{23}$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

The proof of Lemma 3 is implicit in [12, Section 5].

LEMMA 4. *If $P \in \mathscr{P}_n$, then for every $R \geqslant 1,\ q \geqslant 1,\ 0 \leqslant \theta < 2\pi$*

$$\int_0^{2\pi}\int_0^{2\pi} \left|B[P(Re^{i\theta})] + e^{in\alpha} B[R^n P(e^{i\theta}/R)]\right|^q d\theta d\alpha$$

$$\leqslant 2\pi\left[\left|B[R^n e^{in\theta}]\right| + |\lambda_0|\right]^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta. \tag{24}$$

Proof of Lemma 4. Let $M = \max_{|z|=1}|P(z)|$, so that $|P(z)| \leqslant M$ for $|z| \leqslant 1$. If λ is any real or complex number with $|\lambda| > 1$, then by Rouchés theorem $P(z) - \lambda M$ does not vanish in $|z| \leqslant 1$. Hence, if $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then by Lemma 3 and the fact that $B[1] = \lambda_0$, we have

$$|B[P](z) - \lambda\lambda_0 M| \leqslant |B[Q](z) - \lambda M B[z^n]| \text{ for } |z| \geqslant 1. \tag{25}$$

Since $|Q(z)| = |P(z)| \leqslant M$ for $|z| = 1$, therefore by inequality (12), it is possible to choose argument of λ such that

$$|B[Q](z) - \lambda MB[z^n]| = M|\lambda||B[z^n]| - |B[Q](z)|.$$

Hence choosing the argument of λ in the right hand side of inequality (25) suitably, we get

$$|B[P](z)| - |\lambda||\lambda_0|M \leqslant M|\lambda||B[z^n]| - |B[Q](z)|.$$

This gives after making $|\lambda| \to 1$

$$|B[P](z)| + |B[Q](z)| \leqslant \{|B[z^n]| + |\lambda_0|\}M \quad for \quad |z| \geqslant 1. \tag{26}$$

In particular for every $\theta,\ 0 \leqslant \theta < 2\pi$ and $R \geqslant 1$, we have

$$\left|B[P(Re^{i\theta})]\right| + \left|B[R^nP(e^{i\theta}/R)]\right| \leqslant \left\{\left|B[R^ne^{in\theta}]\right| + |\lambda_0|\right\}M.$$

Thus for every α with $0 \leqslant \alpha < 2\pi$, we have

$$\left|B[P(Re^{i\theta})] + e^{in\alpha}B[R^nP(e^{i\theta}/R)]\right| \leqslant \left\{\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|R^n + |\lambda_0|\right\}M. \tag{27}$$

This shows that

$$\Lambda := B[P(Re^{i\theta})] + e^{in\alpha}B[R^nP(e^{i\theta}/R)]$$

is a bounded linear operator on $\mathscr{P}_n$ and in view of (27), the norm of the bounded linear functional

$$\mathscr{L} : P \to \left\{B[P(Re^{i\theta})] + e^{in\alpha}B[R^nP(e^{i\theta}/R)]\right\}_{\theta=0}$$

is

$$\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|R^n + |\lambda_0|.$$

Therefore, by Lemma 1 for $\Theta(t) = t^q,\ q \geqslant 1$, it follows that

$$\int_0^{2\pi} \left|B[P(Re^{i\theta})] + e^{in\alpha}B[R^nP(e^{i\theta}/R)]\right|^q d\theta$$
$$\leqslant \left[\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|R^n + |\lambda_0|\right]^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta. \tag{28}$$

Integrating the two sides of (28) with respect to α, we get

$$\int_0^{2\pi}\int_0^{2\pi} \left|B[P(Re^{i\theta})] + e^{in\alpha}B[R^nP(e^{i\theta}/R)]\right|^q d\theta d\alpha$$
$$\leqslant \int_0^{2\pi} \left[\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|R^n + |\lambda_0|\right]^q d\alpha \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta$$
$$= 2\pi\left[\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right|R^n + |\lambda_0|\right]^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta$$
$$= 2\pi\left[B[R^ne^{in\theta}] + |\lambda_0|\right]^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta.$$

This completes the proof of Lemma 4. □

3. Proofs of the Theorems

Proof of Theorem 1. If $M = \max_{|z|=1} |P(z)|$, then by inequality (12)

$$|B[P](z)| \leqslant M|B[z^n]| \ \ for \ \ |z| \geqslant 1.$$

This in particular gives for every θ, $0 \leqslant \theta < 2\pi$ and $R \geqslant 1$,

$$|B[P(Re^{i\theta})]| \leqslant M|B[R^n e^{in\theta}]|,$$

or

$$\left|B[P(Re^{i\theta})]\right| \leqslant M\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right| R^n. \tag{29}$$

Since B is linear operator (see [12, sec. 5]), therefore $\Lambda = B[P(Re^{i\theta})]$ is a bounded linear operator on $\mathscr{P}_n$. Thus in view of (29), the norm of the bounded linear functional

$$\mathscr{L} : P \to \left\{B[P(Re^{i\theta})]\right\}_{\theta=0}$$

is

$$\left|\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right| R^n.$$

Hence by Lemma 1 for every $q \geqslant 1$, we have

$$\int_0^{2\pi} \left|B[P(Re^{i\theta})]\right|^q d\theta \leqslant \left|\left\{\lambda_0 + \frac{n^2}{2}\lambda_1 + \frac{n^3(n-1)}{8}\lambda_2\right\} R^n\right|^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta.$$

From this inequality (14) follows immediately and this completes the proof of Theorem 1. □

Proof of Theorem 2. Since $P(z) \neq 0$ in $|z| < 1$, by Lemma 3, we have for each θ, $0 \leqslant \theta < 2\pi$ and $R \geqslant 1$,

$$\left|B[P(Re^{i\theta})]\right| \leqslant \left|B[R^n P(e^{i\theta}/R)]\right|.$$

Also for every real θ and $t \geqslant 1$, it can be easily verified that $|1+te^{i\theta}| \geqslant |1+e^{i\theta}|$ and therefore for every $q \geqslant 1$,

$$\int_0^{2\pi} |1+te^{i\theta}|^q d\theta \geqslant \int_0^{2\pi} |1+e^{i\theta}|^q d\theta. \tag{30}$$

Now, taking $t = \frac{\left|B[R^n P(e^{i\theta}/R)]\right|}{\left|B[P(Re^{i\theta})]\right|} \geqslant 1$ and using inequality (30), we have

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left|B[P(Re^{i\theta})]+e^{in\alpha}B[R^nP(e^{i\theta}/R)]\right|^q d\theta d\alpha\\
&\qquad=\int_0^{2\pi}\int_0^{2\pi}|B[P(Re^{i\theta})]|^q\left|1+e^{in\alpha}\frac{B[R^nP(e^{i\theta}/R)]}{B[P(Re^{i\theta})]}\right|^q d\alpha d\theta\\
&\qquad=\int_0^{2\pi}\left\{\left|B[P(Re^{i\theta})]\right|^q\int_0^{2\pi}\left|1+e^{in\alpha}\left|\frac{B[R^nP(e^{i\theta}/R)]}{B[P(Re^{i\theta})]}\right|\right|^q d\alpha\right\}d\theta\\
&\qquad\geqslant\int_0^{2\pi}\left\{\left|B[P(Re^{i\theta})]\right|^q\int_0^{2\pi}|1+e^{in\alpha}|^q d\alpha\right\}d\theta\\
&\qquad=\int_0^{2\pi}\left|B[P(Re^{i\theta})]\right|^q d\theta\int_0^{2\pi}|1+e^{in\alpha}|^q d\alpha.
\end{aligned}\tag{31}$$

Inequality (31) in conjuction with Lemma 4, gives

$$\int_0^{2\pi}\left|B[P(Re^{i\theta})]\right|^q d\theta\leqslant\frac{2\pi\left[\left|B[R^ne^{in\theta}]\right|+|\lambda_0|\right]^q}{\int_0^{2\pi}|1+e^{in\alpha}|^q d\alpha}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

Equivalently

$$\|B\,[P(R\,\cdot)]\,\|_q\leqslant\frac{|B[E_n(R\,\cdot)]|+|\lambda_0|}{\|1+E_n\|_q}\quad\|P\|_q.$$

This completes proof of Theorem 2. □

Proof of Theorem 3. Since $P(z)$ is a self inversive polynomial, we have

$$P(z)=uQ(z),\quad where\ \ Q(z)=z^n\overline{P(1/\bar{z})}\ \ and\ \ |u|=1.$$

This in particular gives

$$|B[P](z)|=|B[Q](z)|\ \ \ for\ \ \ |z|\geqslant 1.$$

That is

$$\left|B[P(Re^{i\theta})]\right|=\left|B[R^nP(e^{i\theta}/R)]\right|,\ \text{for } 0\leqslant\theta<2\pi.\tag{32}$$

Inequality (32) in conjuction with Lemma 4, gives

$$\int_0^{2\pi}\left|B[P(Re^{i\theta})]\right|^q d\theta\leqslant\frac{2\pi\left[|B[R^ne^{in\theta}]|+|\lambda_0|\right]^q}{\int_0^{2\pi}|1+e^{in\alpha}|^q d\alpha}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

Equivalently for $R\geqslant 1$ and $q\geqslant 1$,

$$\|B\,[P(R\,\cdot)]\,\|_q\leqslant\frac{\left|B[E_n(R\,\cdot)]\right|+|\lambda_0|}{\|1+E_n\|_q}\quad\|P\|_q.$$

This completes proof of Theorem 3. □

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