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POLYNOMIAL INVERSE IMAGES AND POLYNOMIAL NUMERICAL HULLS OF NORMAL MATRICES

HAMID REZA AFSHIN, MOHAMMAD ALI MEHRJOOFARD
AND ABBAS SALEMI

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Abstract. Let $A \in M_n$ be a normal matrix and let $k \in \mathbb{N}$. In this note we introduce the notion "Polynomial inverse image of order k". The polynomial numerical hull of order k, denoted by $V^k(A)$ are characterized by the intersection of polynomial inverse images of order k. Also, the locus of $V^{n-1}(A)$ in the complex plane are determined.

1. Introduction

Let M_n be the set of $n \times n$ complex matrices. Motivated by the study of convergence of iterative methods in solving linear systems (e.g., see [4, 5, 7]), researchers studied the *polynomial numerical hull of order* k of a matrix $A \in M_n$, which is defined and denoted by

$$V^k(A) = \{\xi \in \mathbb{C} : |p(\xi)| \leqslant \|p(A)\| \text{ for all } p(z) \in \mathscr{P}_k[\mathbb{C}]\},$$

where $\mathscr{P}_k[\mathbb{C}]$ is the set of complex polynomials with degree at most k. The *joint numerical range* of $(A_1, A_2, \ldots, A_m) \in M_n \times \cdots \times M_n$ is denoted by

$$W(A_1, A_2, \ldots, A_m) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

By the result in [4] (see also [5])

$$V^k(A) = \{\zeta \in \mathbb{C} : (0, \ldots, 0) \in \mathrm{conv}W((A - \zeta I), (A - \zeta I)^2, \ldots, (A - \zeta I)^k)\},$$

where $\mathrm{conv}X$ denotes the convex hull of $X \subseteq \mathbb{C}^k$.

In Section 2, we introduce a new concept "polynomial inverse image of order k". Also, we study the relationship between polynomial inverse image of order k and polynomial numerical hull of order k for a normal matrices. In section 3, by using the polynomial inverse images of $[0, \infty)$, the locus of the polynomial numerical hulls of order $n-1$ are characterized. Additional results are given in Section 4.

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2. Polynomial inverse image of order k

In this section we are introducing the notion polynomial inverse image of order k to study the polynomial numerical hulls of order k. We are using $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ to denote the real and the imaginary parts of $w \in \mathbb{C}$, respectively.

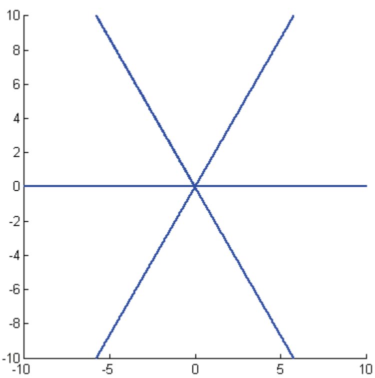
It was shown in [3, Theorem 3.1] that if $A \in M_n(\mathbb{C})$ is normal with $\sigma(A)$ lying on a rectangular hyperbola $\mathscr{R}$ in the complex plane, then $V^2(A)$ is a subset of $\mathscr{R}$ as well. It is readily seen that, if L is a straight line in the complex plane then the set $L^{1/2} = \{z : z^2 \in L\}$ is a rectangular hyperbola. In [1, Theorem 4.3], we obtained that, if A is a normal matrix and S is an arbitrary convex set with $\sigma(A) \subset S^{\frac{1}{k}}$, then $V^k(A) \subset S^{\frac{1}{k}}$. By using the following definition, we are going to extend the above results.

DEFINITION 2.1. Let q be a polynomial of degree k and let $S \subseteq \mathbb{C}$. The set $\{z \in \mathbb{C} : \mathrm{Im}(q(z)) \in S\}$ is called a *polynomial inverse image of order k of S* and is abbreviated by $\mathrm{PII}_k(S)$.

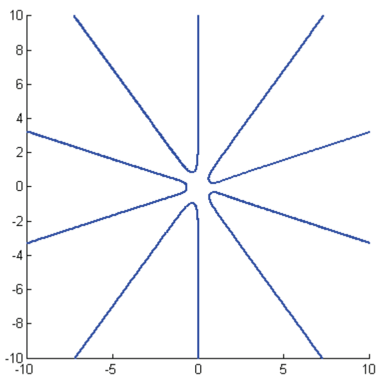
PROPOSITION 2.2. *Every rectangular hyperbola is a* $\mathrm{PII}_2(\{0\})$ *and vice versa.*

Proof. Let $\mathscr{R} = \left\{(x,y) \in \mathbb{R}^2 : r_1\left(x^2 - y^2\right) + r_2xy + r_3x + r_4y + r_5 = 0\right\}$ *be a rectangular hyperbola, where* $r_1, \cdots, r_5 \in \mathbb{R}$, $(r_1, r_2) \neq (0,0)$. *Define* $p(z) = \left(\frac{1}{2}r_2 + ir_1\right)z^2 + (r_4 + ir_3)z + ir_5$. *It is readily seen that* $\mathscr{R} = \{z \in \mathbb{C} : \mathrm{Im}(p(z)) = 0\}$ *is a* $\mathrm{PII}_2(\{0\})$. *By the same method the converse is trivial.* $\square$

We know that $\mathrm{Im}(ip(z)) = \mathrm{Re}(p(z))$, *then* $\{z \in \mathbb{C} : \mathrm{Re}(p(z)) = 0\}$ *is also a* $\mathrm{PII}_k(\{0\})$.



$\left[\mathrm{Im}(z^3)\right]^{-1}(\{0\})$



$\left[\mathrm{Im}(5iz^5 - (1+i)z^4 - 3iz^2 - 3 + 2i)\right]^{-1}(\{0\})$

THEOREM 2.3. *Suppose p is a complex polynomial of degree k and $A \in M_n$ is a normal matrix. Let $S \subset \mathbb{C}$ be a convex set and let $\ell : \mathbb{C} \to \mathbb{C}$ be a real linear transformation such that $\sigma(A) \subset (\ell \circ p)^{-1}(S)$. Then $V^k(A) \subset (\ell \circ p)^{-1}(S)$.*

Proof. Without loss of generality we assume that $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. Let $\mu \in V^k(A)$. By [4], we know that, the joint numerical range $W(A, A^2, \ldots, A^k)$ is convex. So there exists a unit vector $X = (x_1, x_2, \ldots, x_n)^t$ such that $\mu^i = X^* A^i X = \Sigma_{j=1}^n |x_j|^2 a_j^i$, $i = 1, \ldots, n$. Hence $p(\mu) = X^* p(A) X = \Sigma_{j=1}^n |x_j|^2 p(a_j)$. Therefore, $\ell \circ p(\mu) = \Sigma_{j=1}^n |x_j|^2 \ell(p(a_j))$. Since $\ell(p(a_j)) \in S$, $j = 1, \ldots, n$ and S is convex, we obtain that $\ell \circ p(\mu) \in S$ and hence $\mu \in (\ell \circ p)^{-1}(S)$. $\square$

In Theorem 2.3, we consider the linear transformation $\ell : \mathbb{C} \to \mathbb{C}$ by $\ell(z) = \mathrm{Im}(z)$, $\forall z \in \mathbb{C}$ and $S = \{0\}$. Hence, the following holds.

COROLLARY 2.4. *Let $A \in M_n$ be a normal matrix. If $\sigma(A)$ is a subset of a* $\mathrm{PII}_k(\{0\})$, *then* $V^k(A) \subseteq \mathrm{PII}_k(\{0\})$.

Also, if we consider the linear transformation $\ell : \mathbb{C} \to \mathbb{C}$ in Theorem 2.3 by $\ell(z) = z$, $\forall z \in \mathbb{C}$ and $p(z) = z^k$, we obtain the following:

COROLLARY 2.5. [1, Theorem 4.3] *Let $A \in M_n$ be a normal matrix and let $S \subset \mathbb{C}$ be a convex set. If $\sigma(A) \subseteq (S)^{\frac{1}{k}}$, then $V^k(A) \subseteq (S)^{\frac{1}{k}}$.*

If we have 4 points in the complex plane, then there exists a rectangular hyperbola ($\mathrm{PII}_2(\{0\})$) passing through these four points. Now, we attempt to extend this result to $\mathrm{PII}_k(\{0\})$.

THEOREM 2.6. *Let $\{a_1, \ldots, a_{2k}\}$ be a set of complex numbers. Then there exists a $\mathrm{PII}_\ell(\{0\})$, $(1 \leqslant \ell \leqslant k)$, passing through these $2k$ points in the complex plane $\mathbb{C}$.*

Proof. We are looking to find a non-constant complex polynomial $p(z) = \alpha_k z^k + \cdots + \alpha_1 z + \alpha_0$, where $\mathrm{Im} p(a_i) = 0$, $i = 1, \ldots, 2k$. We consider the $2k \times (2k+1)$ matrix $\mathbf{A}$ such that it's i^{th} row $\mathbf{A}_i = \left(1, \mathrm{Re}(a_i), \mathrm{Im}(a_i), \ldots, \mathrm{Re}(a_i^k), \mathrm{Im}(a_i^k)\right)$. We know that the homogeneous system $\mathbf{A}X = 0$ has a nontrivial solution $X = (x_0, x_1, \ldots, x_{2k})^t \in \mathbb{R}^{2k+1}$. Define $\alpha_0 = ix_0$ and $\alpha_j = x_{2j} + ix_{2j-1}$, $j = 1, \ldots, k$. Hence $p(z) = (x_{2k} + ix_{2k-1})z^k + \cdots + (x_2 + ix_1)z + x_0 i$. Let $\ell := \deg(p)$. Then $1 \leqslant \ell \leqslant k$. Direct computation shows that $\mathrm{Im} p((a_i)) = 0$, $i = 1, \ldots, 2k$. Therefore, $\{a_1, \cdots, a_{2k}\} \subseteq [\mathrm{Im} p]^{-1}(\{0\})$. $\square$

The following example shows that in general it is not possible to find a $\mathrm{PII}_k(\{0\})$ passing through any $2k$ points in the complex plane.

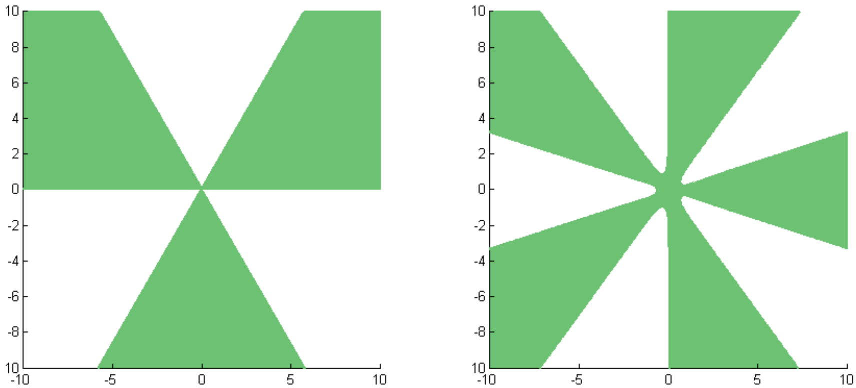
EXAMPLE 2.7. Let $\mathscr{R} = \mathrm{PII}_2(\{0\}) = \left\{z : \mathrm{Im}\left(z^2\right) = 2\right\}$ and suppose that $z_k = k + \frac{i}{k}$, $k = 1, 2, \ldots, 6$ be complex numbers. It is easy to see that $\mathscr{R}$ passing through these 6 points. We will show that there is no $\mathrm{PII}_3(\{0\})$ passing through these 6 points. Assume, if possible that, there exists a polynomial $q(z) = (a_1 + ia_2)z^3 + (b_1 + ib_2)z^2 + (c_1 + ic_2)z + (d_1 + id_2)$ such that $\mathrm{Im} q(z_k) = 0, k = 1, \ldots, 6$ and $(a_1, a_2) \neq (0, 0)$. Therefore,

$$a_2 k^6 + b_2 k^5 + (3a_1 + c_2) k^4 + (2b_1 + d_2) k^3 + (c_1 - 3a_2) k^2 - b_2 k - a_1 = 0, \ k = 1, 2, \ldots, 6. \tag{1}$$

Define $h(z) := a_2z^6 + b_2z^5 + (3a_1 + c_2)z^4 + (2b_1 + d_2)z^3 + (c_1 - 3a_2)z^2 - b_2z - a_1$. By (1), we know that $h(1) = h(2) = \cdots = h(6) = 0$. Then $a_2 \neq 0$. Since, the coefficients of z^5 and $-z$ in the polynomial $h(z)$ are the same, we obtain that $1+2+\cdots+6 = \frac{b_2}{a_2} = -(1\times 2\times\cdots\times 6)\left(1+\frac{1}{2}+\cdots+\frac{1}{6}\right)$, a contradiction.

3. Polynomial inverse image of $[0,\infty)$

Let $A \in M_n$ be a normal matrix. By Corollary 2.4, if $\sigma(A)$ is a subset of a $\mathrm{PII}_k(\{0\})$, then so does $V^k(A)$. But exactly which part of $\mathrm{PII}_k(\{0\})$ belongs to $V^k(A)$ was not determined. In the following we characterize these parts. First, we need the following (see [3, Section 3]).



$\left[\mathrm{Im}(z^3)\right]^{-1}[0,\infty)$ $\left[\mathrm{Im}(5iz^5-(1+i)z^4-3iz^2-3+2i)\right]^{-1}[0,\infty)$

LEMMA 3.1. *Let $A \in M_n$ be a normal matrix such that $\sigma(A)$ is a subset of a polynomial inverse image of $\{0\}$,*

$$\mathrm{PII}_k(\{0\}) = \left\{z : r_{2k}\mathrm{Re}\left(z^k\right) + r_{2k-1}\mathrm{Im}\left(z^k\right) + \cdots + r_2\mathrm{Re}(z) + r_1\mathrm{Im}(z) + r_0 = 0\right\}, \tag{2}$$

where $r_0,\ldots,r_{2k}$ are real numbers and $(r_{2k-1}, r_{2k}) \neq (0,0)$. Then

(a) If $r_{2k-1} \neq 0$, then

$$V^k(A) = \mathrm{PII}_k(\{0\}) \cap \left\{\begin{array}{l} z\in\mathbb{C} : \left(\mathrm{Re}(z),\mathrm{Im}(z),\cdots,\mathrm{Re}\left(z^{k-1}\right),\mathrm{Im}\left(z^{k-1}\right),\mathrm{Re}\left(z^k\right)\right) \\ \in W\left(\mathrm{Re}(A),\mathrm{Im}(A),\cdots,\mathrm{Re}\left(A^{k-1}\right),\mathrm{Im}\left(A^{k-1}\right),\mathrm{Re}\left(A^k\right)\right)\end{array}\right\}$$

(b) if $r_{2k} \neq 0$, then

$$V^k(A) = \mathrm{PII}_k(\{0\}) \cap \left\{\begin{array}{l} z\in\mathbb{C} : \left(\mathrm{Re}(z),\mathrm{Im}(z),\cdots,\mathrm{Re}\left(z^{k-1}\right),\mathrm{Im}\left(z^{k-1}\right),\mathrm{Im}\left(z^k\right)\right) \\ \in W\left(\mathrm{Re}(A),\mathrm{Im}(A),\cdots,\mathrm{Re}\left(A^{k-1}\right),\mathrm{Im}\left(A^{k-1}\right),\mathrm{Im}\left(A^k\right)\right)\end{array}\right\}$$

Let $A = \mathrm{diag}(a_1,\ldots,a_4)$. By [2, Theorem 2.2] we can write $V^2(A)$ as the intersection of 4 $\mathrm{PII}_2([0,\infty))$ sets and the rectangular hyperbola passing through $\sigma(A)$. In the following theorem we extend this result.

THEOREM 3.2. *Let* $A = \operatorname{diag}(a_1, \ldots, a_{2k}) \in M_{2k}(\mathbb{C})$. *Let* $\mathrm{PII}_k(\{0\})$ *as in (2) be the unique polynomial inverse image of order* k *of* $\{0\}$ *passing through* $\sigma(A)$. *Then for any* $1 \leqslant i \leqslant 2k$, *there exist a polynomial inverse image of* $[0,\infty)$ *of order* $1 \leqslant \ell_i \leqslant k$ *such that* $V^k(A) = \bigcap_{i=1}^{2k} \mathrm{PII}_{\ell_i}([0,\infty)) \cap \mathrm{PII}_k(\{0\})$.

Proof. By Lemma 3.1, without loss of generality we assume that

$$V^k(A) = \mathrm{PII}_k(\{0\}) \cap \left\{ \begin{array}{l} \mu \in \mathbb{C} : \left(\mathrm{Re}(\mu), \mathrm{Im}(\mu), \cdots, \mathrm{Re}\left(\mu^{k-1}\right), \mathrm{Im}\left(\mu^{k-1}\right), \mathrm{Re}\left(\mu^k\right)\right) \\ \in W\left(\mathrm{Re}(A), \mathrm{Im}(A), \cdots, \mathrm{Re}\left(A^{k-1}\right), \mathrm{Im}\left(A^{k-1}\right), \mathrm{Re}\left(A^k\right)\right) \end{array} \right\} \tag{3}$$

By [4, Theorem 2.11], we know that $W\left(\mathrm{Re}(A), \mathrm{Im}(A), \cdots, \mathrm{Re}\left(A^{k-1}\right), \mathrm{Im}\left(A^{k-1}\right), \mathrm{Re}\left(A^k\right)\right)$ is convex. Then by (3), $\mu \in V^k(A)$ if and only if $\mu \in \mathrm{PII}_k(\{0\})$ and there exist $\lambda_1, \ldots, \lambda_{2k} \geqslant 0$ such that

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathrm{Re}(a_1) & \mathrm{Re}(a_2) & \cdots & \mathrm{Re}(a_{2k}) \\ \mathrm{Im}(a_1) & \mathrm{Im}(a_2) & \cdots & \mathrm{Im}(a_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}(a_1^k) & \mathrm{Re}(a_2^k) & \cdots & \mathrm{Re}(a_{2k}^k) \end{bmatrix}}_{B} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathrm{Re}(\mu) \\ \mathrm{Im}(\mu) \\ \vdots \\ \mathrm{Re}(\mu^k) \end{bmatrix}.$$

Since $\mathrm{PII}_k(\{0\})$ is the unique polynomial inverse image of order k of $\{0\}$ passing through $\sigma(A)$, we obtain that B is invertible. Let $B_1, \ldots, B_{2k}$ be the rows of the matrix B^{-1}. Thus $\mu \in V^k(A)$ if and only if $\mu \in \mathrm{PII}_k(\{0\})$ and $B_i\left[1, \mathrm{Re}(\mu), \mathrm{Im}(\mu), \ldots, \mathrm{Re}(\mu^k)\right]^t \geqslant 0$, $1 \leqslant i \leqslant 2k$. Define polynomial inverse image of order $1 \leqslant \ell_i \leqslant k$ as follows:

$$\mathrm{PII}_{\ell_i}([0,\infty)) = \left\{ \mu \in \mathbb{C} : B_i \left[1, \mathrm{Re}(\mu), \mathrm{Im}(\mu), \ldots, \mathrm{Re}(\mu^k)\right]^t \geqslant 0 \right\}, \; i = 1, \ldots, 2k.$$

Therefore, $V^k(A) = \bigcap_{i=1}^{2k} \mathrm{PII}_{\ell_i}([0,\infty)) \cap \mathrm{PII}_k(\{0\})$. $\square$

REMARK 3.3. Let $A = \operatorname{diag}(a_1, \cdots, a_n) \in M_n$. We know that if there exist $1 \leqslant i < j \leqslant n$ such that $a_i = a_j$, then $V^{n-1}(A) = \sigma(A)$ [3, Lemma 1.2]. Thus, without loss of generality, we assume that $a_1, \ldots, a_n$ are distinct complex numbers. We are looking to find the locus of the set $V^{n-1}(A) \setminus \sigma(A)$. Note that by [2, Theorem 5.1] and its proof, for a normal matrix A with distinct eigenvalues $a_1, \ldots, a_n$, we have $\mu \in V^{n-1}(A) \setminus \sigma(A)$ if and only if μ is the unique element not in $\sigma(A)$ such that the system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix} X = \begin{pmatrix} 1 \\ \mu \\ \vdots \\ \mu^{n-1} \end{pmatrix} \tag{4}$$

has a nonnegative solution $X = (x_1, \ldots, x_n)^t$. By Cramer's rule, $x_k = p_k(\mu) \geqslant 0$, where $p_k(z) = \frac{\prod_{i \neq k}(z - a_i)}{\prod_{i \neq k}(a_k - a_i)}$, $k = 1, \ldots, n$ are the Lagrange polynomials for $a_1, \ldots, a_n$, respectively.

By Remark 3.3, we have the following:

THEOREM 3.4. *Let $A \in M_n$, $n \geqslant 3$ be a normal matrix with distinct eigenvalues and let the Lagrange polynomials p_k, $k = 1,\dots,n$ be as above. Then $V^{n-1}(A) = \bigcap_{j=1}^{n} p_j^{-1}([0,\infty))$.*

Theorem 3.4 characterize the locus of the set $V^{n-1}(A)$ as the intersection of some $\mathrm{PII}_k([0,\infty))$. In the following examples we are using the Matlab programs to draw the figures (see [1, Theorem 2.5]).

EXAMPLE 3.5. Let $A = \operatorname{diag}(1,-1,i,-i)$. The Lagrange polynomials for $\{1,-1,i,-i\}$ are $p_1(z) = \frac{z^3+z^2+z+1}{4}$, $p_2(z) = \frac{-z^3+z^2-z+1}{4}$, $p_3(z) = \frac{iz^3-z^2-iz+1}{4}$, $p_4(z) = \frac{-iz^3-z^2+iz+1}{4}$ respectively. Then $V^3(A) = \bigcap_{j=1}^{4} p_j^{-1}([0,\infty)) = \{1,-1,0,i,-i\}$, (see Figure (i)).

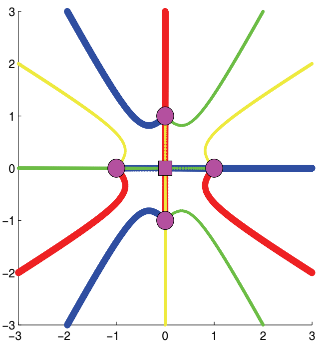


Figure (i)

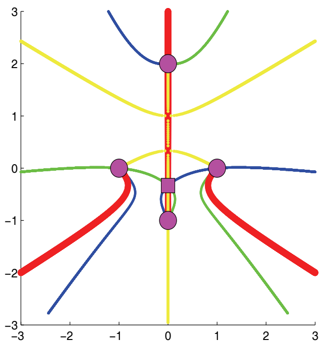


Figure (ii)

EXAMPLE 3.6. Let $B = \operatorname{diag}(1,-1,2i,-i)$. The Lagrange polynomials for $\{1,-1,2i,-i\}$ are $q_1(z) = \frac{z^3+(1-i)z^2+(2-i)z+2}{6-2i}$, $q_2(z) = \frac{z^3-(1+i)z^2+(2+i)z-2}{-6-2i}$, $q_3(z) = \frac{z^3+iz^2-z-i}{-15i}$, $q_4(z) = \frac{z^3-2iz^2-z+2i}{6i}$, respectively. Then $V^3(B) = \bigcap_{j=1}^{4} q_j^{-1}([0,\infty)) = \{1,-1,-i/3,2i,-i\}$, (see Figure (ii)).

REMARK 3.7. Let $A \in M_n$ be a normal matrix. By [4, Theorem 2.11] we know that
$P = W\left(\operatorname{Re}(A),\operatorname{Im}(A),\cdots,\operatorname{Re}\left(A^k\right),\operatorname{Im}\left(A^k\right)\right)$ is a polytope and by Minkowski-Weyl theorem [6], every polytope is a bounded polyhedron. Then there exists an $m \times 2k$ real matrix D and $b \in \mathbb{R}^{2k}$ such that $P = \{X \in \mathbb{R}^{2k} : DX \geqslant b\}$. Also, we know that $\mu \in V^k(A)$ if and only if $\left(\operatorname{Re}(\mu),\operatorname{Im}(\mu),\cdots,\operatorname{Re}(\mu^k),\operatorname{Im}(\mu^k)\right)^t \in P$. Therefore, $V^k(A)$ is the intersection of at most m sets $\mathrm{PII}_{\ell_i}([0,\infty))$ $(1 \leqslant \ell_i \leqslant k)$.

QUESTION. Let $A \in M_n$ be a normal matrix. It would be nice to find the smallest integer m such that $V^k(A)$ is the intersection of m polynomial inverse images of $[0,\infty)$ of orders ℓ_i, $(1 \leqslant \ell_i \leqslant k)$.

4. Additional Results

In this section, we shall characterize the polynomial numerical hulls of order $2k$ for normal matrices such that their spectrum belong to a $\mathrm{PII}_k(\{0\})$.

THEOREM 4.1. *Suppose that $A \in M_n$ is a normal matrix and $\sigma(A)$ is contained in a $\mathrm{PII}_k(\{0\})$. Then $V^{2k}(A) = \sigma(A)$.*

Proof. Assume, if possible that, $\mu \in V^{2k}(A) \setminus \sigma(A)$. Without loss of generality, we assume that $\sigma(A)$ contains n distinct complex numbers and $2k < n$. Let p be a complex polynomial of degree k such that

$$\mathscr{R}_k = \mathrm{PII}_k(\{0\}) = \{z \in \mathbb{C} : \mathrm{Im}(p(z)) = 0\}.$$

Whereas $\sigma(A) \subseteq \mathscr{R}_k$, then $p(\lambda) \in \mathbb{R}$, for all $\lambda \in \sigma(A)$. This means that $p(A)$ is Hermitian. Since $\mu \in V^{2k}(A)$, and $\deg(p) = k$, we obtain that $p(\mu) \in V^2(p(A)) = \sigma(p(A)) = p(\sigma(A))$. Therefore, there exists $\lambda_1 \in \sigma(A)$ such that $p(\mu) = p(\lambda_1)$. Without loss of generality we assume that $A = [\lambda_1] \oplus A_1$, $\lambda_1 \notin \sigma(A_1)$. Therefore, there exist $x = (x_1, x_2)^t \in \mathbb{C}^n$ such that $x_1 \in \mathbb{C}$ and $\mu^i = \lambda_1^i |x_1|^2 + x_2^* A_1^i x_2, i = 1, 2, \ldots, 2k$. Thus, $p(\mu)^i = p(\lambda_1)^i |x_1|^2 + x_2^* p(A_1)^i x_2, i = 1, 2$. Since $\mu \neq \lambda_1$, we obtain that $x_2 \neq 0$ and hence $p(\mu)^i = \frac{x_2^*}{\|x_2\|} p(A_1)^i \frac{x_2}{\|x_2\|}, i = 1, 2$. Therefore, $p(\mu) \in V^2(p(A_1)) = \sigma(p(A_1)) = p(\sigma(A_1))$. Thus, there exists $\lambda_2 \in \sigma(A_1)$ such that $p(\mu) = p(\lambda_2)$. After $k+1$ steps we obtain that $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{k+1}) \oplus A_{k+1}$, where $\{\lambda_1, \ldots, \lambda_{k+1}\} \cap \sigma(A_{k+1}) = \emptyset$. Define $q(z) = p(z) - p(\mu)$. Then $q(\lambda_1) = \cdots = q(\lambda_{k+1}) = 0$. Therefore, the polynomial $q(z)$ of degree k has $k+1$ roots, a contradiction. $\square$

COROLLARY 4.2. *Suppose that $A \in M_n$ be a normal matrix such that A^k is Hermitian. Then $V^{2k}(A) = \sigma(A)$.*

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Hamid Reza Afshin
Department of Mathematics
Vali-E-Asr University of Rafsanjan
Rafsanjan
Iran
e-mail: afshin@mail.vru.ac.ir

Mohammad Ali Mehrjoofard
Department of Mathematics
Vali-E-Asr University of Rafsanjan
Rafsanjan
Iran
e-mail: aahaay@gmail.com

Abbas Salemi
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman
Iran
The SBUK Center of Excellence in
Linear Algebra and Optimization
Iran
e-mail: salemi@mail.uk.ac.ir