Operators and Matrices
Volume 5, Number 1 (2011), 107–118

WEIGHTED CONDITIONAL EXPECTATION OPERATORS

JOHN DAVID HERRON

(*Communicated by L. Rodman*)

Abstract. This paper presents the fundamental operator-theoretic properties of products of conditional expectation and multiplication operators. It is shown that boundedness of such a product need not depend on the boundedness of the multiplication operator. The spectrum is described, as is the unique polar decomposition. It is also shown that compactness implies the existence of an atom in the underlying σ-subalgebra. An algebra containing such operators is shown to be weakly closed and, when the underlying space is of finite measure, its commutant is an algebra of multiplication operators with suitably measurable symbol.

1. Introduction

In this paper we study the class of bounded linear operators on the L^p spaces having the form EM_ω, where E is a conditional expectation operator and M_ω is a (possibly unbounded) multiplication operator. What follows is a brief review of the operators E and M_ω, along with the notational conventions we will be using.

Let $(X,\mathscr{F},\mu)$ be a σ-finite measure space and let $\mathscr{A}$ be a σ-subalgebra of $\mathscr{F}$ such that $(X,\mathscr{A},\mu)$ is also σ-finite. The collection of (equivalence classes modulo sets of zero measure of) $\mathscr{F}$-measurable complex-valued functions on X will be denoted $L^0(\mathscr{F})$, with $L^0(\mathscr{A})$ being likewise defined for $\mathscr{A}$-measurable functions. Moreover, we let $L^p(\mathscr{F}) = L^p(X,\mathscr{F},\mu)$ and $L^p(\mathscr{A}) = L^p(X,\mathscr{A},\mu)$, for $1 \leqslant p \leqslant \infty$. We also adopt the convention that all equations and set-theoretic relationships are assumed to hold almost everywhere relative to μ.

A consequence of the Radon-Nikodym theorem is that to each nonnegative function $f \in L^0(\mathscr{F})$ there exists a unique nonnegative $Ef \in L^0(\mathscr{A})$ such that

$$\int_A f\,d\mu = \int_A Ef\,d\mu$$

for all $A \in \mathscr{A}$. The function Ef is called the *conditional expectation of f with respect to $\mathscr{A}$*. This can be extended to real-valued and complex-valued functions by examining the conditional expectations of the positive and negative parts (in the case of real-valued functions), and the real and imaginary parts (for complex-valued functions). If Ef

Mathematics subject classification (2010): Primary 47A15.
Keywords and phrases: Conditional expectation, multiplication operator.
This paper represents part of the author's doctoral dissertation written at the University of North Carolina at Charlotte under the direction of Professor Alan Lambert. The author would like to extend sincere gratitude to Professors A. Biswas, C. Burnap, A. Lambert, L. Leamy, J. Quinn and B. Weinstock.

Paper OaM-05-07

exists for a function $f \in L^0(\mathscr{F})$, then we say f is *conditionable*. One can show that every L^p function is conditionable; therefore, a linear transformation $E : L^p(\mathscr{F}) \longrightarrow L^p(\mathscr{A})$ can be defined by $f \mapsto Ef$. It is clear that E is an idempotent, and in the case of $p = 2$, it is the orthogonal projection of $L^2(\mathscr{F})$ onto $L^2(\mathscr{A})$. Those properties of E used in our discussion are summarized below. In all cases f and g are conditionable functions.

(1) Monotonicity: If f and g are real-valued with $f \leqslant g$, then $Ef \leqslant Eg$.

(2) If $a \in L^0(\mathscr{A})$, then $E(af) = aEf$.

(3) Conditional version of the Hölder inequality: If p and q are conjugate exponents and $f \in L^p(\mathscr{F})$ and $g \in L^q(\mathscr{F})$, then $E|fg| \leqslant (E|f|^p)^{1/p}(E|g|^q)^{1/q}$.

(4) If $p \geqslant 1$, $(E|f|)^p \leqslant E|f|^p$.

(5) Monotone Convergence: If $\{f_n\}$ is an increasing sequence of nonnegative $\mathscr{F}$-measurable functions, then $\lim_n Ef_n = E(\lim_n f_n)$.

Let $\omega \in L^0(\mathscr{F})$. The corresponding *multiplication operator* M_ω on $L^p(\mathscr{F})$ is defined by $f \mapsto \omega f$. It is well known that M_ω is bounded if and only if $\omega \in L^\infty(\mathscr{F})$, and in the case of boundedness, $\|M_\omega\| = \|\omega\|_\infty$.

Our interest in operators of the form EM_ω stems from the fact that such products (and their adjoints) tend to appear often in the study of those operators related to conditional expectation. This observation was made in [5] within the context of the development of Hilbert C^*-modules and L^2 multipliers. Multiplication-conditional expectation products appear in [2], where it is shown that every contractive projection on certain L^1 spaces can be decomposed into an operator of the form $M_\psi EM_\omega$ and a nilpotent operator. In [3] and [4], operators that are representable as products involving multiplications and conditional expectations are studied (in the langauge of [3] and [4], such operators are said to be *mce-representable*). In [1], the various classes of normality (e.g., normal, hyponormal, $p-$hyponormal) for operators on $L^2(\mathscr{F})$ are studied and multiplication-conditional expectation products are encountered there as well.

Their appearance in certain decompositions and representations, and their utility in studying conditional expectation-related operators seem to suggest that operators formed by conditional expectation-multiplication products warrant a closer study. Such a study is the aim of this paper.

In Theorems 2.1 and 2.2 we have the norm and spectrum. Theorems 3.1 and 3.2 describe the unique polar decompositions of EM_ω and its adjoint $M_{\overline{\omega}}E$. These decompositions will be found to include operators that are themselves conditional expectation-multiplication products. In Theorem 4.1 we have a link between an operator-theoretic property of EM_ω and the underlying structure of $\mathscr{A}$; specifically, if EM_ω is compact and $E|\omega|^2 > 0$, then the σ-subalgebra $\mathscr{A}$ is purely atomic.

The last section deals with the set $\mathscr{W}$ of all bounded operators of the form $EM_\omega + \lambda I$. Here, we show that $\mathscr{W}$ is a weakly closed operator algebra (Theorem 5.1). Moreover, in the case when $\mu X < \infty$, we show that the commutant of $\mathscr{W}$ is the abelian von Neumann algebra $\mathscr{L}^\infty(\mathscr{A}) = \{M_a : a \in L^\infty(\mathscr{A})\}$ (Theorem 5.2).

2. Weighted Conditional Expectation Operators

We now define the class of operator under investigation.

DEFINITION 2.1. Let $(X,\mathscr{F},\mu)$ be a σ-finite measure space and let $\mathscr{A}$ be a σ-subalgebra of $\mathscr{F}$ such that $(X,\mathscr{A},\mu)$ is σ-finite. Let E be the corresponding conditional expectation operator on $L^p(\mathscr{F})$, $1 \leqslant p \leqslant \infty$, relative to $\mathscr{A}$. If $\omega \in L^0(\mathscr{F})$ such that ωf is conditionable and $E(\omega f) \in L^p(\mathscr{A})$ for all $f \in L^p(\mathscr{F})$, then the corresponding *weighted conditional expectation operator* (or *WCE operator*) is the linear transformation $W_\omega : L^p(\mathscr{F}) \longrightarrow L^p(\mathscr{A})$ defined by $f \mapsto E(\omega f)$.

The function ω is called the *weight function* of W_ω and it is not assumed to be bounded (hence, M_ω need not be a bounded operator). In the event it is bounded, however, it is easy to show that W_ω is a bounded operator; perhaps not surprisingly, when $p=1$ the converse is true.

THEOREM 2.1. *Let $W_\omega : L^p(\mathscr{F}) \longrightarrow L^p(\mathscr{A})$ be a WCE operator. If $\omega \in L^\infty(\mathscr{F})$, then W_ω is bounded. If $p=1$, then the converse holds and $\|W_\omega\| = \|\omega\|_\infty$.*

Proof. Clearly, if M_ω is bounded, then EM_ω is bounded and $\|W_\omega\| \leqslant \|M_\omega\|$.
Suppose, then, that $f \in L^1(\mathscr{F})$ and W_ω is a bounded operator on $L^1(\mathscr{F})$. If we write $\omega f = u|\omega f|$, where $|u|=1$, we have

$$\int_X |\omega f|\, d\mu = \int_X E\,|\omega f|\, d\mu = \int_X W_\omega(\overline{u} f) d\mu \leqslant \|W_\omega\|\, \|f\|_1 .$$

From this we conclude that the multiplication operator M_ω on $L^1(\mathscr{F})$ is bounded and $\|\omega\|_\infty = \|M_\omega\| \leqslant \|W_\omega\|$. $\square$

For $p>1$ the situation is more subtle. Rather than being directly dependent upon the behavior of ω, boundedness of W_ω in this more general setting depends on the conditional expectation of the function $|\omega|^p$.

THEOREM 2.2. *Let $W_\omega : L^p(\mathscr{F}) \longrightarrow L^p(\mathscr{A})$ be a WCE operator.*

(1) Let $1<p<\infty$ and q be the conjugate exponent of p. Then, W_ω is bounded if and only if $E\,|\omega|^q \in L^\infty(\mathscr{A})$; if W_ω is bounded, then $\|W_\omega\| = \|E\,|\omega|^q\|_\infty^{1/q}$.

(2) If $p=\infty$, then W_ω is bounded if and only if $E\,|\omega| \in L^\infty(\mathscr{A})$; if W_ω is bounded, then $\|W_\omega\| = \|E\,|\omega|\|_\infty$.

(3) If $\mathscr{A} \neq \mathscr{F}$, then $\sigma(W_\omega) = \{0\} \cup \text{ess range}(E\omega)$.

Proof. (1) Here we prove the special case when $\mu X < \infty$. Suppose W_ω is a bounded operator on $L^p(\mathscr{F})$ and let $f \in L^p(\mathscr{F})$. For each $n \in \mathbb{N}$, define $F_n = \{x \in X : |\omega(x)| \leqslant n\}$. Then, each F_n is $\mathscr{F}$-measurable and $F_n \uparrow X$. Let $G_n = F_n \cap \text{support}\,|\omega|$ and let $A \in \mathscr{A}$. Define

$$f_n = \overline{\omega}\,|\omega|^{q-2}\,\chi_{G_n \cap A}$$

for each positive integer n. Note that $|f_n| \leqslant n^{q-1}$; therefore, $f_n \in L^\infty(\mathscr{F})$ for all n (which in our special case implies $f_n \in L^p(\mathscr{F})$). For each n,

$$\int_X |E(\omega f_n)|^p \, d\mu \leqslant \|W_\omega\|^p \int_X |f_n|^p \, d\mu$$

implies

$$\int_A [E(|\omega|^q \chi_{G_n})]^p d\mu \leqslant \|W_\omega\|^p \int_A E(|\omega|^q \chi_{G_n}) d\mu.$$

Since A is an arbitrary $\mathscr{A}$-measurable set and the integrands are $\mathscr{A}$-measurable functions, we have $[E(|\omega|^q \chi_{G_n})]^p \leqslant \|W_\omega\|^p E(|\omega|^q \chi_{G_n})$. That is,

$$[E(|\omega|^q \chi_{\mathrm{support}|\omega|} \cdot \chi_{F_n})]^p \leqslant \|W_\omega\|^p E(|\omega|^q \chi_{\mathrm{support}|\omega|} \cdot \chi_{F_n}).$$

This inequality in turn gives

$$[E(|\omega|^q \chi_{F_n})]^{p-1} \chi_{\mathrm{support}[E(|\omega|^q \chi_{F_n})]} \leqslant \|W_\omega\|^p$$

or simply

$$E(|\omega|^q \chi_{F_n}) \chi_{\mathrm{support}[E(|\omega|^q \chi_{F_n})]} \leqslant \|W_\omega\|^q.$$

Since $F_n \uparrow X$, the conditional expectation version of the monotone convergence theorem implies $E|\omega|^q \leqslant \|W_\omega\|^q$. In other words, $E|\omega|^q \in L^\infty(\mathscr{A})$ and

$$\|E|\omega|^q\|_\infty^{1/q} \leqslant \|W_\omega\|.$$

Suppose, now, that $E|\omega|^q \in L^\infty(\mathscr{A})$. Using the conditional form of Hölder's inequality we have

$$\begin{aligned} \|W_\omega f\|^p &\leqslant \int_X (E|\omega f|)^p d\mu \\ &\leqslant \int_X \left[(E|\omega|^q)^{1/q} (E|f|^p)^{1/p} \right]^p d\mu \\ &\leqslant \|E|\omega|^q\|_\infty^{p/q} \|f\|_p^p. \end{aligned}$$

Therefore, W_ω is bounded and $\|W_\omega\| \leqslant \|E|\omega|^q\|_\infty^{1/q}$.

As one might expect, extending this result to the case when $(X, \mathscr{F}, \mu)$ is σ-finite involves writing X as a disjoint sequence $\{A_n\}$ of $\mathscr{A}$-measurable sets of finite measure, and then carefully applying the finite-measure result to each $L^p(A_n)$. The details are not difficult but they are lengthly and, for the sake of brevity, are omitted.

(2) Without loss of generality we assume ω is not identically zero on X, otherwise the result holds trivially. Consider the case when W_ω is bounded. Let $A = \{x \in X : (E|\omega|)(x) > \|W_\omega\|\}$ and

$$g = \frac{\overline{\omega}}{|\omega|} \chi_{A \cap \mathrm{support}|\omega|}.$$

If $A \cap \text{support}|\omega|$ has positive measure (i.e., $\|g\|_\infty \neq 0$), then the inequality

$$E|\omega|\chi_A = |W_\omega g| \leqslant \|W_\omega\| \, \|g\|_\infty = \|W_\omega\|$$

produces a contradiction, since $E|\omega| > \|W_\omega\|$ on A. Therefore, the intersection of A and $\text{support}|\omega|$ has zero measure. As a consequence, $|\omega|\chi_A = 0$, and this implies $E|\omega|\chi_A = 0$. Therefore, $\mu A = 0$, and we have $\|E|\omega|\|_\infty \leqslant \|W_\omega\|$.

The converse follows from the fact that $|\omega f| \leqslant |\omega|\,\|f\|_\infty$ implies $|E(\omega f)| \leqslant \|E|\omega|\|_\infty \|f\|_\infty$.

(3) Note that W_ω cannot be surjective, since the range of W_ω is contained in $L^p(\mathscr{A})$. Consequently, $0 \in \sigma(W_\omega)$.

Suppose $\lambda \neq 0$. Define a linear transformation S by

$$Sf = E\left(\frac{\omega}{\lambda(E\omega - \lambda)} f\right) - \frac{f}{\lambda}$$

for any $f \in L^p(\mathscr{F})$. If $\lambda \notin \text{ess range}(E\omega)$, then the function $(E\omega - \lambda)^{-1}$ is bounded and one can show

$$\|Sf\|_p \leqslant \frac{1}{|\lambda|}\left(\left\|(E\omega - \lambda)^{-1}\right\|_\infty \|W_\omega\| + 1\right)\|f\|_p.$$

Conversely, suppose S is bounded. For any $a \in L^p(\mathscr{A})$, $Sa = a/(E\omega - \lambda)$. By assumption, $Sa \in L^p(\mathscr{A})$ for all a. Thus, the multiplication operator M_ψ, with $\psi = (E\omega - \lambda)^{-1}$, is bounded on $L^p(\mathscr{A})$. From this it is easy to see that λ cannot be in the essential range of $E\omega$.

Lastly, a calculation shows that $S(W_\omega - \lambda I) = (W_\omega - \lambda I)S = I$. Hence, $W_\omega - \lambda I$ has a bounded inverse if and only if $\lambda \notin \text{ess range}(E\omega)$ and $\lambda \neq 0$. $\square$

EXAMPLE 2.1. Consider W_ω on the Hilbert space $L^2(\mathscr{F})$. If $\mathscr{A} = \mathscr{F}$, then $W_\omega = M_\omega$ and the standard results for multiplication operators are recovered. At the other extreme, if $(X, \mathscr{F}, \mu)$ is a probability space, then $\mathscr{A} = \{\emptyset, X\}$ is a σ-subalgebra and $\mathscr{A}$-measurable functions are constant on X. In this setting it is not hard to show that W_ω will be bounded if and only if ω is an L^2 function. As a nontrivial example, one that is in some sense between these two extremes, consider $\{A_n\}_{n\in\mathbb{N}}$, a collection of disjoint sets of finite measure whose union is X. Let $\mathscr{A}$ be the σ-algebra generated by this partition. In this case, $\mathscr{A}$-measurable functions are those assuming constant values over each A_n and any WCE operator has the form

$$W_\omega f = \sum_{n=1}^{\infty} \frac{1}{\mu A_n} \int_{A_n} \omega f \, d\mu \cdot \chi_{A_n}$$

for $f \in L^2(\mathscr{F})$. It is clear from Theorem 2.2 (1) that W_ω is bounded if and only if the sequence

$$\beta_n = \frac{1}{\mu A_n} \int_{A_n} |\omega|^2 \, d\mu$$

is bounded. In general the boundedness of the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ does not require ω to be bounded. For instance, let $X=(0,\infty)$, take μ to be Lebesgue measure and let $\mathscr{F}$ be the Lebesgue subsets of X. Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_1=1$ and $a_n=a_{n-1}+n$ for $n>1$. For each n, define the interval $A_n=(a_n-n,a_n]$. Clearly, $\mu A_n=n$. Let $\mathscr{A}$ be the σ-algebra generated by the partition $\{A_n\}_{n\in\mathbb{N}}$. For each n, define a function φ_n on A_n as follows:

$$\varphi_n(x)=\begin{cases} n^2x+n(1-na_n) & \text{if } a_n-1/n<x\leqslant a_n \\ 0 & \text{if } a_n-n<x\leqslant a_n-1/n \end{cases}.$$

Define the weight function ω by

$$\omega(x)=\sum_{n=1}^{\infty}\varphi_n^{1/2}(x)\chi_{A_n}(x).$$

Note that ω is unbounded, since $\varphi_n(a_n)=n$ for each n. If W_ω is the corresponding WCE operator, then boundedness of W_ω depends on the boundedness of the sequence $\{\beta_n\}$ given above. Here, $\beta_n=1/(2n)$, and so W_ω is a bounded operator on $L^2(0,\infty)$.

REMARK 2.1. The proof of Theorem 2.2 (3) provides a formula for the inverse of $W_\omega-\lambda I$ when such an inverse exists. In particular, it was found that the action of $(W_\omega-\lambda I)^{-1}$ could be described by the following equation:

$$(W_\omega-\lambda I)^{-1}f=E\left(\frac{\omega}{\lambda(E\omega-\lambda)}\cdot f\right)-\frac{f}{\lambda}.$$

In other words, $(W_\omega-\lambda I)^{-1}=W_\psi-\gamma I$, where $\psi=\omega[\lambda(E\omega-\lambda)]^{-1}$ and $\gamma=\lambda^{-1}$. So, the collection $\mathscr{W}$ of all bounded operators of the form $W_\omega+\lambda I$ is closed under the formation of inverses. We shall return to the set $\mathscr{W}$ toward the end of the paper.

3. The Polar Decomposition

Recall that any bounded operator T on a Hilbert space can be expressed in terms of its *polar decomposition*: $T=VP$, where V is a partial isometry and P is a positive operator. Moreover, this representation is unique provided $\ker P=\ker V=\ker T$. In this section we show that the unique polar decompositions of W_ω and W_ω^* involve other WCE operators and their adjoints. Before this, however, we need the following two lemmas.

LEMMA 3.1. *Suppose $a\in L^\infty(\mathscr{A})$ such that $aW_\omega f=0$ for all $f\in L^2(\mathscr{F})$. Then, $a=0$ on the support of $E|\omega|^2$.*

Proof. If $aW_\omega f=0$ for all $f\in L^2(\mathscr{F})$, then $W_{a\omega}$ is the zero operator on $L^2(F)$. Therefore,

$$0=\|W_{a\omega}\|=\left\||a|^2E|\omega|^2\right\|_\infty^{1/2}$$

which implies $a=0$ on support$(E|\omega|^2)$. $\square$

LEMMA 3.2. *Let W_ω be a bounded WCE operator $L^2(\mathscr{F})$. Then, W_ω is a partial isometry if and only if $E|\omega|^2 = \chi_A$ for some $A \in \mathscr{A}$.*

Proof. Suppose W_ω is a partial isometry. Then, $W_\omega W_\omega^* W_\omega = W_\omega$. That is, for any $f \in L^2(\mathscr{F})$, $E|\omega|^2 E(\omega f) = E(\omega f)$. Therefore,

$$(E|\omega|^2 - 1)E(\omega f) = 0$$

for all $f \in L^2(\mathscr{F})$. By Lemma 3.1, $E|\omega|^2 - 1 = 0$ on support$(E|\omega|^2)$; in other words, $E|\omega|^2 = \chi_A$, where $A = \text{support}(E|\omega|^2)$.

Conversely, if $E|\omega|^2 = \chi_A$ for some $A \in \mathscr{A}$, then $A = \text{support}(E|\omega|^2)$ and, for any $f \in L^2(\mathscr{F})$,

$$W_\omega W_\omega^* W_\omega f = E|\omega|^2 E(\omega f) = \chi_{\text{support}(E|\omega|^2)} E(\omega f) = E(\omega f) = W_\omega f,$$

where we have made use of the inequality $|E(\omega f)|^2 \leqslant E|\omega|^2 E|f|^2$. □

The very specific nature of partial isometries of the form W_ω seems to suggest that their applicability beyond the study of WCE operators might be limited. However, in [2], contractive projections on $L^1(\mathscr{F})$ are shown to decompose into an operator involving a WCE operator and a nilpotent operator. The weight functions in [2] are defined to be exactly those functions ω such that $E\omega = \chi_A$, where A is an element of the underlying σ-subalgebra.

For any W_ω on $L^2(\mathscr{F})$ define $P_\omega = W_\omega^* W_\omega$ and $Q_\omega = W_\omega W_\omega^*$. In terms of the conditional expectation and multiplications operators, we have $P_\omega = M_{\overline{\omega}} E M_\omega$ and $Q_\omega = M_{E|\omega|^2} E$.

THEOREM 3.1. *The unique polar decomposition of W_ω is $W_\sigma P_\alpha$, where*

$$\sigma = \frac{\omega}{\left(E|\omega|^2\right)^{1/2}} \chi_S \quad \textit{and} \quad \alpha = \frac{\omega}{\left(E|\omega|^2\right)^{1/4}} \chi_S$$

and $S = \text{support}(E|\omega|^2)$.

Proof. A calculation shows that $(W_\omega^* W_\omega)^{1/2} = P_\omega^{1/2} = P_\alpha$. Also,

$$|\sigma|^2 = \frac{|\omega|^2}{E|\omega|^2} \chi_S$$

implies $E|\sigma|^2 = \chi_S$. By Lemma 3.2, W_σ is a partial isometry. Another direct calculation shows that $W_\omega = W_\sigma P_\alpha$ and all that remains is uniqueness.

Let $f \in \ker W_\sigma$. Then, $E(\sigma f) = 0$ implies

$$\frac{\chi_S}{\left(E|\omega|^2\right)^{1/2}} E(\omega f) = 0.$$

From this we have $E(\omega f) = 0$; that is, $f \in \ker W_\omega$. On the other hand, if $E(\omega f) = 0$, then multiplication by $\chi_S(E|\omega|^2)^{-1/2}$ does not change this. Hence, $\ker W_\omega = \ker W_\sigma$. Additionally, $\ker W_\omega = \ker P_\alpha$, since $P_\alpha = (W_\omega^* W_\omega)^{1/2}$. $\square$

An argument similar to that found in the proof of Theorem 3.1 underlies the demonstration of the polar decomposition of W_ω^*.

THEOREM 3.2. *The unique polar decomposition of* W_ω^* *is* $W_\sigma^* Q_\beta$, *where*

$$\sigma = \frac{\omega}{\left(E|\omega|^2\right)^{1/2}} \chi_S \quad \textit{and} \quad \beta = \left(E|\omega|^2\right)^{1/4}$$

and $S = \text{support}(E|\omega|^2)$.

4. Compact WCE Operators

An $\mathscr{F}$-measurable set G is said to be an *atom* if, for every measurable subset $F \subseteq G$, either $\mu F = 0$ or $\mu F = \mu G$. A σ-algebra is said to be *purely atomic* if it is generated by a set of atoms. The following lemma regarding atoms and the dimensionality of $L^2(\mathscr{F})$ is probably not new; however, we state a proof here for convenience.

LEMMA 4.1. *Suppose* $(X, \mathscr{F}, \mu)$ *is a* σ*-finite measure space such that* $L^2(X, \mathscr{F}, \mu)$ *is finite-dimensional. Then,* X *is a finite union of atoms.*

Proof. Let $d = \dim L^2(\mathscr{F}) < \infty$ and let $\{G_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of sets of finite measure with

$$X = \bigcup_{n=1}^{\infty} G_n.$$

For each n define

$$\varphi_n = \frac{1}{\sqrt{\mu G_n}} \chi_{G_n}.$$

It is easy to check that $\{\varphi_n\}$ is an orthonormal set in $L^2(\mathscr{F})$. As such, the set $\{\varphi_n\}$ can contain no more than d vectors. Since there is a clear one-to-one correspondence between $\{G_n\}$ and $\{\varphi_n\}$, it must be that there is only a finite number of distinct sets G_n, $\{G_1, \dots, G_N\}$, where $N \leqslant d$. Thus,

$$X = \bigcup_{n=1}^{N} G_n.$$

If every G_n is an atom, we are done. So, without loss of generality assume G_1 is not an atom. Let $F_1 \subset G_1$ with $0 < \mu F_1 < \mu G_1$. The collection

$$\{F_1, G_1 - F_1, G_2, \dots, G_N\}$$

is a collection of disjoint sets of finite measure whose union is X. As before, consider the functions

$$\varphi_1^{(1)} = \frac{1}{\sqrt{\mu F_1}} \chi_{F_1}$$

$$\varphi_2^{(1)} = \frac{1}{\sqrt{\mu(G_1 - F_1)}} \chi_{G_1 - F_1}$$

and $\varphi_{n+1}^{(1)} = \varphi_n$ for $2 \leqslant n \leqslant N$. The set $\left\{\varphi_n^{(1)}\right\}$ is an orthonormal set consisting of $N+1$ vectors and, consequently, $N+1 \leqslant d$. If both G_1 and F_1 are atoms, the theorem is proved. If not, one may assume F_1 is not an atom and from this develop a collection of $N+2$ disjoint sets having finite measure whose union is X. Again, one concludes $N+2 \leqslant d$. Clearly, this process cannot continue indefinitely. In fact, there cannot exist a sequence $\{F_k\}$ of measurable subsets of G_1 with the property $F_k \subset F_{k-1}$ and $0 < \mu F_k < \mu F_{k-1}$ for $k > d - N$; otherwise, the set of vectors

$$\varphi_1^{(k)} = \frac{1}{\sqrt{\mu F_k}} \chi_{F_k}$$

$$\varphi_2^{(k)} = \frac{1}{\sqrt{\mu(F_{k-1} - F_k)}} \chi_{F_{k-1} - F_k}$$

$$\vdots$$

$$\varphi_{k+1}^{(k)} = \frac{1}{\sqrt{\mu(G_1 - F_1)}} \chi_{G_1 - F_1}$$

and $\varphi_{n+k}^{(k)} = \varphi_n$ for $2 \leqslant n \leqslant N$, is an orthonormal set consisting of $N+k > d$ elements. Hence, X must be a finite union of atoms. □

THEOREM 4.1. *Let $W_\omega : L^2(\mathscr{F}) \longrightarrow L^2(\mathscr{A})$ be a bounded WCE operator such that $E|\omega|^2 > 0$. If W_ω is compact, then $\mathscr{A}$ is purely atomic.*

Proof. If W_ω is compact, then so is $Q_\omega = W_\omega W_\omega^*$. Note that $L^2(X, \mathscr{A})$ is an invariant subspace for Q_ω and

$$Q_\omega \Big|_{L^2(\mathscr{A})} = M_{E|\omega|^2}.$$

Therefore, the essential range of $E|\omega|^2$ is finite or it consists of a countable number of scalars whose limit is zero. Suppose ess range$(E|\omega|^2) = \{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim_n \alpha_n = 0$. For each n define

$$A_n = \left\{x \in X : E|\omega|^2(x) = \alpha_n\right\}.$$

Each A_n is $\mathscr{A}$-measurable, $A_m \cap A_n = \emptyset$ whenever $m \neq n$, and $X = \bigcup_{n=1}^\infty A_n$.

One can show that for each n,

$$\ker\left(M_{E|\omega|^2} - \alpha_n\right) = \chi_{A_n} L^2(X, \mathscr{A}),$$

where $\chi_{A_n}L^2(X,\mathscr{A})$ is the subspace of all functions on X of the form $a\chi_{A_n}$ with $a \in L^2(X,\mathscr{A})$. For each n identify $L^2(A_n,\mathscr{A})$ and $\chi_{A_n}L^2(X,\mathscr{A})$. Since $M_{E|\omega|^2}$ is compact, the eigenspaces $L^2(A_n,\mathscr{A})$ are finite-dimensional. By Lemma 4.1, for each n we have

$$A_n = \bigcup_{m=1}^{M} A_{mn},$$

where each set A_{mn} is an atom. Since for any $A \in \mathscr{A}$ we may write

$$A = \bigcup_{n=1}^{\infty} (A \cap A_n)$$

and each A_n is a finite union of $\mathscr{A}$-measurable atoms, it follows that the σ-algebra $\mathscr{A}$ is generated by a set of atoms.

If ess range$(E|\omega|^2) = \{\alpha_1,\dots,\alpha_N\}$, then the same argument holds, only with the countable collection of sets $\{A_n\}_{n\in\mathbb{N}}$ replaced by a finite collection. $\square$

If the requirement that $E|\omega|^2$ be strictly positive is dropped, then we simply take a nonzero element from the essential range, say α_N, and observe that the corresponding A_N is a finite union of atoms by virtue of the same reasoning as above. In this way, we have the following corollary.

COROLLARY 4.1. *If W_ω is compact and $E|\omega|^2 > 0$ on a set of positive measure, then $\mathscr{A}$ contains an atom.*

5. The Algebra of WCE Operators

Let Ω be the set of all $\mathscr{F}$-measurable functions ω such that $E|\omega|^2$ is bounded. Since $W_\psi + W_\omega = W_{\psi+\omega}$ and $\lambda W_\omega = W_{\lambda\omega}$, it follows that Ω is closed under addition and scalar multiplication. Moreover, if $a \in L^\infty(\mathscr{A})$, then $E|a\omega|^2 = |a|^2 E|\omega|^2$ implies $a\omega \in \Omega$ for all $\omega \in \Omega$. It is also true that although $\omega \in \Omega$ itself need not be bounded, its conditional expectation $E\omega$ is always bounded. This follows from the fact that $|E\omega|^2 \leqslant E|\omega|^2$. These simple observations regarding Ω will prove useful in the study of the algebra of WCE operators.

Let

$$\mathscr{W} = \{W_\omega + \lambda I : \omega \in \Omega \text{ and } \lambda \in \mathbb{C}\}.$$

Note that $\mathscr{W}$ is closed under addition and scalar multiplication. As stated in Remark 2.1, $\mathscr{W}$ is also *invertibly closed*; that is, if $W_\omega + \lambda I$ is invertible, then $(W_\omega + \lambda I)^{-1} \in \mathscr{W}$. Suppose $W_\psi, W_\omega \in \mathscr{W}$. Then, $W_\psi W_\omega = W_{\omega E\psi}$; that is, the product of two WCE operators is again a WCE operator. More generally, for $\gamma, \lambda \in \mathbb{C}$,

$$(W_\psi + \gamma I)(W_\omega + \lambda I) = W_\pi + \alpha I,$$

where $\pi = \gamma\omega + \lambda\psi + \omega E\psi$ and $\alpha = \gamma\lambda$. Therefore, $\mathscr{W}$ is closed under products. These observations, together with the fact that $0 \in \Omega$, imply that $\mathscr{W}$ is a unital operator algebra.

THEOREM 5.1. *The algebra $\mathscr{W}$ is weakly-closed.*

Proof. Let $L_{\overline{\omega}} = W_\omega^*$. We shall call $L_{\overline{\omega}}$ a *left-WCE operator*, since $L_\omega = M_{\overline{\omega}}E$. Let $\{L_{\omega_n}\}$ be a sequence of left-WCE operators such that L_{ω_n} converges weakly to some bounded operator T on $L^2(\mathscr{F})$. Let $a,b \in L^2(\mathscr{A}) \cap L^\infty(\mathscr{A})$. Then,

$$M_a L_{\omega_n} \xrightarrow{\text{weakly}} M_a T,$$

that is, for any $f \in L^2(\mathscr{F})$,

$$\langle a\omega_n Eb, f\rangle \longrightarrow \langle aTb, f\rangle$$

as $n \longrightarrow \infty$. We also have

$$\langle a\omega_n Eb, f\rangle = \langle b\omega_n Ea, f\rangle \longrightarrow \langle bTa, f\rangle.$$

That is, $aTb = bTa$ for all $a,b \in L^2(\mathscr{A}) \cap L^\infty(\mathscr{A})$.

In particular, let $\alpha \in L^2(\mathscr{A}) \cap L^\infty(\mathscr{F})$ such that $\alpha > 0$. Then,

$$Tb = \frac{T\alpha}{\alpha} b.$$

Let $\omega = \alpha^{-1}T\alpha$. Then, $T = L_\omega$ on $L^2(\mathscr{A}) \cap L^\infty(\mathscr{A})$. Since $L^\infty(\mathscr{A}) \cap L^2(\mathscr{A})$ is dense in $L^2(\mathscr{A})$ and T is bounded, $T = L_\omega$ on all of $L^2(\mathscr{A})$.

Now, for any $f,g \in L^2(\mathscr{F})$,

$$\langle \omega_n Ef, g\rangle \longrightarrow \langle Tf, g\rangle$$

and

$$\langle \omega_n E(Ef), g\rangle \longrightarrow \langle TEf, g\rangle.$$

This implies $Tf = TEf$. Therefore, $Tf = TEf = \omega Ef$. Thus, the weak-limit of left-WCE operators is again a left-WCE operator. □

We denote by $\mathscr{L}^\infty(\mathscr{A})$ the algebra of all bounded multiplication operators with $\mathscr{A}$-measurable symbol; that is, $\mathscr{L}^\infty(\mathscr{A}) = \{M_a : a \in L^\infty(\mathscr{A})\}$.

THEOREM 5.2. *If $\mu X < \infty$, then $\mathscr{W}' = \mathscr{L}^\infty(\mathscr{A})$.*

Proof. It is clear that $\mathscr{L}^\infty(\mathscr{A}) \subseteq \mathscr{W}'$. To show the reverse direction, note that when $\mu X < \infty$, $1 \in L^2(\mathscr{F})$.

Let $B \in \mathscr{W}'$ and set $B^*(1) = b$. Since $E = W_1 \in \mathscr{W}$, we have $BE = EB$ (and $B^*E = EB^*$). Therefore,

$$b = B^*(1) = B^*E(1) = EB^*1 = Eb,$$

which implies b is $\mathscr{A}$-measurable. Recall $L_{\overline{\omega}} = W_\omega^*$, where $L_{\overline{\omega}}f = \overline{\omega}Ef$ for any $f \in L^2(\mathscr{F})$. Since B^* commutes with W_ω^*, we have

$$\overline{\omega}E(B^*f) = B^*(\overline{\omega}Ef).$$

Replacing ω with $\overline{\omega}$ and f with 1 gives

$$\omega EB^*(1) = B^*(\omega E(1)).$$

That is, $\omega b = B^*\omega$. Therefore, $B^* = M_b$ on Ω. Note that $L^\infty(\mathscr{F}) \subseteq \Omega$ and $L^\infty(\mathscr{F})$ is dense in $L^2(\mathscr{F})$. Thus, $B^* = M_b$ on a dense set. Since M_b is closed, $B^* = M_b$ on all of $L^2(\mathscr{F})$ and M_b is bounded. Thus, $B = M_a$, where $a = \overline{b}$. $\square$

REFERENCES

[1] C. BURNAP, I.B. JUNG, AND A. LAMBERT, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**, 2 (2005), 381–397.

[2] R. DOUGLAS, *Contractive projections on an L^1 space*, Pacific J. Math., **15**, 2 (1965), 443–462.

[3] J. GROBLER; B. DE PAGTER, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory, **48** (2002), 15–40.

[4] J. GROBLER; D. RAMBANE, *Operators represented by conditional expectations and random measures*, Positivity, **9** (2005), 369–383.

[5] A. LAMBERT, *A Hilbert C^*-module view of some spaces related to probabilistic conditional expectation*, Quaestiones Mathematicae, **22** (1999), 165–170.

(Received January 14, 2010)

John David Herron
224 Harman Hall
Department of Biology, Chemistry and Mathematics
University of Montevallo
Montevallo, Alabama 35115-6000
e-mail: herronjd@montevallo.edu